Quaternion and Clifford–Fourier Transforms and Wavelets
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6 Bochner–Minlos Theorem and Quaternion Fourier Transform

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Abstract. There have been several attempts in the literature to generalize the classical Fourier transform by making use of the Hamiltonian quaternion algebra. The first part of this chapter features certain properties of the asymptotic behaviour of the quaternion Fourier transform. In the second part we introduce the quaternion Fourier transform of a probability measure, and we establish some of its basic properties. In the final analysis, we introduce the notion of positive definite measure, and we set out to extend the classical Bochner–Minlos theorem to the framework of quaternion analysis.

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1. Introduction and Statement of Results

As is well known, the *classical Fourier transform* (FT) has wide applications in engineering, computer sciences, physics and applied mathematics. For instance, the FT can be used to provide signal analysis techniques where the signal from the original time domain is transformed to the frequency domain. Therefore there exists great interest and considerable effort to extend the FT to higher dimensions, and study its properties and interdependencies (see, *e.g.*, [1–7, 9–11, 17, 18, 20–23] and elsewhere). In view of numerous applications in physics and engineering problems, one is particularly interested in higher-dimensional analogues to $\mathbb{R}^n$, in particular, for $n = 4$. To this end, so far quaternion analysis offers the possibility of generalizing the underlying function theory in 2D to 4D, with the advantage of meeting exactly these goals. To aid the reader, see [15, 16, 19, 24, 25] for more complete accounts of this subject and related topics.

The first part of the present work is devoted to the study of the asymptotic behaviour of the *quaternion Fourier transform* (QFT). The QFT was first introduced by Ell in [10]. He proposed a two-sided QFT and studied some of its applications and properties. Later, Bülow [7] (*cf.* [8] and [1,20]) has also conducted a generalization of the real and complex FT using the quaternion algebra based on two complex variables, but, to the best of our knowledge, a detailed study of its asymptotic behaviour has not been carried out yet. The main motivation of the present study is to develop further general numerical methods for partial differential equations and to extend localization theorems for summation of Fourier series in the quaternion analysis setting. In a forthcoming article we shall describe these connections in more detail and illustrate them by some typical examples.

Due to the noncommutativity of the quaternions, there are three different types of QFT: a right-sided QFT, a left-sided QFT, and a two-sided QFT [23]. We will carry out the investigation of the following finite integral (defined from the time domain to the frequency domain)

$$\mathcal{F}_r(f)(\omega_1,\omega_2) := \int_a^b \int_a^b f(x_1,x_2)\, e^{-\boldsymbol{i}\omega_1 x_1}\, e^{-\boldsymbol{j}\omega_2 x_2}\, dx_1 dx_2, \tag{1.1}$$

where the signal $f : [a,b] \times [a,b] \subset \mathbb{R}^2 \longrightarrow \mathbb{H}$ will be taken to be

$$f(x_1,x_2) := [f(x_1,x_2)]_0 + [f(x_1,x_2)]_1\boldsymbol{i} + [f(x_1,x_2)]_2\boldsymbol{j} + [f(x_1,x_2)]_3\boldsymbol{k},$$

$$[f]_l : [a,b] \times [a,b] \longrightarrow \mathbb{R} \qquad (l = 0,1,2,3)$$

satisfying certain conditions, guaranteeing the convergence of the above integral. $\mathcal{F}_r(f)(\omega_1,\omega_2)$ is the *(finite) right-sided Fourier transform* [21] of the quaternion function $f(x_1,x_2)$, and it may be interpreted as a quaternionic extension of the classical FT; the exponential product $e^{-\boldsymbol{i}\omega_1 x_1}e^{-\boldsymbol{j}\omega_2 x_2}$ is called the *(right-sided) quaternion Fourier kernel*, and for $i = 1,2$; x_i will denote the *space* and ω_i the *angular frequency* variables. The previous definition of the QFT varies from the original one only in the fact that we use 2D vectors instead of scalars and that it is defined to be two-dimensional. Here $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are unit pure quaternions (*i.e.*, the quaternions with unit magnitude having no scalar part) that are orthogonal to each other. We point out that the product in (1.1) has to be written in a fixed order since, in general, $e^{-\boldsymbol{i}\omega_1 x_1}e^{-\boldsymbol{j}\omega_2 x_2}$ does not commute with every element of the algebra.

Remark 1.1. Throughout this text we investigate the integral (1.1) only that, for simplicity, we denote by $\mathcal{F}(f)$. Nevertheless, all results can be easily performed from the left-hand side:

$$\mathcal{F}_l(f)(\omega_1,\omega_2) := \int_a^b \int_a^b e^{-\boldsymbol{j}\omega_2 x_2}\, e^{-\boldsymbol{i}\omega_1 x_1}\, f(x_1,x_2)\, dx_1 dx_2,$$

since

$$\mathcal{F}_r(f)(-\omega_1,-\omega_2) = \int_a^b \int_a^b f(x_1,x_2) e^{\boldsymbol{i}\omega_1 x_1} e^{\boldsymbol{j}\omega_2 x_2} dx_1 dx_2$$

$$= \int_a^b \int_a^b f(x_1,x_2)\overline{e^{-j\omega_2 x_2} e^{-i\omega_1 x_1}} dx_1 dx_2$$

$$= \overline{\int_a^b \int_a^b e^{-j\omega_2 x_2} e^{-i\omega_1 x_1} \overline{f(x_1,x_2)} dx_1 dx_2} = \overline{\mathcal{F}_l(\overline{f})}(\omega_1,\omega_2).$$

Lemma 1.2. *The QFT of a* 2D *signal* $f \in L^1([a,b]\times[a,b];\mathbb{H})$ *has the closed-form representation:*

$$\mathcal{F}(f)(\omega_1,\omega_2) := \Phi_0(\omega_1,\omega_2) + \Phi_1(\omega_1,\omega_2) + \Phi_2(\omega_1,\omega_2) + \Phi_3(\omega_1,\omega_2),$$

where the integrals are

$$\Phi_0(\omega_1,\omega_2) = \int_a^b \int_a^b f(x_1,x_2)\cos(\omega_1 x_1)\cos(\omega_2 x_2) dx_1 dx_2,$$

$$\Phi_1(\omega_1,\omega_2) = -\int_a^b \int_a^b f(x_1,x_2) i \sin(\omega_1 x_1)\cos(\omega_2 x_2) dx_1 dx_2,$$

$$\Phi_2(\omega_1,\omega_2) = -\int_a^b \int_a^b f(x_1,x_2) j \cos(\omega_1 x_1)\sin(\omega_2 x_2) dx_1 dx_2,$$

$$\Phi_3(\omega_1,\omega_2) = \int_a^b \int_a^b f(x_1,x_2) k \sin(\omega_1 x_1)\sin(\omega_2 x_2) dx_1 dx_2.$$

For illustrative purposes, we have the following identities:

Corollary 1.3. *The QFT of a* 2D *signal* $f \in L^1([a,b]\times[a,b];\mathbb{H})$ *satisfies the following relations:*

$$\mathcal{F}(f)(\omega_1,\omega_2) + \mathcal{F}(f)(\omega_1,-\omega_2) = 2\left(\Phi_0(\omega_1,\omega_2) + \Phi_1(\omega_1,\omega_2)\right),$$
$$\mathcal{F}(f)(\omega_1,\omega_2) - \mathcal{F}(f)(\omega_1,-\omega_2) = 2\left(\Phi_2(\omega_1,\omega_2) + \Phi_3(\omega_1,\omega_2)\right),$$
$$\mathcal{F}(f)(\omega_1,\omega_2) + \mathcal{F}(f)(-\omega_1,\omega_2) = 2\left(\Phi_0(\omega_1,\omega_2) + \Phi_2(\omega_1,\omega_2)\right),$$
$$\mathcal{F}(f)(\omega_1,\omega_2) - \mathcal{F}(f)(-\omega_1,\omega_2) = 2\left(\Phi_1(\omega_1,\omega_2) + \Phi_3(\omega_1,\omega_2)\right),$$
$$\mathcal{F}(f)(\omega_1,\omega_2) + \mathcal{F}(f)(-\omega_1,-\omega_2) = 2\left(\Phi_0(\omega_1,\omega_2) + \Phi_3(\omega_1,\omega_2)\right),$$
$$\mathcal{F}(f)(\omega_1,\omega_2) - \mathcal{F}(f)(-\omega_1,-\omega_2) = 2\left(\Phi_1(\omega_1,\omega_2) + \Phi_2(\omega_1,\omega_2)\right),$$
$$\mathcal{F}(f)(\omega_1,-\omega_2) + \mathcal{F}(f)(-\omega_1,-\omega_2) = 2\left(\Phi_0(\omega_1,\omega_2) - \Phi_2(\omega_1,\omega_2)\right),$$
$$\mathcal{F}(f)(\omega_1,-\omega_2) - \mathcal{F}(f)(-\omega_1,-\omega_2) = 2\left(\Phi_1(\omega_1,\omega_2) - \Phi_3(\omega_1,\omega_2)\right),$$
$$\mathcal{F}(f)(-\omega_1,\omega_2) + \mathcal{F}(f)(-\omega_1,-\omega_2) = 2\left(\Phi_0(\omega_1,\omega_2) - \Phi_1(\omega_1,\omega_2)\right),$$
$$\mathcal{F}(f)(-\omega_1,\omega_2) - \mathcal{F}(f)(-\omega_1,-\omega_2) = 2\left(\Phi_2(\omega_1,\omega_2) - \Phi_3(\omega_1,\omega_2)\right).$$

Under suitable conditions, the original signal f can be reconstructed from $\mathcal{F}(f)$ by the inverse transform (frequency to time domains).

Definition 1.4. The (*right-sided*) *inverse QFT* of $g \in L^1(\mathbb{R}^2;\mathbb{H})$ is given by

$$\mathcal{F}_r^{-1}(g) : \mathbb{R}^2 \longrightarrow \mathbb{H},$$

$$\mathcal{F}_r^{-1}(g)(x_1,x_2) := \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} g(\omega_1,\omega_2)\, e^{j\omega_2 x_2}\, e^{i\omega_1 x_1}\, d\omega_1 d\omega_2.$$

It has the closed-form representation:

$$\mathcal{F}_r^{-1}(g)(x_1,x_2) := \hat{\Phi}_0(x_1,x_2) + \hat{\Phi}_1(x_1,x_2) + \hat{\Phi}_2(x_1,x_2) + \hat{\Phi}_3(x_1,x_2),$$

where the integrals are

$$\begin{aligned}
\hat{\Phi}_0(x_1,x_2) &= \frac{1}{(2\pi)^2}\int_a^b\int_a^b g(\omega_1,\omega_2)\cos(\omega_1 x_1)\cos(\omega_2 x_2)d\omega_1 d\omega_2,\\
\hat{\Phi}_1(x_1,x_2) &= \frac{1}{(2\pi)^2}\int_a^b\int_a^b g(\omega_1,\omega_2)i\sin(\omega_1 x_1)\cos(\omega_2 x_2)d\omega_1 d\omega_2,\\
\hat{\Phi}_2(x_1,x_2) &= \frac{1}{(2\pi)^2}\int_a^b\int_a^b g(\omega_1,\omega_2)j\cos(\omega_1 x_1)\sin(\omega_2 x_2)d\omega_1 d\omega_2,\\
\hat{\Phi}_3(x_1,x_2) &= -\frac{1}{(2\pi)^2}\int_a^b\int_a^b g(\omega_1,\omega_2)k\sin(\omega_1 x_1)\sin(\omega_2 x_2)d\omega_1 d\omega_2.
\end{aligned}$$

The quaternion exponential product $e^{jx_2\omega_2}\,e^{i\omega_1 x_1}$ is called the *inverse (right-sided) quaternion Fourier kernel.*

Remark 1.5. Again, all computations can easily be converted to other conventions, since

$$\begin{aligned}
\mathcal{F}_r^{-1}(g)(-x_1,-x_2) &= \frac{1}{(2\pi)^2}\int_a^b\int_a^b g(\omega_1,\omega_2)\overline{e^{i\omega_1 x_1}\,e^{j\omega_2 x_2}}d\omega_1 d\omega_2\\
&= \overline{\frac{1}{(2\pi)^2}\int_a^b\int_a^b e^{i\omega_1 x_1}\,e^{j\omega_2 x_2}\overline{g(\omega_1,\omega_2)}d\omega_1 d\omega_2}\\
&:= \overline{\mathcal{F}_l^{-1}(\overline{g})}(x_1,x_2).
\end{aligned}$$

For convenience, below we will denote $\mathcal{F}_r^{-1}$ as $\mathcal{F}^{-1}$.

The present chapter has two main goals. The first consists in studying the asymptotic behaviour of the integral (1.1) under the assumption that f belongs to $L^1((a,b)\times(c,d);\mathbb{H})$ where a, b, c, d can be both finite and infinite points. We shall be interested in the connection between the function $f(x_1,x_2)$, and the behaviour of its Fourier transform $\mathcal{F}(f)(\omega_1,\omega_2)$ at infinity. These properties have an interest on their own for further applications to number theory, combinatorics, signal processing, imaging, computer vision and numerical analysis. The complexity of the underlying computations will need some attention. Central to this viewpoint are certain Fourier transform techniques, which as in the complex case, would be familiar to the reader. Our second goal consists in extending the classical Bochner–Minlos theorem to a noncommutative structure as in the case of quaternion functions. The resulting theorem guarantees the existence and uniqueness of the corresponding probability measure defined on a dual space. This will be done using the concept of *quaternion Fourier transform of a probability measure.* For the reader's convenience and for the sake of easy reference, the chapter is motivated by the results presented in [12, 13].

Although the presented results can be extended to generalized Clifford algebras as well, we will focus the discussion on the QFT for conciseness here. The proof of its generalization to higher dimensions is possible but needs more complicated calculations, which exceed the scope of this manuscript. These results are still under further investigation and will be reported in a forthcoming paper.

2. Preliminaries

At this stage we briefly recall basic algebraic facts about *quaternions* necessary for the sequel. Let $\mathbb{H} := \{p = a + bi + cj + dk : a, b, c, d \in \mathbb{R}\}$ be a four-dimensional associative and noncommutative algebra, where the imaginary units i, j, and k are subject to the Hamiltonian multiplication rules

$$i^2 = j^2 = k^2 = -1;$$
$$ij = k = -ji, \qquad jk = i = -kj, \qquad ki = j = -ik.$$

The *scalar* and *vector parts* of p, $\mathrm{S}(p)$ and $\mathbf{V}(p)$, are defined as the a and $bi+cj+dk$ terms, respectively. For the scalar part the cyclic product rule $\mathrm{S}(pqr) = \mathrm{S}(qrp)$ is valid. Further step is *quaternion conjugation* introduced similarly to that of the complex numbers $\overline{p} = a - bi - cj - dk$. The quaternion conjugation is an anti-linear involution

$$\overline{\overline{p}} = p, \qquad \overline{p+q} = \overline{p} + \overline{q}, \qquad \overline{qp} = \overline{p}\,\overline{q}, \qquad \overline{\lambda p} = \lambda \overline{p} \quad (\forall\, \lambda \in \mathbb{R}).$$

The *norm* of p is defined by $|p| = \sqrt{p\overline{p}} = \sqrt{\overline{p}p} = \sqrt{a^2 + b^2 + c^2 + d^2}$, and it coincides with its corresponding Euclidean norm as a vector in $\mathbb{R}^4$. For every two quaternions p and q the triangle inequalities hold $|p + q| \leq |p| + |q|$, and $|\,|p| - |q|\,| \leq |p \pm q|$. Also, we have $|\mathrm{S}(p)| \leq |p|$ and $|\mathbf{V}(p)| \leq |p|$.

In the sequel, a *quaternion sequence* is a collection of real quaternions p_0, p_1, p_2, ... 'labelled' by nonnegative integers. We shall denote such a sequence by $\{p_n\}$ where $n = 0, 1, 2, \ldots$ and $p_n = p_{0,n} + p_{1,n}i + p_{2,n}j + p_{3,n}k$ are the elements of the sequence, $p_{l,n} \in \mathbb{R}$ $(l = 0, 1, 2, 3)$. To supplement our investigations, we recall the key notion of convergence of a quaternion sequence.

Definition 2.1. The quaternion sequence $\{p_n\}$ is called *convergent* to the quaternion $p = a + bi + cj + dk$ if $\lim_{n \longrightarrow \infty} |p_n - p| = 0$. We will use the traditional notation: $\lim_{n \longrightarrow \infty} p_n = p$.

Lemma 2.2. *Let $\{p_n\}$ and $\{q_n\}$ be two quaternion sequences for which $\lim_{n \longrightarrow \infty} p_n = p$ and $\lim_{n \longrightarrow \infty} q_n = q$, for $p, q \in \mathbb{H}$. Then*

1. $\lim_{n \longrightarrow \infty} (p_n \pm q_n) = p \pm q$;
2. $\lim_{n \longrightarrow \infty} (p_n q_n) = pq$;
3. $\lim_{n \longrightarrow \infty} (\alpha p_n) = \alpha p, \quad \alpha \in \mathbb{H}$.

Now, let s be the space of sequences

$$s := \{\{p_n\} : \lim_{n\longrightarrow\infty} n^t p_n = 0,\ \forall\, t \in \mathbb{N}_0\},$$

where $p_n \in \mathbb{H}$, and let

$$s_m := \{\{p_n\} : \|p\|_m^2 := \sum_{n=0}^{\infty}(1+n^2)^m\, |p_n|^2 < +\infty\}, \quad m \in \mathbb{Z}.$$

Proposition 2.3. *We have* $s = \cap_{m\in\mathbb{Z}} s_m$.

Proof. Let $p \in s$ be arbitrarily chosen and fixed, and $t \in \mathbb{N}_0$. From the definition of the space s we have that $\lim_{n\longrightarrow\infty} n^t p_n = 0$. Then for $C > 0$ a natural number $N = N(C)$ can be found so that for every $n > N$ the following holds

$$n^t |p_n| \le C.$$

Hence, it follows

$$\sum_{n=0}^{\infty} \left(1+n^2\right)^{\left[\frac{t}{2}\right]} |p_n|^2 \le |p_0|^2 + C^2 \sum_{n=1}^{\infty} \left(1+n^2\right)^{\left[\frac{t}{4}\right]} \frac{1}{n^{2t}} < +\infty,$$

and therefore $p \in s_{\left[\frac{t}{4}\right]}$. Since $t \in \mathbb{N}_0$ is arbitrary we conclude that $p \in s_m$ for every $m \in \mathbb{N}_0$, from where $p \in \cap_{m\in\mathbb{N}} s_m$. Since $p \in s$ was arbitrarily chosen it follows that $s \subset \cap_{m\in\mathbb{N}} s_m$. Now, let $p \in \cap_{m\in\mathbb{Z}} s_m$. Then for every $m \in \mathbb{N}_0$ we have that

$$\sum_{n=0}^{\infty} \left(1+n^2\right)^m |p_n|^2 < +\infty,$$

from where $\lim_{n\longrightarrow\infty} \left(1+n^2\right)^m |p_n|^2 = 0$. Therefore $\lim_{n\longrightarrow\infty} n^t p_n = 0$ for every $t \le 2m$. Since m was arbitrarily chosen then $\lim_{n\longrightarrow\infty} n^t p_n = 0$ for every $t \in \mathbb{N}_0$. Consequently, $p \in s$ and since $p \in \cap_{m\in\mathbb{N}} s_m$ was arbitrarily chosen we conclude that $\cap_{m\in\mathbb{N}} s_m \subset s$. □

In the sequel, consider the countable family of semi-norms on s

$$\|p\|_m^2 = \sum_{n=0}^{\infty} \left(1+n^2\right)^m |p_n|^2 .$$

Lemma 2.4. *s is completely a Hausdorff space.*

Proof. If $p \in s$ and $\|p\|_m = 0$ we get

$$\sum\nolimits_{n=0}^{\infty} (1+n^2)^m\, |p_n|^2 = 0.$$

Whence, $p_n = 0$ for every natural n (including zero), and consequently $p = 0$. It follows that s is a Hausdorff space. Now, let $\{p_n^k\} \longrightarrow_{k\longrightarrow\infty} \{p_n\}$, *i.e.*, $\lim_{k\longrightarrow\infty} p_n^k = p_n$ for every $n \in \mathbb{N}_0$. Then

$$\left\|p^k - p\right\|_m \longrightarrow_{k\longrightarrow\infty} 0.$$

It follows that s is completely a space. □

Let us define the metric

$$\rho(p,q) = \sum_{m=0}^{\infty} \frac{\|p-q\|_m}{1+\|p-q\|_m} 2^{-m}$$

for $p, q \in s$. Evidently, the above metric has the translation property. In other words, from the countable family of seminorms we can define a metric with the translation property. From this follows the result that

Lemma 2.5. *s is a Fréchet space.*

Now, let s' denote the topological dual space to s given by $s' = \bigcup_{m\in\mathbb{Z}} s_m$. We will denote the set of all sequences by $\mathbb{R}^{\mathbb{N}_0}$, and we equip the space s' with cylindrical topology. Every element $p' \in s'$ acts on each element $p \in s$ as follows

$$\langle p', p\rangle = \sum_{n=0}^{\infty} p'_n p_n; \qquad \lim_{n\longrightarrow\infty} p'_n = p', \ \lim_{n\longrightarrow\infty} p_n = p.$$

3. Asymptotic Behaviour of the QFT

There are numerous questions that remain untouched about the behaviour of the QFT. We will fill in some of these gaps and discuss some rudiments of the asymptotic behaviour of (1.1).

We begin by proving the following result, which provides an interesting and efficient convergence characterization of the QFT.

Theorem 3.1. *Let $a, b, c, d \in \mathbb{R}$ and $f \in L^1((a,b)\times(c,d);\mathbb{H})$, then $\mathcal{F}(f)$ is uniformly continuous and bounded. Moreover, it satisfies*

$$\lim_{\omega_1\longrightarrow\pm\infty} \mathcal{F}(f)(\omega_1,\omega_2) = 0$$

uniformly in ω_2, and

$$\lim_{\omega_2\longrightarrow\pm\infty} \mathcal{F}(f)(\omega_1,\omega_2) = 0$$

uniformly in ω_1. The results hold with the same proof when the region $(a,b)\times(c,d)$ is replaced by its topological closure $\overline{(a,b)\times(c,d)}$ (a, b, c, d can be $\pm\infty$) or any region G (or $\overline{G}$) in the space $\mathbb{R}^2$.

Proof. It is easy to show that

$$\begin{aligned} |\mathcal{F}(f)(\omega_1,\omega_2)| &\le \int_a^b \int_c^d |f(x_1,x_2)|\, dx_1 dx_2 \\ &= \|f\|_{L^1((a,b)\times(c,d);\mathbb{H})}\,. \end{aligned} \tag{3.1}$$

So, the transform $\mathcal{F}(f)$ is bounded.

To prove the limit assertions, we use a density argument as in the classical case (Riemann–Lebesgue Lemma). We first assume that both f and $\partial_{x_1} f$ are continuous with compact support. Obviously, such functions form a dense subspace in $L^1((a,b)\times(c,d);\mathbb{H})$. By using integration by parts to the x_1-variable, with the iterated integration, we have

$$\begin{aligned}\mathcal{F}(f)(\omega_1,\omega_2) &= \int_a^b \left(f(b,x_2)\,\frac{1}{\mathbf{i}\omega_1}\,e^{-\mathbf{i}\omega_1 b}\,e^{-\mathbf{j}\omega_2 x_2} - f(a,x_2)\,\frac{1}{\mathbf{i}\omega_1}\,e^{-\mathbf{i}\omega_1 a}\,e^{-\mathbf{j}\omega_2 x_2}\right) dx_2 \\ &\quad - \int_a^b\int_a^b \partial_{x_1} f(x_1,x_2)\,\frac{1}{\mathbf{i}\omega_1}\,e^{-\mathbf{i}\omega_1 x_1}\,e^{-\mathbf{j}\omega_2 x_2}\,dx_1dx_2 \\ &= -\int_a^b\int_c^d \partial_{x_1} f(x_1,x_2)\,\frac{1}{\mathbf{i}\omega_1}\,e^{-\mathbf{i}\omega_1 x_1}\,e^{-\mathbf{j}\omega_2 x_2}\,dx_1dx_2.\end{aligned}$$

Therefore,

$$|\mathcal{F}(f)(\omega_1,\omega_2)| \leq \frac{1}{|\omega_1|}\,\|\partial_{x_1} f\|_{L^1((a,b)\times(c,d);\mathbb{H})}\,. \tag{3.2}$$

So,

$$\lim_{\omega_1\longrightarrow\pm\infty} \mathcal{F}(f)(\omega_1,\omega_2) = 0$$

uniformly in ω_2. Similarly we can prove

$$\lim_{\omega_2\longrightarrow\pm\infty} \mathcal{F}(f)(\omega_1,\omega_2) = 0$$

uniformly in ω_1.

Now assume $f\in L^1((a,b)\times(c,d);\mathbb{H})$. For any given $\varepsilon>0$, there exists a function f_ε in the above-mentioned dense class, such that

$$\|f-f_\varepsilon\|_{L^1((a,b)\times(c,d);\mathbb{H})} < \varepsilon.$$

By (3.1), we have

$$\begin{aligned}|\mathcal{F}(f)(\omega_1,\omega_2)| &\leq |\mathcal{F}(f_\varepsilon)(\omega_1,\omega_2)| + \|f-f_\varepsilon\|_{L^1((a,b)\times(c,d);\mathbb{H})} \\ &\leq |\mathcal{F}(f_\varepsilon)(\omega_1,\omega_2)| + \varepsilon.\end{aligned}$$

By using the result just proved for the density class, we have

$$\lim_{\omega_1\rightarrow\pm\infty} |\mathcal{F}(f)(\omega_1,\omega_2)| \leq \varepsilon,$$

uniformly in ω_2. Since ε is arbitrary, we have

$$\lim_{\omega_1\rightarrow\pm\infty} \mathcal{F}(f)(\omega_1,\omega_2) = 0 \tag{3.3}$$

uniformly in ω_2. Similarly,

$$\lim_{\omega_2\rightarrow\pm\infty} \mathcal{F}(f)(\omega_1,\omega_2) = 0, \tag{3.4}$$

uniformly in ω_1.

Now we show the uniform continuity of $\mathcal{F}(f)(\omega_1,\omega_2)$. Given (3.3) and (3.4), since continuous functions are uniform continuous in compact sets, it suffices to show that $\mathcal{F}(f)$ is continuous at every point (ω_1, ω_2). In fact,

$$\begin{aligned}&\mathcal{F}(f)(\omega_1+\rho_1, \omega_2+\rho_2) - \mathcal{F}(f)(\omega_1, \omega_2) \\ &= \int_a^b \int_c^d f(x_1, x_2) \left[e^{-ix_1(\omega_1+\rho_1)} e^{-jx_2(\omega_2+\rho_2)} - e^{-ix_2\omega_1} e^{-jx_2\omega_2} \right] dx_1 dx_2.\end{aligned}$$

For any $\rho_1, \rho_2 > 0$, the integrand is dominated by a constant multiple of $|f(x_1, x_2)|$. Since the factor in the straight brackets tends to zero, by the Lebesgue Dominated Convergence Theorem, we have

$$\lim_{\rho_1, \rho_2 \to 0} \mathcal{F}(f)(\omega_1+\rho_1, \omega_2+\rho_2) - \mathcal{F}(f)(\omega_1, \omega_2) = 0,$$

proving the desired continuity. □

4. Bochner–Minlos Theorem

In this section we extend the classical Bochner–Minlos theorem (named after Bochner and Adol'fovich Minlos) to the framework of quaternion analysis. The resulting theorem guarantees the existence of the corresponding probability measure defined on a dual space. In particular, some interesting properties of the underlying measure are extended in this setting.

Definition 4.1. Let μ be a finite positive measure on $\mathbb{R}^2$. The QFT of μ is the function $\mathcal{F}_{ij}(\mu) : \mathbb{R}^2 \longrightarrow \mathbb{H}$ given by

$$\mathcal{F}_{ij}(\mu)(\omega_1, \omega_2) := \int_{\mathbb{R}^2} e^{-i\omega_1 x_1} e^{-j\omega_2 x_2} d\mu(x_1, x_2),$$

or the function $\mathcal{F}_{ji}(\mu) : \mathbb{R}^2 \longrightarrow \mathbb{H}$ given by

$$\mathcal{F}_{ji}(\mu)(\omega_1, \omega_2) := \int_{\mathbb{R}^2} e^{-j\omega_2 x_2} e^{-i\omega_1 x_1} d\mu(x_1, x_2).$$

Proposition 4.2. *Let μ be a finite positive measure on $\mathbb{R}^2$. The functionals $\mathcal{F}_{ij}(\mu)$ and $\mathcal{F}_{ji}(\mu)$ satisfy the following basic properties:*

1. $\mathcal{F}_{ij}(\mu)(0,0) = \mathcal{F}_{ji}(\mu)(0,0) = 1;$
2. $\mathcal{F}_{ij}(\mu)(-\omega_1, -\omega_2) = \overline{\mathcal{F}_{ji}(\mu)(\omega_1, \omega_2)};$
3. $\mathcal{F}_{ji}(\mu)(-\omega_1, -\omega_2) = \overline{\mathcal{F}_{ij}(\mu)(\omega_1, \omega_2)};$
4. $\mathcal{F}_{ij}(\mu)(-\omega_1, -\omega_2) + \mathcal{F}_{ij}(\mu)(\omega_1, \omega_2)$
   $= 2\int_{\mathbb{R}^2} (\cos(\omega_1 x_1)\cos(\omega_2 x_2) + \boldsymbol{k}\sin(\omega_1 x_1)\sin(\omega_2 x_2))\, d\mu(x_1, x_2);$
5. $\mathcal{F}_{ij}(\mu)(\omega_1, \omega_2) + \mathcal{F}_{ji}(\mu)(-\omega_1, -\omega_2) = 2\int_{\mathbb{R}^2} \cos(\omega_1 x_1)\cos(\omega_2 x_2) d\mu(x_1, x_2).$

Proof. For the first statement, note that

$$\mathcal{F}_{ij}(\mu)(0,0) = \int_{\mathbb{R}^2} d\mu(x_1, x_2) = \mu(\mathbb{R}^2).$$

For Statement 2 a straightforward computation shows that

$$\mathcal{F}_{ij}(\mu)(-\omega_1,-\omega_2) = \int_{\mathbb{R}^2} e^{i\omega_1 x_1} e^{j\omega_2 x_2} d\mu(x_1,x_2) = \int_{\mathbb{R}^2} \overline{e^{-j\omega_2 x_2} e^{-i\omega_1 x_1}} d\mu(x_1,x_2)$$

$$= \overline{\int_{\mathbb{R}^2} e^{-j\omega_2 x_2} e^{-i\omega_1 x_1} d\mu(x_1,x_2)} = \overline{\mathcal{F}_{ji}(\mu)(\omega_1,\omega_2)}.$$

The proof of Statement 3 will be omitted, being similar to the previous one. Now, taking into account that

$$e^{-i\omega_1 x_1} e^{-j\omega_2 x_2} = \cos(\omega_1 x_1)\cos(\omega_2 x_2) - i\sin(\omega_1 x_1)\cos(\omega_2 x_2) - j\cos(\omega_1 x_1)\sin(\omega_2 x_2) + k\sin(\omega_1 x_1)\sin(\omega_2 x_2),$$

and

$$e^{i\omega_1 x_1} e^{j\omega_2 x_2} = \cos(\omega_1 x_1)\cos(\omega_2 x_2) + i\sin(\omega_1 x_1)\cos(\omega_2 x_2) + j\cos(\omega_1 x_1)\sin(\omega_2 x_2) + k\sin(\omega_1 x_1)\sin(\omega_2 x_2),$$

we obtain

$$\begin{aligned}\mathcal{F}_{ij}(\mu)(\omega_1,\omega_2) &+ \mathcal{F}_{ij}(\mu)(-\omega_1,-\omega_2)\\ &= \int_{\mathbb{R}^2} \left(e^{-i\omega_1 x_1} e^{-j\omega_2 x_2} + e^{i\omega_1 x_1} e^{j\omega_2 x_2}\right) d\mu(x_1,x_2)\\ &= 2\int_{\mathbb{R}^2} \left(\cos(\omega_1 x_1)\cos(\omega_2 x_2) + k\sin(\omega_1 x_1)\sin(\omega_2 x_2)\right) d\mu(x_1,x_2).\end{aligned}$$

For the last statement we use the relation

$$e^{jw_2 x_2} e^{i\omega_1 x_1} = \cos(\omega_1 x_1)\cos(\omega_2 x_2) + i\sin(\omega_1 x_1)\cos(\omega_2 x_2) + j\cos(\omega_1 x_1)\sin(\omega_2 x_2) - k\sin(\omega_1 x_1)\sin(\omega_2 x_2).$$

Therefore, it follows

$$\begin{aligned}\mathcal{F}_{ij}(\mu)(\omega_1,\omega_2) &+ \mathcal{F}_{ji}(\mu)(-\omega_1,-\omega_2)\\ &= \int_{\mathbb{R}^2} \left(e^{-i\omega_1 x_1} e^{-j\omega_2 x_2} + e^{j\omega_2 x_2} e^{i\omega_1 x_1}\right) d\mu(x_1,x_2)\\ &= 2\int_{\mathbb{R}^2} \cos(\omega_1 x_1)\cos(\omega_2 x_2) d\mu(x_1,x_2).\end{aligned}$$

□

In the sequel, let us denote by $\mathbb{S}(\mathbb{R}^2)$ the Schwartz space of smooth quaternion functions on $\mathbb{R}^2$. We formulate a first result.

Proposition 4.3. *Let $\phi \in \mathbb{S}(\mathbb{R}^2)$ and μ be a finite positive measure on $\mathbb{R}^2$. Then*

1. $\displaystyle\int_{\mathbb{R}^2} \mathcal{F}_{ij}(\mu)(\omega_1,\omega_2)\phi(\omega_1,\omega_2)d\omega_1 d\omega_2 = \int_{\mathbb{R}^2} \mathcal{F}_{ij}(\phi)(x_1,x_2)d\mu(x_1,x_2);$
2. $\displaystyle\int_{\mathbb{R}^2} \mathcal{F}_{ji}(\mu)(\omega_1,\omega_2)\phi(\omega_1,\omega_2)d\omega_1 d\omega_2 = \int_{\mathbb{R}^2} \mathcal{F}_{ji}(\phi)(x_1,x_2)d\mu(x_1,x_2).$

Proof. For simplicity we just present the computations of the first equality. The proof of the second one is similar. A direct computation shows that

$$\begin{aligned}
&\int_{\mathbb{R}^2} \mathcal{F}_{ij}(\mu)(\omega_1,\omega_2)\phi(\omega_1,\omega_2)d\omega_1 d\omega_2 \\
&\quad= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{-i\omega_1 x_1}e^{-j\omega_2 x_2}d\mu(x_1,x_2)\phi(\omega_1,\omega_2)d\omega_1 d\omega_2 \\
&\quad= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{-i\omega_1 x_1}e^{-j\omega_2 x_2}\phi(\omega_1,\omega_2)d\omega_1 d\omega_2 d\mu(x_1,x_2) \\
&\quad= \int_{\mathbb{R}^2} \mathcal{F}_{ij}(\phi)(x_1,x_2)d\mu(x_1,x_2).
\end{aligned}$$

□

We now analyze some key properties of the above-mentioned functionals.

Proposition 4.4. *Let μ and ν be finite positive measures on $\mathbb{R}^2$. The functionals $\mathcal{F}_{ij}(\mu)$ and $\mathcal{F}_{ji}(\mu)$ are linear, i.e., for every $c,d\in\mathbb{H}$ it holds:*

$$\mathcal{F}_{ij}(\mu c+\nu d) = \mathcal{F}_{ij}(\mu)c + \mathcal{F}_{ij}(\nu)d,$$
$$\mathcal{F}_{ji}(\mu c+\nu d) = \mathcal{F}_{ji}(\mu)c + \mathcal{F}_{ji}(\nu)d.$$

Proposition 4.5. *Let μ be a finite positive measure on $\mathbb{R}^2$. For any $a_i\in\mathbb{R}\setminus\{0\}$ $(i=1,2)$ the following conditions hold:*

1. $\mathcal{F}_{ij}(\mu(a_1x_1,a_2x_2)) = \mathcal{F}_{ij}(\mu(x_1,x_2))\left(\frac{\omega_1}{a_1},\frac{\omega_2}{a_2}\right)$;
2. $\mathcal{F}_{ji}(\mu(a_1x_1,a_2x_2)) = \mathcal{F}_{ji}(\mu(x_1,x_2))\left(\frac{\omega_1}{a_1},\frac{\omega_2}{a_2}\right)$.

Proof. For simplicity we just present the proof of the first condition. A straightforward computation shows that

$$\begin{aligned}
\mathcal{F}_{ij}(\mu(a_1x_1,a_2x_2)) &= \int_{\mathbb{R}^2} e^{-i\omega_1 x_1}e^{-j\omega_2 x_2}d\mu(a_1x_1,a_2x_2) \\
&= \int_{\mathbb{R}^2} e^{-i\frac{\omega_1}{a_1}(a_1x_1)}e^{-j\frac{\omega_2}{a_2}(a_2x_2)}d\mu(a_1x_1,a_2x_2) \\
&= \int_{\mathbb{R}^2} e^{-i\frac{\omega_1}{a_1}y_1}e^{-j\frac{\omega_2}{a_2}y_2}d\mu(y_1,y_2) \\
&= \mathcal{F}_{ij}(\mu(x_1,x_2))\left(\frac{\omega_1}{a_1},\frac{\omega_2}{a_2}\right).
\end{aligned}$$

□

We proceed to define the notion of *positive definitely function* in the context of quaternion analysis.

Definition 4.6. Let f be a quaternion function on $\mathbb{R}^2$ that is continuous and bounded. For every finite positive measure μ on $\mathbb{R}^2$ the function f is said to be *positive definite* if

$$\sum_{k,l=1,k<l}^{N} z_k\overline{z_l}f(\lambda_k-\lambda_l) + \sum_{k,l=1,k<l}^{N}\overline{f(\lambda_k-\lambda_l)}z_l\overline{z_k} + \sum_{k=1}^{N}|z_k|^2\mu(\mathbb{R}^2)\geq 0$$

for every $\lambda_1, \lambda_2, \ldots, \lambda_N \in \mathbb{R}^2$, $z_1, z_2, \ldots, z_N \in \mathbb{H}$. These parameters are measured such that:

1. When $\lambda_1 = \lambda_2 = \cdots = \lambda_N$, and $z_1 = z_2 = \cdots = z_N$ it follows
$$2f(0,0) + \mu(\mathbb{R}^2) \geq 0;$$
2. When $N = 2$, $\lambda_1 = (x_1, x_2)$, and $\lambda_2 = (0,0)$, we have
$$z_1\overline{z}_2 f(x_1, x_2) + f(-x_1, -x_2) z_2 \overline{z}_1 + \left(|z_1|^2 + |z_2|^2\right) \mu\left(\mathbb{R}^2\right) \geq 0,$$
which is valid if $f(-x_1, -x_2) = \overline{f(x_1, x_2)}$.

Proposition 4.7. *The functional $\mathcal{F}_{ij}(\mu)$ is positive definite and bounded.*

Proof. Let $\lambda_1, \lambda_2, \ldots, \lambda_N \in \mathbb{R}^2$ such that $\lambda_i = (\lambda_{1k}, \lambda_{2k})$, and $z_1, z_2, \ldots, z_N \in \mathbb{H}$. Direct computations show that

$$\begin{aligned}
&\sum_{k,l,k<l}^{N} z_k \overline{z}_l \mathcal{F}_{ij}(\mu)(\lambda_k - \lambda_l) + \sum_{k,l=1,k<l}^{N} \overline{\mathcal{F}_{ij}(\mu)(\lambda_k - \lambda_l) z_l \overline{z_k}} + \sum_{k=1}^{N} |z_k|^2 \mu\left(\mathbb{R}^2\right) \\
&= \int_{\mathbb{R}^2} \sum_{k,l=1,k<l}^{N} z_k \overline{z}_l e^{-i(\lambda_{1k}-\lambda_{1l})x_1} e^{-j(\lambda_{2k}-\lambda_{2l})x_2} d\mu(x_1, x_2) \\
&\quad + \int_{\mathbb{R}^2} \sum_{k,l=1,k<l}^{N} e^{-j(\lambda_{2k}-\lambda_{2l})x_2} e^{-i(\lambda_{1k}-\lambda_{1l})x_1} z_l \overline{z_k} d\mu(x_1, x_2) \\
&\quad + \int_{\mathbb{R}^2} \sum_{k=1}^{N} |z_k|^2 d\mu(x_1, x_2) \\
&= \int_{\mathbb{R}^2} \sum_{k,l=1,k<l}^{N} z_k \overline{z}_l e^{-i(\lambda_{1k}-\lambda_{1l})x_1} e^{-j(\lambda_{2k}-\lambda_{2l})x_2} d\mu(x_1, x_2) \\
&\quad + \int_{\mathbb{R}^2} \sum_{k,l=1,k<l}^{N} \overline{z_k \overline{z}_l e^{-i(\lambda_{1k}-\lambda_{1l})x_1} e^{-j(\lambda_{2k}-\lambda_{2l})x_2}} d\mu(x_1, x_2) \\
&\quad + \int_{\mathbb{R}^2} \sum_{k=1}^{N} |z_k|^2 d\mu(x_1, x_2) \\
&= \int_{\mathbb{R}^2} \left[\sum_{k=1}^{N} |z_k|^2 + 2 \sum_{k,l=1,k<l}^{N} \mathrm{S}\left(z_k \overline{z}_l e^{-i(\lambda_{1k}-\lambda_{1l})x_1} e^{-j(\lambda_{2k}-\lambda_{2l})x_2} \right) \right] d\mu(x_1, x_2) \\
&\geq \int_{\mathbb{R}^2} \left(\sum_{k=1}^{N} |z_k|^2 - 2 \sum_{k,l=1,k<l}^{N} |z_k|\,|z_l| \right) d\mu(x_1, x_2) \geq 0.
\end{aligned}$$

Furthermore, it follows that

$$|\mathcal{F}_{ij}(\mu)(\omega_1, \omega_2)| = \left|\int_{\mathbb{R}^2} e^{-i\omega_1 x_1} e^{-j\omega_2 x_2} d\mu(x_1, x_2)\right|$$

$$\leq \int_{\mathbb{R}^2} |e^{-i\omega_1 x_1} e^{-j\omega_2 x_2}| \, d\mu(x_1, x_2) = \mu\left(\mathbb{R}^2\right) < +\infty. \qquad \square$$

Likewise we can prove the following proposition.

Proposition 4.8. *The functional $\mathcal{F}_{ji}(\mu)$ is positive definite and bounded.*

Below we shall assume that μ is a probably measure on s'. For every elements $a \in s$ and $a' \in s'$ we define the functionals g_{ij} and g_{ji} on s as follows:

$$g_{ij} : s \longrightarrow s, \qquad a \longmapsto g_{ij}(a) := \int_{s'} e^{i\langle a', a\rangle} e^{j\langle a', a\rangle} d\mu(a')$$

and

$$g_{ji} : s \longrightarrow s, \qquad a \longmapsto g_{ji}(a) := \int_{s'} e^{j\langle a', a\rangle} e^{i\langle a', a\rangle} d\mu(a').$$

Next we present a generalization of the *classical Bochner–Minlos theorem* on positive definite functions to the case of quaternion functions.

Theorem 4.9 (Bochner–Minlos theorem). *The functional g_{ij} satisfies the following three conditions:*

1. *Normalization: $g_{ij}(0) = 1$;*
2. *Positivity:*
$$\sum_{k,l=1,k<l}^{n} z_k \overline{z}_l g_{ij}(a_k - a_l) + \sum_{k,l=1,k<l}^{n} \overline{g_{ij}(a_k - a_l)} z_l \overline{z_k} + \sum_k |z_k|^2 \geq 0;$$
3. *Continuity: g_{ij} is continuous in the sense of Fréchet topology.*

Proof. We begin the proof by noting that

$$g_{ij}(0) = \int_{s'} d\mu(a') = \mu(s') = 1.$$

For simplicity we will prove Statement 2 in the case $n = 2$ only, *i.e.*, we will prove that

$$z_1 \overline{z}_2 \, g_{ij}(a_1 - a_2) + \overline{g_{ij}(a_1 - a_2)} z_2 \overline{z_1} + |z_1|^2 + |z_2|^2 \geq 0.$$

For the sake of convenience we set $z = a_1 - a_2$. It follows that

$$\begin{aligned} & z_1 \overline{z}_2 \, g_{ij}(z) + \overline{g_{ij}(z)} z_2 \overline{z_1} + |z_1|^2 + |z_2|^2 \\ & = \int_{s'} \left(z_1 \overline{z}_2 e^{i\langle a', z\rangle} e^{j\langle a', z\rangle} + \overline{z_1 \overline{z}_2 e^{i\langle a', z\rangle} e^{j\langle a', z\rangle}} + |z_1|^2 + |z_2|^2 \right) d\mu(a') \\ & = \int_{s'} \left[2\,\mathrm{S}\left(z_1 \overline{z_2} e^{i\langle a', z\rangle} e^{j\langle a', z\rangle} + |z_1|^2 + |z_2|^2 \right) \right] d\mu(a'). \end{aligned} \tag{4.1}$$

Notice that the last equality follows from the relation

$$z_1 \overline{z}_2 \, g_{ij}(z) = \overline{g_{ji}(-z) z_2 \overline{z}_1}.$$

Now, let $z_i = |z_i| \, e^{\frac{\mathrm{V}(z_i)}{|\mathrm{V}(z_i)|}\theta_i}$, with $\theta_i = \arg(z_i)$ $(i = 1, 2)$. Then

$$\mathrm{S}(z_1 \overline{z}_2 \, g_{ij}(z)) \geq -|z_1|\,|z_2|.$$

From here and (4.1) we obtain

$$z_1\overline{z}_2\, g_{ij}(z) + g_{ji}(-z)z_2\overline{z}_1 + |z_1|^2 + |z_2|^2$$
$$\geq \int_{s'} \left(|z_1|^2 + |z_2|^2 - 2\,|z_1|\,|z_2|\right) d\mu(a') \geq 0.$$

Using induction we may conclude that Statement 2 is valid for every natural n. For the proof of the remaining statement, let $\lim_{n\longrightarrow\infty} a_n = a$ be understood in the sense of the topology of Fréchet. Then $\lim_{n\longrightarrow\infty} a_n = a$ holds in the usual sense. From here and from the definition of the functional g_{ij} we conclude that $\lim_{n\longrightarrow\infty} g_{ij}(a_n) = g_{ij}(a)$. Therefore for every $\epsilon > 0$ a natural number $N = N(\epsilon) > 0$ can be found so that for $k > N$ the following holds

$$\|g(a_n) - g(a)\|_k < \epsilon.$$

Consequently, it follows that

$$\rho\left(g(a_n), g(a)\right) = \sum_{N+1}^{\infty} \frac{\|g(a_n) - g(a)\|_k}{1 + \|g(a_n) - g(a)\|_k} 2^{-k} \leq \epsilon \sum_{N+1}^{\infty} 2^{-k}. \qquad \square$$

Proposition 4.10. *For every element $a \in s$ the functionals g_{ij} and g_{ji} satisfy the additional properties:*

1. $g_{ij}(-a) = \overline{g_{ji}(a)}$;
2. $g_{ji}(-a) = \overline{g_{ij}(a)}$.

Though the significance of our approach to concrete applications, such as the characterization of measurement configurations for functional spaces, was the main reason for restricting ourselves to the quaternionic case, doubtless, the reduction in calculations for proving the results played an important role too. As was already mentioned, it is possible to perform an analogous study to generalized Clifford algebras following the same ideas. Further investigations will be presented in a forthcoming paper.

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