Research Article

Mathematical Methods in the Applied Sciences

Received XXXX

(www.interscience.wiley.com) DOI: 10.1002/sim.0000
MOS subject classification: 30G35; 11E88; 42C05; 35S30

Generalized prolate spheroidal wave functions for offset linear canonical transform in Clifford analysis

K. Kou*, J. Morais and Y. Zhang

Prolate spheroidal wave functions (PSWFs) possess many remarkable properties. They are orthogonal basis of both square integrable space of finite interval and the Paley-Wiener space of bandlimited functions on the real line. No other system of classical orthogonal functions is known to obey this unique property. This raises the question of whether they possess these properties in Clifford analysis. The aim of the article is to answer this question and extend the results to more flexible integral transforms, such as offset linear canonical transform. We also illustrate how to use the generalized CPSWFs (for offset Clifford linear canonical transform) we derive to analyze the energy preservation problems. CPSWFs is new in literature and has some consequences that are now under investigation. Copyright © 2009 John Wiley & Sons, Ltd.

Keywords: Clifford analysis, Fourier transform, linear canonical transform, offset linear canonical transform, prolate spheroidal wave functions.

1. Introduction

Clifford analysis offers both a generalization of complex analysis in the plane and a refinement of classical harmonic analysis in several real variables. The rich structure of this function theory involves the study of functions defined in open subsets of $\mathbb{R}^m$ $(m > 2)$ with values in a Clifford algebra. Yet Clifford analysis became a source of insight and inspiration, and an inexhaustible store of new ideas and methods, both theoretical and applied. For details, the reader is referred to [4] and [8].

The connection between PSWFs and the energy concentration problem was first introduced in the early-sixties by Slepian and Pollak [21]. They are also known as Slepian functions and solutions of a Sturm-Liouville problem for solving elliptic boundary value problems in spheroidal geometry. The PSWFs have been extensively used for a variety of physical and engineering applications, such as wave scattering, signal processing, and antenna theory, see for instance, [11, 12]. Recently, there has been a growing interest in developing numerical methods using PSWFs as basis functions, see [3, 6, 28, 26, 27], etc. Most notably, a series of papers by Slepian et al. [11, 21, 22] and recent works in [2, 3, 13, 19, 20, 30, 31, 32] have shown that the PSWFs are optimal tools for approximating bandlimited functions. They are preferable to classical polynomial bases (such as Legendre and Chebychev polynomials). Higher dimensional extension of PSWFs was first studied by Slepian [22] and recent work on higher dimensional prolate speroidal wavelets [27]. The prolate spheroidal wavelets were constructed to retain the energy concentration property and were shown to have many desirable properties lacking in other wavelet systems. Within the scope of this paper it is therefore of some interests to check if it is possible to develop a counterpart of the PSWFs to a noncommutative structure as in the case of Clifford functions, and study under what circumstances similar properties and interdependencies can be obtained.

PSWFs are known to be useful for analyzing the properties of the Fourier transform (FT). The fractional Fourier transform (FRFT) is a generalization of the FT, and the linear canonical transform (LCT) is a further generalization of the FRFT. They are all useful mathematical tools and are widely used for spectrum analysis, signal processing, and optical system analysis. Offset FTs are similar to the original FTs, except that the kernel $\exp(-i\omega x)$ is replaced by $\exp(-i(\omega-\tau)(x-\eta))$. That is, the kernel is generalized by appending a space-shifted term and a frequency-modulated term. We can also define offset FRFTs and offset LCTs in a similar way. Offset FTs, FRFTs, and LCTs are more flexible than the original ones. They are especially useful for

Department of Mathematics, Faculty of Science and Technology, University of Macau, Macau.
**Correspondence to: K. Kou, Department of Mathematics, Faculty of Science and Technology, University of Macau, Macau.*

Copyright © 2009 John Wiley & Sons, Ltd.

analyzing optical systems with prisms or shifted lenses. We derive the Clifford LCT (CLCT) and offset CLCT and study the generalized PSWFs for CLCT and offset CLCT and the connection with energy concentration problem.

The rest of the paper is organized as follows. In the next section, we recall some related properties of PSWFs in one dimension and collect some basic concepts in Clifford analysis. This is followed by consideration of PSWFs in Clifford analysis in Section 3. The mean square convergence of the Clifford Slepian series to the bandlimited functions are also discussed. More importantly, the maximization problem of the CPSWFs are analyzed. In Section 4, we introduce the CLCT and offset CLCT and derive the generalize CPSWFs for the finite-extension fractional Clifford Fourier transform (fi-CFRFT), the finite-extension Clifford linear canonical transform (fi-CLCT), and the finite-extension offset Clifford linear canonical transform (fi-offset CLCT). We also illustrate how to use these generalized PSWFs to analyze the energy preservation ratio.

2. Preliminaries

2.1. Prolate spheroidal wave functions

In this subsection, we review and extend some relevant properties of the Prolate spheroidal wave functions (PSWFs), most of which can be founded e.g. in [10, 13, 18, 21].

The PSWFs $\phi_{n,T,W}:\mathbb{R}\to\mathbb{C}$ constitute an orthogonal basis of the space of W-bandlimited signals with finite energy, that is, for continuous functions in $L^2(\mathbb{R})$ whose Fourier transform has support in $[-W,W]$. They are maximally concentrated on the interval $[-T,T]$ and depend on parameters T and W. PSWFs are characterized as the eigenfunctions of an integral operator with kernel arising from the sinc functions $K(y):=\frac{\sin(\pi y)}{\pi y}$:

$$\frac{W}{\pi}\int_{-T}^{T}\phi_{n,T,W}(x)K\left(\frac{W}{\pi}(y-x)\right)dx=\gamma_{n,T,W}\,\phi_{n,T,W}(y),\quad |y|\le T. \tag{1}$$

It is easy to show that the symmetrical kernel K is positive definite, so that from [7] we know that (1) has solutions in $L^2([-T,T])$ only for a discrete set of real positive values of $\gamma_{n,T,W}$ say $\gamma_{0,T,W}\le\gamma_{1,T,W}\le\ldots$ and that $\lim_{n\to\infty}\gamma_{n,T,W}=0$. The corresponding solutions, or eigenfunctions, $\phi_{0,T,W}(y),\phi_{1,T,W}(y),\ldots$ can be chosen to be real and orthogonal on $(-T,T)$.

The variational problem that led to (1) only requires that equation to hold for $|y|\le T$. With $\phi_{n,T,W}(x)$ on the left-hand side of (1) gives for $|x|\le T$, however, the left is well defined for all y. We use this to extend the range of definition of the $\phi_{n,T,W}$'s and so define

$$\phi_{n,T,W}(y):=\frac{W}{\pi\gamma_{n,T,W}}\int_{-T}^{T}\phi_{n,T,W}(x)K\left(\frac{W}{\pi}(y-x)\right)dx,\quad |y|>T.$$

The eigenfunctions $\phi_{n,T,W}$ are now defined for all y. In addition to the equation (1), the $\{\phi_{n,T,W}\}$ satisfy an integral equation over $(-\infty,\infty)$

$$\frac{W}{\pi}\int_{-\infty}^{\infty}\phi_{n,T,W}(x)K\left(\frac{W}{\pi}(y-x)\right)dx=\left(\phi_{n,T,W}*K_1\right)(y)=\phi_{n,T,W}(y)$$

with the kernel $K_1(x):=\frac{W}{\pi}K\left(\frac{W}{\pi}x\right)$. This leads to a dual orthogonality

$$\int_{-T}^{T}\phi_{n,T,W}(x)\phi_{M,T,W}(x)dx=\gamma_{n,T,W}\,\delta_{nm}, \tag{2}$$

$$\int_{-\infty}^{\infty}\phi_{n,T,W}(x)\phi_{M,T,W}(x)dx=\delta_{nm} \tag{3}$$

and the fact that they constitute an orthogonal basis of $L^2([-T,T])$, as well as an orthonormal basis of the subspaces B_W of $L^2(\mathbb{R})$, the Paley-Wiener space of all W-bandlimited functions.

Remark 2.1 *The bandwidth parameter is given by $c=TW$, where T is the interval on which the function is known and W is the finite bandwidth or cutoff frequency of $\phi_n(x)$ of a given order n. In the case in which the roles of T, W are not emphasized, we simplify the notations as $\phi_n(x)$ and γ_n.*

With (3) we can represent any $f\in L^2(\mathbb{R})$ in terms of Slepian series [13]

$$f(x)=\sum_{n=0}^{\infty}a_n\phi_n(x),$$

where the coefficients are given by

$$a_n=\int_{\mathbb{R}}f(x)\phi_n(x)dx.$$

 Copyright © 2009 John Wiley & Sons, Ltd.
Prepared using mmaauth.cls

This obeys the Parseval's equality

$$\sum_{n=0}^{\infty} |a_n|^2 = \int_{\mathbb{R}} |f(x)|^2 dx.$$

The mean square convergence of the series is given by [21]

$$\lim_{N\to\infty} \int_{\mathbb{R}} \left| f(x) - \sum_{n=0}^{N} a_n \phi_n(x) \right|^2 dx = 0.$$

From this $f(x)$ can be approximated for all $x \in \mathbb{R}$ and with finite N terms

$$f(x) \simeq \sum_{n=0}^{N} a_n \phi_n(x). \tag{4}$$

This kind of approximation is good for functions whose energy is distributed over the infinite time domain. For practical applications we are more concerned with data over finite domains. Multiplying ϕ_n on both sides of (4) and using (2), then the coefficients determined by a finite interval employ the following

$$a_n = \lambda_n^{-1} \int_{-T}^{T} f(x)\phi_n(x) dx. \tag{5}$$

Due to the orthogonal nature of ϕ_n, equations (4) and (5) can be viewed in much the same way as other orthogonal expansions.

2.2. Clifford analysis

The present section collects some definitions and basic algebraic facts of a special Clifford algebra of signature $(0,m)$ and its associated function theory, which will be needed throughout the text. Let $\{e_1, e_2, \ldots, e_m\}$ be an orthonormal basis of the Euclidean vector space $\mathbb{R}^m$ with a product according to the multiplication rules:

$$e_i e_j + e_j e_i = -2\delta_{i,j}, \quad (i,j = 1,\ldots,m),$$

where $\delta_{i,j}$ is the Kronecker symbol. Whence, the set $\{e_A : A \subseteq \{1,\ldots,m\}\}$ with $e_A = e_{h_1} e_{h_2} \ldots e_{h_r}$, $1 \le h_1 < \ldots < h_r \le m$, and $e_\phi = e_0 = 1$ forms a basis of the 2^m-dimensional Clifford algebra $Cl_{0,m}$ over $\mathbb{R}$.

Any Clifford number $a \in Cl_{0,m}$ may thus be written as $a = \sum_A a_A e_A, a_A \in \mathbb{R}$, or still as $a = \sum_{k=0}^{m} [a]_k$, where $[a]_k = \sum_{|A|=k} a_A e_A$ is so-called k-vector $(k = 0,1,\ldots,m)$. The real vector space $\mathbb{R}^{m+1}$ will be embedded in $Cl_{0,m}$ by identifying the element $(x_0, x_1, \ldots, x_m) \in \mathbb{R}^{m+1}$ with the algebra's element

$$x = x_0 + \mathbf{x} \in \mathbb{R}^{m+1} := \mathrm{span}_{\mathbb{R}}\{1, e_1, \ldots, e_m\} \subset Cl_{0,m}.$$

The elements of $\mathbb{R}^{m+1}$ are usually called paravectors, and $x_0 :=$ Sc(x) and $e_1 x_1 + \cdots + e_m x_m := \mathbf{x}$ are the so-called scalar and vector parts of x. The conjugate of x is $\bar{x} = x_0 - \mathbf{x}$, and the norm $|x|$ of x is defined by $|x|^2 = x\bar{x} = \bar{x}x = x_0^2 + x_1^2 + \cdots + x_m^2$.

We consider $\mathbb{R}^{m+1}$-valued functions defined in $\mathbb{R}^m$, i.e. functions of the form $f : \mathbb{R}^m \longrightarrow \mathbb{R}^{m+1}$ so that $f(\mathbf{x}) = \sum_{k=0}^{m} f_k(\mathbf{x}) e_k$, where f_A are real-valued functions defined in $\mathbb{R}^m$. Properties (like integrability, continuity or differentiability) that are ascribed to f have to be fulfilled by all components f_A.

Let $L^2(\mathbb{R}^m; \mathbb{R}^{m+1})$ denote the linear Hilbert space of square integrable $\mathbb{R}^{m+1}$-valued functions defined in $\mathbb{R}^m$. We further introduce an inner product for two functions $f, g : \mathbb{R}^m \longrightarrow \mathbb{R}^{m+1}$ as follows:

$$< f, g >_{L^2(\mathbb{R}^m;\mathbb{R}^{m+1})} := \int_{\mathbb{R}^m} f(\mathbf{x}) \overline{g(\mathbf{x})} \, d\sigma(\mathbf{x}), \tag{6}$$

where $d\sigma$ denotes the Lebesgue measure on $\mathbb{R}^m$. Depending on the case, our integrals will be over $\mathbb{R}^m$ or an open subset of $\mathbb{R}^m$ with a piecewise smooth boundary, and we shall always mention this explicitly.

The reader should also note that the norm induced by this inner product,

$$\|f\|^2_{L^2(\mathbb{R}^m;\mathbb{R}^{m+1})} :=< f, f >_{L^2(\mathbb{R}^m;\mathbb{R}^{m+1})}$$

coincides with the usual L^2-norm for f, considered as a vector-valued function.

The Clifford Fourier transform (CFT) of $f \in L^1(\mathbb{R}^m; \mathbb{R}^{m+1}) := \{f : \mathbb{R}^m \to \mathbb{R}^{m+1} | \int_{\mathbb{R}^m} |f(\mathbf{x})| d\sigma\mathbf{x} < \infty\}$ is given by [5]

$$\mathscr{F}(f) : \mathbb{R}^m \to Cl_{0,m}, \qquad \mathscr{F}(f)(\omega) := \int_{\mathbb{R}^m} f(\mathbf{x}) \mathbf{e}(\omega, \mathbf{x}) \, d\sigma(\mathbf{x}), \tag{7}$$

where the kernel function

$$\mathbf{e} : \mathbb{R}^m \times \mathbb{R}^m \to Cl_{0,m}, \qquad \mathbf{e}(\omega, \mathbf{x}) := e^{-e_1 \omega_1 x_1} \cdots e^{-e_m \omega_m x_m} = \prod_{i=1}^{m} e^{-e_i \omega_i x_i}. \tag{8}$$

Copyright © 2009 John Wiley & Sons, Ltd.

We point out that the product in (8) has to be performed in a fixed order since, in general, $\mathbf{e}(\omega,\mathbf{x})$ does not commute with every element of $\mathbb{R}^{m+1}$.

The inverse Clifford Fourier transform of $g \in L^1(\mathbb{R}^m;\mathbb{R}^{m+1})$ is given by

$$\mathscr{F}^{-1}(g):\mathbb{R}^m \to Cl_{0,m}, \qquad \mathscr{F}^{-1}(g)(\mathbf{x}) = (2\pi)^{-m}\int_{\mathbb{R}^m} g(\omega)\overline{\mathbf{e}(\omega,\mathbf{x})}\,d\sigma(\omega), \tag{9}$$

where $\overline{\mathbf{e}(\omega,\mathbf{x})} = e^{e_m\omega_m x_m}\cdots e^{e_1\omega_1 x_1}$. In this notation Parseval's theorem is

$$\int_{\mathbb{R}^m} f(\mathbf{x})\overline{g(\mathbf{x})}d\sigma(\mathbf{x}) = (2\pi)^{-m}\int_{\mathbb{R}^m}\mathscr{F}(f)(\omega)\overline{\mathscr{F}(g)(\omega)}d\sigma(\omega) \tag{10}$$

In practice, we perform the *finite Clifford Fourier transform (fi-CFT)* instead of the CFT:

$$\mathscr{F}(f_{\mathbf{T}})(\omega) = \int_{\mathbb{R}^m} f_{\mathbf{T}}(\mathbf{x})\mathbf{e}(\omega,\mathbf{x})\,d\sigma(\mathbf{x}), \tag{11}$$

where

$$f_{\mathbf{T}}(\mathbf{x}) := f(\mathbf{x})\chi_{\mathbf{T}}(\mathbf{x}), \qquad \chi_{\mathbf{T}}(\mathbf{x}) := \begin{cases} 1, & \text{if } \mathbf{x}\in\mathbf{T};\\ 0, & \text{otherwise;}\end{cases} \tag{12}$$

and

$$\mathbf{T} := [-T_1,T_1]\times\cdots\times[-T_m,T_m]\subset\mathbb{R}^m$$

is the *space interval*;

$$\omega\in\mathbf{W} := [-W_1,W_1]\times\cdots\times[-W_m,W_m]\subset\mathbb{R}^m$$

is the *frequency interval*. For $i=1,\ldots,m$, T_i and W_i are positive real numbers throughout the paper.

Since the intervals are finite, after the fi-CFT is done, some energy $\|f\|_{L^2(\mathbb{R}^m;\mathbb{R}^{m+1})}$ is lost. For the original CFT of $f\in L^1\bigcap L^2(\mathbb{R}^m;\mathbb{R}^{m+1})$, the energy preservation property is satisfied:

$$(2\pi)^{-m}\|\mathscr{F}(f)\|^2_{L^2(\mathbb{R}^m;\mathbb{R}^{m+1})} = \|f\|^2_{L^2(\mathbb{R}^m;\mathbb{R}^{m+1})}. \tag{13}$$

However, for the fi-CFT of $f\in L^1\bigcap L^2(\mathbb{R}^m;\mathbb{R}^{m+1})$, the energy preservation ratio of $\mathscr{F}(f_{\mathbf{T}})(\omega)$ on space interval $\mathbf{T}$ to $f(\mathbf{x})$ is small than or equals to 1:

$$0 < \text{energy preservation ratio} := \frac{\|\mathscr{F}(f_{\mathbf{T}})\|^2_{L^2(\mathbf{W};\mathbb{R}^{m+1})}}{\|f\|^2_{L^2(\mathbf{T};\mathbb{R}^{m+1})}} = (2\pi)^{-m}\left(\frac{\int_{\mathbf{W}}|\mathscr{F}(f_{\mathbf{T}})(\omega)|^2 d\sigma(\omega)}{\int_{\mathbf{T}}|f(\mathbf{x})|^2\,d\sigma(\mathbf{x})}\right)\le 1, \tag{14}$$

since

$$(2\pi)^{-m}\|\mathscr{F}(f_{\mathbf{T}})\|^2_{L^2(\mathbf{W};\mathbb{R}^{m+1})} \le (2\pi)^{-m}\|\mathscr{F}(f_{\mathbf{T}})\|^2_{L^2(\mathbb{R}^m;\mathbb{R}^{m+1})} = \|f_{\mathbf{T}}\|^2_{L^2(\mathbb{R}^m;\mathbb{R}^{m+1})} = \|f\|^2_{L^2(\mathbf{T};\mathbb{R}^{m+1})}.$$

Then one may ask under what conditions the energy ratio is maximal. As shown by Slepian and Pollak [21], and Landau and Pollak [11, 12] in the one dimensional case, these problems could be successfully analyzed by the PSWFs. With these outcomes in mind, higher dimensional extensions of these problems may be analyzed by Clifford prolate spheroidal wave functions (CPSWFs) (Definition 3.1 in Section 3).

Remark 2.2 *In the present article, we consider the case where the intervals **T** and **W** are symmetric only. The non-symmetric case will be discussed in the upcoming research.*

3. Definitions and properties of CPSWFs

3.1. CPSWFs

Considerable simplification occurs when $\mathbf{W}$ is a scaled version of $\mathbf{T}$. We write $\mathbf{W}=c\mathbf{T}$ where $x\in c\mathbf{T}$ if and only if $x/c\in\mathbf{T}$ with c a positive constant. We restrict our attention henceforth to this case. Let us now introduce the Clifford prolate spheroidal wave functions in the finite Clifford Fourier transform (fi-CFT) setting.

Definition 3.1 *(CPSWF, fi-CFT form)*
*Let **W** and **T** be given. Let $\psi_{n,\mathbf{T},\mathbf{W}}:\mathbb{R}^m\to Cl_{0,m}$, $n=0,1,2,\ldots$, be a complete set of solution of the integral equation*

$$\int_{\mathbf{T}}\psi_{n,\mathbf{T},\mathbf{W}}(\mathbf{y})\overline{\mathbf{e}(c\mathbf{y},\mathbf{x})}d\sigma(\mathbf{y}) := \mu_{n,\mathbf{T},\mathbf{W}}\psi_{n,\mathbf{T},\mathbf{W}}(\mathbf{x}), \tag{15}$$

and let $\mu_{n,\mathbf{T},\mathbf{W}}$ be the complex parameter corresponding to $\psi_{n,\mathbf{T},\mathbf{W}}, n=0,1,\ldots$. The function $\psi_{n,\mathbf{T},\mathbf{W}}$ is called the Clifford prolate spheroidal wave function (CPSWF).

 Copyright © 2009 John Wiley & Sons, Ltd.
Prepared using mmaauth.cls

Remark 3.1 *From the symmetry of* $\boldsymbol{T}$*, it follows that if* $\psi_{n,\boldsymbol{T},\boldsymbol{W}}(\mathbf{x})$ *is a solution of (15), so also is* $\psi_{n,\boldsymbol{T},\boldsymbol{W}}(-\mathbf{x})$*, so that both*

$$\psi_e(\mathbf{x}) := \psi_{n,\boldsymbol{T},\boldsymbol{W}}(\mathbf{x}) + \psi_{n,\boldsymbol{T},\boldsymbol{W}}(-\mathbf{x})$$

and

$$\psi_o(\mathbf{x}) := \psi_{n,\boldsymbol{T},\boldsymbol{W}}(\mathbf{x}) - \psi_{n,\boldsymbol{T},\boldsymbol{W}}(-\mathbf{x})$$

are solutions as well. The eigenfunctions of (15) can be chosen to be either even or odd functions of $\mathbf{x}$*. Considerable simplification occurs when* $\psi_{n,\boldsymbol{T},\boldsymbol{W}}(\mathbf{x})$ *is even or odd with* n*. We restrict our attention henceforth to this case.*

Example 3.1 *Let* $\boldsymbol{T} = [-1,1]\times\cdots\times[-1,1]$ *and* $\boldsymbol{W} = [-c,c]\times\cdots\times[-c,c]$*,* c *is positive real number. Consider the* m*-vector signal*

$$f(\mathbf{x}) = \phi_{0,T,W}(x_0)e_0\, \phi_{1,T,W}(x_1)e_1 \cdots \phi_{m,T,W}(x_m)e_m.$$

It is easy to see that the signal f *is a CPSWF which satisfies (15) with* $\mu_{n,\boldsymbol{T},\boldsymbol{W}} = \gamma_0\gamma_1\cdots\gamma_m$*, since for* $j = 0,1,\ldots,m$

$$\int_{-1}^{1} \phi_j(y_j) e^{e_j c y_j x_j} dy_j = \gamma_j \phi_j(x_j).$$

Here we used (28) in [30].

We shall show that solution of this equation (15) is completely equivalent to solution of the eigenvalue equation

$$\int_{\mathbf{T}} K_{\mathbf{W}}(\mathbf{x}-\mathbf{y})\psi_{n,\mathbf{T},\mathbf{W}}(\mathbf{y})d\sigma(\mathbf{y}) = \lambda_{n,\mathbf{T},\mathbf{W}}\,\psi_{n,\mathbf{T},\mathbf{W}}(\mathbf{x}), \tag{16}$$

where

$$K_{\mathbf{W}}(\mathbf{x}) := (2\pi)^{-m}\int_{\mathbf{W}} \overline{\mathbf{e}(\omega,\mathbf{x})}d\sigma(\omega) = \prod_{i=1}^{m}\left(\frac{\sin(W_i x_i)}{\pi x_i}\right) = \prod_{i=1}^{n}\left[\left(\frac{W_i}{\pi}\right)K\left(\frac{W_i x_i}{\pi}\right)\right]$$

and K is the sinc function. The complex parameters $\lambda_{n,\mathbf{T},\mathbf{W}}$'s are called the eigenvalues corresponding to $\psi_{n,\mathbf{T},\mathbf{W}}, n = 0,1,\ldots$.

By the definition of $K_{\mathbf{W}}$, it is even and real-valued, so it can commute with any Clifford number. The left-hand side of (16) becomes

$$\begin{aligned}\int_{\mathbf{T}} \psi_{n,\mathbf{T},\mathbf{W}}(\mathbf{y})K_{\mathbf{W}}(\mathbf{x}-\mathbf{y})d\sigma(\mathbf{y}) &= (2\pi)^{-m}\int_{\mathbf{T}}\int_{\mathbf{W}} \psi_{n,\mathbf{T},\mathbf{W}}(\mathbf{y})\overline{\mathbf{e}(\omega,\mathbf{x}-\mathbf{y})}d\sigma(\omega)d\sigma(\mathbf{y}) \\ &= (2\pi)^{-m}\int_{\mathbf{W}}\left(\int_{\mathbf{T}} \psi_{n,\mathbf{T},\mathbf{W}}(\mathbf{y})\overline{\mathbf{e}(\omega,\mathbf{y})}d\sigma(\mathbf{y})\right)\mathbf{e}(\omega,\mathbf{x})d\sigma(\omega).\end{aligned} \tag{17}$$

The last equality holds since

$$\int_{\mathbf{W}} \overline{\mathbf{e}(\omega,\mathbf{x}-\mathbf{y})}d\sigma(\omega) = \int_{\mathbf{W}} \overline{\mathbf{e}(\omega,\mathbf{y})}\mathbf{e}(\omega,\mathbf{x})d\sigma(\omega).$$

Using (15) equation (17) becomes

$$\begin{aligned}\int_{\mathbf{T}} \psi_{n,\mathbf{T},\mathbf{W}}(\mathbf{y})K_{\mathbf{W}}(\mathbf{x}-\mathbf{y})d\sigma(\mathbf{y}) &= (2\pi)^{-m}\mu_{n,\mathbf{T},\mathbf{W}}\int_{\mathbf{W}} \psi_{n,\mathbf{T},\mathbf{W}}\left(\frac{\omega}{c}\right)\mathbf{e}(\omega,\mathbf{x})d\sigma(\omega) \\ &= \left(\frac{c}{2\pi}\right)^m \mu_{n,\mathbf{T},\mathbf{W}}\int_{\mathbf{W}/c} \psi_{n,\mathbf{T},\mathbf{W}}(\omega)\mathbf{e}(c\omega,\mathbf{x})d\sigma(\omega) \\ &= \left(\frac{c}{2\pi}\right)^m (-1)^{\frac{m(m+1)}{2}}\mu_{n,\mathbf{T},\mathbf{W}}\int_{\mathbf{T}} \psi_{n,\mathbf{T},\mathbf{W}}(\omega)\overline{\mathbf{e}(-\mathbf{x},c\omega)}d\sigma(\omega) \\ &= (-1)^{\frac{m(m+1)}{2}}\left(\frac{c}{2\pi}\right)^m \left(\mu_{n,\mathbf{T},\mathbf{W}}\right)^2 \psi_{n,\mathbf{T},\mathbf{W}}(-\mathbf{x}) \\ &= \lambda_{n,\mathbf{T},\mathbf{W}}\psi_{n,\mathbf{T},\mathbf{W}}(\mathbf{x}),\end{aligned}$$

where $\lambda_{n,\mathbf{T},\mathbf{W}} := (-1)^{j+\frac{m(m+1)}{2}}(c/2\pi)^m\left(\overline{\mu_{n,\mathbf{T},\mathbf{W}}}\right)^2$. Here we have used $\mathbf{e}(\mathbf{x},c\omega) = (-1)^{\frac{m(m+1)}{2}}\overline{\mathbf{e}(-\mathbf{x},c\omega)}$ and the fact that $\psi_{n,\mathbf{T},\mathbf{W}}$ is even and odd with n. It gives the right-hand side of (16).

We have now shown that the solutions of (15) and (16) are equivalent. The following is the alternative definition of CPSWF in the low pass filtering form.

Definition 3.2 *(CPSWF, low pass filtering form)*
Let $\boldsymbol{W}$ *and* $\boldsymbol{T}$ *be given. Let* $\psi_{n,\boldsymbol{T},\boldsymbol{W}}: \mathbb{R}^m \to Cl_{0,m}$*,* $n = 0,1,2,\ldots$*, be a complete set of solution of the integral equation*

$$\int_{\mathbf{T}} K_{\boldsymbol{W}}(\mathbf{x}-\mathbf{y})\psi_{n,\boldsymbol{T},\boldsymbol{W}}(\mathbf{y})d\sigma(\mathbf{y}) = \lambda_{n,\boldsymbol{T},\boldsymbol{W}}\psi_{n,\boldsymbol{T},\boldsymbol{W}}(\mathbf{x}), \tag{18}$$

Copyright © 2009 John Wiley & Sons, Ltd.

where

$$K_{\mathbf{W}}(\mathbf{x}) := (2\pi)^{-m} \int_{\mathbf{W}} \overline{\mathbf{e}(\omega, \mathbf{x})} d\sigma(\omega) = \prod_{i=1}^{m} \left(\frac{\sin(W_i x_i)}{\pi x_i} \right) = \prod_{i=1}^{n} \left[\left(\frac{W_i}{\pi} \right) K \left(\frac{W_i x_i}{\pi} \right) \right] \tag{19}$$

and

$$\lambda_{n,\mathbf{T},\mathbf{W}} := (-1)^{j+\frac{m(m+1)}{2}} (c/2\pi)^m \left(\overline{\mu_{n,\mathbf{T},\mathbf{W}}} \right)^2 . \tag{20}$$

The function $\psi_{n,\mathbf{T},\mathbf{W}}$ is called the Clifford prolate spheroidal wave function (CPSWF). Let $\lambda_{n,\mathbf{T},\mathbf{W}}$ be the eigenvalue corresponding to $\psi_{n,\mathbf{T},\mathbf{W}}, n = 0, 1, \ldots$.

Remark 3.2 *The low pass filtering form of CPSWF (18) is the analog of (1).*

Remark 3.3 *In the considerations to follow we will suppress this dependence and simply write $\mu_n := \mu_{n,\mathbf{T},\mathbf{W}}$, $\lambda_{n,\mathbf{T},\mathbf{W}}$ and $\psi_n(\mathbf{x}) := \psi_{n,\mathbf{T},\mathbf{W}}(\mathbf{x})$.*

The kernel (19) of (18) is positive definite, since

$$\int_{\mathbf{T}} \int_{\mathbf{T}} K_{\mathbf{W}}(\mathbf{x} - \mathbf{y}) f(\mathbf{x}) \overline{f(\mathbf{y})} d\sigma(\mathbf{x}) d\sigma(\mathbf{y}) = (2\pi)^{-m} \int_{\mathbf{W}} \left| \int_{\mathbf{T}} \mathbf{e}(\omega, \mathbf{x}) f(\mathbf{x}) d\sigma(\mathbf{x}) \right|^2 d\sigma(\omega) > 0,$$

whenever

$$\int_{\mathbf{T}} |f(\mathbf{x})|^2 d\sigma(\mathbf{x}) > 0.$$

This follows from the fact that if it were equal to zero for some nonzero $f \in \mathscr{B}_{\mathbf{W}}$, then f would be zero on $\mathbf{T}$, a distinct impossibility since $\mathscr{B}_{\mathbf{W}}$ is composed of monogenic functions in $\mathbb{R}^{m+1}$. A similar argument shows it to be self-adjoint.

Hence there exists a sequence of positive real eigenvalues $\lambda_0 \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq 0$, with their corresponding normalized eigenfunctions $\psi_0, \psi_1, \psi_2, \cdots$, refer to [25].

Remark 3.4 *In one dimension, the eigenvalues all have multiplicity one, but the proof of this uses the differential equation of the PSWF as well [21]. This is also true for symmetric regions in higher dimensions, but in general, we can conclude only that each eigenvalue has finite multiplicity. These eigenfunctions constitute the m-dimensional PSWFs associated with the sets $\mathbf{W}$ and $\mathbf{T}$; they are orthogonal on $\mathbf{T}$, and, in fact, constitute an orthogonal basis of $L^2(\mathbf{T}; \mathbb{R}^{m+1})$ since 0 is not an eigenvalue of such a positive operator. They are also orthogonal on $\mathbb{R}^m$ which can be shown by the orthogonality over finite region $\mathbf{T}$.*

The $\psi_n(\mathbf{x})$ can be normalized so that the following statements hold:

(i) The ψ_n are W-bandlimited, orthonormal on $\mathbb{R}^m$, i.e.,

$$\int_{\mathbb{R}^m} \psi_l(\mathbf{x}) \overline{\psi_n(\mathbf{x})} d\sigma(\mathbf{x}) = \delta_{l,n} \tag{21}$$

and complete in the space of $\mathbf{W}$-bandlimited signals

$$\mathscr{B}_{\mathbf{W}} := \left\{ f \in L^2(\mathbb{R}^m; \mathbb{R}^{m+1}) |\ \text{supp}\ \mathscr{F}(f) \subset \mathbf{W} \subset \mathbb{R}^m \right\},$$

with finite energy.

(ii) The ψ_n are orthogonal and complete in $L^2(\mathbf{T}; \mathbb{R}^{m+1})$:

$$\int_{\mathbf{T}} \psi_l(\mathbf{x}) \overline{\psi_n(\mathbf{x})} d\sigma(\mathbf{x}) = \lambda_n \delta_{l,n}. \tag{22}$$

Remark 3.5 *Since $\|\psi_n\|^2_{L^2(\mathbb{R}^m;\mathbb{R}^{m+1})} = 1$ and $\|\psi_n\|^2_{L^2(\mathbf{T};\mathbb{R}^{m+1})} = \lambda_n$, a small value of λ_n implies that $\psi_n(\mathbf{x})$ has most of its energy outside the interval $\mathbf{T}$ while a value of λ_n near 1 implies that $\psi_n(\mathbf{x})$ is concentrated largely in $\mathbf{T}$.*

To prove (21), using low pass filtering definition of CPSWF (18) and $K_{\mathbf{W}}$ is real-valued, we have

$$\overline{\psi_{n,}(\mathbf{x})} = \frac{1}{\lambda_n} \int_{\mathbf{T}} K_{\mathbf{W}}(\mathbf{x} - \mathbf{y}) \overline{\psi_{n,\mathbf{T},\mathbf{W}}(\mathbf{y})} d\sigma(\mathbf{y}), \qquad \mathbf{x} \in \mathbb{R}^m.$$

Consider

$$\begin{aligned} & \int_{\mathbb{R}^m} \psi_{l,\mathbf{T},\mathbf{W}}(\mathbf{x}) \overline{\psi_{n,\mathbf{T},\mathbf{W}}(\mathbf{x})} d\sigma(\mathbf{x}) \\ = \ & \frac{1}{\lambda_l \lambda_n} \int_{\mathbf{T}} \int_{\mathbf{T}} \psi_{l,\mathbf{T},\mathbf{W}}(\mathbf{y}) \overline{\psi_{l,\mathbf{T},\mathbf{W}}(\mathbf{t})} \left(\int_{\mathbb{R}^m} K_{\mathbf{W}}(\mathbf{x} - \mathbf{y}) K_{\mathbf{W}}(\mathbf{x} - \mathbf{t}) d\sigma(\mathbf{x}) \right) d\sigma(\mathbf{y}) d\sigma(\mathbf{t}). \end{aligned}$$

Note that $K_{\mathbf{T}}$ is an even and real-valued function and that from (19) and Parseval's theorem (26) it follows that

$$\int_{\mathbb{R}^m} K_{\mathbf{W}}(\mathbf{x}-\mathbf{y})K_{\mathbf{W}}(\mathbf{x}-\mathbf{t})d\sigma(\mathbf{x}) = K_{\mathbf{W}}(\mathbf{y}-\mathbf{t}).$$

Therefore, one then finds

$$\begin{aligned} & \int_{\mathbb{R}^m} \psi_{l,\mathbf{T},\mathbf{W}}(\mathbf{x})\overline{\psi_{n,\mathbf{T},\mathbf{W}}(\mathbf{x})}d\sigma(\mathbf{x}) \\ = & \frac{1}{\lambda_l\lambda_n}\int_{\mathbf{T}} \psi_{l,\mathbf{T},\mathbf{W}}(\mathbf{y})\left(\int_{\mathbf{T}} K_{\mathbf{W}}(\mathbf{y}-\mathbf{t})\overline{\psi_{l,\mathbf{T},\mathbf{W}}(\mathbf{t})}d\sigma(\mathbf{t})\right)d\sigma(\mathbf{y}) \\ = & \frac{1}{\lambda_l}\int_{\mathbf{T}} \psi_{l,\mathbf{T},\mathbf{W}}(\mathbf{y})\overline{\psi_n(\mathbf{y})}d\sigma(\mathbf{y}) \end{aligned}$$

The orthogonality of $\psi_{n,\mathbf{T},\mathbf{W}}$ over $\mathbf{T}$ thus implies orthogonality over $\mathbb{R}^m$ as well.

It is sometimes desired to extrapolate a bandlimited function known only on the space interval **T** to values outside this interval using the bandlimited functions CPSWFs ψ_n. The mean-square convergence of the *(left-sided) Clifford Slepian series* (CSS)$\sum_{n=0}^{N} a_n\psi_n(\mathbf{x})$ is useful in many applications.

Lemma 3.1 *Suppose $f \in \mathscr{B}_{\mathbf{W}}$. Then*

$$\lim_{N\to\infty}\int_{\mathbb{R}^m}\left|f(\mathbf{x})-\sum_{n=0}^{N} a_n\psi_n(\mathbf{x})\right|^2 d\sigma(\mathbf{x}) = 0, \tag{23}$$

where the coefficients of CSS are

$$a_n = \lambda_n^{-1}\int_{\mathbf{T}} f(\mathbf{x})\,\overline{\psi_n(\mathbf{x})}\,d\sigma(\mathbf{x}) \ \in \mathbb{R}^{m+1}. \tag{24}$$

The coefficients in (23) can be determined by (24) from values of f in the space interval $\mathbf{T}$.

Proof. From (21), ψ_n constitutes an orthonormal basis in $L^2(\mathbb{R}^m;\mathbb{R}^{m+1})$. For any $\mathbf{x}\in\mathbb{R}^{\mathbf{m}}$,

$$f(\mathbf{x}) = \sum_{n=0}^{\infty} a_n\,\psi_n(\mathbf{x}), \tag{25}$$

where

$$a_n = \int_{\mathbb{R}^m} f(\mathbf{x})\,\overline{\psi_n(\mathbf{x})}\,d\sigma(\mathbf{x}).$$

Using Parseval's identity (13), f may be characterized by its coefficients

$$\sum_{n=0}^{\infty}|a_n|^2 = \int_{\mathbb{R}^m}|f(\mathbf{x})|^2\,d\sigma(\mathbf{x}), \tag{26}$$

and the convergence in (25) is in the mean square sense

$$\lim_{N\to\infty}\int_{\mathbb{R}^m}\left|f(\mathbf{x})-\sum_{n=0}^{N} a_n\psi_n(\mathbf{x})\right|^2 d\sigma(\mathbf{x}) = 0.$$

Now multiply (25) by $\overline{\psi_l(\mathbf{x})}$ from the right-hand side, integrate in **T** and use (22). A direct computation shows that

$$a_n = \lambda_n^{-1}\int_{\mathbf{T}} f(\mathbf{x})\,\overline{\psi_n(\mathbf{x})}\,d\sigma(\mathbf{x}).$$

The above result suggests approximating $f(\mathbf{x})$ for all $\mathbf{x}\in\mathbb{R}^m$ by

$$f_N(\mathbf{x}) = \sum_{n=0}^{N} a_n\,\psi_n(\mathbf{x}), \tag{27}$$

where the coefficients a_n are given by (24). The approximation (27) is itself bandlimited. Whence, the mean square error is

$$\int_{\mathbb{R}^m}|f(\mathbf{x})-f_N(\mathbf{x})|^2\,d\sigma(\mathbf{x}) = \sum_{n=N+1}^{\infty}|a_n|^2 \tag{28}$$

Copyright © 2009 John Wiley & Sons, Ltd.

and by (26) can be made as small as desired by making N sufficiently large. In the sense of (28), the extrapolation remains good for all $\mathbf{x}\in\mathbb{R}^{\mathbf{m}}$. The error in the fit of f_N to f in **T** is given by

$$\int_{\mathbf{T}} |f(\mathbf{x})-f_N(\mathbf{x})|^2\,d\sigma(\mathbf{x}) = \sum_{n=N+1}^{\infty} |a_n|^2\lambda_n. \tag{29}$$

As the λ_n's approach zero rapidly for sufficiently large n, it may happen that (29) is small for values of N for which (28) is still large.

Remark 3.6 *Suppose now $f\in L^2(\mathbb{R}^m;\mathbb{R}^{m+1})$ is known in the space interval **T** but $f\notin\mathscr{B}_{\mathbf{W}}$. From the proof of Lemma 3.1, it follows that f may still be represented by (23) with the coefficient given by (24), but this representation is valid now for $\mathbf{x}\in\mathbf{T}$ only. If indeed $f\notin\mathscr{B}_{\mathbf{W}}$, the series (23) will certainly not converge in mean square over the whole $\mathbb{R}^m$ space.*

3.2. Maxima of energy preservation problem

CPSWFs are helpful for calculating the power preservation ratio (14) of the fi-CFT.

Theorem 3.1 *If $f\in L^1\bigcap L^2(\mathbf{T};Cl_{o,m})$ and $f_{\mathbf{T}}=f(x)\chi_{\mathbf{T}}(x)$, the characteristic function on **T**, then the maximum value of energy preservation ratio*

$$\mathscr{R}_{\mathscr{F}} := \frac{(2\pi)^{-m}\int_{\mathbf{W}}|\mathscr{F}(f_{\mathbf{T}})(\omega)|^2\,d\sigma(\omega)}{\int_{\mathbf{T}}|f(\mathbf{x})|^2\,d\sigma(\mathbf{x})} \tag{30}$$

of fi-CFT can be obtained if f is a multiple of $\psi_0(\mathbf{x})$, the eigenfunction of (16) belonging to the largest eigenvalue λ_0. That is, $f(\mathbf{x})=a_0\,\psi_0(\mathbf{x})$, where the constant $a_0\in\mathbb{R}^{m+1}$.

Proof. By the definition of fi-CFT,

$$\begin{aligned}\int_{\mathbf{W}}|\mathscr{F}(f_{\mathbf{T}})(\omega)|^2\,d\sigma(\omega) &= \int_{\mathbf{W}}\mathscr{F}(f_{\mathbf{T}})(\omega)\,\overline{\mathscr{F}(f_{\mathbf{T}})(\omega)}\,d\sigma(\omega)\\ &= \int_{\mathbf{W}}\left(\int_{\mathbb{R}^m} f_{\mathbf{T}}(\mathbf{x})\,\mathbf{e}(\omega,\mathbf{x})\,d\sigma(\mathbf{x})\right)\left(\int_{\mathbb{R}^m}\overline{\mathbf{e}(\omega,\mathbf{y})}\;\overline{f_{\mathbf{T}}(\mathbf{y})}\,d\sigma(\mathbf{y})\right)d\sigma(\omega).\end{aligned}$$

Taking into account that

$$\int_{\mathbf{W}}\mathbf{e}(\omega,\mathbf{x})\,\overline{\mathbf{e}(\omega,\mathbf{y})}\,d\sigma(\omega) = (2\pi)^m K_{\mathbf{W}}(\mathbf{x}-\mathbf{y})$$

and, since $K_{\mathbf{W}}$ is real it computes with any Clifford member of $\mathbb{R}^{m+1}$,

$$\int_{\mathbf{W}}|\mathscr{F}(f_{\mathbf{T}})(\omega)|^2\,d\sigma(\omega) = (2\pi)^m\int_{\mathbb{R}^m}\int_{\mathbb{R}^m}K_{\mathbf{W}}(\mathbf{y}-\mathbf{x})\,f_{\mathbf{T}}(\mathbf{x})\overline{f_{\mathbf{T}}(\mathbf{y})}\,d\sigma(\mathbf{x})\,d\sigma(\mathbf{y}).$$

By Lemma 3.1, for $\mathbf{x}\in\mathbf{T}$, $f(\mathbf{x})=\sum_{n=0}^{\infty}a_n\psi_n(\mathbf{x})$, $K_{\mathbf{W}}$ is real and even and (16) in Definition 3.1, we have

$$\begin{aligned}\int_{\mathbb{R}^m}\int_{\mathbb{R}^m}K_{\mathbf{W}}(\mathbf{y}-\mathbf{x})\,f_{\mathbf{T}}(\mathbf{x})\overline{f_{\mathbf{T}}(\mathbf{y})}\,d\sigma(\mathbf{x})\,d\sigma(\mathbf{y}) &= \int_{\mathbf{T}}\int_{\mathbf{T}}K_{\mathbf{W}}(\mathbf{y}-\mathbf{x})\,f(\mathbf{x})\overline{f(\mathbf{y})}\,d\sigma(\mathbf{x})\,d\sigma(\mathbf{y})\\ &= \sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\int_{\mathbf{T}}a_n\psi_n(\mathbf{x})\left(\int_{\mathbf{T}}K_{\mathbf{W}}(\mathbf{x}-\mathbf{y})\,\overline{\psi_l(\mathbf{y})}\,d\sigma(\mathbf{y})\right)\overline{a_l}\,d\sigma(\mathbf{x})\\ &= \sum_{n=0}^{\infty}\sum_{l=0}^{\infty}a_n\,\lambda_l\left(\int_{\mathbf{T}}\psi_n(\mathbf{x})\,\overline{\psi_l(\mathbf{x})}\,d\sigma(\mathbf{x})\right)\overline{a_l}.\end{aligned}$$

Using (22) in property (ii),

$$(2\pi)^{-m}\int_{\mathbf{W}}|\mathscr{F}(f_{\mathbf{T}})(\omega)|^2\,d\sigma(\omega) = \sum_{n=0}^{\infty}|a_n|^2\,\lambda_n^2.$$

Similarly,

$$\begin{aligned}\int_{\mathbf{T}}|f(\mathbf{x})|^2\,d\sigma(\mathbf{x}) &= \int_{\mathbf{T}}f(\mathbf{x})\,\overline{f(\mathbf{x})}\,d\sigma(\mathbf{x})\\ &= \int_{\mathbf{T}}\left(\sum_{n=0}^{\infty}a_n\,\psi_n(\mathbf{x})\right)\left(\sum_{l=0}^{\infty}\overline{a_l\,\psi_l(\mathbf{x})}\right)d\sigma(\mathbf{x}).\end{aligned}$$

Since λ_n are real and applying (22) again, the above equals to

$$\int_{\mathbf{T}} |f(\mathbf{x})|^2 d\sigma(\mathbf{x}) = \sum_{n=0}^{\infty}\sum_{l=0}^{\infty} a_n \left(\int_{\mathbf{T}} \psi_n(\mathbf{x}) \overline{\psi_l(\mathbf{x})} d\sigma(\mathbf{x}) \right) \overline{a_l} = \sum_{n=0}^{\infty} |a_n|^2 \lambda_n.$$

Thus the energy preservation ratio of $\mathscr{F}(f)(\omega)$ to $f(\mathbf{x})$ is given by

$$\frac{(2\pi)^{-m} \int_{\mathbf{W}} |\mathscr{F}(f_{\mathbf{T}})(\omega)|^2 d\sigma(\omega)}{\int_{\mathbf{T}} |f(\mathbf{x})|^2 d\sigma(\mathbf{x})} = \frac{\sum_{n=0}^{\infty} |a_n|^2 \lambda_n^2}{\sum_{n=0}^{\infty} |a_n|^2 \lambda_n}.$$

Since $|\lambda_0| > |\lambda_n|$, if $n \neq 0$, to make the energy ratio maximal, $f(\mathbf{x})$ should be a multiple of $\psi_0(\mathbf{x})$, the eigenfunction of (16) belonging to the largest eigenvalue λ_0; so that $f(\mathbf{x}) = a_0 \psi_0(\mathbf{x})$.

4. Generalized CPSWFs and applications

In the present section we generalize the theory of CPSWFs and discuss how to use them to deal with the energy preservation problems associated with the finite-extension Clifford linear canonical transforms (fi-CLCTs), and the finite-extension offset Clifford linear canonical transforms (fi-offset CLCTs). This finite transforms are more general than the fi-CFTs.

4.1. Fi-CLCT and offset fi-CLCT

The linear canonical transform (LCT), as a generalization of the Fourier transform, was first proposed in the 70s by Moshinsky and Collins [14, 29]. It has recently received much attention in signal processing and optics [15, 17, 24]. Higher dimensional extension of LCT in the Quaternion analysis setting was first studied in [9]. In this paper, we generalized the LCT into the Clifford analysis setting.

Let us introduce the *Clifford linear canonical transform (CLCT)* of signal $f \in L^1(\mathbb{R}^m;\mathbb{R}^{m+1})$.

Definition 4.1 *For $i \in \mathbb{N}$, let the matrix parameter $A_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in \mathbb{C}^{2\times 2}$ with det $(A_i) = 1$. The Clifford linear canonical transform (CLCT) $\mathscr{L}_{A_i}(f) : \mathbb{R}^m \to Cl_{0,m}$ of a signal $f \in L^1(\mathbb{R}^m;\mathbb{R}^{m+1})$ is defined as*

$$\mathscr{L}_{A_i}(f)(\mathbf{u}) := \begin{cases} \int_{\mathbb{R}^m} f(\mathbf{x}) \mathscr{E}_{A_i}(\mathbf{x},\mathbf{u})\, d\sigma(\mathbf{x}), & b_i \neq 0 \\ f(d_i\mathbf{u}) d_i^{-\frac{m}{2}} \left[\prod_{j=1}^{m} \exp\left(e_j \frac{c_i d_i}{2} |\mathbf{u}|^2 \right) \right], & b_i = 0 \end{cases} \tag{31}$$

For $b_i \neq 0$, the kernel of the CLCT, $\mathscr{E}_{A_i} : \mathbb{R}^m \times \mathbb{R}^m \to Cl_{0,m}$, is

$$\mathscr{E}_{A_i}(\mathbf{x},\mathbf{u}) := \prod_{j=1}^{m} \left((2\pi b e_j)^{-\frac{m}{2}} \exp\left[e_j \left(\frac{a_i}{2b_i} |\mathbf{x}|^2 + \frac{1}{b_i} \langle \mathbf{x},\mathbf{u} \rangle + \frac{d_i}{2b_i} |\mathbf{u}|^2 \right) \right] \right). \tag{32}$$

When a_i, b_i, c_i, d_i are restricted to be real numbers, then $f \in L^1(\mathbb{R}^m;\mathbb{R}^{m+1})$ guarantees the existence of the CLCT. When some of a_i, b_i, c_i, d_i are complex numbers, then the above integrability condition should be replaced by

$$\int_{\mathbb{R}^m} |f(\mathbf{x})\, e^{\sigma_i |\mathbf{x}|^2}|\, d\sigma(\mathbf{x}) < \infty, \quad \text{with } \sigma_i = \text{Im}\left(\frac{a_i}{2b_i} \right).$$

and in the case f is in a larger class of function space. Nervelessly, for typographical convenience, unless otherwise stated, we shall restrict ourselves to the case of real parameters, in which case the corresponding CLCT is a unitary operator in $L^2(\mathbb{R}^m;\mathbb{R}^{m+1})$.

Note that when $b_i = 0$, the CLCT of a signal is essentially a chirp multiplication and is of no particular interest for our objective in this work. Hence, without loss of generality, we restrict our discussion to the case $b_i \neq 0$.

The CLCT converts to its special cases when taking different parameters a_i, b_i, c_i, d_i. For example, when $A_i = (1, b_i, 0, 1)$ the CLCT becomes the Gauss-Weierstrass transform or chirp convolution; when $A_{\alpha_i} = \begin{pmatrix} \cos\alpha_i & \sin\alpha_i \\ -\sin\alpha_i & \cos\alpha_i \end{pmatrix}$, it becomes the *Clifford fractional Fourier transform* (CFRFT)

$$\mathscr{L}_{A_{\alpha_i}}(f)(\mathbf{u}) = \int_{\mathbb{R}^m} f(\mathbf{x}) \mathscr{E}_{A_{\alpha_i}}(\mathbf{x},\mathbf{u})\, d\sigma(\mathbf{x}), \tag{33}$$

$$\mathscr{E}_{A_{\alpha_i}}(\mathbf{x},\mathbf{u}) = \prod_{j=1}^{m} \left(\frac{1 - e_j \cot\alpha_i}{2\pi} \right)^{\frac{m}{2}} \exp\left[e_j \left(\frac{\cot\alpha_i}{2} |\mathbf{x}|^2 + \csc\alpha_i \langle \mathbf{x},\mathbf{u} \rangle + \frac{\cot\alpha_i}{2} |\mathbf{u}|^2 \right) \right]$$

Copyright © 2009 John Wiley & Sons, Ltd.

multiplication with the fixed phase factor $(e^{e_j\alpha_i})^{\frac{m}{2}}$. When $\alpha_i=\pi/2$, the CFRFT becomes the CFT. When $\alpha_i=-\pi/2$, the CFRFT becomes the inverse CFT. The CFRFT and the CLCT are more flexible than the original CFT.

The CLCT can be further generalized into the *offset Clifford linear canonical transform* (offset CLCT). It has two extra vector parameters τ_i and $\rho_i\in\mathbb{R}^m$, which represent the space and frequency offsets. The definition is given as follows.

Definition 4.2 *For $i\in\mathbb{N}$, let the matrix parameter $B_i=\begin{pmatrix} a_i & b_i & \tau_i\\ c_i & d_i & \rho_i\end{pmatrix}$ satisfying $a_id_i-b_ic_i=1$. The offset Clifford linear canonical transform (offset CLCT) $\mathscr{L}_{B_i}(f):\mathbb{R}^m\to Cl_{0,m}$ of a signal $f\in L^1(\mathbb{R}^m;\mathbb{R}^{m+1})$is defined as*

$$\mathscr{L}_{B_i}(f)(\mathbf{u}) \quad := \quad \begin{cases} \int_{\mathbb{R}^m} f(\mathbf{x})\mathscr{E}_{B_i}(\mathbf{x},\mathbf{u})\,d\sigma(\mathbf{x}), & b_i\neq 0,\\ f\left(d_i(\mathbf{u}-\tau_i)\right)d_i^{-\frac{m}{2}}\prod_{j=1}^m \exp\left[e_j\left(\frac{c_id_i}{2}|\mathbf{u}-\tau_i|^2-\langle\rho_i,\mathbf{u}\rangle\right)\right], & b_i=0.\end{cases} \tag{34}$$

For $b_i\neq 0$, the kernel of the CLCT, $\mathscr{E}_{B_i}:\mathbb{R}^m\times\mathbb{R}^m\to Cl_{0,m}$, is

$$\mathscr{E}_{B_i}(\mathbf{x},\mathbf{u}) \quad := \quad \prod_{j=1}^m (2\pi b_ie_j)^{-\frac{m}{2}}\exp\left(e_j\langle\rho_i,\mathbf{u}\rangle\right)\exp\left[e_j\left(\frac{a_i}{2b_i}|\mathbf{x}|^2+\frac{1}{b_i}\langle\mathbf{x},\mathbf{u}-\tau_i\rangle+\frac{d_i}{2b_i}|\mathbf{u}-\tau_i|^2\right)\right]. \tag{35}$$

Notice that, in equations (31), (33) and (34), the space intervals are $\mathbf{x}\in\mathbb{R}^m$. It is more practical to use the finite-extension Clifford fractional Fourier transform (fi-CFRFT), the finite-extension Clifford linear canonical transform (fi-CLCT) and the offset fi-CLCT to replace the CFRFT, the CLCT, and the offset CLCT with infinite intervals. Since CFRFT is a special case of CLCT we give the definitions of fi-CLCT and offset fi-CLCT only. Let $f_{\mathbf{T}}$ be given in (12), $\mathscr{E}_{A_i}$ be given in (32), $\mathscr{E}_{B_i}$ be given in (35), $\mathbf{u}\in\mathbf{W}$ and $b_i\neq 0$.

(i) Fi-CLCT:

$$\mathscr{L}_{A_i}(f_{\mathbf{T}})(\mathbf{u})=\int_{\mathbb{R}^m} f_{\mathbf{T}}(\mathbf{x})\mathscr{E}_{A_i}(\mathbf{x},\mathbf{u})\,d\sigma(\mathbf{x}). \tag{36}$$

(ii) Offset fi-CLCT:

$$\mathscr{L}_{B_i}(f_{\mathbf{T}})(\mathbf{u})=\int_{\mathbb{R}^m} f_{\mathbf{T}}(\mathbf{x})\mathscr{E}_{B_i}(\mathbf{x},\mathbf{u})\,d\sigma(\mathbf{x}). \tag{37}$$

Now we are ready to study the energy preservation problems of fi-CLCTs and offset fi-CLCTs.

4.2. Generalized CPSFs for fi-CLCTs

Here we derive the function set that can analyze the energy preservation property of the fi-CLCT.

Definition 4.3 *(Generalized CPSF, low pass filtering form)*
For $i\in\mathbb{N}$, let the matrix parameter $A_i=\begin{pmatrix} a_i & b_i\\ c_i & d_i\end{pmatrix}\in\mathbb{C}^{2\times 2}$ with det $(A_i)=1$, let $\varphi_{n,\boldsymbol{T},\boldsymbol{W}}:\mathbb{R}^m\to\mathbb{R}^{m+1}$ be a set of solution satisfies the integral equation

$$\int_{\boldsymbol{T}} K^{A_i}_{\boldsymbol{W}}(\mathbf{x}-\mathbf{y})\,\varphi_{n,\mathbf{T},\boldsymbol{W}}(\mathbf{y})\,d\sigma(\mathbf{y})=\eta_{n,\mathbf{T},\boldsymbol{W}}\varphi_{n,\boldsymbol{T},\boldsymbol{W}}(\mathbf{x}). \tag{38}$$

It is called the generalized Clifford prolate spheroidal wave function (GCPSWF) for fi-CLCT. Here the kernel function $K^{A_i}_{W}:\boldsymbol{T}\times\boldsymbol{T}\to\mathbb{R}^{m+1}$ is defined by

$$K^{A_i}_{\boldsymbol{W}}(\mathbf{x}-\mathbf{y}):=K_{\frac{\boldsymbol{W}}{|b_i|}}(\mathbf{x},\mathbf{y})\left\{\prod_{j=1}^m\exp\left[e_j\frac{a_i}{2b_i}\left(|\mathbf{y}|^2-|\mathbf{x}|^2\right)\right]\right\} \tag{39}$$

and $K_{\boldsymbol{W}}$ is given in (19). The complex parameters $\eta_{n,\boldsymbol{T},\boldsymbol{W}}$'s are called the eigenvalues corresponding to $\varphi_{n,\boldsymbol{T},\boldsymbol{W}}$, $n=0,1,\ldots$.

Comparing (38) and (39) with (16) in Definition 3.1, and if $\psi_{n,\mathbf{T},\frac{\mathbf{W}}{|b_i|}}(\mathbf{x})$ is the CPSWF of the original fi-CFT that satisfies

$$\int_{\mathbf{T}} K_{\frac{\mathbf{W}}{|b_i|}}(\mathbf{x}-\mathbf{y})\,\psi_{n,\mathbf{T},\frac{\mathbf{W}}{|b_i|}}(\mathbf{y})\,d\sigma(\mathbf{y})=\lambda_{n,\mathbf{T},\frac{\mathbf{W}}{|b_i|}}\,\psi_{n,\mathbf{T},\frac{\mathbf{W}}{|b_i|}}(\mathbf{x}), \tag{40}$$

then the GCPSWFs and the corresponding eigenvalues for fi-CLCT are

$$\varphi_{n,\mathbf{T},\mathbf{W}}(\mathbf{x})=\left(\prod_{j=1}^m\exp\left(e_j\frac{a_i}{2b_i}|\mathbf{x}|^2\right)\right)\psi_{n,\mathbf{T},\frac{\mathbf{W}}{|b_i|}}(\mathbf{x})$$

and

$$\eta_{n,\mathbf{T},\mathbf{W}} = \lambda_{n,\mathbf{T},\frac{\mathbf{W}}{|b_i|}},$$

respectively. This is due to (39), $K_{\frac{\mathbf{W}}{|b_i|}}$ is real and using (40), it follows

$$\begin{aligned}
\int_{\mathbf{T}} K_{\mathbf{W}}^{A_i}(\mathbf{x}-\mathbf{y})\varphi_{n,\mathbf{T},\mathbf{W}}(\mathbf{y})\,d\sigma(\mathbf{y}) &= \left[\prod_{j=1}^{m} \exp\left(e_j \frac{a_i}{2b_i}|\mathbf{x}|^2\right)\right] \int_{\mathbf{T}} K_{\frac{\mathbf{W}}{|b_i|}}(\mathbf{x},\mathbf{y}) \left[\prod_{j=1}^{m} \exp\left(e_j \frac{a_i}{2b_i}|\mathbf{y}|^2\right)\right] \varphi_{n,\mathbf{T},\mathbf{W}}(\mathbf{y})d\sigma(\mathbf{y}) \\
&= \left[\prod_{j=1}^{m} \exp\left(e_j \frac{a_i}{2b_i}|\mathbf{x}|^2\right)\right] \int_{\mathbf{T}} K_{\frac{\mathbf{W}}{|b_i|}}(\mathbf{x},\mathbf{y})\psi_{n,\mathbf{T},\frac{\mathbf{W}}{|b_i|}}(\mathbf{y})d\sigma(\mathbf{y}) \\
&= \eta_{n,\mathbf{T},\mathbf{W}}\,\varphi_{n,\mathbf{T},\mathbf{W}}(\mathbf{x}).
\end{aligned}$$

As in the case of the original CPSWFs, all $\eta_{n,\mathbf{T},\mathbf{W}}$'s are positive and smaller than 1. We can sort $\varphi_{n,\mathbf{T},\mathbf{W}}(\mathbf{x})$ and $\eta_{n,\mathbf{T},\mathbf{W}}$ such that $\eta_{l,\mathbf{T},\mathbf{W}} \geq \eta_{n,\mathbf{T},\mathbf{W}}$ if $l > n$:

$$1 \geq \eta_{0,\mathbf{T},\mathbf{W}} \geq \eta_{1,\mathbf{T},\mathbf{W}} \geq \eta_{2,\mathbf{T},\mathbf{W}} \geq \cdots > 0.$$

The GCPSWFs for the fi-CLCT are orthogonal in the intervals of $\mathbf{x} \in \mathbf{T}$.

$$\int_{\mathbf{T}} \varphi_{n,\mathbf{T},\mathbf{W}}(\mathbf{x})\,\overline{\varphi_{l,\mathbf{T},\mathbf{W}}(\mathbf{x})}\,d\sigma(\mathbf{x}) = \int_{\mathbf{T}} \psi_{n,\mathbf{T},\frac{\mathbf{W}}{|b_i|}}(\mathbf{x})\overline{\psi_{n,\mathbf{T},\frac{\mathbf{W}}{|b_i|}}(\mathbf{x})}d\sigma(\mathbf{x}) = \eta_{n,\mathbf{T},\frac{\mathbf{W}}{|b_i|}}\,\delta_{l,n}. \tag{41}$$

Furthermore, they are also orthogonal in $\mathbf{x} \in \mathbb{R}^m$:

$$\int_{\mathbb{R}^m} \varphi_{n,\mathbf{T},\mathbf{W}}(\mathbf{x})\,\overline{\varphi_{l,\mathbf{T},\mathbf{W}}(\mathbf{x})}\,d\sigma(\mathbf{x}) = \int_{\mathbb{R}^m} \psi_{n,\mathbf{T},\frac{\mathbf{W}}{|b_i|}}(\mathbf{x})\overline{\psi_{n,\mathbf{T},\frac{\mathbf{W}}{|b_i|}}(\mathbf{x})}d\sigma(\mathbf{x}) = \delta_{l,n}.$$

Since the original CPSWFs $\{\psi_0(\mathbf{x}), \psi_1(\mathbf{x}), \ldots\}$ are complete, so the GCPSWFs $\{\varphi_0(\mathbf{x}), \varphi_1(\mathbf{x}), \ldots\}$ are also complete. Now $\{\varphi_n(\mathbf{x}) : n \in \mathbb{N}_0\}$ forms an orthogonal and complete function set, any function $f \in L^2(\mathbb{R}^m; \mathbb{R}^{m+1})$ can be expressed as a linear combination of $\varphi_0(\mathbf{x})$, $\varphi_1(\mathbf{x})$, ...:

$$f(\mathbf{x}) = \sum_{n=0}^{\infty} b_n\,\varphi_n(\mathbf{x}), \qquad \mathbf{x} \in \mathbf{T} \tag{42}$$

where coefficients of GCPSWS are

$$b_n = \eta_n^{-1} \int_{\mathbf{T}} f(\mathbf{x})\,\overline{\varphi_n(\mathbf{x})}\,d\sigma(\mathbf{x}) \ \in \mathbb{R}^{m+1}.$$

4.3. Maxima of energy preservation problem for fi-CLCT

The GCPSWFs are useful for studying the power preservation ratio of the fi-CLCT.

Theorem 4.1 *If $f \in L^1 \bigcap L^2(\mathbf{T}; Cl_{o,m})$, then the maximum value of energy preservation ratio*

$$\mathscr{R}_{\mathscr{L}}^{A_i} := \frac{\int_{\mathbf{W}} |\mathscr{L}_{A_i}(f_{\mathbf{T}})(\mathbf{u})|^2\,d\sigma(\mathbf{u})}{\int_{\mathbf{T}} |f(\mathbf{x})|^2\,d\sigma(\mathbf{x})} \tag{43}$$

of fi-CLCT can be obtained if f is a multiple of $\varphi_0(\mathbf{x})$, the eigenfunction of (38) belonging to the largest eigenvalue η_0. That is, $f(\mathbf{x}) = a_0\,\varphi_0(\mathbf{x})$, where the constant $a_0 \in \mathbb{R}^{m+1}$.

Proof. From (36) and (32), it follows that

$$\begin{aligned}
\int_{\mathbf{W}} |\mathscr{L}_{A_i}(f_{\mathbf{T}})(\mathbf{u})|^2\,d\sigma(\mathbf{u}) &= \int_{\mathbf{W}} \mathscr{L}_{A_i}(f_{\mathbf{T}})(\mathbf{u})\,\overline{\mathscr{L}_{A_i}(f_{\mathbf{T}})(\mathbf{u})}\,d\sigma(\mathbf{u}) \\
&= \int_{\mathbf{W}} \left(\int_{\mathbf{T}} f(\mathbf{x})\mathscr{E}_{A_i}(\mathbf{x},\mathbf{u})\,d\sigma(\mathbf{x})\right)\left(\int_{\mathbf{T}} \overline{\mathscr{E}_{A_i}(\mathbf{y},\mathbf{u})}\,\overline{f(\mathbf{y})}\,d\sigma(\mathbf{y})\right)d\sigma(\mathbf{u}).
\end{aligned}$$

A direct computation shows that

$$\int_{\mathbf{W}} \mathscr{E}_{A_i}(\mathbf{x},\mathbf{u})\overline{\mathscr{E}_{A_i}(\mathbf{y},\mathbf{u})}d\sigma(\mathbf{u}) = K_{\mathbf{W}}^{A_i}(\mathbf{x},\mathbf{y}),$$

Copyright © 2009 John Wiley & Sons, Ltd.

where $K_{\mathbf{W}}^{A_i}$ is given in (39). It follows that

$$\int_{\mathbf{W}} |\mathscr{L}_{A_i}(f_{\mathbf{T}})(\mathbf{u})|^2 d\sigma(\mathbf{u}) = \int_{\mathbf{T}}\int_{\mathbf{T}} f(\mathbf{x}) K_{\mathbf{W}}^{A_i}(\mathbf{x},\mathbf{y})\overline{f(\mathbf{y})} d\sigma(\mathbf{x}) d\sigma(\mathbf{y}).$$

For $\mathbf{x}\in\mathbf{T}$, by (42), $f(\mathbf{x}) = \sum_{n=0}^{\infty} a_n\varphi_n(\mathbf{x})$, we have

$$\int_{\mathbf{T}}\int_{\mathbf{T}} f(\mathbf{x}) K_{\mathbf{W}}^{A_i}(\mathbf{x},\mathbf{y})\overline{f(\mathbf{y})} d\sigma(\mathbf{x}) d\sigma(\mathbf{y}) = \int_{\mathbf{T}}\int_{\mathbf{T}} \left(\sum_{n=0}^{\infty} b_n\varphi_n(\mathbf{x})\right) K_{\mathbf{W}}^{A_i}(\mathbf{x},\mathbf{y}) \left(\sum_{l=0}^{\infty} \overline{b_l\varphi_l(\mathbf{y})}\right) d\sigma(\mathbf{x}) d\sigma(\mathbf{y}).$$

By (38) and (41), we have

$$\begin{aligned}
\int_{\mathbf{W}} |\mathscr{L}_{A_i}(f_{\mathbf{T}})(\mathbf{u})|^2 d\sigma(\mathbf{u}) &= \int_{\mathbf{T}} \left(\sum_{n=0}^{\infty} b_n\varphi_n(\mathbf{x})\right) \sum_{l=0}^{\infty} \left(\int_{\mathbf{W}} K_{\mathbf{W}}^{A_i}(\mathbf{x},\mathbf{y})\overline{\varphi_l(\mathbf{y})} d\sigma(\mathbf{y})\right) \overline{b_l} d\sigma(\mathbf{x}). \\
&= \sum\sum b_n\eta_l \left(\int_{\mathbf{T}} \varphi_n(\mathbf{x})\overline{\varphi_l(\mathbf{x})} d\sigma(\mathbf{x})\right) \overline{b_l} \\
&= \sum_{n=0}^{\infty} |b_n|^2\eta_n^2.
\end{aligned}$$

After few manipulations and by (41) again, it follows

$$\begin{aligned}
\int_{\mathbf{T}} |f(\mathbf{x})|^2 d\sigma(\mathbf{x}) &= \int_{\mathbf{T}} f(\mathbf{x})\overline{f(\mathbf{x})} d\sigma(\mathbf{x}) \\
&= \sum_{n=0}^{\infty}\sum_{l=0}^{\infty} b_n \left(\int_{\mathbf{T}} \varphi_n(\mathbf{x})\overline{\varphi_l(\mathbf{x})} d\sigma(\mathbf{x})\right) \overline{b_l} \\
&= \sum_{n=0}^{\infty} |b_n|^2\eta_n.
\end{aligned}$$

Therefore the energy preservation ratio for the fi-CLCT:

$$\mathscr{R}_{\mathscr{L}}^{A_i} = \frac{\int_{\mathbf{W}} |\mathscr{L}_{A_i}(f_{\mathbf{T}})(\mathbf{u})|^2 d\sigma(\mathbf{u})}{\int_{\mathbf{T}} |f(\mathbf{x})|^2 d\sigma(\mathbf{x})} = \frac{\sum_{n=0}^{\infty} |b_n|^2\eta_n^2}{\sum_{n=0}^{\infty} |b_n|^2\eta_n}. \tag{44}$$

Thus we can use the GCPSWFs to analyze the energy preservation property of the fi-CLCT. Notice that the maximal and minimal values of the energy preservation ratio are $\max(\mathscr{R}_{\mathscr{L}}^{A_i}) = \eta_0 \approx 1$, so that $g(\mathbf{x}) = b_0\varphi_0(\mathbf{x})$ and $\min(\mathscr{R}_{\mathscr{L}}^{A_i}) = \eta_n \approx 0$, so that $g(\mathbf{x}) = b_n\varphi_n(\mathbf{x})$, respectively.

Remark 4.1 *When* $A_{\alpha_i} = \begin{pmatrix} \cos\alpha_i & \sin\alpha_i \\ -\sin\alpha_i & \cos\alpha_i \end{pmatrix}$, *the fi-CLCT becomes the finite-extension Clifford fractional Fourier transform (fi-CFRFT). Thus the GCPSWFs and the corresponding eigenvalues of fi-CFRFT are*

$$\overline{\varphi_{n,\boldsymbol{T},\boldsymbol{W}}(\mathbf{x})} = \left[\prod_{j=1}^{m} \exp\left(e_j \frac{\cot\alpha_i}{2}|\mathbf{x}|^2\right)\right] \overline{\psi_{n,\boldsymbol{T},|\csc\alpha_i|\boldsymbol{W}}(\mathbf{x})},$$

and

$$\eta_{n,\boldsymbol{T},\boldsymbol{W}} = \lambda_{n,\boldsymbol{T},|\csc\alpha_i|\boldsymbol{W}},$$

respectively. Here $\psi_{n,\boldsymbol{T},|\csc\alpha_i|\boldsymbol{W}}(\mathbf{x})$ *is the original CPSWF that satisfies*

$$\int_{\boldsymbol{T}} K_{|\csc\alpha_i|\boldsymbol{W}}(\mathbf{x}-\mathbf{y})\,\overline{\psi_{n,\boldsymbol{T},|\csc\alpha_i|\boldsymbol{W}}(\mathbf{y})}\,d\sigma(\mathbf{y}) = \lambda_{n,\boldsymbol{T},|\csc\alpha_i|\boldsymbol{W}}\overline{\psi_{n,\boldsymbol{T},|\csc\alpha_i|\boldsymbol{W}}(\mathbf{x})}.$$

The energy preservation ratio of fi-CFRFT can be obtained if f *is a multiple of* $\varphi_0 = \psi_{0,\boldsymbol{T},|\csc\alpha_i|\boldsymbol{W}}$ *as given in Theorem 4.1.*

4.4. Generalized CPSWFs for offset fi-CLCTs

In this subsection we further generalize the results in Section 4.2 into the case of the offset fi-CLCTs.

Definition 4.4 *(Generalized CPSWFs for offset fi-CLCTs, low pass filtering form)*
For $i \in \mathbb{N}$, let the matrix parameter $B_i = \begin{pmatrix} a_i & b_i & \tau_i \\ c_i & d_i & \rho_i \end{pmatrix}$ satisfying $a_i d_i - b_i c_i = 1$. The solutions $\theta_{n,\boldsymbol{T},\boldsymbol{W}} : \mathbb{R}^m \to \mathbb{R}^{m+1}$, are continuous functions that satisfy the integral equation

$$\int_{\boldsymbol{T}} K_{\boldsymbol{W}}^{B_i}(\mathbf{x}-\mathbf{y})\, \overline{\theta_n(\mathbf{y})}\, d\sigma(\mathbf{y}) = \kappa_{n,\boldsymbol{T},\boldsymbol{W}} \overline{\theta_{n,\boldsymbol{T},\boldsymbol{W}}(\mathbf{x})}, \tag{45}$$

are called the generalized Clifford prolate spheroidal wave functions (GCPSWFs) for fi-CLCTs. Here the kernel function $K_{\boldsymbol{W}}^{B_i} : \boldsymbol{T} \times \boldsymbol{T} \to \mathbb{R}^{m+1}$ is defined by

$$K_{\boldsymbol{W}}^{B_i}(\mathbf{x}-\mathbf{y}) = K_{\boldsymbol{W}}^{A_i}(\mathbf{x}-\mathbf{y}) \left[\prod_{j=1}^{m} \exp\left(\frac{e_j}{b_i} < \tau, \mathbf{y} - \mathbf{x} > \right) \right], \tag{46}$$

and $K_{\boldsymbol{W}}^{A_i}$ is given in (39). The complex parameters κ_n's are called the eigenvalues of GCPSWFs for offset fi-CLCT.

Following a process similar to that in Section 4.2, then the GCPSWFs and the corresponding eigenvalues for offset fi-CLCT are

$$\overline{\theta_n(\mathbf{x})} = \left\{ \prod_{j=1}^{m} \exp\left[-e_j \left(\frac{a_i}{2b_i} |\mathbf{x}|^2 + \frac{1}{b_i} \langle \tau, \mathbf{x} \rangle \right) \right] \right\} \overline{\psi_{n,\mathbf{T},\frac{\mathbf{W}}{|b_i|}}(\mathbf{x})}$$

and

$$\kappa_n = \eta_n = \lambda_{n,\mathbf{T},\frac{\mathbf{W}}{|b_i|}},$$

respectively. Notice that the GPSWFs for the offset fi-CLCT has six parameters; however, only three of them (a_i, b_i and τ_i) affect the form of GPSWFs. The GPSWFs of the offset fi-CLCT satisfy the orthogonal functions set in $\mathbf{x} \in \mathbf{T}$ and $\mathbf{x} \in \mathbb{R}^m$ as well. We can also use the way used in Theorem 4.1 to analyze the energy preservation ratio of the offset fi-CLCT.

In one dimensional case, many finite-sized optical systems that cannot be represented by the finite-extension Fourier transform can still be modeled by the finite-extension linear canonical transform or the offset finite-extension linear canonical transform, please refer [18] for more details. We expect that the generalized finite transforms have the same properties. We may use them to analyze the properties of the more generalized finite-sized optical system in the higher dimensional case in the upcoming paper.

5. Conclusion

In this paper we present an extension of continuous PSWFs for the fi-CFT in the Clifford analysis setting. They are called the CPSWFs. We use these CPSWFs to analyze the energy preservation problem. Then the fi-CLCT (a generalization of the fi-CFT) and offset fi-CLCT (the fi-CLCT with two extra offset parameters) are derived. The GPSWFs for the fi-CLCT and the offset fi-CLCT are then studied. In addition, we use the GPSWFs to analyze the energy preservation ratio. Interestingly, the used methods also allow a generalization of the famous uncertainty principle and the Paley-Wiener theorem, including the case of Clifford functions that satisfy higher dimensional generalizations of Cauchy-Riemann or Dirac systems. Besides their obvious importance these results will not be discussed in the present article. Further investigations will be reported in a forthcoming paper.

6. Acknowledgement

The first author acknowledges financial support from the research grant of the University of Macau No. MYRG142(Y1-L2)-FST11-KKI. Partial support from the Foundation for Science and Technology (FCT) via the post-doctoral grant SFRH/BPD/66342/2009 is also acknowledged by the second author. Partial support by the Academic Human Resources Development in Institutions of Higher Learning under the Jurisdiction of Beijing Municipality (IHLB,201008257) and Scientific Research Common Program of Beijing Municipal Commission of Education(KM 200810011005).

References

1. C. J. Bouwkamp. *On spheroidal wave functions of order zero.* J. Math. Phys., 26 (1947), 79-92.

Copyright © 2009 John Wiley & Sons, Ltd.

2. J. P. Boyd. *Approximation of an analytic function on a finite real interval by a bandlimited function and conjectures on properties of prolate spheroidal functions*. Appl. Comput. Harmon. Anal., 15(2):168176, 2003.
3. J. P. Boyd. *Prolate spheroidal wavefunctions as an alternative to Chebyshev and Legendre polynomials for spectral element and pseudospectral algorithms*. J. Comput. Phys., 199(2):688716, 2004.
4. F. Brackx, R. Delanghe and F. Sommen. *Clifford Analysis*. Pitman Publishing, Boston-London-Melbourne, 1982.
5. T. Bülow, M. Felsberg and G. Sommer. *Non-commutative hypercomplex Fourier transforms of multidimensional signals*, in G. Sommer (ed.), Geom. Comp. with Cliff. Alg., Theor. Found. and Appl. in Comp. Vision and Robotics, Springer, 2001, pp. 187-207, Springer.
6. Q. Y. Chen, D. Gottlieb and J. S. Hesthaven. *Spectral methods based on prolate spheroidal wave functions for hyperbolic PDEs*. SIAM J. Numer. Anal., 43(5):1912-1933, (2005).
7. R. Courant and D. Hilbert. *Methods of Mathematical Physics*, Vol. 1, Interscience Publishers, New York, (1955), 122-134.
8. R. Delanghe, F. Sommen and V. Soucek. *Clifford Agebras and Spinor-Valued Functions*. Kluwer Academic Publishers Group, Dordrecht, 1992.
9. K. Kou and J. Ou. *An Uncertainty Principle for Quaternion Linear Canonical Transform*, to appear.
10. C. Flammer. *Spheroidal Wave Functions*. Stanford University Press, Stanford, CA, 1957.
11. H. J. Landau and H. O. Pollak. *Prolate Spheroidal Wave Functions, Fourier Analysis and Uncertainty, II*, Bell System Tech. J., Vol. 40, 1961, pp. 65-84.
12. H. J. Landau and H. O. Pollak. *Prolate Spheroidal Wave Functions, Fourier Analysis and Uncertainty, III*, Bell System Tech. J., Vol. 41, 1962, pp. 1295-1336.
13. I. C. Moore and M. Cada. *Prolate spheroidal wave functions, an introduction to the Slepian series and its properties*. Appl. Comput. Harmon. Anal., 16(3):208230, 2004.
14. M. Moshinsky and C. Quesne. *Linear canonical transforms and their unitary representations*, J. Math. Phys. Vol. 12, 1971, pp. 1772-1783.
15. H. M. Ozaktas, M. A. Kutay and Z. Zalevsky. *The Fractional Fourier Transform with Applications in Optics and Signal Processing*, Wiley, New York (2000).
16. A. Papoulis. *Signal Analysis*, McGrawHill, New York (1977).
17. S. Pei and J. Ding. *Eigenfunctions of the offset Fourier, fractional Fourier, and linear canonical transforms*, J. Opt. Soc. Am., A 20, 2003, pp. 522-532.
18. S. Pei and J. Ding. *Generalized prolate spheroidal wave functions for optical finite fractional Fourier and Linear canonical transforms*. J. Opt., Soc. Am. A/Vol. 22(3), 2005.
19. V. Rokhlin and H. Xiao. *Approximate formulae for certain prolate spheroidal wave functions valid for large values of both order and band-limit*. Appl. Comput. Harmon. Anal., 22(1):105123, 2007.
20. Y. Shkolnisky, M. Tygert, and V. Rokhlin. *Approximation of bandlimited functions*. Appl. Comput. Harmon. Anal., 21(3):413420, 2006.
21. D. Slepian and H. O. Pollak. *Prolate Spheroidal Wave Functions, Fourier Analysis and Uncertainty, I*, Bell System Tech. J., Vol. 40, 1961, pp. 43-64.
22. D. Slepian. *Prolate spheroidal wave functions, Fourier analysis and uncertainity. IV. Extensions to many dimensions; generalized prolate spheroidal functions.* Bell System Tech. J., 43:30093057, 1964. MR0181766 (31:5993).
23. D. Slepian. *Some asymptotic expansions for prolate spheroidal wave function.* J. Math. Phys. (44), 1965, pp. 99-140.
24. R. Tao, B. Deng and Y. Wang. *Fractional Fourier Transform and its Applications*, Tsinghua University Press, Beijing (2009).
25. G. Walter. *Prolate Spheroidal Wavelets in Higher Dimensions.* J. Integ. Equat., 18(3): 415-435, (2006).
26. G. Walter. *Prolate spheroidal wavelets: Translation, convolution, and differentiation made easy.* J. Fourier Anal. Appl., 11(1): 73-84, (2005).
27. G. Walter. *Wavelets based on prolate spheroidal wave functions.* J. Fourier Anal. Appl., 10(1):1-26 (2004).
28. G. Walter and T. Soleski. *A new friendly method of computing prolate spheroidal wave functions and wavelets.* Appl. Comput. Harmon. Anal., 19(3):432-443, (2005).
29. K. B. Wolf. *Integral Transforms in Science and Engineering*, New York: Plenum Press (1979).
30. H. Xiao, V. Rokhlin, and N. Yarvin. *Prolate spheroidal wavefunctions, quadrature and interpolation. Inverse Problems*, 17(4):805838, 2001. Special issue to celebrate Pierre Sabatier's 65th birthday (Montpellier, 2000).
31. H. Xiao. *Prolate spheroidal wave functions, quadrature, interpolation, and asymptotic formulae*. Ph.D. Thesis, Yale University, 2001.
32. A. I. Zayed. *A generalization of the prolate spheroidal wave functions*. Proc. Amer. Math. Soc., 135(7):21932203 (electronic), 2007.

 Copyright © 2009 John Wiley & Sons, Ltd.
Prepared using mmaauth.cls