The Sine Transform Operator in the Banach Space of Symmetric Matrices and Its Application in Image Restoration *

Kit-Ian KOU, Xiao-Qing JIN and Vai-Kuong SIN

Faculty of Science and Technology

University of Macau, Macau

Abstract. In this paper, we study an operator s which maps every n-by-n symmetric matrix A_n to a matrix $s(A_n)$ that minimizes $\|B_n - A_n\|_F$ over the set of all matrices B_n that can be diagonalized by the sine transform. The matrix $s(A_n)$, called the optimal sine transform preconditioner, is defined for any n-by-n symmetric matrices A_n. The cost of constructing $s(A_n)$ is the same as that of optimal circulant preconditioner $c(A_n)$ which is defined in [8]. The $s(A_n)$ has been proved in [6] to be a good preconditioner in solving symmetric Toeplitz systems with the preconditioned conjugate gradient (PCG) method. In this paper, we discuss the algebraic and geometric properties of the operator s, and compute its operator norms in Banach spaces of symmetric matrices. Some numerical tests and an application in image restoration are also given.

Keywords. Toeplitz matrix, Optimal sine transform preconditioner, Sine transform operator, PCG method, Image restoration

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1 Introduction

In this paper, we discuss the optimal sine transform preconditioner which is defined in [6] from the operator point of view. Let $(\mathcal{F}_{n\times n}, \|\cdot\|)$ be the Banach space of all n-by-n symmetric matrices over the real field, equipped with a matrix norm $\|\cdot\|$. Let $(\mathcal{B}_{n\times n}, \|\cdot\|)$ be the subspace of all n-by-n symmetric matrices which can be diagonalized by the sine transform, i.e.,

$$\mathcal{B}_{n\times n} = \{S_n \Lambda_n S_n \mid \Lambda_n \text{ is an } n\text{-by-}n \text{ real diagonal matrix}\} \tag{1}$$

where S_n is the discrete sine transform matrix. Let s be an operator defined on $(\mathcal{F}_{n\times n}, \|\cdot\|)$ such that for any $A_n \in \mathcal{F}_{n\times n}$, $s(A_n)$ is the minimizer of $\|B_n - A_n\|_F$ over all $B_n \in \mathcal{B}_{n\times n}$. Here $\|\cdot\|_F$ denotes the Frobenius norm. The s is an operator from $(\mathcal{F}_{n\times n}, \|\cdot\|)$ into its subspace $(\mathcal{B}_{n\times n}, \|\cdot\|)$. We call s the sine transform operator and $s(A_n)$ the optimal sine transform preconditioner respectively. We remark that the idea can be easily extended to the block matrix A_{mn} by considering the minimizer $s(A_{mn})$ of $\|B_{mn} - A_{mn}\|_F$ over all $B_{mn} \in \mathcal{B}_{mn\times mn}$ where

$$\mathcal{B}_{mn\times mn} = \{(S_m \otimes S_n)\Lambda_{mn}(S_m \otimes S_n) \mid \Lambda_{mn} \text{ is diagonal}\}, \tag{2}$$

see [3] and [14].

Since only the sine transform will be involved, all the computations can be done in real arithmetic. We note that the matrix-vector product $S_n v$ can be done in $O(n \log n)$ real operations by using the fast sine transform (FST), see [16]. The matrix $s(A_n)$ is equal to A_n itself if A_n is a tridiagonal Toeplitz matrix. It is shown by Huckle in [11] that the minimizer $s(A_n)$ is given by $S_n \Lambda S_n$ where Λ is the diagonal matrix with diagonal entries

$$\Lambda_{jj} = (S_n A_n S_n)_{jj}, \quad j = 1, \ldots, n. \tag{3}$$

Computing all the diagonal entries of Λ by using (3) will require $O(n^2 \log n)$ operations. However, R. Chan, M. Ng and C. Wong in [6] show that the minimizer $s(A_n)$ can be obtained in $O(n^2)$ operations for general symmetric matrices and the cost can be reduced to $O(n)$ operations when A_n is a Toeplitz matrix. These operation counts are the same as that for obtaining optimal circulant preconditioner $c(A_n)$ which is defined by T. Chan in [**?**]. These preconditioners have been proved to be good preconditioners in solving symmetric Toeplitz systems with the PCG method. For the history of the iterative Toeplitz solvers and the applications of the optimal sine transform preconditioners, we refer to [1, 2, 4, 6, 7, 8, 11, 12, 13, 15].

The purpose of this paper is to discuss some other aspects of this operator s. The outline of the paper is as follows. In section 2, we prove some algebraic and geometric properties of the operator s. In section 3, we compute its operator norms in different Banach spaces of symmetric matrices. Some numerical results are given in section 4 and an application in image restoration is given in the last section.

2 The Sine Transform Operator

Firstly, let S_n denote the n-by-n discrete sine transform matrix with the (i, j)th entry given by

$$\sqrt{\frac{2}{n+1}} \sin\left(\frac{\pi i j}{n+1}\right), \quad 1 \leq i, j \leq n.$$

We remark that S_n is symmetric and orthogonal, i.e., $S_n = S_n^t$ and $S_n^2 = I_n$. Also, for any n-vector v, the matrix-vector multiplication $S_n v$ can be performed in $O(n \log n)$ real operations by using the FST, see [16]. Hence the number of operations required for the FST is less than that of the FFT. It was shown in [1] and [12] that a matrix belongs to $\mathcal{B}_{n\times n}$ defined by (1) if and only if the matrix can be expressed as a special sum of a Toeplitz matrix and a Hankel matrix. We recall that a matrix $A_n = [a_{i,j}]$ is said to be Toeplitz if $a_{i,j} = a_{i-j}$ and Hankel if $a_{i,j} = a_{i+j}$. For any A_n in $\mathcal{F}_{n\times n}$, let $\delta(A_n)$ denote the diagonal matrix whose diagonal is equal to the diagonal of the matrix A_n. The following Lemma gives a method for finding $s(A_n)$, see [11].

Lemma 2.1. *Let $A_n = [a_{ij}] \in \mathcal{F}_{n\times n}$ and $s(A_n)$ be the minimizer of $\|B_n - A_n\|_F$ over all $B_n \in \mathcal{B}_{n\times n}$. Then $s(A_n)$ is uniquely determined by A_n and is given by*

$$s(A_n) = S_n \delta(S_n A_n S_n) S_n \tag{4}$$

where S_n is the sine transform matrix. Hence the eigenvalues of $s(A_n)$ are given by $\delta(S_n A_n S_n)$.

Now, we want to discuss some properties of the sine transform operator. The following Theorem is on the algebraic properties of the operator s.

Theorem 2.1. *We have*

- *(i) For all $A_n, C_n \in \mathcal{F}_{n\times n}$ and α, β are real scalars,*
$$s(\alpha A_n + \beta C_n) = \alpha s(A_n) + \beta s(C_n).$$
Moreover,
$$s^2(A_n) = s(s(A)) = s(A_n)$$
for $\forall\, A_n \in \mathcal{F}_{n\times n}$. Thus s is a linear projection operator.
- *(ii) Let $A_n \in \mathcal{F}_{n\times n}$, then*
$$tr(s(A_n)) = tr(A_n) = \sum_{j=0}^{n-1} \lambda_j(A_n)$$
where $\lambda_j(A_n)$ are the eigenvalues of A_n.
- *(iii) Let $A_n \in \mathcal{F}_{n\times n}$ and $B_n \in \mathcal{B}_{n\times n}$, then*
$$s(B_n A_n) = B_n \cdot s(A_n) \quad \text{and} \quad s(A_n B_n) = s(A_n) \cdot B_n.$$
- *(iv) Let $A_n \in \mathcal{F}_{n\times n}$, then*
$$s(A_n^2) - [s(A_n)]^2 \geq 0.$$

Proof: The proofs of (i) and (ii) are trivial, therefore we omit them.

The proof of (iii) is as follows. For $A_n \in \mathcal{F}_{n\times n}$ and $B_n \in \mathcal{B}_{n\times n}$, B_n could be written as $B_n = S_n \Lambda_n S_n$ where Λ_n is a diagonal matrix. By using (4), we have
$$\begin{aligned} s(B_n A_n) &= S_n \delta(S_n(B_n A_n)S_n)S_n \\ &= S_n \delta[S_n(S_n \Lambda_n S_n)A_n S_n]S_n \\ &= S_n \delta[\Lambda_n(S_n A_n S_n)]S_n \\ &= S_n \Lambda_n \delta(S_n A_n S_n)S_n \\ &= S_n \Lambda_n S_n \cdot S_n \delta(S_n A_n S_n)S_n \\ &= B_n \cdot s(A_n). \end{aligned}$$

By similar arguments, one can prove that $s(A_nB_n) = s(A_n) \cdot B_n$.

The proof of (iv) is as follows. Let $A_n = [a_{ij}]$, $[S_n]_{ij} = \sqrt{2/(n+1)}\xi_{ij}$ with $\xi_{ij} = \sin\left(\frac{\pi ij}{n+1}\right)$ and let

$$\begin{aligned} D_n &= s(A_n^2) - [s(A_n)]^2 \\ &= S_n[\delta(S_nA_n^2S_n) - (\delta(S_nA_nS_n))^2]S_n. \end{aligned}$$

Then we have, for all $k = 1, \dots, n$,

$$[\delta(S_nA_n^2S_n)]_{kk} = [\delta((S_nA_n)(S_nA_n)^T)]_{kk} = \left(\frac{2}{n+1}\right)\sum_{p=1}^{n}\left(\sum_{q=1}^{n} a_{pq}\xi_{qk}\right)^2$$

and

$$[(\delta(S_nA_nS_n))^2]_{kk} = \left[\left(\frac{2}{n+1}\right)\sum_{p=1}^{n}\sum_{q=1}^{n}\xi_{kp}a_{pq}\xi_{qk}\right]^2 = \left(\frac{2}{n+1}\right)^2\left(\sum_{p=1}^{n}\sum_{q=1}^{n} a_{pq}\xi_{kp}\xi_{qk}\right)^2.$$

Hence the kth eigenvalue of D_n is given by

$$\begin{aligned} \lambda_k(D_n) &= \left(\tfrac{2}{n+1}\right)\textstyle\sum_{p=1}^{n}\left(\sum_{q=1}^{n} a_{pq}\xi_{qk}\right)^2 - \left(\tfrac{2}{n+1}\right)^2\left(\sum_{p=1}^{n}\sum_{q=1}^{n} a_{pq}\xi_{kp}\xi_{qk}\right)^2 \\ &= \left(\tfrac{2}{n+1}\right)\textstyle\sum_{p=1}^{n}\left(\sum_{q=1}^{n} a_{pq}\xi_{qk}\right)^2 - \left(\tfrac{2}{n+1}\right)^2\left(\sum_{p=1}^{n}(\sum_{q=1}^{n} a_{pq}\xi_{qk})\xi_{kp}\right)^2 \\ &= \left(\tfrac{2}{n+1}\right)\textstyle\sum_{p=1}^{n} c_{pk}^2 - \left(\tfrac{2}{n+1}\right)^2(\sum_{p=1}^{n} c_{pk}\xi_{kp})^2 \end{aligned}$$

where $c_{pk} = \sum_{q=1}^{n} a_{pq}\xi_{qk}$. Then by the Cauchy-Schwartz inequality and the orthogonal property of S_n, i.e.,

$$\sum_{p=1}^{n}\xi_{ip}\xi_{pj} = \begin{cases} 0 & \text{for } i \neq j, \\ \frac{n+1}{2} & \text{for } i = j, \end{cases}$$

we have

$$\begin{aligned} \left(\tfrac{2}{n+1}\right)^2\left(\textstyle\sum_{p=1}^{n} c_{pk}\xi_{kp}\right)^2 &\leq \left(\tfrac{2}{n+1}\right)^2\left(\textstyle\sum_{p=1}^{n} c_{pk}^2\right)\cdot\left(\textstyle\sum_{p=1}^{n}\xi_{kp}^2\right) \\ &= \left(\tfrac{2}{n+1}\right)^2\textstyle\sum_{p=1}^{n} c_{pk}^2\left(\tfrac{n+1}{2}\right) \\ &= \left(\tfrac{2}{n+1}\right)\textstyle\sum_{p=1}^{n} c_{pk}^2. \end{aligned}$$

Therefore,

$$\lambda_k(D_n) = \left(\frac{2}{n+1}\right)\sum_{p=1}^{n} c_{pk}^2 - \left(\frac{2}{n+1}\right)^2 \left(\sum_{p=1}^{n} c_{pk}\xi_{kp}\right)^2 \geq 0, \quad k = 1, \ldots, n.$$

Thus $D_n = s(A_n^2) - [s(A_n)]^2$ is positive semidefinite. $\square$

Next, we are going to give some geometric properties of the sine transform operator. For all $A_n, C_n \in \mathcal{F}_{n\times n}$, let $\langle A_n, C_n\rangle_F \equiv (\frac{1}{n})\mathrm{tr}(A_n C_n)$. Obviously, $\langle A_n, C_n\rangle_F$ is an inner product in $\mathcal{F}_{n\times n}$ and $\langle A_n, A_n\rangle_F = (\frac{1}{n})\|A_n\|_F^2$. We show below that $A_n - s(A_n)$ is perpendicular to $\mathcal{B}_{n\times n}$.

Theorem 2.2. *Let $A_n \in \mathcal{F}_{n\times n}$, then we have*

- *(i)* $\langle A_n - (A_n), B_n\rangle_F = 0, \quad \forall\, B_n \in \mathcal{B}_{n\times n}$.
- *(ii)* $\langle A_n, s(A_n)\rangle_F = \frac{1}{n}\|s(A_n)\|_F^2$.
- *(iii)* $\|A_n - s(A_n)\|_F^2 = \|A_n\|_F^2 - \|s(A_n)\|_F^2$.

Proof: For (i), $\forall\, B_n \in \mathcal{B}_{n\times n}$, we have

$$\begin{aligned}\langle A_n - s(A_n), B_n\rangle_F &= (1/n)\mathrm{tr}([A_n - s(A_n)]B_n) \\ &= (1/n)\mathrm{tr}(A_n B_n) - (1/n)\mathrm{tr}(s(A_n)B_n).\end{aligned}$$

Then, we want to prove that

$$\mathrm{tr}(s(A_n)B_n) = \mathrm{tr}(A_n B_n).$$

Since $B_n = S_n\Lambda_n S_n$, we have by using (4),

$$\begin{aligned}\mathrm{tr}(s(A_n)B_n) &= \mathrm{tr}[s(A_n)(S_n\Lambda_n S_n)] \\ &= \mathrm{tr}[S_n\delta(S_n A_n S_n)S_n \cdot (S_n\Lambda_n S_n)] \\ &= \mathrm{tr}[S_n\delta(S_n A_n S_n)\Lambda_n S_n] \\ &= \mathrm{tr}(\delta(S_n A_n S_n)\Lambda_n) \\ &= \mathrm{tr}((S_n A_n S_n)\Lambda_n) \\ &= \mathrm{tr}(A_n(S_n\Lambda_n S_n)) \\ &= \mathrm{tr}(A_n B_n).\end{aligned}$$

Now, the proof of (ii) follows directly from (i). By using (i) and (ii), one can prove (iii) easily. $\square$

3 The Spectral Properties of Operator s

In this section, we discuss some spectral properties of the sine transform operator s. The proof of the following Lemma is similar to that of Theorem 1 in Chan, Jin and Yeung [4] or that of Theorem 2 in Huckle [11].

Lemma 3.1. *For $A_n \in \mathcal{F}_{n\times n}$, we have*

$$\lambda_{\min}(A_n) \leq \lambda_{\min}(s(A_n)) \leq \lambda_{\max}(s(A_n)) \leq \lambda_{\max}(A_n)$$

where $\lambda_{\max}(\cdot)$ and $\lambda_{\min}(\cdot)$ denote the largest and the smallest eigenvalues respectively. In particular,

$$\|s(A_n)\|_2 \leq \|A_n\|_2 \tag{5}$$

and if A_n is positive definite, then $s(A_n)$ is also positive definite.

For the spectrum properties of the operator s, we have

Theorem 3.1. *For all $n \geq 1$, we have*

- *(i)* $\|s\|_F \equiv \sup_{\|A_n\|_F=1} \|s(A_n)\|_F = 1.$
- *(ii)* $\|s\|_2 \equiv \sup_{\|A_n\|_2=1} \|s(A_n)\|_2 = 1.$

Proof: To prove (i), we notice that if $A_n = (1/n)I$, then

$$\|s(A_n)\|_F = (1/n)\|I\|_F = 1.$$

For general A_n in $\mathcal{F}_{n\times n}$, by Lemma 2.3 (iii), we have

$$\|s(A_n)\|_F^2 = \|A_n\|_F^2 - \|A_n - s(A_n)\|_F^2 \leq \|A_n\|_F^2.$$

Thus $\|s(A_n)\|_F \leq \|A_n\|_F$. Hence $\|s\|_F = 1$ for all n.

To prove (ii), we have by (5),

$$\|s(A_n)\|_2 \leq \|A_n\|_2$$

for all A_n in $\mathcal{F}_{n\times n}$. Since $\|s(I)\|_2 = \|I\|_2 = 1$, we have $\|s\|_2 = 1$ for all n. $\square$

4 The Numerical Results

In this section, we use the PCG method (see [9]) with the optimal sine transform preconditioner $s(A_n)$ to solve symmetric Toeplitz system

$$A_n(f)x = b.$$

Here f is the generating function of the system. We compare the $s(A_n)$ with the optimal circulant preconditioner $c(A_n)$ which is defined in [8]. Three systems with different generating functions were tested. The generating functions are

$$\begin{array}{ll} \text{(i)} & f(\theta) = \theta^6 + 1, \\ \text{(ii)} & f(\theta) = |\theta|^3, \\ \text{(iii)} & f(\theta) = \theta^4 + (\sin\theta)^2. \end{array}$$

When the PCG method is applied to such kind of systems, the stopping criteria for the method is set at

$$\frac{\|r_k\|_2}{\|r_0\|_2} < 10^{-7}$$

where r_k is the residual vector at the kth iteration. The right hand side vector b is chosen to be the vector of all ones and the zero vector is the initial guess. Table 1 shows the number of iterations required for convergence and I means no preconditioner was used. Figures 1, 2, 3 show the relative errors at each iteration for different systems with $n = 256$.

$f(\theta)$	θ^6+1			$\|\theta\|^3$			$\theta^4+(\sin\theta)^2$		
n	I	$s(A_n)$	$c(A_n)$	I	$s(A_n)$	$c(A_n)$	I	$s(A_n)$	$c(A_n)$
32	28	10	15	22	9	13	24	8	21
64	59	9	13	59	10	18	55	8	17
128	106	7	11	176	11	25	142	8	18
256	168	6	9	562	13	36	334	7	19
512	203	6	7	1934	14	83	752	7	18
1024	218	6	8	>2000	15	190	1625	7	18

Table 1: Number of iterations for different systems.

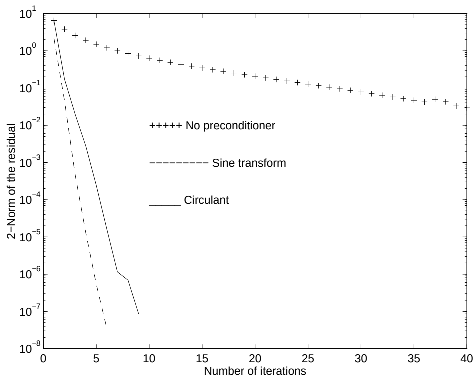

Figure 1: Relative errors at each iteration for the system with (i).

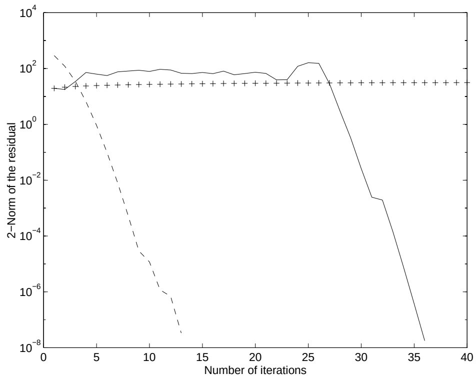

Figure 2: Relative errors at each iteration for the system with (ii).

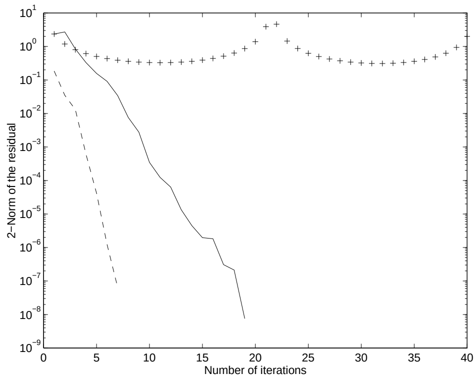

Figure 3: Relative errors at each iteration for the system with (iii).

5 An Application in Image Restoration

In this section, we consider the application of the sine transform preconditioner proposed in previous sections for solving the linear systems arising from image restoration. The mathematical model of the linear image restoration problem is given as follows, see [10],

$$g(\xi,\delta)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}t(\xi-\alpha,\delta-\beta)f(\alpha,\beta)d\alpha d\beta+\eta(\xi,\delta) \tag{6}$$

where $g(\xi,\delta)$ is the recorded (or degraded) image, $f(\alpha,\beta)$ is the ideal (or original) image, the vector $\eta(\xi,\delta)$ represents additive noise. The function t is called the point spread function (PSF) and represents the degradation of the image. Since the PSF here is a function t of $\xi-\alpha$ and $\delta-\beta$, the t is said to be spatially invariant. The integral in (6) is a two-dimensional convolution.

In the digital implementation of (6), the integral is discretized by using some quadrature rule, to obtain the discrete scalar model

$$g(i,j)=\sum_{k=1}^{m}\sum_{l=1}^{n}t(i-k,j-l)f(k,l)+\eta(i,j).$$

In matrix form, we have the following linear algebraic system of the image restoration problem,

$$g=T_{mn}f+\eta \tag{7}$$

where g,η and f are n^2-vectors and T_{mn} is an n^2-by-n^2 block-Toeplitz-Toeplitz-block matrix. This is the square image formulation. The PCG method is proposed as main tool to solve (7), see [3] and [5]. We will use the optimal block sine transform preconditioner $s(T_{mn})$ which is defined to be the minimizer of

$$\|T_{mn}-B_{mn}\|_F$$

over all $B_{mn}\in\mathcal{B}_{mn\times mn}$ where $\mathcal{B}_{mn\times mn}$ is defined by (2).

The following example is constructed. First, we generate an original 256×256 image of an ocean reconnaissance satellite, see Figure 4 (left). Then we consider the spatially invariant discretized PSF matrix T_{mn} with the entries given by

$$t_{i-j,k-l} = \begin{cases} \exp\{-800(i-j)^2 - 800(k-l)^2\}, & \text{if } \sqrt{(i-j)^2+(k-l)^2} < 1/8, \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

The matrix for this example is 65536×65536. The observed image is constructed by forming the vector

$$g = T_{mn} f + \eta$$

where T_{mn} is defined by (8), f is a vector formed by row ordering the original image. By unstacking the vector g, we obtain the blurred noisy (observed) image, see Figure 4 (right). The noise function η has normal distribution and is scaled such that $\|\eta\|_2/\|T_{mn}f\|_2 = 10^{-4}$.

Our goal is: given g and T_{mn}, to recover an approximation to the original image f. Since T_{mn} is ill conditioning, we use the method of Tikhonov regularization, see [5], and then use the PCG method with the optimal block sine transform preconditioner $s(T_{mn})$. We compare the number of iterations required for convergence of the $s(T_{mn})$ with the optimal block circulant preconditioner $c(T_{mn})$ which is defined in [3] and [5]. The stopping criteria is $\frac{\|r_k\|_2}{\|r_0\|_2} < 10^{-4}$. The convergence results for the image restoration with no preconditioner I_{mn}, our preconditioner $s(T_{mn})$, and preconditioner $c(T_{mn})$ require 221, 67 and 91 iterations respectively, see Figures 5, 6 and 7(right).

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Figure 4: Original image (left) and observed image (right).

Figure 5: Restored Images with I_{mn}: 20 (left) and 221 iterations (right).

Figure 6: Restored Image with $s(T_{mn})$: 20 (left) and 67 iterations (right).

Figure 7: Restored Image with $c(T_{mn})$: 20 (left) and 91 iterations (right).

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