L^p POLYHARMONIC DIRICHLET PROBLEMS IN REGULAR DOMAINS I: THE UNIT DISC

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Abstract. In this article, we consider a class of Dirichlet problems with L^p boundary data for polyharmonic functions in the unit disc. Using higher order Poisson kernels due to Begehr et al ([3,9]), we give the integral representation solutions of the problems.

1. Introduction

There has been a great amount of research on various boundary value problems (simplified as BVPs) for polyanalytic, polyharmonic and metaanalytic functions etc.. Those include Riemann, Hilbert, Dirichlet, Neumann, Schwarz and Robin problems [1–9, 11]. One of the main purposes is to obtain integral representation solutions of BVPs in various settings such as Hölder continuous, continuous and Sobolev boundary data and so on. All of those are to generalize the classical integral representations for analytic and harmonic functions. Of which Dirichlet problems for polyharmonic functions attract considerable interest.

The purpose of this article is to solve the following polyharmonic Dirichlet problems (or simply, PHD problems) with L^p boundary data in the unit disc, i.e.,

$$\begin{cases} \Delta^n u = 0 \text{ in } \mathbb{D}, \\ \Delta^j u = f_j \text{ on } \mathbb{T}, \end{cases} \tag{1.1}$$

where $\mathbb{D}$ is the unit disc, $\mathbb{T}$ is the unit circle, $f_j \in L^p(\mathbb{T})$, $n \in \mathbb{N}$, $0 \leq j < n$, and $p \geq 1$. Sometimes this class of problems are also called Riquier problems (see [2, 14, 15]). In [3], Begehr, Du and Wang consider the PHD problems with Hölder continuous boundary data, i.e., all f_j are Hölder continuous on $\mathbb{T}$. The main technique of the proof of [3] is to transfer the PHD problems to the classical Riemann problems for analytic functions on the unit circle. That is done by means of certain decompositions of polyanalytic and polyharmonic functions by the authors, as well as Schwarz reflection principle. By introducing a class of kernel functions, integral representation solutions of the problems are given. Those kernel functions are higher order analogs of the Poisson kernel for the unit disc (So we call them higher order Poisson kernels). Explicit formulas for the kernels are not obtained due to the complexity until in [4, 8, 9] where by establishing a new decomposition theorem for polyharmonic functions, Du et al not only give explicit representations of the higher order Poisson kernels in terms of certain vertical sums, but also extend the integral representation results in [3] to the continuous boundary data setting. Even

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so, there the explicit expressions are complicate and difficult to understand. In the present paper, we will reformulate the explicit expressions of the higher order Poisson kernels in an easily accessible way, and further extend the results of [4,8,9] to the L^p boundary data setting.

2. Higher order Poisson kernels

Definition 2.1. Let D be a simply connected (bounded or unbounded) domain in the plane with smooth boundary ∂D, and $H(D)$ denote the set of analytic functions in D. If f is a continuous function defined on $D \times \partial D$ with $f(\cdot, t) \in H(D)$ for any fixed $t \in \partial D$, and $f(z, \cdot) \in C(\partial D)$ for any fixed $z \in D$, then f is said to be in $H \times C$ on $D \times \partial D$, and written as $f \in (H \times C)(D \times \partial D)$. Likewise, $(C^\infty \times C)(D \times \partial D)$, $(H \times L^p)(D \times \partial D)$, etc., may be similarly defined.

Definition 2.2. A sequence $\{ g_n(z,t) \}_{n=1}^\infty$ of real-valued functions of two variables on $\mathbb{D} \times \mathbb{T}$ is called a sequence of higher order Poisson kernels, or more precisely, $g_n(z,t)$ is an nth order Poisson kernel, if it satisfies the following conditions.
1. For any positive $n \in \mathbb{N}$, $g_n \in (C^\infty \times C)(\mathbb{D} \times \mathbb{T})$; the non-tangential boundary value

$$\lim_{\substack{z \to s \\ z \in \mathbb{D}, s \in \mathbb{T}}} g_n(z,t) = g_n(s,t)$$

exists for all t with $t \neq s$, where s is any fixed point on $\mathbb{T}$. Moreover, $g_n(\cdot, s)$ can be continuously extended to $\overline{\mathbb{D}} \setminus \{s\}$ for all $s \in \mathbb{T}$;
2. $g_1(z,t) = \frac{1}{1-z\bar{t}} + \frac{1}{1-\bar{z}t} - 1$ and $(\partial_z \partial_{\bar{z}}) g_n(z,t) = g_{n-1}(z,t)$ for $n > 1$;
3. $\lim_{z \to s,\, z\in\mathbb{D},\, s\in\mathbb{T}} \frac{1}{2\pi i} \int_{\mathbb{T}} g_1(z,t)\gamma(t)\frac{dt}{t} = \gamma(s)$ a.e. for any $\gamma \in L^p(\mathbb{T})$, $p \geq 1$;
4. $\lim_{z \to s,\, z\in\mathbb{D},\, s\in\mathbb{T}} \frac{1}{2\pi i} \int_{\mathbb{T}} g_2(z,t)\gamma(t)\frac{dt}{t} = 0$ for any $\gamma \in L^p(\mathbb{T})$, $p \geq 1$; and
5. $\lim_{z \to s,\, z\in\mathbb{D},\, s\in\mathbb{T}} g_n(z,t) = 0$ uniformly in $t \in \mathbb{T}$ for any fixed $s \in \mathbb{T}$, $n \geq 3$,
where all the limits are non-tangential [18].

Higher order Poisson kernels are the key in our approach to solve the PHD problems (1.1). In what follows, we show that a sequence of the higher order Poisson kernels defined in Definition 2.2 always exist in terms of an axiomatic way. To do so, firstly, we establish some lemmas as follows.

Lemma 2.3. *Let D be a simply connected bounded domain in the plane with smooth boundary ∂D. If $f \in (H \times C)(D \times \partial D)$. For any fixed $z_0 \in D$, the primitive function*

$$F(z,t) = \int_{z_0}^{z} f(\zeta, t) d\zeta \tag{2.1}$$

is also $H \times C$ on $D \times \partial D$.

Proof. Since $f \in (H \times C)(D \times \partial D)$, it is trivial to show the analyticity of $F(z,t)$ with respect to z in D for each fixed $t \in \partial D$ [17]. Clearly,

$$
\begin{aligned}
|F(z',t') - F(z,t)| &= \left| \int_{z_0}^{z'} f(\zeta,t')d\zeta - \int_{z_0}^{z} f(\zeta,t)d\zeta \right| \\
&\leq \int_{\gamma[z_0,z]} |f(\zeta,t') - f(\zeta,t)||d\zeta| \\
&\quad + \int_{\gamma[z,z']} |f(\zeta,t') - f(z,t)||d\zeta| \\
&\quad + \int_{\gamma[z,z']} |f(\zeta,t) - f(z,t)||d\zeta|,
\end{aligned} \tag{2.2}
$$

where $\gamma[z_0,z]$ is any simple curve from z_0 to z in D. Fix any $z \in D$, denote $d_z = \mathrm{dist}\{z, \partial D\}$ and $D_z = \{\zeta \in D : d_\zeta \geq d_z/2\}$. Since $f \in C(D \times \partial D)$, f is uniformly continuous on $D_z \times \partial D$ by the compactness of D_z and ∂D. Therefore, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$
|f(\zeta',t') - f(\zeta,t)| < \epsilon \tag{2.3}
$$

whenever $|\zeta' - \zeta| < \delta$ and $|t' - t| < \delta$ for any $\zeta', \zeta \in D_z$ and $t', t \in \partial D$. Thus, from (2.2), F is continuous at any $(z,t) \in D \times \partial D$. Then $F \in C(D \times \partial D)$. □

Lemma 2.4. *Let D be a simply connected bounded domain in the plane with smooth boundary ∂D. If $f \in (H \times C)(D \times \partial D)$ and the non-tangential boundary value*

$$
\lim_{\substack{z \to s \\ z \in D, s \in \partial D}} f(z,t) = f(s,t) \tag{2.4}
$$

exists on ∂D except $s \in \partial D$, and $f(s,\cdot) \in L^p(\partial D)$ for any fixed $s \in \partial D$. For any fixed $t \in \partial D$, $f(\cdot,t)$ can be continuously extended to $\overline{D} \setminus \{t\}$. Moreover,

$$
\lim_{\substack{z \to s \\ z \in \overline{D}, s \in \partial D}} |f(z,s)| = +\infty \quad \text{and} \quad \lim_{\substack{z \to s \\ z \in \overline{D}, s \in \partial D}} |(z-s)^\epsilon f(z,s)| = 0 \tag{2.5}
$$

for any $s \in \partial D$ and any $\epsilon > 0$, where the main analytic branch of $(z-s)^\epsilon$ is always chosen in the complex plane cut along a straight line from s to infinity in $\mathbb{C} \setminus \overline{D}$ and $1^\epsilon = 1$. For any fixed $z_0 \in D$, set the primitive function

$$
F(z,t) = \int_{z_0}^{z} f(\zeta,t)d\zeta, \ z \in D, \ t \in \partial D,
$$

then F enjoys the same properties as f does.

Proof. By the assumption, $f(\cdot,t)$ can be continuously extended to $\overline{D} \setminus \{t\}$ for any fixed $t \in \partial D$. Therefore, for any $z_0 \in D$ and fixed $t \in \partial D$, we can define

$$
F(s,t) = \int_{z_0}^{s} f(\zeta,t)d\zeta \text{ whenever } s \neq t. \tag{2.6}
$$

Let $n_s = (x_s, y_s)$ be the unit inner normal vector at s of ∂D and $N_s = x_s + iy_s$. For any $0 < \alpha < \pi/2$, denote

$$
\vee_s^\alpha(D) = \{z \in D : \arccos \frac{|\mathrm{Re}\{(z-s)\overline{N}_s\}|}{|z-s|} < \alpha\} \tag{2.7}
$$

as a pseudo-cone with vertex s and opening angle 2α in D. Due to the fact that $f(z,t)$ is continuous on $\vee_s^\alpha(D)\cup\{s\}, s\neq t$, it is clear that

$$|F(z,t)-F(s,t)|\leq\int_{\gamma[z,s]}|f(\zeta,t)||d\zeta|\leq l\{\gamma[z,s]\}\max_{\zeta\in\gamma[z_0,s]}|f(\zeta,t)|, \tag{2.8}$$

where $z\in\vee_s^\alpha(D)$ and $\gamma[z,s]\subset\gamma[z_0,s]\subset\overline{D}$ for any $z_0\in D$ and $0<\alpha<\pi/2$. Thus $F(z,t)$ has the non-tangential boundary value $F(s,t)$ given by (2.6).

By the first limit of (2.5), it is easy to get that

$$\lim_{\substack{z\to s\\ z\in\overline{D},s\in\partial D}}|F(z,s)|=+\infty.$$

For any fixed $z_0\in D$, define

$$L(z,s)=|z-s|^\epsilon\max_{\zeta\in\gamma[z_0,z]}|f(\zeta,s)|, \tag{2.9}$$

where $z\in\overline{D}\setminus\{s\}$ and $\epsilon>0$. Since by (2.5)

$$\begin{aligned} L(z,s)=|z-s|^\epsilon|f(z^*,s)| &= \left|\frac{z-s}{z^*-s}\right|^\epsilon|(z^*-s)^\epsilon f(z^*,s)| \\ &\leq |z^*-s|^\epsilon|f(z^*,s)| \end{aligned} \tag{2.10}$$

in which $z^*=z^*(z)\in\gamma[z_0,z]$ as z is sufficiently close to s. Note that in this case z^* is also sufficiently close to s. If not, $z^*(z)\to z_1\in\vee_s^\alpha(D)$ as $z\to s$. However, by the continuity of $f(\cdot,s)$, $f(z^*,s)\to f(z_1,s)$ as $z^*\to z_1$. This observation leads to the following contradiction to the first limit of (2.5),

$$|f(z,s)|\leq\max_{\zeta\in\gamma[z_0,z]}|f(\zeta,s)|=|f(z^*(z),s)|\leq|f(z_1,s)|+1<+\infty$$

for any z sufficiently closing to s. Therefore,

$$\lim_{\substack{z\to s\\ z\in\overline{D},s\in\partial D}}L(z,s)=0. \tag{2.11}$$

Note that

$$\begin{aligned} |(z-s)^\epsilon F(z,s)| &= \left|(z-s)^\epsilon\int_{z_0}^{z}f(\zeta,s)d\zeta\right| \\ &\leq |z-s|^\epsilon\int_{\gamma[z_0,z]}|f(\zeta,s)||d\zeta| \\ &\leq l\{\gamma[z_0,z]\}|z-s|^\epsilon\max_{\zeta\in\gamma[z_0,z]}|f(\zeta,s)| \\ &\leq l\{\gamma[z_0,s]\}L(z,s). \end{aligned} \tag{2.12}$$

It immediately follows that

$$\lim_{\substack{z\to s\\ z\in\overline{D},s\in\partial D}}|(z-s)^\epsilon F(z,s)|=0. \tag{2.13}$$

For any $s\in\partial D$, $0<\alpha<\pi/2$ and $\epsilon>0$, define

$$U_{s,\alpha,\epsilon}(z,t)=\begin{cases}(z-s)^\epsilon f(z,t), & t\neq s,\\ 0, & t=s,\end{cases} \tag{2.14}$$

where $z \in \overline{D} \setminus \{s\}$ and $t \in \partial D$. Note that $f \in C(D \times \partial D)$ and $f(\cdot, s) \in C(\overline{D} \setminus \{s\})$ for any fixed $s \in \partial D$, then from (2.5), $U_{s,\alpha,\epsilon} \in C(\overline{D} \times \partial D)$. Thus, for $p \geq 1$, take $0 < \epsilon < 1/2p$, there exists $\varrho > 0$ such that

$$|(z-s)^{\epsilon} f(z,t)| \leq 1 \tag{2.15}$$

for any $z \in \overline{D}$ and $t \in \partial D$ satisfying $|z-s| \leq \varrho$ and $|t-s| \leq \varrho$ respectively. Therefore, by Minkowski's inequality for integrals [12, 16],

$$\begin{aligned} ||F(s,\cdot)||_p &= \left(\int_{\partial D} |F(s,t)|^p |dt| \right)^{1/p} \\ &\leq \int_{\gamma_{[z_0,s]}} \left| \int_{\mathcal{C}(\partial D)_\varrho \cup (\partial D)_\varrho} |f(\zeta,t)|^p \, |dt| \right|^{1/p} |d\zeta| \\ &\leq \int_{\gamma_{[z_0,s]}} \left| \int_{\mathcal{C}(\partial D)_\varrho} |f(\zeta,t)|^p \, |dt| \right|^{1/p} |d\zeta| \\ &\quad + \int_{\gamma_{[z_0,z_1]}} \left| \int_{(\partial D)_\varrho} |f(\zeta,t)|^p \, |dt| \right|^{1/p} |d\zeta| \\ &\quad + \int_{\gamma_{[z_1,s]}} \left| \int_{(\partial D)_\varrho} \frac{1}{|\zeta - s|^{\epsilon p}} |dt| \right|^{1/p} |d\zeta| \\ &< +\infty, \end{aligned} \tag{2.16}$$

where $\gamma_{[z_0,s]}$ is any simple curve from z_0 to s in D, $(\partial D)_\varrho = \{t \in \partial D : |t-s| \leq \varrho\}$, $\mathcal{C}(\partial D)_\varrho = \{t \in \partial D : |t-s| > \varrho\}$ and $z_1 \in \gamma_{[z_0,s]}$ satisfying $|z_1 - s| \leq \varrho$. (Note that $0 < \epsilon p < 1/2$ in (2.16).)

Finally, by Lemma 2.3, $F \in (H \times C)(D \times \partial D)$. □

Lemma 2.5. *Let $D, \partial D, f$ be as in Lemma 2.4. Then for any $\gamma \in L^p(\mathbb{R})$, $p \geq 1$,*

$$\lim_{\substack{z \to s \\ z \in D, s \in \partial D}} \int_{\partial D} (z-s) f(z,t) \gamma(t) dt = 0. \tag{2.17}$$

Proof. If $p > 1$, then $q = \frac{p}{p-1} > 1$. By Hölder's inequality, both $f(z,\cdot)\gamma(\cdot)$ and $f(s,\cdot)\gamma(\cdot)$ belong to $L^1(\partial D)$ for any fixed $z \in D$ and $s \in \partial D$. Define $U_{s,\alpha,\epsilon}$ as in (2.14), the same assumption as (2.5) implies that $U_{s,\alpha,\epsilon} \in C(\overline{D} \times \partial D)$. For $p = 1$, note that $(z-s) f(z,\cdot)\gamma(\cdot) = U_{s,\alpha,1}(z,\cdot)\gamma(\cdot)$, therefore $(z-s) f(z,\cdot)\gamma(\cdot)$ also belongs to $L^1(\partial D)$ for any $\gamma \in L^p(\partial D)$, $p \geq 1$.

Due to $U_{s,\alpha,\epsilon} \in C(\vee_s^\alpha(D) \times \partial D)$, for $0 < \epsilon < 1$, there exists $\eta > 0$ such that

$$|U_{s,\alpha,\epsilon}(z,t)| < 1 \tag{2.18}$$

for any $z \in \overline{D}$ and $t \in \partial D$ satisfying $|z-s| \leq \eta$ and $|t-s| \leq \eta$ respectively. Fix such η, by splitting,

$$\begin{aligned} \int_{\partial D} (z-s) f(z,t) \gamma(t) dt &= (z-s)^{1-\epsilon} \int_{|t-s| \leq \eta, \ t \in \partial D} U_{s,\alpha,\epsilon}(z,t) \gamma(t) dt \\ &\quad + (z-s) \int_{|t-s| \geq \eta, \ t \in \partial D} f(z,t) \gamma(t) dt \end{aligned} \tag{2.19}$$

in which $z \in \triangledown_{s,\eta}^{\alpha}(D) = \{z \in \vee_s^{\alpha}(D) : |z-s| \leq \eta\}$. Moreover, from (2.4) and (2.5),

$$\lim_{\substack{z \to s \\ z \in \overline{D}, s \in \partial D}} (z-s) f(z,t)\gamma(t) = 0, \ \ a.e. \ t \in \partial D. \tag{2.20}$$

Due to (2.18)-(2.20) and the fact $f \in C(\triangledown_{s,\eta}^{\alpha}(D) \times \{t \in \partial D : |t-s| \geq \eta\})$ for arbitrary $\alpha > 0$, by Lebesgue's dominated convergence theorem, we establish (2.17). □

By all the above proved lemmas and the decomposition theorem of polyharmonic functions due to Du et al in [9], we have the following

Theorem 2.6. *If $\{ g_n(z,t) \}_{n=1}^{\infty}$ is the sequence of higher order Poisson kernels defined on $\mathbb{D} \times \mathbb{T}$, i.e., $\{ g_n(z,t) \}_{n=1}^{\infty}$ fulfills the aforementioned properties 1-5 in Definition 2.2, then, for $n > 1$, there exist functions $g_{n,0}(z,t), g_{n,1}(z,t), \ldots, g_{n,n-1}(z,t)$ defined on $\mathbb{D} \times \mathbb{T}$ such that*

$$g_n(z,t) = 2\mathrm{Re}\Big\{ \sum_{j=0}^{n-1} \overline{z}^j g_{n,j}(z,t) \Big\}, \ z \in \mathbb{D}, \, t \in \mathbb{T} \tag{2.21}$$

with

$$\partial_z g_{n,j}(z,t) = j^{-1} g_{n-1,j-1}(z,t) \tag{2.22}$$

for $1 \leq j \leq n-1$,

$$\partial_z^k g_{n,j}(0,t) = 0 \tag{2.23}$$

for $0 \leq k \leq j-1$ with respect to $t \in \mathbb{T}$ and

$$g_{n,0}(z,t) = -\sum_{j=1}^{n-1} z^{-j} g_{n,j}(z,t). \tag{2.24}$$

Moreover,

$$g_1(z,t) = \frac{1}{1-z\overline{t}} + \frac{1}{1-\overline{z}t} - 1, \tag{2.25}$$

which is the classical Poisson kernel for the unit disc. All of the above $g_{n,j} \in (H \times C)(\mathbb{D} \times \mathbb{T})$, the non-tangential boundary value

$$\lim_{\substack{z \to s \\ z \in \mathbb{D}, s \in \mathbb{T}}} g_{n,j}(z,t) = g_{n,j}(s,t) \tag{2.26}$$

exists on $\mathbb{T}$ except $t \in \mathbb{T}$ and $g_{n,j}(s,\cdot) \in L^p(\mathbb{T})$, $p \geq 1$ for any fixed $s \in \mathbb{T}$. For any fixed $t \in \mathbb{T}$, $g_{n,j}(\cdot,t)$ can be continuously extended to $\overline{\mathbb{D}} \setminus \{t\}$. Moreover,

$$\lim_{\substack{z \to s \\ z \in \overline{\mathbb{D}}, s \in \mathbb{T}}} |g_{n,j}(z,s)| = +\infty \ \text{and} \ \lim_{\substack{z \to s \\ z \in \overline{\mathbb{D}}, s \in \mathbb{T}}} |(z-s)^{\epsilon} g_{n,j}(z,s)| = 0 \tag{2.27}$$

for any $s \in \mathbb{T}$, any $\epsilon > 0$ and $n \geq 2$, where the main analytic branch of $(z-s)^{\epsilon}$ is always chosen in the complex plane cut along a straight line from s to infinity in $\mathbb{C} \setminus \overline{\mathbb{D}}$ and $1^{\epsilon} = 1$.

Proof. The property 2 indicates that $g_n(\cdot,t)$ is polyharmonic in $\mathbb{D}$ for every $t \in \mathbb{T}$. By the decomposition theorem of polyharmonic functions in [9], (2.21) holds with (2.22)-(2.24), where $g_{n,j}(z,t)$ is defined on $\mathbb{D} \times \mathbb{T}$, and analytic in $\mathbb{D}$ with respect to z and has at least the jth order zero at $z = 0$ for each $t \in \mathbb{T}$ for $n \geq 2$.

When g_1 is given by (2.25). It is well known that the classical Poisson kernel given by (2.25) satisfies the properties 1 and 3 in Definition 2.2, see [18]. Starting from $g_{1,0}(z,t)=\frac{1}{1-z\bar{t}}-\frac{1}{2}=\sum_{k=1}^{\infty}(z\bar{t})^k+\frac{1}{2}$ and using (2.22)-(2.24), all $g_{n,j}$ and g_n can be inductively obtained (see the algorithm given after this theorem).

The following facts are noted. Firstly, $g_{1,0}\in(H\times C)(\mathbb{D}\times\mathbb{T})$; secondly, the nontangential boundary value

$$\lim_{\substack{z\longrightarrow s\\ z\in\mathbb{D},s\in\mathbb{T}}} g_{1,0}(z,t)=g_{1,0}(s,t) \tag{2.28}$$

exists on $\mathbb{T}$ except $t\in\mathbb{T}$; thirdly, for any fixed $t\in\mathbb{T}$, $g_{1,0}(\cdot,t)$ can be continuously extended to $\overline{\mathbb{D}}\setminus\{t\}$. Therefore, by Lemmas 2.3-2.4 and an induction argument, for any $n\in\mathbb{N}$ and $0\le j\le n-1$, $g_{n,j}\in(H\times C)(\mathbb{D}\times\mathbb{T})$; the non-tangential boundary value

$$\lim_{\substack{z\longrightarrow s\\ z\in\mathbb{D},s\in\mathbb{T}}} g_{n,j}(z,t)=g_{n,j}(s,t)$$

exists on $\mathbb{T}$ except $t\in\mathbb{T}$; and $g_{n,j}(\cdot,t)$ can be continuously extended to $\overline{\mathbb{D}}\setminus\{t\}$ for any fixed $t\in\mathbb{T}$. Obviously, $g_{n,j}(z,t)$ has at least a jth order zero at $z=0$ for each $t\in\mathbb{T}$. Moreover, $g_{n,j}(s,\cdot)\in L^p(\mathbb{T})$, $p\ge 1$ for any fixed $s\in\mathbb{T}$;

$$\lim_{\substack{z\longrightarrow s\\ z\in\overline{\mathbb{D}},s\in\mathbb{T}}} |g_{n,j}(z,s)|=+\infty \ \ and \ \ \lim_{\substack{z\longrightarrow s\\ z\in\overline{\mathbb{D}},s\in\mathbb{T}}} |(z-s)^{\epsilon}g_{n,j}(z,s)|=0$$

for any $s\in\mathbb{T}$, $\epsilon>0$, $n\ge 2$ and $1\le j\le n-1$ in terms of Lemma 2.4 since $g_{2,1}$ has the same properties by using straightforward calculations.

By (2.21) and (2.24),

$$\begin{aligned} g_n(z,t) &= 2\mathrm{Re}\Big\{(\overline{z}-z^{-1})\sum_{j=1}^{n-1}\sum_{l=0}^{j-1}\overline{z}^{j-1-l}z^{-l}g_{n,j}(z,t)\Big\} \\ &= 2\mathrm{Re}\Big\{\sum_{j=1}^{n-1}\sum_{l=0}^{j-1}\overline{z}^{j-1-l}z^{-l}\big[(\overline{z}-s^{-1})-(z^{-1}-s^{-1})\big]g_{n,j}(z,t)\Big\}, \end{aligned}$$

where $z\in\mathbb{D}$ and $t\in\mathbb{T}$ for any fixed $s\in\mathbb{T}$.

From the above facts, by a similar argument as used for Lemma 2.5 and Minkowski's inequality, all g_n satisfy the properties 1, 2, 4 ,5 and the nontangential limit

$$\lim_{\substack{z\longrightarrow s\\ z\in\mathbb{D},s\in\mathbb{T}}} \frac{1}{2\pi i}\int_{\mathbb{T}} g_n(z,t)\gamma(t)\frac{dt}{t}=0 \tag{2.29}$$

for any $n\ge 2$ and $\gamma\in L^p(\mathbb{T})$, $p\ge 1$. □

In fact, following from Theorem 2.6, we can establish an algorithm to obtain all explicit expressions of higher order Poisson kernels as follows.

For $n=1$,

$$g_{1,0}(z,t)=\frac{1}{1-z\bar{t}}-\frac{1}{2}=\sum_{k=1}^{\infty}(z\bar{t})^k+\frac{1}{2}=\sum_{k=2}^{\infty}(z\bar{t})^{k-1}+\frac{1}{2}, \tag{2.30}$$

therefore

$$g_1(z,t)=2\mathrm{Re}\{g_{1,0}(z,t)\}=\sum_{k=2}^{\infty}\big((z\bar{t})^{k-1}+(\overline{z}t)^{k-1}\big)+1. \tag{2.31}$$

For $n=2$,

$$g_{2,1}(z,t)=\int_0^z g_{1,0}(\zeta,t)\mathrm{d}\zeta=\sum_{k=2}^{\infty}\frac{1}{k}z^k\bar{t}^{k-1}+\frac{1}{2}z=-\frac{1}{\bar{t}}\log(1-z\bar{t})-\frac{1}{2}z \tag{2.32}$$

and

$$g_{2,0}(z,t)=-z^{-1}g_{2,1}(z,t)=-\Big[\sum_{k=2}^{\infty}\frac{1}{k}(z\bar{t})^{k-1}+\frac{1}{2}\Big]=\frac{1}{z\bar{t}}\log(1-z\bar{t})+\frac{1}{2}, \tag{2.33}$$

therefore

$$\begin{aligned} g_2(z,\tau)&=2\mathrm{Re}\{g_{2,0}(z,t)+\overline{z}g_{2,1}(z,t)\}\\ &=-(1-|z|^2)\Big[\sum_{k=2}^{\infty}\frac{1}{k}\big((z\bar{t})^{k-1}+(\overline{z}t)^{k-1}\big)+1\Big]\\ &=(1-|z|^2)\Big[\frac{1}{z\bar{t}}\log(1-z\bar{t})+\frac{1}{\overline{z}t}\log(1-\overline{z}t)+1\Big]. \end{aligned} \tag{2.34}$$

For $n=3$,

$$g_{3,2}(z,t)=\frac{1}{2}\int_0^z g_{2,1}(\zeta,t)\mathrm{d}\zeta=\frac{1}{2!}\Big[\sum_{k=2}^{\infty}\frac{1}{k(k+1)}z^{k+1}\bar{t}^{k-1}+\frac{1}{2\cdot 2!}z^2\Big], \tag{2.35}$$

$$g_{3,1}(z,t)=\int_0^z g_{2,0}(\zeta,t)\mathrm{d}\zeta=-\Big[\sum_{k=2}^{\infty}\frac{1}{k^2}z^k\bar{t}^{k-1}+\frac{1}{2}z\Big] \tag{2.36}$$

and

$$\begin{aligned} g_{3,0}(z,t)&=-[z^{-1}g_{3,1}(z,t)+z^{-2}g_{3,2}(z,t)]\\ &=\Big[\sum_{k=2}^{\infty}\frac{1}{k^2}(z\bar{t})^{k-1}+\frac{1}{2}\Big]-\frac{1}{2!}\Big[\sum_{k=2}^{\infty}\frac{1}{k(k+1)}(z\bar{t})^{k-1}+\frac{1}{2\cdot 2!}\Big], \end{aligned} \tag{2.37}$$

so

$$\begin{aligned} g_3(z,t)=&2\mathrm{Re}\{g_{3,0}(z,t)+\overline{z}g_{3,1}(z,t)+\overline{z}^2g_{3,2}(z,t)\}\\ =&-\frac{1-|z|^4}{2!}\Big[\sum_{k=2}^{\infty}\frac{1}{k(k+1)}\big((z\bar{t})^{k-1}+(\overline{z}t)^{k-1}\big)+\frac{1}{2!}\Big]\\ &+(1-|z|^2)\Big[\sum_{k=2}^{\infty}\frac{1}{k^2}\big((z\bar{t})^{k-1}+(\overline{z}t)^{k-1}\big)+1\Big]. \end{aligned} \tag{2.38}$$

Similarly, we have

$$\begin{aligned} g_4(z,t)=&-\frac{1-|z|^6}{3!}\Big[\sum_{k=2}^{\infty}\frac{1}{k(k+1)(k+2)}\big((z\bar{t})^{k-1}+(\overline{z}t)^{k-1}\big)+\frac{1}{3!}\Big]\\ &+\frac{1-|z|^4}{2!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^2(k+1)}\big((z\bar{t})^{k-1}+(\overline{z}t)^{k-1}\big)+\frac{1}{2!}\Big]\\ &+(1-|z|^2)\Big\{\Big[\frac{1}{2!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^2(k+1)}\big((z\bar{t})^{k-1}+(\overline{z}t)^{k-1}\big)+\frac{1}{2!}\Big]\\ &-\sum_{k=2}^{\infty}\frac{1}{k^3}\big((z\bar{t})^{k-1}+(z\bar{t})^{k-1}\big)+1\Big]\Big\} \end{aligned} \tag{2.39}$$

and

$$
\begin{aligned}
g_5(z,t) = & -\frac{1-|z|^8}{4!}\Big[\sum_{k=2}^{\infty}\frac{1}{k(k+1)(k+2)(k+3)}\big((z\bar{t})^{k-1}+(\overline{z}t)^{k-1}\big)+\frac{1}{4!}\Big] \\
& + \frac{1-|z|^6}{3!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^2(k+1)(k+2)}\big((z\bar{t})^{k-1}+(\overline{z}t)^{k-1}\big)+\frac{1}{3!}\Big] \\
& + \frac{1-|z|^4}{2!}\Big\{\frac{1}{2!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^2(k+1)^2}\big((z\bar{t})^{k-1}+(\overline{z}t)^{k-1}\big)+\frac{1}{2!\cdot 2!}\Big] \\
& - \Big[\sum_{k=2}^{\infty}\frac{1}{k^3(k+1)}\big((z\bar{t})^{k-1}+(\overline{z}t)^{k-1}\big)+\frac{1}{2!}\Big]\Big\} \\
& + (1-|z|^2)\Big\{\frac{1}{3!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^2(k+1)(k+2)}\big((z\bar{t})^{k-1}+(\overline{z}t)^{k-1}\big)+\frac{1}{3!}\Big] \\
& - \frac{1}{2!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^3(k+1)}\big((z\bar{t})^{k-1}+(\overline{z}t)^{k-1}\big)+\frac{1}{2!}\Big] \\
& - \Big\{\frac{1}{2!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^3(k+1)}\big((z\bar{t})^{k-1}+(\overline{z}t)^{k-1}\big)+\frac{1}{2!}\Big] \\
& - \Big[\sum_{k=2}^{\infty}\frac{1}{k^4}\big((z\bar{t})^{k-1}-(\overline{z}t)^{k-1}\big)+1\Big]\Big\}\Big\}.
\end{aligned} \tag{2.40}
$$

To get all explicit expressions for general g_n, $n \in \mathbb{N}_+$ consists of all positive integers, we need introduce some notations and notions as follows.

For any $k \in \mathbb{N}_+$, denote

$$
\frac{1}{\mathbb{N}_k} = \left\{\frac{1}{k}, \frac{1}{k+1}, \dots\right\}. \tag{2.41}
$$

Introduce a class of index transformation operators, which are defined as follows:

$$
\begin{aligned}
I_j^{(k)} : \quad & \left(\frac{1}{\mathbb{N}_k}\right)^{\mathbb{N}^\infty} \longrightarrow \left(\frac{1}{\mathbb{N}_k}\right)^{\mathbb{N}^\infty} \\
& \frac{c}{k^{l_0}\cdots(k+j)^{l_j}\cdots} \longmapsto \frac{1}{j+1}\,\frac{c}{k^{l_0}\cdots(k+j)^{l_j+1}\cdots},
\end{aligned} \tag{2.42}
$$

where $0 \le j < \infty$, and the set $\left(\frac{1}{\mathbb{N}_k}\right)^{\mathbb{N}^\infty} = \left\{c\prod_{p=0}^{\infty}\frac{1}{(k+p)^{l_p}} : l_p \in \mathbb{N}, c \in \mathbb{R}\right\}$, in which only finitely many indices l_p of any element $\prod_{p=0}^{\infty}\frac{1}{(k+p)^{l_p}}$ are nonzero.

Moreover, define a formal operation of vector operators:

$$
(T_1, T_2, \dots, T_q) \circ (h_1, h_2, \dots, h_q) = (T_1 h_1, T_2 h_2, \dots, T_q h_q), \tag{2.43}
$$

where $q \in \mathbb{N}_+$, T_q is an operator on some appropriate linear space H_q and $h_q \in H_q$.

Set

$$\mathcal{I}_n^{(k)} = \Big(I_{n-1}^{(k)}, \underbrace{I_{n-2}^{(k)}}_{2^0 \text{ term}}, \underbrace{I_{n-3}^{(k)}, I_{n-3}^{(k)}}_{2^1 \text{ terms}}, \ldots, \underbrace{I_p^{(k)}, \ldots, I_p^{(k)}}_{2^{n-2-p} \text{ terms}}, \ldots, \underbrace{I_2^{(k)}, \ldots, I_2^{(k)}}_{2^{n-4} \text{ terms}}, \underbrace{I_1^{(k)}, \ldots, I_1^{(k)}}_{2^{n-3} \text{ terms}}, \underbrace{I_0^{(k)}, \ldots, I_0^{(k)}}_{2^{n-2} \text{ terms}}\Big), \tag{2.44}$$

and

$$Z_n = \Big(1-|z|^{2n}, \underbrace{1-|z|^{2(n-1)}}_{2^0 \text{ term}}, \ldots, \underbrace{1-|z|^{2(p+1)}, \ldots, (1-|z|^{2(p+1)}}_{2^{n-2-p} \text{ terms}}, \ldots, \underbrace{1-|z|^4, \ldots, 1-|z|^4}_{2^{n-3} \text{ terms}}, \underbrace{1-|z|^2, \ldots, 1-|z|^2}_{2^{n-2} \text{ terms}}\Big) \tag{2.45}$$

for any $n \geq 2$, $n \in \mathbb{N}_+$. With the above preliminaries, we can give the following definitions:

$$c_0^{(k)} := 1 = \frac{1}{k^0}, \tag{2.46}$$

$$c_1^{(k)} := -\mathcal{I}_1^{(k)} c_0^{(k)} = -I_0^{(k)} c_0^{(k)} = -\frac{1}{k}, \tag{2.47}$$

and for any $n \geq 2$,

$$c_n^{(k)} := \mathcal{I}_n^{(k)} \circ \left(c_{n-1}^{(k)}, -c_{n-1}^{(k)}\right). \tag{2.48}$$

Further, we define the following formal summation operation:

$$\sum\nolimits_{k=1}^{\infty} : (a_1(k), a_2(k), \ldots, a_s(k)) \longrightarrow \left(\sum_{k=1}^{\infty} a_1(k), \sum_{k=1}^{\infty} a_2(k), \ldots, \sum_{k=1}^{\infty} a_s(k)\right) \tag{2.49}$$

where $k, s \in \mathbb{N}_+$, and a_p is a function with respect to $k \in \mathbb{N} \setminus \{0,1\}$, $1 \leq p \leq s$.

By a straightforward calculation, we have

Lemma 2.7. *Let $\vartheta_k(z,t) = (z\bar{t})^{k-1} + (\overline{z}t)^{k-1}$, $k \in \mathbb{N}_+$, then*

$$\partial_z \partial_{\overline{z}} \left(|z|^{2j} \vartheta_k(z,t)\right) = j(k+j-1)|z|^{2(j-1)} \vartheta_k(z,t) \tag{2.50}$$

for any $k, j \in \mathbb{N}_+$, and

$$\partial_z \partial_{\overline{z}} \vartheta_k(z,t) = 0. \tag{2.51}$$

Theorem 2.8. *Let*

$$g_1(z,t) = \sum_{k=2}^{\infty} \left((z\bar{t})^{k-1} + (\overline{z}t)^{k-1}\right) + 1$$

and

$$g_n(z,t) = Z_{n-1} \left[\sum\nolimits_{k=1}^{\infty} \left(c_{n-1}^{(k)} \left(\left(1 - \frac{\delta_{k1}}{2}\right) \vartheta_k(z,t) I_{2^{n-2}}\right)\right)\right]^T \tag{2.52}$$

as $n \geq 2$, where δ_{k1} is the Kronecker's sign, $\vartheta_k(z,t)$ are given as in Lemma 2.7, Z_{n-1}, $c_{n-1}^{(k)}$ and the summation operation $\sum_{k=1}^{\infty}$ be defined as above, $k \in \mathbb{N}_+$, $[\cdots]^T$ denotes the transpose operation, and $I_{2^{n-2}}$ is the identical matrix of order 2^{n-2}, then

$\{g_n(z,t)\}_{n=1}^{\infty}$ *is a sequence of higher order Poisson kernels defined as in Definition 2.2.*

Proof. Firstly, in terms of the definitions of $\vartheta_k(z,t)$, Z_{n-1}, $c_{n-1}^{(k)}$ and the summation operation $\sum_{k=1}^{\infty}$, together with (2.31) and (2.52), it immediately follows that for any $n \in \mathbb{N}_+$, $g_n \in (C^\infty \times C)(\mathbb{D} \times \mathbb{T})$; the non-tangential boundary value

$$\lim_{\substack{z \longrightarrow s \\ z\in\mathbb{D}, s\in\mathbb{T}}} g_n(z,t) = g_n(s,t)$$

exists for all t with $t \neq s$, where s is any fixed point on $\mathbb{T}$. Moreover, $g_n(\cdot, s)$ can be continuously extended to $\overline{\mathbb{D}} \setminus \{s\}$ for all $s \in \mathbb{T}$, i.e., the property 1 in Definition 2.2 holds.

Secondly, by direct calculations, we have

$$g_1(z,t) = \frac{1}{1-z\overline{t}} + \frac{1}{1-\overline{z}t} - 1$$

which is the classical Poisson kernel for the unit disc, and

$$g_2(z,t) = -(1-|z|^2)\Big[\sum_{k=2}^{\infty} \frac{1}{k}\big((z\overline{t})^{k-1} + (\overline{z}t)^{k-1}\big) + 1\Big].$$

So the property 3 in Definition 2.2 holds (see [18]), $\Delta g_1(z,t) = 0$, and it is easy to get $\Delta g_2(z,t) = g_1(z,t)$.

Now we turn to g_n, $n \geq 3$. Denote

(2.53)
$$\begin{aligned} c_{n-1}^{(k)} =& ((c_{n-1}^{(k)})_{2^0}, (c_{n-1}^{(k)})_{2^0+2^0}, (c_{n-1}^{(k)})_{2^1+1}, (c_{n-1}^{(k)})_{2^1+2^1}, \ldots, (c_{n-1}^{(k)})_{2^j+1}, \\ & (c_{n-1}^{(k)})_{2^j+2}, \ldots, (c_{n-1}^{(k)})_{2^j+2^j}, \ldots, (c_{n-1}^{(k)})_{2^{n-3}+1}, \ldots, (c_{n-1}^{(k)})_{2^{n-3}+2^{n-3}}). \end{aligned}$$

Then, by the definition (2.48) of $c_n^{(k)}$, we have

$$\begin{aligned} c_n^{(k)} =& ((c_n^{(k)})_{2^0}, (c_n^{(k)})_{2^0+2^0}, (c_n^{(k)})_{2^1+1}, (c_n^{(k)})_{2^1+2^1}, \ldots, (c_n^{(k)})_{2^j+1}, \\ & (c_n^{(k)})_{2^j+2}, \ldots, (c_n^{(k)})_{2^j+2^j}, \ldots, (c_n^{(k)})_{2^{n-2}+1}, \ldots, (c_n^{(k)})_{2^{n-2}+2^{n-2}}) \end{aligned} \tag{2.54}$$

with

$$(c_n^{(k)})_{2^0} = \frac{1}{n(k+n-1)}(c_{n-1}^{(k)})_{2^0}, \tag{2.55}$$

(2.56)
$$(c_n^{(k)})_{2^p+q} = \frac{1}{(n-1-p)(k+n-2-p)}(c_{n-1}^{(k)})_{2^p+q},\ 0 \leq p \leq n-3;\ 1 \leq q \leq 2^p$$

and

$$(c_n^{(k)})_{2^{n-2}+r} = -\frac{1}{k}(c_{n-1}^{(k)})_r,\ 1 \leq r \leq 2^{n-2}. \tag{2.57}$$

Thus, by (2.52), for $n \geq 2$,

$$
\begin{aligned}
(2.58)\quad g_{n+1}(z,t) =& \big(1-|z|^{2n}\big)\sum_{k=1}^{\infty}\Big(1-\frac{\delta_{k1}}{2}\Big)(c_n^{(k)})_{2^0}\vartheta_k(z,t)\\
&+\sum_{p=0}^{n-2}\sum_{q=1}^{2^p}\left[\Big(1-|z|^{2(n-1-p)}\Big)\sum_{k=1}^{\infty}\Big(1-\frac{\delta_{k1}}{2}\Big)(c_n^{(k)})_{2^p+q}\vartheta_k(z,t)\right]\\
=& \big(1-|z|^{2n}\big)\sum_{k=1}^{\infty}\Big(1-\frac{\delta_{k1}}{2}\Big)(c_n^{(k)})_{2^0}\vartheta_k(z,t)\\
&+\sum_{p=0}^{n-3}\sum_{q=1}^{2^p}\left[\Big(1-|z|^{2(n-1-p)}\Big)\sum_{k=1}^{\infty}\Big(1-\frac{\delta_{k1}}{2}\Big)(c_n^{(k)})_{2^p+q}\vartheta_k(z,t)\right]\\
&+\sum_{r=1}^{2^{n-2}}\left[\big(1-|z|^{2}\big)\sum_{k=1}^{\infty}\Big(1-\frac{\delta_{k1}}{2}\Big)(c_n^{(k)})_{2^{n-2}+r}\vartheta_k(z,t)\right].
\end{aligned}
$$

Noting (2.55)-(2.57), by Lemma 2.7, we have

(2.59)

$$
\begin{aligned}
\partial_z\partial_{\overline{z}}g_{n+1}(z,t) =& -|z|^{2(n-1)}\sum_{k=1}^{\infty}\Big(1-\frac{\delta_{k1}}{2}\Big)\Big[n(k+n-1)(c_n^{(k)})_{2^0}\Big]\vartheta_k(z,t)\\
&+\sum_{p=0}^{n-3}\sum_{q=1}^{2^p}\Big[-|z|^{2(n-2-p)}\sum_{k=1}^{\infty}\Big(1-\frac{\delta_{k1}}{2}\Big)\Big[(n-1-p)\\
&\times(k+n-2-p)(c_n^{(k)})_{2^p+q}\Big]\vartheta_k(z,t)\Big]\\
&+\sum_{r=1}^{2^{n-2}}\left[-\sum_{k=1}^{\infty}\Big(1-\frac{\delta_{k1}}{2}\Big)\Big[k(c_n^{(k)})_{2^{n-2}+r}\Big]\vartheta_k(z,t)\right]\\
=& -|z|^{2(n-1)}\sum_{k=1}^{\infty}\Big(1-\frac{\delta_{k1}}{2}\Big)(c_{n-1}^{(k)})_{2^0}\vartheta_k(z,t)\\
&+\sum_{p=0}^{n-3}\sum_{q=1}^{2^p}\Big[-|z|^{2(n-2-p)}\sum_{k=1}^{\infty}\Big(1-\frac{\delta_{k1}}{2}\Big)(c_{n-1}^{(k)})_{2^p+q}\vartheta_k(z,t)\Big]\\
&+\sum_{r=1}^{2^{n-2}}\left[\sum_{k=1}^{\infty}\Big(1-\frac{\delta_{k1}}{2}\Big)(c_{n-1}^{(k)})_{r}\vartheta_k(z,t)\right]\\
=& \Big(1-|z|^{2(n-1)}\Big)\sum_{k=1}^{\infty}\Big(1-\frac{\delta_{k1}}{2}\Big)(c_{n-1}^{(k)})_{2^0}\vartheta_k(z,t)\\
&+\sum_{p=0}^{n-3}\sum_{q=1}^{2^p}\Big[\Big(1-|z|^{2(n-2-p)}\Big)\sum_{k=1}^{\infty}\Big(1-\frac{\delta_{k1}}{2}\Big)(c_{n-1}^{(k)})_{2^p+q}\vartheta_k(z,t)\Big]\\
=& g_n(z,t).
\end{aligned}
$$

Therefore, the property 2 in Definition 2.2 holds.

Thirdly, by a simple calculation, we have (also see [3, 8]),

$$
\begin{aligned}
g_2(z,t) =& -(1-|z|^2)\Big[\sum_{k=2}^{\infty}\frac{1}{k}\big((z\bar{t})^{k-1}+(\overline{z}t)^{k-1}\big)+1\Big] \\
=&(1-|z|^2)\left[\frac{1}{z\bar{t}}\log(1-z\bar{t})+\frac{1}{\overline{z}t}\log(1-\overline{z}t)+1\right] \\
=&(1-|z|^2)\left[2\mathrm{Re}\left\{\frac{1}{z\bar{t}}\log(1-z\bar{t})\right\}+1\right].
\end{aligned} \tag{2.60}
$$

For any $z\in\vee_s^\alpha(\mathbb{D})$ (see the definition (2.7)), $|z-s|\le 1$ and $t\in\mathbb{T}$,

$$
\begin{aligned}
|g_2(z,t)| \le&(1-|z|^2)\left[2\sum_{k=2}^{\infty}\frac{1}{k}|z|^{k-1}+1\right] \\
\le&(1-|z|^2)\left[2\sum_{k=2}^{\infty}|z|^{k-1}+1\right] \\
\le&(1+|z|)^2\le 4.
\end{aligned} \tag{2.61}
$$

Therefore, for any $\gamma\in L^p(\mathbb{T})$ as $p\ge 1$,

$$
|g_2(z,t)\gamma(t)|\le 4|\gamma(t)|, \tag{2.62}
$$

where $z\in\triangledown_{s,1}^\alpha(\mathbb{D})=\{z\in\vee_s^\alpha(\mathbb{D}): |z-s|\le 1\}$ and $t\in\mathbb{T}$. On the other hand, let $z=re^{i\varphi}$ and $t=e^{i\theta}$, then

$$
g_2(z,t) =2(1+1/r)(1-r)\Big[\cos(\varphi-\theta)\log|1-re^{i(\varphi-\theta)}| \tag{2.63}
$$

$$
+\sin(\varphi-\theta)\arg(1-re^{i(\varphi-\theta)})\Big]+(1-r^2). \tag{2.64}
$$

Since $|\arg(1-re^{i(\varphi-\theta)})|<\frac{\pi}{2}$ and $|\log(1-re^{i(\varphi-\theta)})|\le|\log(1-r)|$ for any $r\in[0,1)$ and $\varphi,\,\theta\in[0,2\pi)$, therefore for any $s,t\in\mathbb{T}$ and $z\in\triangledown_{s,1}^\alpha(\mathbb{D})$,

$$
|g_2(z,t)|\le 2(1+1/r)\left[|(1-r)\log(1-r)|+\frac{\pi}{2}(1-r)\right]+(1-r^2). \tag{2.65}
$$

Due to the fundamental limit

$$
\lim_{r\to 1-}(1-r)\log(1-r)=0, \tag{2.66}
$$

we have the non-tangential limit

$$
\lim_{\substack{z\to s,\\ z\in\mathbb{D},\, s\in\mathbb{T}}} g_2(z,t)=0 \tag{2.67}
$$

uniformly in $t\in\mathbb{T}$ for any fixed $s\in\mathbb{T}$. Thus the non-tangential limit

$$
\lim_{\substack{z\to s,\\ z\in\mathbb{D},\, s\in\mathbb{T}}} g_2(z,t)\gamma(t)=0 \tag{2.68}
$$

uniformly in $t\in\mathbb{T}$ for any fixed $s\in\mathbb{T}$ and any $\gamma\in L^p(\mathbb{T})$, $p\ge 1$. Due to (2.62) and (2.68), by Lebesgue's dominated convergence theorem,

$$
\lim_{\substack{z\to s\\ z\in\mathbb{D},\, s\in\mathbb{T}}}\frac{1}{2\pi i}\int_{\mathbb{T}} g_2(z,t)\gamma(t)\frac{dt}{t}=0
$$

for any $\gamma\in L^p(\mathbb{T})$, $p\ge 1$. That is, the property 4 in Definition 2.2 holds.

Finally, in terms of (2.53), we can express

$$
\begin{aligned}
g_n(z,t) &= \Big(1-|z|^{2(n-1)}\Big)\sum_{k=1}^{\infty}\Big(1-\frac{\delta_{k1}}{2}\Big)(c_{n-1}^{(k)})_{2^0}\vartheta_k(z,t) \\
&+\sum_{p=0}^{n-3}\sum_{q=1}^{2^p}\Big[\Big(1-|z|^{2(n-2-p)}\Big)\sum_{k=1}^{\infty}\Big(1-\frac{\delta_{k1}}{2}\Big)(c_{n-1}^{(k)})_{2^p+q}\vartheta_k(z,t)\Big].
\end{aligned}
\tag{2.69}
$$

By the definition of $c_n^{(k)}$, it is easy to get that

$$
\Big|(c_{n-1}^{(k)})_r\Big| \le 1/k^2 \tag{2.70}
$$

for any $n \ge 3$, $1 \le r \le 2^{n-2}$ and $k \in \mathbb{N}_+$. Note that

$$
|\vartheta_k(z,t)| \le 2 \tag{2.71}
$$

for any $z \in \mathbb{D}$, $t \in \mathbb{T}$ and $k \in \mathbb{N}_+$, by (2.69), we get that

$$
|g_n(z,t)| \le 2^{n-1}(1-|z|^{2n-2})\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{2^{n-2}\pi^2}{3}(1-|z|^{2n-2}) \tag{2.72}
$$

for any $z \in \mathbb{D}$ and $t \in \mathbb{T}$. Then the non-tangential limit

$$
\lim_{\substack{z\longrightarrow s,\\ z\in\mathbb{D},\, s\in\mathbb{T}}} g_n(z,t) = 0
$$

uniformly in $t \in \mathbb{T}$ for any fixed $s \in \mathbb{T}$ and $n \ge 3$, viz., the property 5 in Definition 2.2 holds. Thus we complete the proof of the theorem. □

Remark 2.9. From the above arguments, comparing with other kernels g_n, $n > 2$, we get an unobvious fact: the non-tangential limit

$$
\lim_{\substack{z\longrightarrow s,\\ z\in\mathbb{D},\, s\in\mathbb{T}}} g_2(z,t) = 0
$$

uniformly in $t \in \mathbb{T}$ for any fixed $s \in \mathbb{T}$.

Remark 2.10. In [8, 9], by defining a vertical sum, viz.,

$$
\sum\begin{cases} a_1 \\ a_2 \\ \vdots \\ a_n \end{cases} =: a_1 + a_2 + \cdots + a_n,
$$

Du et al give the explicit expressions for all g_n, $n \in \mathbb{N}$. For example, the above g_5 can be expressed as

$$g_5(z,t) = \sum \begin{cases} (1-|z|^2)\sum \begin{cases} \Big[\sum_{k=2}^{\infty}\frac{1}{k^4}\big((z\bar{t})^{k-1}+(\overline{z}t)^{k-1}\big)+1\Big] \\ -\frac{1}{2!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^3(k+1)}\big((z\bar{t})^{k-1}+(\overline{z}t)^{k-1}\big)+\frac{1}{2!}\Big] \\ -\frac{1}{2!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^3(k+1)}\big((z\bar{t})^{k-1}+(\overline{z}t)^{k-1}\big)+\frac{1}{2!}\Big] \\ \frac{1}{3!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^2(k+1)(k+2)}\big((z\bar{t})^{k-1}+(\overline{z}tu)^{k-1}\big)+\frac{1}{3!}\Big] \end{cases} \\ -\dfrac{1-|z|^4}{2!}\sum \begin{cases} \Big[\sum_{k=2}^{\infty}\frac{1}{k^3(k+1)}\big((z\bar{t})^{k-1}+(\overline{z}t)^{k-1}\big)+\frac{1}{2!}\Big] \\ -\frac{1}{2!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^2(k+1)^2}\big((z\bar{t})^{k-1}+(\overline{z}t)^{k-1}\big)+\frac{1}{2!\cdot 2!}\Big] \end{cases} \\ \dfrac{1-|z|^6}{3!}\Big[\displaystyle\sum_{k=2}^{\infty}\frac{1}{k^2(k+1)(k+2)}\big((z\bar{t})^{k-1}+(z\bar{t})^{k-1}\big)+\frac{1}{3!}\Big] \\ -\dfrac{1-|z|^8}{4!}\Big[\displaystyle\sum_{k=2}^{\infty}\frac{1}{k(k+1)(k+2)(k+3)}\big((z\bar{t})^{k-1}+(\overline{z}t)^{k-1}\big)+\frac{1}{4!}\Big]. \end{cases}$$

However, for sufficiently large n, there the expressions of g_n are complicate and difficult to understand. In Theorem 2.8, we reformulate these expressions in an easily accessible way.

3. Polyharmonic Dirichlet problems in the unit disc

In this section, we solve the PHD problems (1.1), i.e.,

$$\begin{cases} \Delta^n u = 0 \text{ in } \mathbb{D}, \\ \Delta^j u = f_j \text{ on } \mathbb{T}, \end{cases}$$

where $\mathbb{D}$ is the unit disc, $\mathbb{T}$ is the unit circle, $f_j \in L^p(\mathbb{T})$, $n \in \mathbb{N}$, $0 \le j < n$ and $p \ge 1$.

To do so, we need the following

Lemma 3.1. *Let D be a simply connected bounded domain in the plane with smooth boundary ∂D. If $f \in (H \times L^1)(D \times \partial D)$, then*

$$\frac{\partial}{\partial z}\left(\int_{\partial D} f(z,t)dt\right) = \int_{\partial D}\frac{\partial f}{\partial z}(z,t)dt. \tag{3.1}$$

Proof. Fix $z_0 \in D$, take any sequence $\{z_l\}_{l=1}^{+\infty}$ such that $\lim_{l\to+\infty} z_l = z_0$ and $z_l \ne z_0$ for all l. Since $f \in (H \times L^1)(D \times \partial D)$, denote

$$\begin{aligned} D_l(z_0,t) &= \frac{f(z_l,t)-f(z_0,t)}{z_l - z_0} \\ &= \frac{1}{2\pi i}\int_{|\zeta - z_0| = R}\frac{f(\zeta,t)}{(\zeta - z_l)(\zeta - z_0)}d\zeta, \end{aligned} \tag{3.2}$$

then by the assumptions,

$$
\begin{aligned}
|D_l(z_0,t)| &\le \frac{1}{2\pi i}\int_{|\zeta-z_0|=R}\frac{|f(\zeta,t)|}{|\zeta-z_l|}\frac{d\zeta}{\zeta-z_0} \\
&\le \frac{2}{R}\max_{|\zeta-z_0|=R}\{|f(\zeta,t)|\} \\
&\le \frac{2}{R}\max_{|\zeta-z_0|=R,t\in\partial D}\{|f(\zeta,t)|\}
\end{aligned}
\tag{3.3}
$$

uniformly for $t\in\partial D$ whenever $z_l\in\{\zeta:|\zeta-z_0|<R/2\}\subset\{\zeta:|\zeta-z_0|<R\}\subset D$. Since

$$
\lim_{l\to+\infty} D_l(z_0,t)=\frac{\partial f}{\partial z}(z_0,t),\ t\in\partial D, \tag{3.4}
$$

by Lebesgue's dominated convergence theorem,

$$
\lim_{l\to+\infty}\int_{\partial D} D_l(z_0,t)dt=\int_{\partial D}\frac{\partial f}{\partial z}(z_0,t)dt, \tag{3.5}
$$

i.e.,

$$
\lim_{l\to+\infty}\frac{\int_{\partial D} f(z_l,t)dt-\int_{\partial D} f(z_0,t)dt}{z_l-z_0}=\int_{\partial D}\frac{\partial f}{\partial z}(z_0,t)dt. \tag{3.6}
$$

Since z_0 and the sequence $\{z_l\}_{l=1}^{+\infty}$ are arbitrary, we have

$$
\frac{\partial}{\partial z}\left(\int_{\partial D} f(z,t)dt\right)=\int_{\partial D}\frac{\partial f}{\partial z}(z,t)dt. \qquad \square
$$

From Lemma 3.1, we can obtain an important theorem concerning the differentiability of integrals of higher order Poisson kernels as follows.

Theorem 3.2. *Let $\{\, g_n(z,t)\,\}_{n=1}^{\infty}$ be the sequence of higher order Poisson kernels defined on $\mathbb{D}\times\mathbb{T}$, stated as in Theorem 2.6, then for any $n>1$ and $\gamma\in L^p(\mathbb{T})$, $p\ge 1$,*

$$
\frac{\partial^2}{\partial z\partial\overline{z}}\left(\frac{1}{2\pi i}\int_{\mathbb{T}} g_n(z,t)\gamma(t)\frac{dt}{t}\right)=\frac{1}{2\pi i}\int_{\mathbb{T}} g_{n-1}(z,t)\gamma(t)\frac{dt}{t}. \tag{3.7}
$$

Proof. By Theorem 2.6, for any $n>1$,

$$
\begin{aligned}
g_n(z,t) &= 2\mathrm{Re}\Big\{\sum_{j=0}^{n-1}\overline{z}^j g_{n,j}(z,t)\Big\} \\
&= 2\mathrm{Re}\Big\{\sum_{j=1}^{n-1}[\overline{z}^j-z^{-j}]g_{n,j}(z,t)\Big\},
\end{aligned}
\tag{3.8}
$$

where all $g_{n,j}(z,t)$ fulfill

$$
j\,\partial_z g_{n,j}(z,t)=g_{n-1,j-1}(z,t). \tag{3.9}
$$

Thus

$$
\begin{aligned}
(3.10)\qquad \frac{1}{2\pi i}\int_{\mathbb{T}} g_n(z,t)\gamma(t)\frac{dt}{t} &= 2\mathrm{Re}\Big\{\sum_{j=0}^{n-1}\overline{z}^j\frac{1}{2\pi i}\int_{\mathbb{T}} g_{n,j}(z,t)\gamma(t)\frac{dt}{t}\Big\}\\
&= 2\mathrm{Re}\Big\{\sum_{j=1}^{n-1}[\overline{z}^j - z^{-j}]\\
&\quad \times\frac{1}{2\pi i}\int_{\mathbb{T}} g_{n,j}(z,t)\gamma(t)\frac{dt}{t}\Big\}.
\end{aligned}
$$

Similarly, by Lemma 3.1,

$$
\begin{aligned}
\frac{\partial}{\partial z}\left(\frac{1}{2\pi i}\int_{\mathbb{T}} g_n(z,t)\gamma(t)\frac{dt}{t}\right) &= \sum_{j=1}^{n-1}\Big\{[\overline{z}^j - z^{-j}]\\
&\quad\times\frac{1}{2\pi i}\int_{\mathbb{T}}\partial_z g_{n,j}(z,t)\gamma(t)\frac{dt}{t}\\
&\quad + j\, z^{-j-1}\frac{1}{2\pi i}\int_{\mathbb{T}} g_{n,j}(z,t)\gamma(t)\frac{dt}{t}\\
&\quad + j\, z^{j-1}\frac{1}{2\pi i}\int_{\mathbb{T}}\overline{g_{n,j}(z,t)}\,\overline{\gamma(t)}\frac{dt}{t}\Big\}.
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
\frac{\partial^2}{\partial z\partial\overline{z}}\left(\frac{1}{2\pi i}\int_{\mathbb{T}} g_n(z,t)\gamma(t)\frac{dt}{t}\right) &= \sum_{j=1}^{n-1}\Big\{\overline{z}^{j-1}\frac{1}{2\pi i}\int_{\mathbb{T}}\Big(j\,\partial_z g_{n,j}(z,t)\Big)\gamma(t)\frac{dt}{t}\\
&\quad + z^{j-1}\frac{1}{2\pi i}\int_{\mathbb{T}}\overline{\Big(j\,\partial_z g_{n,j}(z,t)\Big)}\,\overline{\gamma(t)}\frac{dt}{t}\Big\}\\
&= 2\mathrm{Re}\Big\{\sum_{j=1}^{n-1}\overline{z}^j\frac{1}{2\pi i}\int_{\mathbb{T}} g_{n-1,j-1}(z,t)\gamma(t)\frac{dt}{t}\Big\}\\
&= 2\mathrm{Re}\Big\{\sum_{j=0}^{n-2}\overline{z}^j\frac{1}{2\pi i}\int_{\mathbb{T}} g_{n-1,j}(z,t)\gamma(t)\frac{dt}{t}\Big\}\\
&= \frac{1}{2\pi i}\int_{\mathbb{T}} g_{n-1}(z,t)\gamma(t)\frac{dt}{t}.
\end{aligned}
$$

□

Now we can give the main result for polyharmonic Dirichlet problems in the unit disc as follows.

Theorem 3.3. *Let* $\{\, g_n(z,t)\,\}_{n=1}^{\infty}$ *be the sequence of higher order Poisson kernels defined on* $\mathbb{D}\times\mathbb{T}$*, stated as in Theorem 2.6 with* g_n *given as in Theorem 2.8, then for any* $n>1$*, the PHD problem (1.1) is solvable and its general solution is given by*

$$
(3.11)\qquad u(z) = \sum_{j=1}^{n}\frac{4^j}{2\pi i}\int_{\mathbb{T}} g_j(z,t)f_{j-1}(t)\frac{dt}{t} + u_h(z),\ z\in\mathbb{D},
$$

where $u_h(z)$ denotes the general solution of the accompanying homogeneous PHD problem

$$\begin{cases} \Delta^n u = 0 \text{ in } \mathbb{D}, \\ \Delta^j u = 0 \text{ on } \mathbb{T}. \end{cases} \tag{3.12}$$

Proof. Note the inductive property of higher order Poisson kernels stated as in Definition 2.2, and let the polyharmonic operators Δ^l, $1 \le l \le n-1$, act on the two sides of (3.11); By Theorem 3.2, we have

$$\Delta^l u(z) = \sum_{j=l+1}^{n} \frac{4^{j-l}}{2\pi i} \int_{\mathbb{T}} g_{j-l}(z,t) f_{j-1}(t) \frac{dt}{t} + \Delta^l u_h(z) \tag{3.13}$$

since the Laplacian $\Delta = \frac{1}{4}\frac{\partial^2}{\partial z \partial \bar{z}}$. Thus, since $\Delta^l u_h = 0$ on $\mathbb{T}$, the non-tangential boundary value

$$\Delta^l u(s) = f_l(s), \ s \in \mathbb{T}, \ 0 \le l \le n-1 \tag{3.14}$$

follows from (2.29) and the nice property of g_1, i.e.,

$$\lim_{\substack{z \to s \\ z \in \mathbb{D}, s \in \mathbb{T}}} \frac{1}{2\pi i} \int_{\mathbb{T}} g_1(z,t) \gamma(t) dt = \gamma(s) \tag{3.15}$$

for any $\gamma \in L^p(\mathbb{T})$, $p \ge 1$. Similarly, letting the polyharmonic operators Δ^n act on the two sides of (3.11), we have $\Delta^n u(z) = 0$ for any $z \in \mathbb{D}$. Thus (3.11) is a solution of the PHD problem (1.1).

Denote

$$u^*(z) = \sum_{j=1}^{n} \frac{4^j}{2\pi i} \int_{\mathbb{T}} g_j(z,t) f_{j-1}(t) \frac{dt}{t}. \tag{3.16}$$

The above argument shows that u^* is a special solution of the PHD problem (1.1). Since u_h is the general solution of the accompanying homogenous PHD problem (3.12), then it is immediate from linear algebra that (3.11) is the general solution of the PHD problem (1.1). □

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