Paley-Wiener Theorem In $\mathbf{R}^n$ With Clifford Analysis Setting*

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We prove the Paley-Wiener Theorem in the Clifford algebra setting. As an application we derive the corresponding result for conjugate harmonic functions.

Key Words: Fourier Analysis, Paley-Wiener Theorem, Conjugate Harmonic System, Clifford Analysis.

0. INTRODUCTION

Higher dimensional extensions of Paley-Wiener Theorem have been studied, for instance, in [16], [14], [6], [1] and [11]. In [16] a corresponding extension is obtained by imbedding $\mathbf{R}^n$ into $\mathbf{C}^n$ and by reducing it to the one complex variable case. The present work uses the imbedding of $\mathbf{R}^n$ into the real-Clifford algebra $\mathbf{R}^{(n)}$ (see the notation in §1). The latter imbedding provides $\mathbf{R}^n$ with a global complex structure in analogy with the imbedding of $\mathbf{R}$ into the complex plane. Under this frame we present in this note the precise analogue of the classical Paley-Wiener Theorem which has been targeted by others. In [1] results of the same kind are obtained of which either stronger conditions are imposed (see 30.10, [1]) or weaker conclusions, namely, in the distribution sense, are obtained (see 30.19, [1]). In [11] a set of results are obtained in which the pointwise estimate in the usual Paley-Wiener Theorem is replaced by an integral inequality.

It is well known that the classical Paley-Wiener Theorem has important applications to a wide range of topics in function theory of one complex variable and approximation of one real variable, etc. As example, in the

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Shannon sampling and interpolation using the sinc functions, the sampling scale is determined by the constant R (see §2 Theorem 2.1) appearing in the exponential part of the estimate for the holomorphic function under study ([17]). Owing to the analogous complex structure in $\mathbf{R}^n$ induced by the Dirac operator (see §1), the Paley-Wiener Theorem proved in this note offers the same applications to topics in several real variables.

In §1 we provide the basic knowledge of Clifford analysis used in the paper. In §2 we formulate and prove the Paley-Wiener Theorem. Our proof is guided by the one for the classical Paley-Wiener Theorem cited in [19]. In §3 we show that the concept monogenic functions is a natural way to represent conjugate harmonic systems. As an application, we present a new result on conjugate harmonic systems.

Some alternative proofs of the classical Paley-Wiener Theorem invoke Phragmén-Lindelöf Theorem in one complex variable (see, for instance, [16] or [3]). The proof of the latter theorem involves products of complex analytic functions and makes use of the fact that the product of two analytic functions is still analytic. This is failed in the Clifford setting. In general, products of monogenic functions are no longer monogenic. It would be interesting, however, to see the generalization of Phragmén-Liwdelöf Theorem in the Clifford analysis setting, and accordingly, a proof of Paley-Wiener Theorem using the generalized Phragmén-Liwdelöf Theorem.

1. PRELIMINARIES

Most of the basic knowledge and notation recalled in this section are referred to [1], [4] and [2].

Let $\mathbf{e}_1, ..., \mathbf{e}_n$ be *basic elements* satisfying $\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i = -2\delta_{ij}$, where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise, $i, j = 1, 2, \cdots, n$. Let

$$\mathbf{R}^n = \{\underline{x} = x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n : x_j \in \mathbf{R}, j = 1, 2, \cdots, n\}$$

be identical with the usual Euclidean space $\mathbf{R}^n$, and

$$\mathbf{R}_1^n = \{x_0 + \underline{x} : x_0 \in \mathbf{R}, \underline{x} \in \mathbf{R}^n\}.$$

An element in $\mathbf{R}_1^n$ is called a *vector*. The real (complex) Clifford algebra generated by $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n$, denoted by $\mathbf{R}^{(n)}$ ($\mathbf{C}^{(n)}$), is the associative algebra generated by $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n$, over the real (complex) field $\mathbf{R}$ ($\mathbf{C}$). A general element in $\mathbf{R}^{(n)}$, therefore, is of the form $x = \sum_S x_S\mathbf{e}_S$, where

$\mathbf{e}_S = \mathbf{e}_{i_1}\mathbf{e}_{i_2}\cdots\mathbf{e}_{i_l}$, and S runs over all the ordered subsets of $\{1, 2, \cdots, n\}$, namely

$$S = \{1 \leq i_1 < i_2 < \cdots < i_l \leq n\}, \quad 1 \leq l \leq n.$$

The natural inner product between x and y in $\mathbf{C}^{(n)}$, denoted by $< x, y >$, is the complex number $\sum_S x_S \overline{y_S}$, where $x = \sum_S x_S \mathbf{e}_S$ and $y = \sum_S y_S \mathbf{e}_S$. The norm associated with this inner product is

$$|x| = < x, x >^{\frac{1}{2}} = (\sum_S |x_S|^2)^{\frac{1}{2}}.$$

If $x, y, ..., u$ are vectors, then

$$|xy \cdots u| = |x||y| \cdots |u|.$$

The conjugate of a vector $x = x_0 + \underline{x}$, is defined as $\overline{x} = x_0 - \underline{x}$. It is easy to verify that $0 \neq x \in \mathbf{R}_1^n$ implies

$$x^{-1} = \frac{\overline{x}}{|x|^2}.$$

The *unit sphere* $\{x \in \mathbf{R}_1^n : |x| = 1\}$ is denoted by S^n. We use $B(x, r)$ for the open ball in $\mathbf{R}_1^n$ centered at x with radius r.

In below we will study functions defined in $\mathbf{R}^n$ or $\mathbf{R}_1^n$ taking values in $\mathbf{C}^{(n)}$. So, they are of the form $f(x) = \sum_S f_S(x)\mathbf{e}_S$, where f_S are complex-valued functions. We will be using the *Dirac operator*

$$D = D_0 + \underline{D},$$

where $D_0 = \frac{\partial}{\partial x_0}$ and $\underline{D} = \frac{\partial}{\partial x_1}\mathbf{e}_1 + \cdots + \frac{\partial}{\partial x_n}\mathbf{e}_n$. To be symmetric, we also write $D_0 = \frac{\partial}{\partial x_0} = \frac{\partial}{\partial x_0}\mathbf{e}_0$, with $\mathbf{e}_0 = 1$. We define the "left" and "right" roles of the operators D by

$$Df = \sum_{i=0}^{n} \sum_S \frac{\partial f_S}{\partial x_i} \mathbf{e}_i \mathbf{e}_S$$

and

$$fD = \sum_{i=0}^{n} \sum_S \frac{\partial f_S}{\partial x_i} \mathbf{e}_S \mathbf{e}_i.$$

If $Df = 0$ in a domain (open and connected) Ω, then we say that f is *left-monogenic* in Ω; and, if $fD = 0$ in Ω, we say that f in *right-monogenic* in Ω. If f is both left- and right-monogenic, then we say that f is *monogenic*.

The Cauchy's Theorem holds in the present case: Let Ω be a domain of Lipschitz boundary $\partial\Omega$ and g be right- and f be left-monogenic in a neighborhood of $\Omega \cup \partial\Omega$, then

$$\int_{\partial\Omega} g(y)n(y)f(y)d\sigma(y) = 0,$$

where $n(y)$ is the outward unit normal to the surface $\partial\Omega$ at y and $d\sigma(y)$ is the area measure. We also have Cauchy's Formulas. Under the above assumptions,

$$g(x) = \frac{1}{\omega_n}\int_{\partial\Omega} g(y)n(y)E(y-x)d\sigma(y), \quad x \in \Omega$$

and

$$f(x) = \frac{1}{\omega_n}\int_{\partial\Omega} E(y-x)n(y)f(y)d\sigma(y), \quad x \in \Omega,$$

where

$$E(x) = \frac{\overline{x}}{|x|^{n+1}},$$

is the *Cauchy kernel*, and $\omega_n = 2\pi^{\frac{n+1}{2}}/\Gamma(\frac{n+1}{2})$ is the area of the n-dimensional unit sphere S^n in $\mathbf{R}_1^n$.

We will use the Taylor expansion of left-monogenic functions: If f is left-monogenic in $B(0,r)$, then

$$f(x) = \sum_{k=0}^{\infty}\frac{1}{\omega_n}\int_{\partial B(0,r)} P^{(k)}(y^{-1}x)E(y)n(y)f(y)d\sigma(y), \tag{1}$$

where

$$P^{(k)}(y^{-1}x) = |y^{-1}x|^k C^+_{n+1,k}(\xi,\eta), \tag{2}$$

and

$$\begin{aligned} C^+_{n+1,k}(\xi,\eta) = \tfrac{1}{1-n}\Big[&-(n+k-1)G_k^{\frac{n-1}{2}}(<\xi,\eta>)+ \\ &(1-n)G_{k-1}^{\frac{n+1}{2}}(<\xi,\eta>)(<\xi,\eta> - \overline{\xi}\eta)\Big], \end{aligned} \tag{3}$$

where $x = |x|\xi$, $y = |y|\eta$ and G_k^ν is the Gegenbauer polynomial of degree k associated with ν (see [DSS]).

The function in (2) being a function of $y^{-1}x$ can be seen from (3) and the relations

$$< \xi, \eta >= \frac{< y^{-1}x, 1 >}{|y^{-1}x|} \quad \text{and} \quad \overline{\xi}\eta = \left(\frac{y^{-1}x}{|y^{-1}x|}\right)^{-1}. \tag{4}$$

We note that in (1) the integral region $\partial B(0, r)$ can be changed to any $\partial B(0, \rho)$ with $0 < \rho < r$. The validity of this change can be proved by using other versions of the Taylor series expansion (for example, use the one in [18], also see [DSS]).

The Taylor expansion (1) is originated by [10] and, independently by [9], that was followed by various versions later on (see [1] and [DSS] for instance). The form (1) is taken from [DSS] combined with a further study on the form in [8] and [7].

We correspondingly have Taylor expansions at points different from the origin, and those for right-monogenic functions. We also have Laurent expansions of one-sided or two-sided monogenic functions on annulus. In the present paper, we only use Taylor expansions at the origin, and we will be based on the following facts:

- $|P^{(k)}(y^{-1}x)| \leq C_n k^n \frac{|x|^k}{|y|^k}$ (established by combining estimates (8) and (9) on page 431 of [10]), where C_n stands for a constant depending on the dimension n.
- $P^{(k)}(y^{-1}x)$ is a polynomial in x of degree k (see [18] or [2]).

Fourier transform in $\mathbf{R}^n$ is defined by

$$\mathcal{F}(f)(\underline{\xi}) = \int_{\mathbf{R}^n} e^{-i<\underline{x},\underline{\xi}>} f(\underline{x}) d\underline{x}$$

and the inverse Fourier transform is defined by

$$\mathcal{F}^{-1}(g)(\underline{x}) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} e^{i<\underline{x},\underline{\xi}>} g(\underline{\xi}) d\underline{\xi}.$$

Here $\underline{\xi} = \xi_1\mathbf{e}_1 + \cdots + \xi_n\mathbf{e}_n$. To extend the Fourier transform to $\mathbf{R}_1^n$, we need first to extend the exponential function $e^{i<\underline{x},\underline{\xi}>}$. Denote, for $x = x_0 + \underline{x}$,

$$e(x, \underline{\xi}) = e^{i<\underline{x},\underline{\xi}>} e^{-x_0|\underline{\xi}|} \chi_+(\underline{\xi}) + e^{i<\underline{x},\underline{\xi}>} e^{x_0|\underline{\xi}|} \chi_-(\underline{\xi}),$$

where

$$\chi_{\pm}(\underline{\xi}) = \frac{1}{2}(1 \pm i\frac{\underline{\xi}}{|\underline{\xi}|}).$$

It is easy to verity that

$$\chi_{-}\chi_{+} = \chi_{+}\chi_{-} = 0, \quad \chi_{\pm}^2 = \chi_{\pm}, \quad \chi_{+} + \chi_{-} = 1.$$

The function $e(x, \underline{\xi})$ is obviously an extension of $e(\underline{x}, \underline{\xi}) = e^{i<\underline{x},\underline{\xi}>}$ onto $\mathbf{R}_1^n \times \mathbf{R}^n$. It is easy to verity that $e(x, \underline{\xi})$ is monogenic in $x \in \mathbf{R}_1^n$ for any fixed $\underline{\xi}$. Generalizations of the exponential function of this kind can be first found in F. Sommen's work ([12] and [13]), and then in [4], where $\underline{\xi}$ is further extended to $\underline{\zeta} = \underline{\xi} + i\underline{\eta} \in \mathbf{C}^n$.

It is well known that, if $f \in L^2(\mathbf{R}^n)$, then $f = f^+ + f^-$, where f^+ is the boundary value of a function in the Hardy space H^2 in the upper-half-space, and f^- is the boundary value of a function in the Hardy space H^2 in the lower-half-space (see [4], also [5]). The Hardy functions, still denoted by f^+ and f^-, in the upper and lower half spaces are, in fact, given by

$$f^{\pm}(x) = \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} e^{i<\underline{x},\underline{\xi}>} e^{\mp x_0|\underline{\xi}|}\chi_{\pm}(\underline{\xi})\mathcal{F}(f|_{\mathbf{R}^n})(\underline{\xi})d\underline{\xi}, \quad \pm x_0 > 0,$$

respectively.

In [1] a Clifford valued generalized function theory is developed. In below we will adopt the definition that T is called a *tempered distribution*, if T is a continuous linear functional from $\mathcal{S}(\mathbf{R}^n)$ to $\mathbf{C}^{(n)}$, where $\mathcal{S}(\mathbf{R}^n)$ is the Schwarz class of rapidly decreasing functions. This is equivalent with the one defined in [1] using modules, and enables us to quickly define Fourier transforms on tempered distributions, by

$$\mathcal{F}(T)(\varphi) = T(\mathcal{F}(\varphi)), \quad \forall \varphi \in \mathcal{S}(\mathbf{R}^n),$$

which is just to perform Fourier transform on each of the components of the distribution. We will use the following results:

$$\mathcal{F}(1) = (2\pi)^n\delta, \quad \mathcal{F}^{-1}(\xi_1^{\alpha_1}\cdots\xi_n^{\alpha_n}) = i^{-|\alpha|}D^{\alpha}\delta,$$

where $\alpha = (\alpha_1, \cdots, \alpha_n)$, $D^{\alpha} = (\frac{\partial}{\partial x_1})^{\alpha_1}\cdots(\frac{\partial}{\partial x_n})^{\alpha_n}$ and δ is the usual Dirac δ function.

2. PALEY-WIENER THEOREM

The Theory is stated as follows

THEOREM 2.1. *Let $f : \mathbf{R}_1^n \to \mathbf{C}^{(n)}$ be left-monogenic in $\mathbf{R}_1^n$, $f|_{\mathbf{R}^n} \in L^2(\mathbf{R}^n)$, and $R > 0$ be a positive number. Then the following two assertions are equivalent:*

(i) There exists a constant C such that

$$|f(x)| \le Ce^{R|x|}, \quad \forall x \in \mathbf{R}_1^n.$$

(ii) supp $\mathcal{F}(f|_{\mathbf{R}^n}) \subset B(0,R)$.

Moreover, if one of the above conditions holds, then we have

$$f(x) = \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} e(x,\underline{\xi})\mathcal{F}(f|_{\mathbf{R}^n})(\underline{\xi})d\underline{\xi}, \quad x \in \mathbf{R}_1^n.$$

Proof. (ii) $\Rightarrow$ (i): Assume that (ii) holds. Let

$$F(x) = \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} e(x,\underline{\xi})\mathcal{F}(f|_{\mathbf{R}^n})(\underline{\xi})d\underline{\xi}.$$

Denote by $\chi_{B(0,R)}$ the characteristic function of $B(0,R)$. Since supp $\mathcal{F}(f|_{\mathbf{R}^n}) \subset B(0,R)$, we have

$$F(x) = \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} e(x,\underline{\xi})\chi_{B(0,R)}(\underline{\xi})\mathcal{F}(f|_{\mathbf{R}^n})(\underline{\xi})d\underline{\xi}.$$

Hölder inequality then implies

$$|F(x)| \le Ce^{R|x_0|}\|\chi_{B(0,R)}\|_2\|\mathcal{F}(f|_{\mathbf{R}^n})\|_2 \le Ce^{R|x|}.$$

Since $f(\underline{x}) = F(\underline{x})$ in $\mathbf{R}^n$ and both functions are left-monogenic in $\mathbf{R}_1^n$, we conclude that $f(x) = F(x)$. Thus $f(x)$ is of the desired estimate.

(i) $\Rightarrow$ (ii): Assume that (i) holds. Consider

$$G^+(x) = \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} e^{i<\underline{x},\underline{\xi}>}e^{-x_0|\underline{\xi}|}\chi_+(\underline{\xi})f(\underline{\xi})d\underline{\xi}, \quad x_0 > 0, \tag{5}$$

which is well defined as $f \in L^2(\mathbf{R}^n)$. It is easy to show that $G^+(x)$ is left-monogenic for $x_0 > 0$. Substituting $f(\underline{\xi})$ by its Taylor series (1), the identity (5) may be rewritten as

$$\begin{aligned} G^+(x) &= \tfrac{1}{(2\pi)^n}\int_{\mathbf{R}^n} e^{i<\underline{x},\underline{\xi}>}e^{-x_0|\underline{\xi}|}\chi_+(\underline{\xi}) \\ &\qquad \left(\sum_{k=0}^\infty \tfrac{1}{\omega_n}\int_{\partial B(0,r)} P^{(k)}(y^{-1}\underline{\xi})E(y)n(y)f(y)d\sigma(y)\right)d\underline{\xi} \\ &= \lim_{N\to\infty} \tfrac{1}{(2\pi)^n}\int_{\mathbf{R}^n} e^{i<\underline{x},\underline{\xi}>}e^{-x_0|\underline{\xi}|}\chi_+(\underline{\xi})\chi_{B(0,N)}(\underline{\xi}) \\ &\qquad \left(\sum_{k=0}^\infty \tfrac{1}{\omega_n}\int_{\partial B(0,r)} P^{(k)}(y^{-1}\underline{\xi})E(y)n(y)f(y)d\sigma(y)\right)d\underline{\xi}, \end{aligned}$$

where r is any positive number. Owing to the uniform convergence property of the series for $|\underline{\xi}| \leq N$, we have

$$\begin{aligned} G^+(x) = \lim_{N\to\infty} \tfrac{1}{(2\pi)^n} \textstyle\sum_{k=0}^{\infty} \tfrac{1}{\omega_n} \int_{\partial B(0,r)} \Big(\int_{\mathbf{R}^n} e^{i<\underline{x},\underline{\xi}>} e^{-x_0|\underline{\xi}|} \chi_+(\underline{\xi}) \\ \chi_{B(0,N)}(\underline{\xi}) P^{(k)}(y^{-1}\underline{\xi}) d\underline{\xi} \Big) E(y)n(y)f(y)d\sigma(y). \end{aligned} \tag{6}$$

We now prove that for $x_0 > 0$, we can exchange the order of taking limit $N \to \infty$ and taking summation $\sum_{k=0}^{\infty}$, by showing that the series is dominated by an absolutely convergent one independent of N for $x_0 > R$. We consequently have

$$\begin{aligned} G^+(x) = \textstyle\sum_{k=0}^{\infty} \tfrac{1}{(2\pi)^n \omega_n} \int_{\partial B(0,r)} \Big(\int_{\mathbf{R}^n} e^{i<\underline{x},\underline{\xi}>} e^{-x_0|\underline{\xi}|} \chi_+(\underline{\xi}) \\ P^{(k)}(y^{-1}\underline{\xi}) d\underline{\xi} \Big) E(y)n(y)f(y)d\sigma(y), \quad x_0 > R. \end{aligned} \tag{7}$$

In fact, using the bounds of $P^{(k)}(y^{-1}\underline{\xi})$, and that of $f(y)$, and the spherical coordinates, we have

$$\begin{aligned} &\tfrac{1}{(2\pi)^n \omega_n} \Big| \int_{\partial B(0,r)} \Big(\int_{\mathbf{R}^n} e^{i<\underline{x},\underline{\xi}>} e^{-x_0|\underline{\xi}|} \chi_+(\underline{\xi}) P^{(k)}(y^{-1}\underline{\xi}) d\underline{\xi} \Big) \\ &\qquad\qquad\qquad\qquad E(y)n(y)f(y)d\sigma(y) \Big| \\ \leq\ & C_n k^n \textstyle\int_{\mathbf{R}^n} e^{-x_0|\underline{\xi}|} |\underline{\xi}|^k r^{-k} r^{-n} r^n e^{Rr} d\underline{\xi} \\ =\ & C_n k^n \tfrac{e^{Rr}}{r^k} \textstyle\int_0^{\infty} e^{-x_0 s} s^{k+n-1} ds \\ =\ & C_n k^n \tfrac{e^{Rr}}{r^k} \tfrac{(k+n-1)!}{x_0^{k+n}}. \end{aligned}$$

The last inequality holds for any $r > 0$. Taking the minimum value of the last expression with respect to r, we have that, the series in $G^+(x)$ is dominated by

$$\begin{aligned} & C_n \textstyle\sum_{k=0}^{\infty} k^n (k+n-1)! (\tfrac{e}{k})^k R^k \tfrac{1}{x_0^{n+k}} \\ =\ & \tfrac{C_n}{x_0^n} \textstyle\sum_{k=0}^{\infty} d_k \tfrac{1}{x_0^k}, \end{aligned} \tag{8}$$

where

$$d_k = k^n (k+n-1)! \left(\frac{e}{k}\right)^k R^k.$$

Using Stirling's formula, we conclude that

$$\overline{\lim_{k\to\infty}} (d_k)^{\frac{1}{k}} = R.$$

Using Hadamard's criterion, the convergence radius of the associated power series is R^{-1}. Correspondingly, the series (8) converges for $x_0 > R$. Now we have justified that we can exchange the limit procedure $N \to \infty$ and the summation $\sum_{k=0}^{\infty}$ in (6) if $x_0 > R$, and thus (7) holds for $x_0 > R$.

Let $\varphi_m(\underline{\xi})$ be a sequence of functions in $C_0^\infty(\mathbf{R}^n)$ such that $\varphi_m(\underline{\xi}) = 0$ if $|\underline{\xi}| \le \frac{1}{m}$ and $\varphi_m(\underline{\xi}) = 1$ if $|\underline{\xi}| \ge \frac{2}{m}$ and $0 \le \varphi_m(\underline{\xi}) \le 1$ otherwise. Obviously, $\varphi_m \to 1$ distributionally. We rewrite $G^+(x)$ as

$$\begin{aligned} G^+(x) = \tfrac{1}{(2\pi)^n} \textstyle\sum_{k=0}^{\infty} \tfrac{1}{\omega_n} \int_{\partial B(0,r)} \Big(&\lim_{m\to\infty} \textstyle\int_{\mathbf{R}^n} e^{i<\underline{x},\underline{\xi}>} e^{-x_0|\underline{\xi}|} \chi_+(\underline{\xi}) \\ &\varphi_m(\underline{\xi}) P^{(k)}(y^{-1}\underline{\xi}) d\underline{\xi} \Big) E(y) n(y) f(y) d\sigma(y), \quad x_0 > R. \end{aligned}$$

Since $e^{i<\underline{x},\cdot>} e^{-x_0|\cdot|} \chi_+(\cdot)\varphi_m(\cdot) \in \mathcal{S}(\mathbf{R}^n)$, the inside integral can be rewritten using the notation of distribution:

$$\begin{aligned} & P^{(k)}(y^{-1}(\cdot)) \left(e^{i<\underline{x},\cdot>} e^{-x_0|\cdot|} \chi_+(\cdot)\varphi_m(\cdot) \right) \\ = \ & \mathcal{F}^{-1} \left(P^{(k)}(y^{-1}(\cdot)) \left(\mathcal{F}(e^{i<\underline{x},\cdot>} e^{-x_0|\cdot|} \chi_+(\cdot)\varphi_m(\cdot)) \right) \right) \\ = \ & i^{-k} \left(P^{(k)}(y^{-1}\underline{D})\delta \right) \left(\mathcal{F}(e^{i<\underline{x},\cdot>} e^{-x_0|\cdot|} \chi_+(\cdot)) * \mathcal{F}(\varphi_m) \right). \end{aligned} \tag{9}$$

Now

$$\mathcal{F}(e^{i<\underline{x},\cdot>} e^{-x_0|\cdot|} \chi_+(\cdot)) = \frac{1}{2} \mathcal{F}(e^{i<\underline{x},\cdot>} e^{-x_0|\cdot|}) + \frac{1}{2} \mathcal{F} \left(e^{i<\underline{x},\cdot>} e^{-x_0|\cdot|} \frac{i(\cdot)}{|\cdot|} \right),$$

where

$$\begin{aligned} \tfrac{1}{2} \mathcal{F}(e^{i<\underline{x},\cdot>} e^{-x_0|\cdot|})(\underline{\zeta}) &= \tfrac{1}{2} \textstyle\int_{\mathbf{R}^n} e^{-i<\underline{\zeta},\underline{\xi}>} e^{i<\underline{x},\underline{\xi}>} e^{-x_0|\underline{\xi}|} d\underline{\xi} \\ &= \tfrac{1}{2} \textstyle\int_{\mathbf{R}^n} e^{-i<\underline{\zeta}-\underline{x},\underline{\xi}>} e^{-x_0|\underline{\xi}|} d\underline{\xi} \\ &= \tilde{c} \tfrac{x_0}{(x_0^2 + |\underline{\zeta}-\underline{x}|^2)^{\frac{n+1}{2}}}, \end{aligned}$$

where $\tilde{c} = 2^{n-1}\pi^{\frac{n-1}{2}} \Gamma(\frac{n+1}{2})$.

It is observed that

$$\begin{aligned} \tfrac{1}{2} \mathcal{F} \left(e^{i<\underline{x},\cdot>} e^{-x_0|\cdot|} \tfrac{i(\cdot)}{|\cdot|} \right) (\underline{\zeta}) &= \tfrac{1}{2} \textstyle\int_{x_0}^{\infty} \underline{D}_{\underline{x}} \mathcal{F}(e^{i<\underline{x},\cdot>} e^{-t|\cdot|})(\underline{\zeta}) dt \\ &= \tilde{c} \textstyle\int_{x_0}^{\infty} \underline{D}_{\underline{x}} \left(\tfrac{t}{(t^2 + |\underline{\zeta}-\underline{x}|^2)^{\frac{n+1}{2}}} \right) dt \\ &= \tilde{c} \tfrac{\underline{\zeta}-\underline{x}}{(x_0^2 + |\underline{\zeta}-\underline{x}|^2)^{\frac{n+1}{2}}}. \end{aligned}$$

Hence

$$\mathcal{F}(e^{i<\underline{x},\cdot>}e^{-x_0|\cdot|}\chi_+(\cdot))(\underline{\zeta}) = \tilde{c}\frac{\overline{x-\underline{\zeta}}}{|x-\underline{\zeta}|^{n+1}} = -\tilde{c}E(\underline{\zeta}-x). \tag{10}$$

(Note that this computation may be omitted if one directly uses the corresponding result in [4].) Therefore, (9) becomes

$$\begin{aligned} & -\tilde{c}i^{-k}\left(P^{(k)}(y^{-1}\underline{D})\delta\right)(E(\cdot - x) * \mathcal{F}(\varphi_m)) \\ = & -\tilde{c}i^{-k}(-1)^k\delta\left((P^{(k)}(y^{-1}\underline{D})E)(\cdot - x) * \mathcal{F}(\varphi_m)\right) \\ = & -\tilde{c}i^k\left(\left(P^{(k)}(y^{-1}\underline{D})E\right)(\cdot - x) * \mathcal{F}(\varphi_m)\right)(0). \end{aligned}$$

As $\mathcal{F}(\varphi_m) \to (2\pi)^n\delta$, and we finally conclude that

$$\int_{\mathbf{R}^n} e^{i<\underline{x},\underline{\xi}>}e^{-x_0|\underline{\xi}|}\chi_+(\underline{\xi})P^{(k)}(y^{-1}\underline{\xi})d\underline{\xi} = -(2\pi)^n\tilde{c}i^k(P^{(k)}(y^{-1}\underline{D})E)(-x),$$

and thus for $x_0 > R$, we have

$$G^+(x) = -\tilde{c}\sum_{k=0}^{\infty}\frac{i^k}{\omega_n}\int_{\partial B(0,r)}\left(P^{(k)}(y^{-1}\underline{D})E\right)(-x)E(y)n(y)f(y)d\sigma(y). \tag{11}$$

We next point out that the series expression of $G^+(x)$ in (11) for $x_0 > R$ can be monogenically extended to all $x \in \mathbf{R}_1^n$ such that $|x| > R$.

In fact, owing to the estimate

$$\left|\left(P^k(y^{-1}\underline{D})E\right)(-x)\right| \le C_n k^n \frac{1}{|x|^{n+k}}\frac{1}{|y|^k},$$

if we proceed as before to estimate a general entry of the series (11), we obtain that the series (11) is dominated by

$$\tilde{c}\sum_{k=0}^{\infty}k^n(k+n-1)!\left(\frac{e}{k}\right)^k R^k\frac{1}{|x|^{n+k}}.$$

The same argument then implies that the series (11) converges uniformly in any compact set in the region $|x| > R$ and thus the sum function is left-monogenic for $|x| > R$.

Now we define

$$G^-(x) = \frac{1}{(2\pi)^n}\int_{-\infty}^{\infty}e^{i<\underline{x},\underline{\xi}>}e^{x_0|\underline{\xi}|}\chi_-(\underline{\xi})f(\underline{\xi})d\underline{\xi}, \quad x_0 < 0, \tag{12}$$

that is left-monogenic for $x_0 < 0$. Using the same procedure we can first show that for $-x_0 > R$,

$$G^-(x) = \tilde{c} \sum_{k=0}^{\infty} \frac{i^k}{\omega_n} \int_{\partial B(0,r)} \left(P^{(k)}(y^{-1}\underline{D})E\right)(-x)E(y)n(y)f(y)d\sigma(y), \quad (13)$$

and then $G^-(x)$ can be monogenically extended using the series expansion (13) to $|x| > R$. We will be content with only pointing out how the negative sign in the beginning of formula (11) drops off from the calculation.

When we compute $\mathcal{F}(e^{i<\underline{x},\cdot>}e^{x_0|\cdot|}\chi_-(\cdot))$, with $x_0' = -x_0 > 0$, we first write it as $\mathcal{F}(e^{i<\underline{x},\cdot>}e^{-x_0'|\cdot|}\chi_-(\cdot))$. Then, as before, we have

$$\frac{1}{2}\mathcal{F}(e^{i<\underline{x},\cdot>}e^{-x_0'|\cdot|})(\underline{\zeta}) = \tilde{c}\frac{x_0'}{(x_0'^2 + |\underline{\zeta} - \underline{x}|^2)^{\frac{n+1}{2}}}.$$

We accordingly have

$$\frac{1}{2}\mathcal{F}\left(e^{i<\underline{x},\cdot>}e^{-x_0'|\cdot|}\left(-i\frac{(\cdot)}{|\cdot|}\right)\right)(\underline{\zeta}) = -\tilde{c}\frac{\underline{\zeta} - \underline{x}}{(x_0'^2 + |\underline{\zeta} - \underline{x}|^2)^{\frac{n+1}{2}}}.$$

Putting together, we have

$$\mathcal{F}(e^{i<\underline{x},\cdot>}e^{x_0|\cdot|}\chi_-(\cdot))(\underline{\zeta}) = -\tilde{c}\frac{\overline{x - \underline{\zeta}}}{|x - \underline{\zeta}|^{n+1}} = \tilde{c}E(\underline{\zeta} - x). \quad (14)$$

Now we show that G^+ and G^- have the alternative forms

$$G^+(x_0 + \underline{x}) = \frac{1}{\omega_n}\int_{\mathbf{R}^n} E((x_0 + \underline{x}) - \underline{\zeta})\mathcal{F}(f|_{\mathbf{R}^n})(-\underline{\zeta})d\underline{\zeta}$$

and

$$G^-(-x_0 + \underline{x}) = -\frac{1}{\omega_n}\int_{\mathbf{R}^n} E((-x_0 + \underline{x}) - \underline{\zeta})\mathcal{F}(f|_{\mathbf{R}^n})(-\underline{\zeta})d\underline{\zeta},$$

respectively. In fact, owing to Parseval's identity

$$\int_{\mathbf{R}^n} h(\underline{\xi})g(\underline{\xi})d\underline{\xi} = \int_{\mathbf{R}^n} \mathcal{F}(h)(\underline{\zeta})\mathcal{F}(g)(-\underline{\zeta})d\underline{\zeta},$$

and the identity (10), we have

$$\begin{aligned} G^+(x_0+\underline{x}) &= \tfrac{1}{(2\pi)^n}\int_{\mathbf{R}^n} e^{i<\underline{x},\underline{\xi}>}e^{-x_o|\underline{\xi}|}\chi_+(\underline{\xi})f(\underline{\xi})d\underline{\xi} \\ &= \tfrac{1}{(2\pi)^n}\int_{\mathbf{R}^n} \mathcal{F}\left(e^{i<\underline{x},\cdot>}e^{-x_o|\cdot|}\chi_+(\cdot)\right)(\underline{\zeta})\mathcal{F}(f|_{\mathbf{R}^n})(-\underline{\zeta})d\underline{\zeta} \\ &= \tfrac{1}{\omega_n}\int_{\mathbf{R}^n} E((x_0+\underline{x})-\underline{\zeta})\mathcal{F}(f|_{\mathbf{R}^n})(-\underline{\zeta})d\underline{\zeta}. \end{aligned}$$

The last step uses the relation $\frac{1}{\omega_n}=\frac{\tilde{c}}{(2\pi)^n}$. The expression for G^- can be proved similarly using (14). The Plemelj formula (see [4]) then gives

$$\lim_{x_0\to 0+}\left(G^+(x_0+\underline{x})+G^-(-x_0+\underline{x})\right)=\mathcal{F}(f|_{\mathbf{R}^n})(-\underline{x}).$$

This, together with the series expressions (11) and (13) for $|x|>R$, gives $\mathcal{F}(f|_{\mathbf{R}^n})(\underline{x})=0$ for $|\underline{x}|>R$. Therefore supp $\mathcal{F}(f|_{\mathbf{R}^n})\subset B(0,R)$.

To show

$$f(x)=\frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} e(x,\underline{\xi})\mathcal{F}(f|_{\mathbf{R}^n})(\underline{\xi})d\underline{\xi},\quad x\in\mathbf{R}^n_1.$$

We notice that the left-hand-side is equal to the right-hand-side if $x_0=0$. Since both sides are left-monogenic in $\mathbf{R}^n_1$ and coincident in $\mathbf{R}^n$, they have to be equal. ▮

3. AN APPLICATION TO CONJUGATE HARMONIC SYSTEM IN $\mathbf{R}^n_1$

If an ordered set of $n+1$ functions $u_0(x_0,x_1,\cdots,x_n)$, $u_1(x_0,x_1,\cdots,x_n)$, $\cdots$, $u_n(x_0,x_1,\cdots,x_n)$, satisfies the relations

$$\begin{cases} \sum_{j=0}^n \frac{\partial u_j}{\partial x_j}=0 \\ \frac{\partial u_k}{\partial x_j}=\frac{\partial u_j}{\partial x_k},\quad 0\le k<j\le n, \end{cases}$$

then it is called a *conjugate harmonic system*. For the relevant literature we refer to [15] and [16]. In below we denote by U the following vector-valued function:

$$U=-u_0+u_1\mathbf{e}_1+\cdots+u_n\mathbf{e}_n.$$

PROPOSITION 3.1. *An ordered set of functions $u_0, u_1, \cdots, u_n$ is a conjugate harmonic system if and only if the corresponding vector-valued function U is monogenic.*

Proof. Denote $\underline{u} = u_1\mathbf{e}_1 + \cdots + u_n\mathbf{e}_n$. Then

$$\begin{aligned} DU &= (D_0 + \underline{D})(-u_0 + \underline{u}) \\ &= (-D_0u_0 - \textstyle\sum_{j=1}^n \frac{\partial u_j}{\partial x_j}) + (D_0\underline{u} - \underline{D}u_0) + \sum_{k<j}\left(\frac{\partial u_k}{\partial x_j} - \frac{\partial u_j}{\partial x_k}\right)\mathbf{e}_j\mathbf{e}_k. \end{aligned}$$

So, $DU = 0$ if and only if

$$\begin{cases} \sum_{j=0}^n \frac{\partial u_j}{\partial x_j} = 0 \\ \frac{\partial u_k}{\partial x_j} = \frac{\partial u_j}{\partial x_k}, \quad 0 \le k < j \le n. \end{cases}$$

The right-monogenity is proved similarly. ∎

The proposition indicates that the Clifford algebra frame of $\mathbf{R}^n$ is a natural one to study Hardy spaces in relation to the space (see [5]). The following is an immediate consequence of Theorem 2.1.

THEOREM 3.1. *Let $u_0, u_1, \cdots, u_n$ be a conjugate harmonic system in $\mathbf{R}^n_1$. Let $U|_{\mathbf{R}^n} \in L^2(\mathbf{R}^n)$. Then*

$$|U(x)| \le Ce^{R|x|}$$

if and only if

$$\mathit{supp}\ \mathcal{F}(U)(0,\cdot) \subset B(0,R),$$

where $\mathcal{F}(U)(0,\underline{\xi}) = \mathcal{F}(U|_{\mathbf{R}^n})(\underline{\xi})$. Moreover, if one of the above conditions holds, then

$$U(x) = \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} e(x,\underline{\xi})\mathcal{F}(U)(0,\underline{\xi})d\underline{\xi}.$$

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