Acta Mathematica Sinica, English Series
Published online: Apr. 19, 2006
DOI: 10.1007/s10114-005-0754-7
Http://www.ActaMath.com

Acta Mathematica Sinica, English Series

The Mehler Formula for the Generalized Clifford–Hermite Polynomials

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Abstract The Mehler formula for the Hermite polynomials allows for an integral representation of the one-dimensional Fractional Fourier transform. In this paper, we introduce a multi-dimensional Fractional Fourier transform in the framework of Clifford analysis. By showing that it coincides with the classical tensorial approach we are able to prove Mehler's formula for the generalized Clifford–Hermite polynomials of Clifford analysis.

Keywords Clifford analysis, Fractional Fourier transform, Hermite polynomials
MR(2000) Subject Classification 30G35, 42B10

1 Introduction

The Fractional Fourier transform (abbreviated FrFT) may be considered as a fractional power of the classical Fourier transform. It has been intensely studied during the last decade, and may have partially aroused men's attention because of their vivid interest in time-frequency analysis methods of signal processing.

In the one-dimensional case, one obtains an integral representation for the FrFT by means of Mehler's formula for the Hermite polynomials (see [1]). In this paper, we will proceed the other way around. First we introduce a multi-dimensional FrFT in the framework of Clifford analysis making use of the generalized Clifford–Hermite polynomials. Clifford analysis may be regarded as a direct and elegant generalization to higher dimensions of the theory of holomorphic functions in the complex plane, centred around the notion of a monogenic function, i.e. a null solution of the Dirac operator. Then we show that our FrFT coincides with the classical tensorial FrFT in higher dimension. In this way we are able to prove Mehler's formula for the generalized Clifford–Hermite polynomials.

The outline of the paper is as follows. To make it self-contained, a section on definitions and basic properties of Clifford analysis is included (Section 2). In Section 3 we describe the classical

Received April 15, 2004, Accepted January 4, 2005
The work is supported by Research Grant of the University of Macau No. RG021/03-045/QT/FST
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FrFT. Subsequently, we introduce the FrFT in the framework of Clifford analysis (Section 4). First we discuss the generalized Clifford–Hermite polynomials, a multi-dimensional generalization to Clifford analysis of the classical Hermite polynomials on the real line (Subsection 4.1). By means of these generalized Clifford–Hermite polynomials we then construct an orthonormal basis of eigenfunctions of the Fourier transform for the space of square integrable functions (Subsection 4.2). In Subsection 4.3 we define the FrFT in Clifford analysis and show that it can be written as an operator exponential. From this operator exponential form it becomes clear that our FrFT coincides with the tensorial higher-dimensional FrFT. This allows us to derive the so-called Mehler formula for the generalized Clifford–Hermite polynomials (Section 5).

2 Clifford Analysis

Clifford analysis (see e.g. [2] and [3]) offers a function theory which is a higher-dimensional analogue of the theory of the holomorphic functions of one complex variable.

Consider functions defined in $\mathbb{R}^m$ $(m > 1)$ and taking values in the Clifford algebra $\mathbb{R}_m$ or its complexification $\mathbb{C}_m$. If $(e_1, \ldots, e_m)$ is an orthonormal basis of $\mathbb{R}^m$, then a basis for $\mathbb{R}_m$ is given by $(e_A : A \subset \{1, \ldots, m\})$, where $e_\emptyset = 1$ is the identity element. The non-commutative multiplication in the Clifford algebra $\mathbb{R}_m$ is governed by the rules:

$$e_j^2 = -1,\ j = 1, \ldots, m. \quad e_j e_k + e_k e_j = 0,\ j \neq k,\ j, k = 1, \ldots, m.$$

Conjugation is defined as the anti-involution for which $\overline{e_j} = -e_j,\ j = 1, \ldots, m$ with the additional rule $\overline{i} = -i$ in the case of $\mathbb{C}_m$.

The Euclidean space $\mathbb{R}^m$, respectively $\mathbb{R}^{m+1}$, is embedded in the Clifford algebras $\mathbb{R}_m$ and $\mathbb{C}_m$ by identifying $(x_1, \ldots, x_m) \in \mathbb{R}^m$ with the vector variable $\underline{x}$ given by $\underline{x} = \sum_{j=1}^m e_j x_j$, and $(x_0, x_1, \ldots, x_m) \in \mathbb{R}^{m+1}$ with $x_0 + \underline{x}$.

An $\mathbb{R}_m$- or $\mathbb{C}_m$-valued function $F(x_1, \ldots, x_m)$, respectively $F(x_0, x_1, \ldots, x_m)$, is called left monogenic in an open region of $\mathbb{R}^m$, respectively $\mathbb{R}^{m+1}$, if in that region:

$$\partial_{\underline{x}} F = 0, \quad \text{respectively} \quad (\partial_{x_0} + \partial_{\underline{x}})F = 0.$$

Here $\partial_{\underline{x}}$ is the Dirac operator in $\mathbb{R}^m$: $\partial_{\underline{x}} = \sum_{j=1}^m e_j \partial_{x_j}$, which splits the Laplacian in $\mathbb{R}^m$: $\Delta_m = -\partial_{\underline{x}}^2$, whereas $\partial_{x_0} + \partial_{\underline{x}}$ is the Cauchy–Riemann operator in $\mathbb{R}^{m+1}$ for which $\Delta_{m+1} = (\partial_{x_0} + \partial_{\underline{x}})(\partial_{x_0} + \overline{\partial_{\underline{x}}})$. The notion of right-monogenicity is defined in a similar way by letting the Dirac operator or the Cauchy–Riemann operator act from the right.

Let h be a positive function on $\mathbb{R}^m$. Then we consider the inner product

$$(f, g) = \int_{\mathbb{R}^m} h(\underline{x}) \overline{f}(\underline{x}) g(\underline{x}) dV(\underline{x}),$$

where $dV(\underline{x})$ stands for the Lebesgue measure on $\mathbb{R}^m$, and the associated norm $\|f\|^2 = [(f, f)]_0$. Here $[\lambda]_0$ stands for the scalar part of the Clifford number λ. The unitary right Clifford-module of measurable functions on $\mathbb{R}^m$, for which $\|f\|^2 < \infty$, is a right Hilbert Clifford-module, which we denote by $L_2(\mathbb{R}^m, h)$.

Finally, the Fourier transform of $f(\underline{x})$ will be denoted by $\mathscr{F}(f)(\underline{y})$; it is given by:

$$\mathscr{F}(f)(\underline{y}) = \left(\frac{1}{\sqrt{2\pi}}\right)^m \int_{\mathbb{R}^m} \exp\left(-i\langle \underline{x}, \underline{y}\rangle\right) f(\underline{x}) dV(\underline{x}),$$

where $\langle \underline{x}, \underline{y}\rangle = \sum_{j=1}^m x_j y_j$. The function $\exp\left(-\frac{|\underline{x}|^2}{2}\right)$ is an eigenfunction of this Fourier transform:

$$\mathscr{F}\left(\exp\left(-\frac{|\underline{x}|^2}{2}\right)\right)(\underline{y}) = \exp\left(-\frac{|\underline{y}|^2}{2}\right). \tag{1}$$

3 The Classical Fractional Fourier Transform

The idea of fractional powers of the Fourier operator appears in the mathematical literature as early as in 1929 (see [4–6]). It has been rediscovered in quantum mechanics, optics and signal processing. The boom in publications started in the early years of the 1990s and it is still going on. A recent state of the art can be found in [7].

The FrFT on the real line is first defined on a basis for the space of rapidly decreasing functions $\mathscr{S}(\mathbb{R})$. For this basis one uses a complete orthonormal set of eigenfunctions of the Fourier transform given by

$$(\mathscr{F} f)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) \exp(-i\xi x) dx, \quad f \in \mathscr{S}(\mathbb{R}).$$

A possible choice for these eigenfunctions leads to the normalized Hermite–Gauss functions:

$$\phi_n(x) = \frac{2^{1/4}}{\sqrt{2^n n!}} \exp\left(-\frac{x^2}{2}\right) H_n(x),$$

where $H_n(x) = (-1)^n \exp(x^2) \frac{d^n}{dx^n}\big(\exp(-x^2)\big)$ are the Hermite polynomials associated with the weight function $\exp(-x^2)$. These eigenfunctions satisfy the orthonormality relation $\langle \phi_n, \phi_m \rangle = \delta_{n,m}$ with $\langle f, g \rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) g(x) dx$ and the eigenvalue equation $\mathscr{F} \phi_n = \exp\big(-in\frac{\pi}{2}\big)\phi_n$. The eigenvalue for ϕ_n is thus given by $\lambda_n = \lambda^n$, with $\lambda = \exp\big(-i\frac{\pi}{2}\big)$ representing a rotation over an angle $\frac{\pi}{2}$.

The FrFT generalizes the concept of rotating over an angle that is $\frac{\pi}{2}$ in the classical Fourier transform situation. Like the classical Fourier transform corresponds to a rotation in the time-frequency plane over an angle $\alpha = \frac{\pi}{2}$, the FrFT corresponds to a rotation over an arbitrary angle $\alpha = a\frac{\pi}{2}$ with $a \in \mathbb{R}$.

Consequently the FrFT is defined by

$$\mathscr{F}^a \phi_n = \exp\left(-ina\frac{\pi}{2}\right)\phi_n = \lambda_n^a \phi_n = \lambda_a^n \phi_n, \tag{2}$$

with $\lambda_a = \exp\big(-ia\frac{\pi}{2}\big) = \exp(-i\alpha)$ causing a rotation over an angle $\alpha = a\frac{\pi}{2}$. Thus the classical Fourier transform corresponds to $\mathscr{F}^1$. Note also that, for $\alpha = a = 0$, we get the identity operator $\mathscr{F}^0 = I$ and for $\alpha = \pi$ or $a = 2$ we get the parity operator $\mathscr{F}^2 = \exp(-i\pi\mathscr{H})$.

The FrFT can be written as an operator exponential $\mathscr{F}^a = \exp(-i\alpha\mathscr{H})$, so that

$$\exp(-i\alpha\mathscr{H})\left(\exp\left(-\frac{x^2}{2}\right)H_n(x)\right) = \exp(-in\alpha)\exp\left(-\frac{x^2}{2}\right)H_n(x).$$

Differentiating the above relation with respect to α, setting $\alpha = 0$ and then using the differential equation $H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0$, one can easily verify that the operator $\mathscr{H}$ is given by $\mathscr{H} = -\frac{1}{2}\big(\frac{d^2}{dx^2} - x^2 + 1\big)$.

As the set of normalized Hermite–Gauss functions ϕ_n constitutes an orthonormal basis for $L^2(\mathbb{R})$, each function $f \in L^2(\mathbb{R})$ can be expanded in terms of these eigenfunctions ϕ_n:

$$f = \sum_{n=0}^{\infty} a_n \phi_n,$$

where the coefficients a_n are given by

$$a_n = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \phi_n(x) f(x) dx = \frac{1}{\sqrt{2^n n! \pi\sqrt{2}}} \int_{-\infty}^{+\infty} H_n(x) \exp\left(-\frac{x^2}{2}\right) f(x) dx. \tag{3}$$

Applying the FrFT yields

$$\mathscr{F}^a f = \sum_{n=0}^{\infty} a_n \exp\left(-ina\frac{\pi}{2}\right)\phi_n. \tag{4}$$

The calculation of transforms by means of the series (4) is usually not practical. In order to obtain the integral representation of the operator $\mathscr{F}^a$, a formula due to Mehler is used:

$$\sum_{n=0}^{\infty} \frac{\exp(-in\alpha) H_n(\xi) H_n(x)}{2^n n! \sqrt{\pi}} = \frac{\exp\left(\frac{2x\xi\exp(-i\alpha) - \exp(-2i\alpha)(\xi^2 + x^2)}{1 - \exp(-2i\alpha)}\right)}{\sqrt{\pi(1 - \exp(-2i\alpha))}}.$$

Inserting a_n from equation (3) into equation (4) and using Mehler's formula, one obtains

$$(\mathscr{F}^a f)(\xi) = \frac{1}{\sqrt{\pi}\sqrt{1-\exp(-2i\alpha)}} \int_{-\infty}^{+\infty} \exp\left(\frac{2x\xi\exp(-i\alpha) - \exp(-2i\alpha)(\xi^2+x^2)}{1-\exp(-2i\alpha)}\right) \exp\left(-\frac{\xi^2+x^2}{2}\right) f(x)dx. \quad (5)$$

Note that, for $0 < |\alpha| < \pi$, this expression can also be written as

$$(\mathscr{F}^a f)(\xi) = \frac{\exp\left(-\frac{i}{2}\left(\frac{\pi}{2}\hat{\alpha}-\alpha\right)\right)\exp\left(\frac{i}{2}\xi^2\cot(\alpha)\right)}{\sqrt{2\pi|\sin(\alpha)|}} \int_{-\infty}^{+\infty} \exp\left(-i\frac{x\xi}{\sin(\alpha)} + \frac{i}{2}x^2\cot(\alpha)\right) f(x)dx,$$

where $\hat{\alpha} = \mathrm{sgn}(\sin(\alpha))$.

It was already mentioned that $(\mathscr{F}^0 f)(\xi) = f(\xi)$ and $(\mathscr{F}^{\pm\pi} f)(\xi) = f(-\xi)$. Furthermore, when $|\alpha| > \pi$, the definition is taken modulo 2π and reduced to the interval $[-\pi, \pi]$.

The FrFT can be extended to higher dimensions by taking tensor products. If $K_a(\xi, x)$ denotes the kernel of the one-dimensional FrFT, i.e.

$$(\mathscr{F}^a f)(\xi) = \int_{-\infty}^{+\infty} K_a(\xi, x) f(x)dx,$$

then one defines the m-dimensional FrFT as follows:

$$(\mathscr{F}^{a_1,\dots,a_m} f)(\xi_1,\dots,\xi_m) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} K_{a_1,\dots,a_m}(\xi_1,\dots,\xi_m; x_1,\dots,x_m) f(x_1,\dots,x_m)dx_1\cdots dx_m,$$

where

$$K_{a_1,\dots,a_m}(\xi_1,\dots,\xi_m; x_1,\dots,x_m) = K_{a_1}(\xi_1, x_1)\cdots K_{a_m}(\xi_m, x_m).$$

4 The Fractional Fourier Transform in the Framework of Clifford Analysis

4.1 Properties of the Generalized Clifford–Hermite polynomials

In order to obtain a basis for the weighted Hilbert module $L_2\left(\mathbb{R}^m, \exp\left(-\frac{|\underline{x}|^2}{2}\right)\right)$, Sommen introduced the generalized Clifford–Hermite polynomials $H_{\ell,m,k}(\underline{x})$ (see [8]), which satisfy the recurrence relation:

$$H_{\ell+1,m,k}(\underline{x})P_k(\underline{x}) = (\underline{x} - \partial_{\underline{x}})\left(H_{\ell,m,k}(\underline{x})P_k(\underline{x})\right). \quad (6)$$

Here $P_k(\underline{x})$ denotes a left monogenic homogeneous polynomial of degree k, inner spherical monogenic of order k for short. The function $H_{\ell,m,k}(\underline{x})$ is a polynomial of degree ℓ in the variable $\underline{x}$ with real coefficients depending on k. $H_{2\ell,m,k}(\underline{x})$ contains only even powers of $\underline{x}$, while $H_{2\ell+1,m,k}(\underline{x})$ contains only odd ones. Moreover, the generalized Clifford–Hermite polynomials satisfy the Rodrigues formula

$$H_{\ell,m,k}(\underline{x})P_k(\underline{x}) = \exp\left(\frac{|\underline{x}|^2}{2}\right)(-\partial_{\underline{x}})^\ell\left(\exp\left(-\frac{|\underline{x}|^2}{2}\right)P_k(\underline{x})\right)$$

and the orthogonality relation

$$\int_{\mathbb{R}^m} \exp\left(-\frac{|\underline{x}|^2}{2}\right)\overline{H_{\ell,m,k_1}(\underline{x})P_{k_1}(\underline{x})}H_{t,m,k_2}(\underline{x})P_{k_2}(\underline{x})dV(\underline{x}) = \gamma_{\ell,k_1}\ \delta_{\ell,t}\ \delta_{k_1,k_2},$$

with

$$\gamma_{2p,k} = \frac{2^{2p+m/2+k}p!\pi^{m/2}\Gamma\left(\frac{m}{2}+k+p\right)}{\Gamma\left(\frac{m}{2}\right)}$$

and

$$\gamma_{2p+1,k} = \frac{2^{2p+m/2+k+1}p!\pi^{m/2}\Gamma\left(\frac{m}{2}+k+p+1\right)}{\Gamma\left(\frac{m}{2}\right)}.$$

Furthermore, the set

$$\left\{\frac{1}{(\gamma_{\ell,k})^{1/2}}H_{\ell,m,k}(\underline{x})P_k^{(j)}(\underline{x}); \ell\in\mathbb{N}, k\in\mathbb{N}, j\leq\dim(M_\ell^+(k))\right\}$$

constitutes an orthonormal basis for $L_2\big(\mathbb{R}^m, \exp\big(-\frac{|\underline{x}|^2}{2}\big)\big)$. Here

$$\big\{P_k^{(j)}(\underline{x}); j = 1, 2, \ldots, \dim(M_\ell^+(k))\big\}$$

denotes an orthonormal basis of $M_\ell^+(k)$, the space of left inner spherical monogenics of order k.

By a straightforward, but lengthy, calculation the following fundamental properties of the generalized Clifford–Hermite polynomials may be proved:

Proposition 1 *The generalized Clifford–Hermite polynomials satisfy:*

(i) $\partial_{\underline{x}}(\underline{x} H_{\ell,m,k}(\underline{x})P_k(\underline{x})) = a_{\ell,m,k} H_{\ell,m,k}(\underline{x})P_k(\underline{x}) + \underline{x}\ \partial_{\underline{x}}(H_{\ell,m,k}(\underline{x})P_k(\underline{x}))$;

(ii) $\partial_{\underline{x}}(H_{\ell,m,k}(\underline{x})P_k(\underline{x})) = -C_{\ell,m,k} H_{\ell-1,m,k}(\underline{x})P_k(\underline{x})$;

(iii) $\partial_{\underline{x}}^2(H_{\ell,m,k}(\underline{x})P_k(\underline{x})) - \underline{x}\ \partial_{\underline{x}}(H_{\ell,m,k}(\underline{x})P_k(\underline{x})) = C_{\ell,m,k} H_{\ell,m,k}(\underline{x})P_k(\underline{x})$, *with*

$$a_{\ell,m,k} = \begin{cases} -(m+2k), & \text{for } \ell \text{ even}, \\ m+2k-2, & \text{for } \ell \text{ odd}, \end{cases} \quad \text{and} \quad C_{\ell,m,k} = \begin{cases} \ell, & \text{for } \ell \text{ even}, \\ \ell - 1 + m + 2k, & \text{for } \ell \text{ odd}. \end{cases}$$

Formula (iii) of Proposition 1 generalizes the differential equation

$$H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0$$

satisfied by the classical Hermite polynomials on the real line.

Furthermore, combining formula (ii) of Proposition 1 and formula (6) yields

$$H_{\ell+1,m,k}(\underline{x})P_k(\underline{x}) - \underline{x}H_{\ell,m,k}(\underline{x})P_k(\underline{x}) - C_{\ell,m,k}H_{\ell-1,m,k}(\underline{x})P_k(\underline{x}) = 0,$$

which is a generalization of the recurrence relation $H_{n+1}(x)+2nH_{n-1}(x)-2xH_n(x) = 0$ satisfied by the classical Hermite polynomials.

4.2 Orthonormal Basis of Eigenfunctions of the Fourier Transform for $L_2(\mathbb{R}^m)$

In this subsection we will construct an orthonormal basis of eigenfunctions of the Fourier transform for $L_2(\mathbb{R}^m)$. We start with the following result:

Proposition 2 *The set*

$$\left\{\frac{1}{(\gamma_{\ell,k})^{1/2}} H_{\ell,m,k}(\underline{x})P_k^{(j)}(\underline{x}) \exp\left(-\frac{|\underline{x}|^2}{4}\right); \ell \in \mathbb{N}, k \in \mathbb{N}, j \le \dim(M_\ell^+(k))\right\}$$

constitutes an orthonormal basis for $L_2(\mathbb{R}^m)$.

In view of (1), we carry out the substitution $\underline{x} \to \sqrt{2}\ \underline{x}$, which leads to the following orthonormal basis for $L_2(\mathbb{R}^m)$:

$$\left\{\frac{2^{m/4}}{(\gamma_{\ell,k})^{1/2}} H_{\ell,m,k}(\sqrt{2}\underline{x})P_k^{(j)}(\sqrt{2}\underline{x}) \exp\left(-\frac{|\underline{x}|^2}{2}\right); \ell \in \mathbb{N},\ k \in \mathbb{N},\ j \le \dim(M_\ell^+(k))\right\}.$$

It may be proved by induction that this basis consists of eigenfunctions of the Fourier transform.

Proposition 3 *For all inner spherical monogenics $P_k(\underline{x})$ of order $k \in \mathbb{N}$ and all $\ell \in \mathbb{N}$, one has*

$$\mathscr{F}\left(\exp\left(-\frac{|\underline{x}|^2}{2}\right) H_{\ell,m,k}(\sqrt{2}\underline{x})P_k(\sqrt{2}\underline{x})\right)(\underline{y})$$
$$= \exp\left(-i(\ell+k)\frac{\pi}{2}\right) \exp\left(-\frac{|\underline{y}|^2}{2}\right) H_{\ell,m,k}(\sqrt{2}\underline{y})P_k(\sqrt{2}\underline{y}).$$

4.3 The Fractional Fourier Transform: Definition and Operator Exponential Form

In what follows, we denote for $\ell, k \in \mathbb{N},\ j \le \dim(M_\ell^+(k))$:

$$\phi_{\ell,k,j}(\underline{x}) = \frac{2^{m/4}}{(\gamma_{\ell,k})^{1/2}} H_{\ell,m,k}(\sqrt{2}\underline{x})P_k^{(j)}(\sqrt{2}\underline{x}) \exp\left(-\frac{|\underline{x}|^2}{2}\right).$$

In the previous section (Proposition 3), we have shown that

$$(\mathscr{F}(\phi_{\ell,k,j}(\underline{x})))(\underline{y}) = \exp\left(-i(\ell+k)\frac{\pi}{2}\right)\phi_{\ell,k,j}(\underline{y}).$$

Corresponding to the definition on the real line (see (2)), we define the multi-dimensional FrFT in Clifford analysis by:

$$\begin{aligned}(\mathscr{F}_C^a(\phi_{\ell,k,j}(\underline{x})))(\underline{y}) &= \exp\left(-i(\ell+k)a\frac{\pi}{2}\right)\phi_{\ell,k,j}(\underline{y}); \quad a \in \mathbb{R} \\ &= \exp\big(-i(\ell+k)\alpha\big)\phi_{\ell,k,j}(\underline{y}),\end{aligned}$$

with $\alpha = a\frac{\pi}{2}$.

Now we will show that, similarly to the classical case, the FrFT $\mathscr{F}_C^a$ can be written as an operator exponential.

Proposition 4 *The Fractional Fourier transform $\mathscr{F}_C^a$ can be written as an operator exponential $\mathscr{F}_C^a = \exp(-i\alpha\mathscr{H}_C) = \exp(-ia\frac{\pi}{2}\mathscr{H}_C)$, where the operator $\mathscr{H}_C$ is given by $\mathscr{H}_C = \frac{1}{2}(\partial_{\underline{x}}^2 - \underline{x}^2 - mI) = -\frac{1}{2}(\Delta_m - |\underline{x}|^2 + mI)$ with I the identity operator.*

Proof First we remark that the operator exponential $\exp(-i\alpha\mathscr{H}_C)$ is defined as a series

$$\exp(-i\alpha\mathscr{H}_C) = \sum_{n=0}^{\infty}(-i\alpha)^n\frac{\mathscr{H}_C^n}{n!}.$$

Differentiating the relation

$$\begin{aligned}&\exp(-i\alpha\mathscr{H}_C)\bigg(H_{\ell,m,k}(\sqrt{2}\underline{x})P_k^{(j)}(\sqrt{2}\underline{x})\exp\left(-\frac{|\underline{x}|^2}{2}\right)\bigg) \\ &= \exp\big(-i(\ell+k)\alpha\big)H_{\ell,m,k}(\sqrt{2}\underline{x})P_k^{(j)}(\sqrt{2}\underline{x})\exp\left(-\frac{|\underline{x}|^2}{2}\right)\end{aligned}$$

with respect to α, and setting α equal to zero, yields

$$\mathscr{H}_C\bigg(H_{\ell,m,k}(\sqrt{2}\underline{x})P_k^{(j)}(\sqrt{2}\underline{x})\exp\left(-\frac{|\underline{x}|^2}{2}\right)\bigg) = (\ell+k)H_{\ell,m,k}(\sqrt{2}\underline{x})P_k^{(j)}(\sqrt{2}\underline{x})\exp\left(-\frac{|\underline{x}|^2}{2}\right).$$

Now we will verify that the operator $\mathscr{H}_C$ is indeed given by $\mathscr{H}_C = \frac{1}{2}(\partial_{\underline{x}}^2 - \underline{x}^2 - mI)$. We have

$$\begin{aligned}&(\partial_{\underline{x}}^2 - \underline{x}^2 - mI)\bigg(\exp\left(-\frac{|\underline{x}|^2}{2}\right)H_{\ell,m,k}(\sqrt{2}\underline{x})P_k^{(j)}(\sqrt{2}\underline{x})\bigg) \\ &= -\exp\left(-\frac{|\underline{x}|^2}{2}\right)\partial_{\underline{x}}\big(\underline{x}H_{\ell,m,k}(\sqrt{2}\underline{x})P_k^{(j)}(\sqrt{2}\underline{x})\big) \\ &\quad - \underline{x}\exp\left(-\frac{|\underline{x}|^2}{2}\right)\partial_{\underline{x}}\big(H_{\ell,m,k}(\sqrt{2}\underline{x})P_k^{(j)}(\sqrt{2}\underline{x})\big) \\ &\quad + \exp\left(-\frac{|\underline{x}|^2}{2}\right)\partial_{\underline{x}}^2\big(H_{\ell,m,k}(\sqrt{2}\underline{x})P_k^{(j)}(\sqrt{2}\underline{x})\big) \\ &\quad - m\exp\left(-\frac{|\underline{x}|^2}{2}\right)H_{\ell,m,k}(\sqrt{2}\underline{x})P_k^{(j)}(\sqrt{2}\underline{x}).\end{aligned}$$

By means of formula (i) of Proposition 1, this becomes

$$\begin{aligned}&(\partial_{\underline{x}}^2 - \underline{x}^2 - mI)\bigg(\exp\left(-\frac{|\underline{x}|^2}{2}\right)H_{\ell,m,k}(\sqrt{2}\underline{x})P_k^{(j)}(\sqrt{2}\underline{x})\bigg) \\ &= \exp\left(-\frac{|\underline{x}|^2}{2}\right)\big\{\partial_{\underline{x}}^2\big(H_{\ell,m,k}(\sqrt{2}\underline{x})P_k^{(j)}(\sqrt{2}\underline{x})\big) - 2\underline{x}\ \partial_{\underline{x}}\big(H_{\ell,m,k}(\sqrt{2}\underline{x})P_k^{(j)}(\sqrt{2}\underline{x})\big) \\ &\qquad - (a_{\ell,m,k}+m)H_{\ell,m,k}(\sqrt{2}\underline{x})P_k^{(j)}(\sqrt{2}\underline{x})\big\}.\end{aligned}$$

Formula (iii) of Proposition 1 finally leads to

$$\begin{aligned}&\frac{1}{2}(\partial_{\underline{x}}^2-\underline{x}^2-mI)\left(\exp\left(-\frac{|\underline{x}|^2}{2}\right)H_{\ell,m,k}(\sqrt{2}\underline{x})P_k^{(j)}(\sqrt{2}\underline{x})\right)\\&=\frac{1}{2}(2C_{\ell,m,k}-a_{\ell,m,k}-m)H_{\ell,m,k}(\sqrt{2}\underline{x})P_k^{(j)}(\sqrt{2}\underline{x})\exp\left(-\frac{|\underline{x}|^2}{2}\right)\\&=(\ell+k)H_{\ell,m,k}(\sqrt{2}\underline{x})P_k^{(j)}(\sqrt{2}\underline{x})\exp\left(-\frac{|\underline{x}|^2}{2}\right),\end{aligned}$$

since for all $\ell\in\mathbb{N}$, $2C_{\ell,m,k}-a_{\ell,m,k}-m=2\ell+2k$.

From Proposition 4 we observe that, surprisingly, our FrFT coincides with the classical tensorial higher-dimensional FrFT $\mathscr{F}^{a_1,\dots,a_m}$ with $a_1=a_2=\cdots=a_m=a$.

5 The Mehler Formula for the Generalized Clifford–Hermite Polynomials

A Clifford algebra-valued square integrable function $f(\underline{x})$ can be expanded in terms of the eigenfunctions $\{\phi_{\ell,k,j}\}$:

$$f(\underline{x})=\sum_{\ell=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=1}^{\dim(M_\ell^+(k))}\phi_{\ell,k,j}(\underline{x})\ a_{\ell,k,j},$$

where the Clifford algebra-valued coefficients $a_{\ell,k,j}$ are given by

$$a_{\ell,k,j}=\langle\phi_{\ell,k,j},f\rangle=\int_{\mathbb{R}^m}\overline{\phi}_{\ell,k,j}(\underline{x})f(\underline{x})dV(\underline{x}). \tag{7}$$

By applying the operator $\mathscr{F}_C^a$, we get

$$\begin{aligned}\big(\mathscr{F}_C^a f(\underline{x})\big)(\underline{y})&=\sum_{\ell=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=1}^{\dim(M_\ell^+(k))}\big(\mathscr{F}_C^a\phi_{\ell,k,j}(\underline{x})\big)(\underline{y})\ a_{\ell,k,j}\\&=\sum_{\ell=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=1}^{\dim(M_\ell^+(k))}\exp\left(-i(\ell+k)\alpha\right)\ \phi_{\ell,k,j}(\underline{y})\ a_{\ell,k,j}.\end{aligned}$$

We thus have obtained the definition of the FrFT $\mathscr{F}_C^a$ in the form of a series.

By replacing $a_{\ell,k,j}$ in the series by their integral expression (7), it is turned into

$$\begin{aligned}&\big(\mathscr{F}_C^a f(\underline{x})\big)(\underline{y})\\&=\sum_{\ell=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=1}^{\dim(M_\ell^+(k))}\exp\left(-i(\ell+k)\alpha\right)\frac{2^{m/4}}{(\gamma_{\ell,k})^{1/2}}H_{\ell,m,k}(\sqrt{2}\underline{y})P_k^{(j)}(\sqrt{2}\underline{y})\exp\left(-\frac{|\underline{y}|^2}{2}\right)\\&\quad\int_{\mathbb{R}^m}\frac{2^{m/4}}{(\gamma_{\ell,k})^{1/2}}\overline{H_{\ell,m,k}(\sqrt{2}\underline{x})P_k^{(j)}(\sqrt{2}\underline{x})}\exp\left(-\frac{|\underline{x}|^2}{2}\right)f(\underline{x})dV(\underline{x})\\&=2^{m/2}\int_{\mathbb{R}^m}\left\{\sum_{\ell=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=1}^{\dim(M_\ell^+(k))}\frac{\exp\left(-i(\ell+k)\alpha\right)}{\gamma_{\ell,k}}H_{\ell,m,k}(\sqrt{2}\underline{y})P_k^{(j)}(\sqrt{2}\underline{y})\right.\\&\qquad\left.\overline{H_{\ell,m,k}(\sqrt{2}\underline{x})P_k^{(j)}(\sqrt{2}\underline{x})}\right\}\exp\left(-\frac{|\underline{x}|^2+|\underline{y}|^2}{2}\right)f(\underline{x})dV(\underline{x}).\end{aligned} \tag{8}$$

On the other hand, from the previous section we know that our FrFT coincides with $\mathscr{F}^{a,\dots,a}$. Consequently, by means of (5), we have

$$\begin{aligned}(\mathscr{F}_C^a f(\underline{x}))(\underline{y})&=\int_{\mathbb{R}^m}K_a(y_1,x_1)\cdots K_a(y_m,x_m)f(\underline{x})dV(\underline{x})\\&=\left(\frac{1}{\sqrt{\pi}\sqrt{1-\exp\left(-2i\alpha\right)}}\right)^m\int_{\mathbb{R}^m}\exp\left(\frac{2x_1y_1\exp\left(-i\alpha\right)-\exp\left(-2i\alpha\right)(y_1^2+x_1^2)}{1-\exp\left(-2i\alpha\right)}\right)\end{aligned}$$

$$\exp\left(-\frac{y_1^2+x_1^2}{2}\right)\cdots\ \exp\left(\frac{2x_my_m\exp(-i\alpha)-\exp(-2i\alpha)(y_m^2+x_m^2)}{1-\exp(-2i\alpha)}\right)$$
$$\exp\left(-\frac{y_m^2+x_m^2}{2}\right)f(\underline{x})dV(\underline{x})$$
$$=\left(\frac{1}{\sqrt{\pi}\sqrt{1-\exp(-2i\alpha)}}\right)^m\int_{\mathbb{R}^m}\exp\left(\frac{2\langle\underline{x},\underline{y}\rangle\exp(-i\alpha)}{1-\exp(-2i\alpha)}\right)$$
$$\exp\left(-\frac{(|\underline{x}|^2+|\underline{y}|^2)\exp(-2i\alpha)}{1-\exp(-2i\alpha)}\right)\exp\left(-\frac{|\underline{x}|^2+|\underline{y}|^2}{2}\right)f(\underline{x})dV(\underline{x}). \tag{9}$$

Comparing (8) and (9) yields the following result:

Theorem 1 *The Mehler formula for the generalized Clifford–Hermite polynomials takes the following form*:

$$\sum_{\ell=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=1}^{\dim(M_\ell^+(k))}\frac{\exp(-i(\ell+k)\alpha)}{\gamma_{\ell,k}}H_{\ell,m,k}(\sqrt{2}\underline{y})P_k^{(j)}(\sqrt{2}\underline{y})\overline{H_{\ell,m,k}(\sqrt{2}\underline{x})P_k^{(j)}(\sqrt{2}\underline{x})}$$
$$=\left(\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{1-\exp(-2i\alpha)}}\right)^m\ \exp\left(\frac{2\langle\underline{x},\underline{y}\rangle\exp(-i\alpha)-(|\underline{x}|^2+|\underline{y}|^2)\exp(-2i\alpha)}{1-\exp(-2i\alpha)}\right).$$

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