Shannon Sampling and Estimation of Band-Limited Functions in the Several Complex Variables Setting *

Kit-Ian Kou and Tao Qian

Faculty of Science and Technology, University of Macau, Macau

kikou@umac.mo and fsttq@mac.mo

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Abstract. In this work we develop the n-dimensional sinc function theory in the several complex variables setting. In terms of the corresponding Paley-Wiener theorem the exact sinc interpolation and quadrature are proved. Exponential convergence rate of the error estimates for band-limited functions in n-dimensional strips are obtained.

Key words. Shannon sampling, sinc function, harmonic analysis

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1 Introduction

Systematic studies on sinc function of one complex variable may be traced to the work of Stenger ([10]). Numerical methods based on the sinc function were subsequently studied in [12]. The results arising from the numerical methods were further applied to solutions of boundary value problems ([11], [4]), and solutions of integral equations ([2], [9]). Sinc methods play an important role in numerical computation due to their high accuracy and stability.

The purpose of this paper is to establish a higher dimensional analogue of the one-dimensional theory ([5], [13]). For functions in the Paley-Wiener classes we will prove exact interpolation and quadrature, and for those not in those classes but in n-dimensional strips we obtain error estimates of approximation by cardinal functions. As in the one-dimensional case, we prove that the higher dimensional approximation has exponential convergence rate.

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Sinc methods are particularly powerful when the problem has singularities, in which case the error of an n-point approximation converges at an incredible $\mathcal{O}(e^{-c\sqrt{n}})$ rate, whereas in such circumstances polynomial methods, at best, converge at the $\mathcal{O}(n^{-c})$ rate, where c is some positive number. Thus sinc methods are particularly suitable for solving partial differential and integral equations; for such problems the resulting system of algebraic equations usually can be written out explicitly, and is relatively small in size, the determination of the close to optimal mesh spacing is automatic and the use of the fast Fourier transform based on one of the sinc approximations, can be conveniently applied to yield a highly efficient code. The numerical solution of partial differential equations with the Sinc-Galerkin method has previously been developed for the model elliptic equation in two dimensions ([1, 6]). The paper ([7]) extends the method to the steady state problem in three dimensions and the model time dependent problems in at least two dimensions. Numerical results for the test model problems in those papers illustrate the exponential convergence rate of the method even in the presence of singularities.

While most of the above mentioned applications have counterparts in the several variables setting, the present paper is only devoted to the theoretical development. We provide all necessary details of the proofs of which some are missing in the one variable case.

2 Exact Interpolation and Quadrature on $\mathbb{R}^n$

Set

$$\mathbb{R}^n = \{\underline{x} = (x_1, \dots, x_n) : x_j \in \mathbb{R}, j = 1, 2, \cdots, n\}$$

and

$$\mathbb{C}^n = \{\underline{z} = (z_1, \dots, z_n) : z_j = x_j + \mathbf{i}y_j, x_j, y_j \in \mathbb{R}, j = 1, 2, \dots, n\}.$$

The norms in $\mathbb{R}^n$ and $\mathbb{C}^n$ are defined by $|\underline{x}| = (\sum_{j=1}^n x_j^2)^{1/2}$ and $|\underline{z}| = (\sum_{j=1}^n |z_j|^2)^{1/2}$, respectively. Let $B(\underline{a}, R)$ be the open ball in $\mathbb{R}^n$ centered at $\underline{a}$ with radius R, that is

$$B(\underline{a}, R) = \{\underline{x} \in \mathbb{R}^n : |\underline{x} - \underline{a}| < R\}.$$

The Fourier transform of an integrable function $f \in L^1(\mathbb{R}^n)$ is defined by

$$\mathcal{F}(f)(\underline{\xi}) = \int_{\mathbb{R}^n} e^{-\mathbf{i}\underline{\xi}\cdot\underline{x}} f(\underline{x}) d\underline{x}$$

and the inverse Fourier transform of g, if applicable, is

$$\mathcal{F}^{-1}(g)(\underline{x}) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{\mathbf{i}\underline{x}\cdot\underline{\xi}} g(\underline{\xi}) d\underline{\xi}.$$

The sinc function in $\mathbb{C}^n$ is defined to be

$$\underline{\text{sinc}}\ \underline{z} \equiv \prod_{j=1}^{n} \text{sinc}\ z_j,$$

where $\underline{z} = (z_1, \cdots, z_n)$, and

$$\text{sinc}\ z_j = \begin{cases} \frac{\sin(\pi z_j)}{\pi z_j}, & z_j \neq 0; \\ 1 \quad, & z_j = 0. \end{cases}$$

Thus $\underline{\text{sinc}}\ \underline{z}$ is an entire function in $\mathbb{C}^n$.

For a set A, the notation $\mathcal{X}_A$ denotes the characteristic function of A. A direct computation shows that

$$\text{sinc}\ x_j = \mathcal{F}_1^{-1}(\mathcal{X}_{[-\pi,\pi]})(x_j), \qquad [-\pi,\pi] \subset \mathbb{R},$$

where $\mathcal{F}_1$ and $\mathcal{F}_1^{-1}$ denote respectively the Fourier transform and its inverse on $\mathbb{R}$.

Hence,

$$\begin{aligned} \underline{\text{sinc}}\ \underline{x} &= \prod_{j=1}^{n} \text{sinc}\ x_j = \prod_{j=1}^{n} \left[\mathcal{F}_1^{-1}(\mathcal{X}_{[-\pi,\pi]})(x_j)\right] = \mathcal{F}^{-1}\left(\prod_{j=1}^{n} \mathcal{X}_{[-\pi,\pi]}((\cdot)_j)\right)(\underline{x}) \\ &= \mathcal{F}^{-1}(\mathcal{X}_{[-\pi,\pi]^n})(\underline{x}). \end{aligned} \tag{1}$$

Definition 2.1 For any given $R > 0$ the *Paley-Wiener class* $PW(R)$ is defined to be the class of entire functions f on $\mathbb{C}^n$ that $f|_{\mathbb{R}^n} \in L^2(\mathbb{R}^n)$ and satisfies

$$|f(\underline{z})| \leq K e^{R|\underline{z}|}, \quad \forall \underline{z} \in \mathbb{C}^n.$$

An entire functions satisfying the last inequality is said to be of *the exponential type R*.

The one dimensional Paley-Wiener theorem asserts that the Fourier transforms of the restrictions in $\mathbb{R}$ of a function in $PW(R)$ is supported in $[-R, R]$, and, conversely, if the Fourier transform is supported in $[-R, R]$, then the original function may be extended to become a function in $PW(R)$. There are a number of generalizations of the Paley-Wiener theorem in the higher dimensional spaces (see [14] and [8]). The study in this paper is based on the following theorem that is a special case of the general result in [14]. Note that a recent generalization in the Clifford analysis setting is given in [3].

For a function f defined in $\mathbb{C}^n$ we will denote $\mathcal{F}(f) = \mathcal{F}(f|_{\mathbb{R}^n})$.

Theorem 2.1 *(Paley-Wiener Theorem) Let $f \in L^2(\mathbb{R}^n)$. Then f may be holomorphically extended to an entire function of exponential type R if and only if supp $\mathcal{F}(f) \subset B(\underline{0}, R)$. Moreover, in the case we have*

$$f(\underline{z}) = \frac{1}{(2\pi)^n} \int_{B(\underline{0},R)} e^{\mathbf{i}\underline{z}\cdot\underline{x}} \mathcal{F}(f)(\underline{x}) d\underline{x}. \tag{2}$$

The following theorems characterize the Paley-Wiener class $PW(R)$.

Theorem 2.2 *If $f \in PW(\pi/h)$, then for all $\underline{z} \in \mathbb{C}^n$,*

$$f(\underline{z}) = \frac{1}{h^n} \int_{\mathbb{R}^n} f(\underline{t}) \ \underline{sinc} \left(\frac{\underline{z} - \underline{t}}{h} \right) d\underline{t}. \tag{3}$$

Proof Since $f \in PW(\pi/h)$, the Paley-Wiener Theorem 2.1 shows that

$$\begin{aligned}
f(\underline{z}) &= \frac{1}{(2\pi)^n} \int_{B(\underline{0},\pi/h)} e^{\mathbf{i}\underline{z}\cdot\underline{x}} \mathcal{F}(f)(\underline{x}) d\underline{x} \\
&= \frac{1}{(2\pi)^n} \int_{[-\frac{\pi}{h},\frac{\pi}{h}]^n} e^{\mathbf{i}\underline{z}\cdot\underline{x}} \left[\int_{\mathbb{R}^n} e^{-\mathbf{i}\underline{x}\cdot\underline{t}} f(\underline{t}) d\underline{t} \right] d\underline{x} \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(\underline{t}) \left[\int_{[-\frac{\pi}{h},\frac{\pi}{h}]^n} e^{\mathbf{i}\underline{x}\cdot(\underline{z}-\underline{t})} d\underline{x} \right] d\underline{t} \\
&= \frac{1}{h^n} \int_{\mathbb{R}^n} f(\underline{t}) \left[\frac{1}{(2\pi)^n} \int_{[-\pi,\pi]^n} e^{\mathbf{i}\underline{x}\cdot(\underline{z}-\underline{t})/h} d\underline{x} \right] d\underline{t} \\
&= \frac{1}{h^n} \int_{\mathbb{R}^n} f(\underline{t}) \ \underline{\mathrm{sinc}} \left(\frac{\underline{z} - \underline{t}}{h} \right) d\underline{t},
\end{aligned}$$

where change of order of integration is justified by the Fubini Theorem, based on the assumption that the function f is square integrable, as well as the fact that sinc function is also square integrable. □

It is a consequence of (1) that $\underline{\mathrm{sinc}}\ (\underline{z}/h)$ belongs to $PW(\sqrt{n}\pi/h)$. Further examples of functions in $PW(\sqrt{n}\pi/h)$ may be constructed using the following theorem.

Theorem 2.3 *If $g \in L^2(\mathbb{R}^n)$, then the function*

$$p(\underline{z}) = h^n \int_{\mathbb{R}^n} g(\underline{x}) \ \underline{sinc}\left(\frac{\underline{z}-\underline{x}}{h}\right) d\underline{x}$$

belongs to $PW(\sqrt{n}\pi/h)$.

Proof The proof is just the reverse of that of Theorem 2.2. Applying Parseval's Theorem, we have

$$\begin{aligned} p(\underline{z}) &= \frac{h^n}{(2\pi)^n} \int_{\mathbb{R}^n} \mathcal{F}(g)(\underline{t}) \mathcal{F}\left(\ \underline{\mathrm{sinc}}\left(\frac{\underline{z}-\underline{\cdot}}{h}\right)\right)(-\underline{t}) d\underline{t} \\ &= \frac{h^n}{(2\pi)^n} \int_{\mathbb{R}^n} \mathcal{F}(g)(\underline{t}) \left[h^{-n} \mathcal{X}_{[-\frac{\pi}{h},\frac{\pi}{h}]^n}(\underline{t}) e^{\mathbf{i}\underline{t}\cdot\underline{z}}\right] d\underline{t} \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{\mathbf{i}\underline{t}\cdot\underline{z}} \left[\mathcal{F}(g)(\underline{t}) \mathcal{X}_{[-\frac{\pi}{h},\frac{\pi}{h}]^n}(\underline{t})\right] d\underline{t}. \end{aligned}$$

This shows that

$$\mathrm{supp}\ \mathcal{F}(p) = \mathrm{supp}\ \left[\mathcal{F}(g)\mathcal{X}_{[-\frac{\pi}{h},\frac{\pi}{h}]^n}\right] \subset \left[-\frac{\pi}{h}, \frac{\pi}{h}\right]^n \subset B(\underline{0}, \sqrt{n}\pi/h).$$

From the Paley-Wiener Theorem, we obtain that $p(\underline{z}) \in PW(\sqrt{n}\pi/h)$. □

Definition 2.2 Let $h > 0$ be given and f be a function defined on $\mathbb{R}^n$. The *cardinal function of* f is defined to be

$$C(f;\underline{x}) \equiv \sum_{\underline{k}\in\mathbb{Z}^n} f(h\underline{k}) \ \underline{\mathrm{sinc}}\left(\frac{\underline{x}-h\underline{k}}{h}\right). \tag{4}$$

When the series is convergent and the sum is independent of the order of taking summation, we may write

$$C(f;\underline{x}) = \lim_{M\to\infty} \sum_{|\underline{k}|\le M} f(h\underline{k}) \ \underline{\mathrm{sinc}}\left(\frac{\underline{x}-h\underline{k}}{h}\right),$$

where $|\underline{k}| = (\sum_{j=1}^n k_j^2)^{1/2}$.

The *truncated cardinal series* is denoted by

$$C_{N_1,\cdots,N_n}^{M_1,\cdots,M_n}(f;\underline{x}) \equiv \sum_{k_1=-N_1}^{M_1} \cdots \sum_{k_n=-N_n}^{M_n} f(h\underline{k}) \ \underline{\mathrm{sinc}}\left(\frac{\underline{x}-h\underline{k}}{h}\right).$$

If $N_j = M_j$, $j = 1, \cdots, n$, then write $C_{N_1,\cdots,N_n} = C_{N_1,\cdots,N_n}^{M_1,\cdots,M_n}$. The cardinal function in the one-dimensional case was first discussed in [15]. Note that the above defined cardinal function and cut-off cardinal functions all depend on a parameter $h > 0$.

The following theorem gives the exact sinc interpolation and quadrature formula for functions in $PW(\pi/h)$.

Theorem 2.4 *Let $f \in PW(\pi/h)$. We have*

(i)

$$\frac{1}{(2\pi)^n} \int_{[-\frac{\pi}{h},\frac{\pi}{h}]^n} |\mathcal{F}(f)(\underline{x})|^2 \, d\underline{x} = \int_{\mathbb{R}^n} |f(\underline{t})|^2 d\underline{t} = \sum_{\underline{k}\in\mathbb{Z}^n} |f(h\underline{k})|^2.$$

(ii) In the uniform convergence sense, for all $\underline{z} \in \mathbb{C}^n$,

$$f(\underline{z}) = C(f;\underline{z}) = \sum_{\underline{k}\in\mathbb{Z}^n} f(h\underline{k}) \ \underline{sinc}\left(\frac{\underline{z}-h\underline{k}}{h}\right) \tag{5}$$

where

$$f(h\underline{k}) = \frac{1}{h^n} \int_{\mathbb{R}^n} f(\underline{t}) \ \underline{sinc}\left(\frac{\underline{t}-h\underline{k}}{h}\right) d\underline{t}.$$

(iii) If additionally $f \in L^1(\mathbb{R}^n)$, and

$$\lim_{N\to\infty} \sum_{\substack{|k_j|\le N \\ j=1,\cdots,n}} f(h\underline{k})$$

exists, then

$$\int_{\mathbb{R}^n} f(\underline{t}) d\underline{t} = h^n \sum_{\underline{k}\in\mathbb{Z}^n} f(h\underline{k}).$$

Proof (i) From the Paley-Wiener Theorem 2.1,

$$f(\underline{t}) = \frac{1}{(2\pi)^n} \int_{B(0,\pi/h)} e^{\mathbf{i}\underline{x}\cdot\underline{t}} \mathcal{F}(f)(\underline{x}) d\underline{x} = \frac{1}{(2\pi)^n} \int_{[-\frac{\pi}{h},\frac{\pi}{h}]^n} e^{\mathbf{i}\underline{x}\cdot\underline{t}} \mathcal{F}(f)(\underline{x}) d\underline{x},$$

consider the Fourier expansion of $\mathcal{F}(f)$ in the cube $[-\pi/h, \pi/h]^n$. We have

$$h^n f(h\underline{k}) = \frac{1}{(2R)^n} \int_{[-R,R]^n} e^{\mathbf{i}\pi(\underline{x}\cdot\underline{k})/R} \mathcal{F}(f)(\underline{x}) d\underline{x} = c_{\underline{k}},$$

where $R = \pi/h$, and the coefficients $c_{\underline{k}}$ are the Fourier coefficients of $\mathcal{F}(f)$. The Plancherel Theorem of Fourier series is

$$\int_{[-R,R]^n} |\mathcal{F}(f)(\underline{x})|^2 d\underline{x} = (2R)^n \sum_{\underline{k}\in\mathbb{Z}^n} |c_{\underline{k}}|^2,$$

and the Plancherel Theorem on L^2-functions in $\mathbb{R}^n$ reads

$$\int_{\mathbb{R}^n} |\mathcal{F}(f)(\underline{x})|^2 \, d\underline{x} = \int_{[-R,R]^n} |\mathcal{F}(f)(\underline{x})|^2 d\underline{x} = (2\pi)^n \int_{\mathbb{R}^n} |f(\underline{t})|^2 d\underline{t}.$$

Putting together, we have

$$\frac{1}{(2\pi)^n}\int_{[-\frac{\pi}{h},\frac{\pi}{h}]^n} |\mathcal{F}(f)(\underline{x})|^2\, d\underline{x} = \int_{\mathbb{R}^n} |f(\underline{t})|^2 d\underline{t} = \sum_{\underline{k}\in\mathbb{Z}^n} |f(h\underline{k})|^2.$$

To prove (ii), let $\phi_\epsilon(\underline{x})$ be an infinitely differentiable function such that $\operatorname{supp}(\phi_\epsilon)\subset[-\pi/h,\pi/h]^n$, $|\phi_\epsilon|\le 1$ and $\phi_\epsilon\to\chi_{[-\pi/h,\pi/h]^n}$ a.e. as $\epsilon\to 0$.

Define

$$I_\epsilon f(\underline{z}) \quad = \quad \frac{1}{(2\pi)^n}\int_{[-\frac{\pi}{h},\frac{\pi}{h}]^n} \phi_\epsilon(\underline{x})e^{\mathbf{i}\underline{x}\cdot\underline{z}}\mathcal{F}(f)(\underline{x})d\underline{x}. \tag{6}$$

For any fixed $\epsilon>0$ and $\underline{z}\in\mathbb{C}^n$, expand $\phi_\epsilon(\underline{x})e^{\mathbf{i}\underline{x}\cdot\underline{z}}$ on the cube $[-\pi/h,\pi/h]^n$ into its Fourier series in n-variables, we have

$$\phi_\epsilon(\underline{x})e^{\mathbf{i}\underline{x}\cdot\underline{z}} = \sum_{\underline{k}\in\mathbb{Z}^n} e^{\mathbf{i}h\underline{k}\cdot\underline{x}}S_\epsilon(\underline{k},\underline{z}), \tag{7}$$

where the terms $S_\epsilon(\underline{k},\underline{z})$ are the Fourier coefficients of $\phi_\epsilon(\underline{x})e^{\mathbf{i}\underline{x}\cdot\underline{z}}$ on the cube.

By the definition of ϕ_ϵ, the function $\phi_\epsilon(\underline{x})e^{\mathbf{i}\underline{x}\cdot\underline{z}}$ has continuous first and second derivatives and vanishes on the boundary of the cube $[-\pi/h,\pi/h]^n$, and thus is $2\pi/h$-periodic in each variable. A routine argument shows that

$$\left|e^{\mathbf{i}h\underline{k}\cdot\underline{x}}S_\epsilon(\underline{k},\underline{z})\right| \le \frac{C(\epsilon;\underline{z})}{\underline{k}^2}, \qquad \forall \underline{x}\in\mathbb{R}^n.$$

So the series (7) is convergent uniformly in terms of $\underline{x}$ for fixed ϵ and $\underline{z}$.

Inserting the series (7) into (6) and exchanging the order of integration and summation, justified by the uniform convergence, we have

$$\begin{aligned} I_\epsilon f(\underline{z}) \quad &= \quad \frac{1}{(2\pi)^n}\int_{[-\frac{\pi}{h},\frac{\pi}{h}]^n}\sum_{\underline{k}\in\mathbb{Z}^n} e^{\mathbf{i}h\underline{k}\cdot\underline{x}}S_\epsilon(\underline{k},\underline{z})\mathcal{F}(f)(\underline{x})d\underline{x} \\ &= \quad \sum_{\underline{k}\in\mathbb{Z}^n}\left[\frac{1}{(2\pi)^n}\int_{[-\frac{\pi}{h},\frac{\pi}{h}]^n} e^{\mathbf{i}h\underline{k}\cdot\underline{x}}\mathcal{F}(f)(\underline{x})d\underline{x}\right]S_\epsilon(\underline{k},\underline{z}). \end{aligned} \tag{8}$$

Since $f\in PW(\pi/h)$, the Paley-Wiener Theorem 2.1 gives

$$f(h\underline{k}) = \frac{1}{(2\pi)^n}\int_{B(0,\pi/h)} e^{\mathbf{i}h\underline{k}\cdot\underline{x}}\mathcal{F}(f)(\underline{x})d\underline{x} = \frac{1}{(2\pi)^n}\int_{[-\frac{\pi}{h},\frac{\pi}{h}]^n} e^{\mathbf{i}h\underline{k}\cdot\underline{x}}\mathcal{F}(f)(\underline{x})d\underline{x}.$$

Therefore the integral $I_\epsilon f$ in (8) becomes

$$I_\epsilon f(\underline{z}) = \sum_{\underline{k}\in \mathbb{Z}^n} f(h\underline{k}) S_\epsilon(\underline{k}, \underline{z}). \tag{9}$$

We are to take limit $\epsilon \to 0$. Since $\mathcal{F}(f) \in L^2(B(0, \pi/h)) \subset L^1(B(0, \pi/h))$, we have

$$\left|\phi_\epsilon(\underline{x}) e^{\mathbf{i}\underline{x}\cdot\underline{z}} \mathcal{F}(f)(\underline{x})\right| \leq \left|e^{\mathbf{i}\underline{x}\cdot\underline{z}} \mathcal{F}(f)(\underline{x})\right| \in L^1([-\pi/h, \pi/h]^n).$$

Moreover, by the definition of ϕ_ϵ, we have

$$\lim_{\epsilon\to 0} \phi_\epsilon(\underline{x}) e^{\mathbf{i}\underline{x}\cdot\underline{z}} \mathcal{F}(f)(\underline{x}) = e^{\mathbf{i}\underline{x}\cdot\underline{z}} \mathcal{F}(f)(\underline{x}).$$

Using Legesgue's Dominated Convergence Theorem, the left hand side of (9) tends to

$$\frac{1}{(2\pi)^n} \int_{[-\frac{\pi}{h}, \frac{\pi}{h}]^n} e^{\mathbf{i}\underline{x}\cdot\underline{z}} \mathcal{F}(f)(\underline{x}) d\underline{x},$$

that is $f(\underline{z})$ by invoking the Paley-Wiener Theorem 2.1.

Now consider the series on the right hand side of (9). For any positive number $M \in \mathbb{R}$, using the Cauchy-Schwarz inequality, we have

$$\left|\sum_{|\underline{k}|>M} f(h\underline{k}) S_\epsilon(\underline{k}, \underline{z})\right| \leq \left(\sum_{|\underline{k}|>M} |f(h\underline{k})|^2\right)^{1/2} \left(\sum_{|\underline{k}|>M} |S_\epsilon(\underline{k}, \underline{z})|^2\right)^{1/2}.$$

Note that the function $\phi_\epsilon(\underline{x}) e^{i\underline{x}\cdot\underline{z}} \in L^2([-\pi/h, \pi/h]^n)$. Therefore the sequence of its Fourier coefficients $\{S_\epsilon(\underline{k}, \underline{z})\} \in l^2$. The Bessel inequality gives

$$\left(\sum_{|\underline{k}|>M} |S_\epsilon(\underline{k}, \underline{z})|^2\right)^{1/2} \leq \frac{h^{n/2}}{(2\pi)^{n/2}} \left\|\phi_\epsilon(\underline{x}) e^{\mathbf{i}\underline{x}\cdot\underline{z}}\right\|_{L^2([-\pi/h,\pi/h]^n)} \leq \frac{h^{n/2}}{(2\pi)^{n/2}} \left\|e^{\mathbf{i}\underline{x}\cdot\underline{z}}\right\|_{L^2([-\pi/h,\pi/h]^n)} < \infty.$$

Therefore the series in (9) is convergent uniformly in ϵ.

Note that

$$\lim_{\epsilon\to 0} S_\epsilon(\underline{k}, \underline{z}) = \underline{\text{sinc}}\left(\frac{\underline{z} - h\underline{k}}{h}\right).$$

Taking limit $\epsilon \to 0$ on the right-hand-side of (9). Since we can exchange the limit procedure with the summation owing to the uniform convergence in ϵ, the relation (5) follows. The assertion for the coefficients $f(h\underline{k})$ in (ii) follows from (3) of Theorem 2.2.

Now we prove (iii). As in the proof of (i), the Fourier series of $\mathcal{F}(f)$ is

$$S(\underline{x}) = \sum_{\underline{k}\in\mathbb{Z}^n} h^n f(h\underline{k}) e^{-\mathbf{i}h\underline{k}\cdot\underline{x}}.$$

The notation $S(\underline{x})$ is for the series, as well as for the limit value when the series is convergent. The convergence of $S(\underline{x})$ at $\underline{x} = \underline{0}$ is now an assumption, and hence

$$S(\underline{0}) = \lim_{N\to\infty} \sum_{\substack{|k_j|\leq N \\ j=1,\cdots,n}} h^n f(h\underline{k}) = \sum_{\underline{k}\in\mathbb{Z}^n} h^n f(h\underline{k}).$$

Since $f \in L^1$, we have

$$\mathcal{F}(f)(\underline{0}) = \int_{\mathbb{R}^n} f(\underline{t}) d\underline{t}.$$

Thus, to prove (iii) is to prove

$$\mathcal{F}(f)(\underline{0}) = S(\underline{0}).$$

Denote by $C(\underline{x})$ the Cesáro mean series corresponding to $S(\underline{x})$. The notation also stands for the limit value when the series is convergent. It is an easy exercise in mathematical analysis that

$$C(\underline{x}) = S(\underline{x})$$

whenever $S(\underline{x})$ is convergent, and therefore,

$$C(\underline{0}) = S(\underline{0}).$$

Fejér's Theorem asserts that at the continuous points of $\mathcal{F}(f)$ we always have

$$C(\underline{x}) = \mathcal{F}(f)(\underline{x}).$$

The assumption $f \in L^1$ implies that $\mathcal{F}(f)$ is uniformly continuous at all points, and therefore for all points in $[-\pi/h, \pi/h]^n$ the above relation holds. In particular, we have

$$C(\underline{0}) = \mathcal{F}(f)(\underline{0}).$$

We finally arrive

$$\mathcal{F}(f)(\underline{0}) = S(\underline{0}). \qquad \square$$

Remark The quadrature relation (iii) is an immediate consequence of the Poisson summation formula. In fact, in our case the latter formula reads

$$\sum_{\underline{k}\in\mathbb{Z}^n} 1/h^n \mathcal{F}(f)\,(2\pi\underline{k}/h) = \sum_{\underline{k}\in\mathbb{Z}^n} f(h\underline{k}).$$

Since the support of $\mathcal{F}(f)$ is contained in $[-\frac{\pi}{h},\frac{\pi}{h}]^n$, the left-hand-side is equal to $h^{-n}\mathcal{F}(f)(\underline{0})$ that gives the assertion (iii). Besides the conditions $f\in PW(\pi/h)$ and $f\in L^1(\mathbb{R}^n)$, however, extra conditions on f are needed to guarantee the Poisson summation formula to hold. For instance, if the space dimension $n=1$, then f having bounded variation is sufficient (see Chapter II, §13, [16]).

3 The Difference Between $f(\underline{x})$ and $C(f;\underline{x})$

In this section we deal with functions holomorphic in strips in $\mathbb{C}^n$ containing $\mathbb{R}^n$. We shall give an integral error formula in approximating such functions by their cardinal functions $C(f;\underline{x})$.

Definition 3.1 Let d be a positive number, $\mathcal{D}_d$ and $\mathcal{D}_d^n$ denote the *infinite strip regions* containing $\mathbb{R}$ and $\mathbb{R}^n$, in respectively $\mathbb{C}$ and $\mathbb{C}^n$, given by

$$\mathcal{D}_d = \{z\in\mathbb{C}: z=x+iy, |y|\le d\}$$

and

$$\mathcal{D}_d^n = \{\underline{z}=(z_1,\ldots,z_n)\in\mathbb{C}^n : z_j = x_j+iy_j,\ |y_j|\le d,\ 1\le j\le n\}.$$

Define

$$d_j = d-\frac{1}{N_j},\quad j=1,...,n,$$

where for each fixed j, N_j represents a positive integer tending to ∞.

Definition 3.2 Let $\{i_1,\cdots,i_l\}\cup\{j_1,\cdots,j_m\}\cup\{q_1,\cdots,q_h\}$ be a partition of $\{1,2,\cdots,n\}$, denoted by $\mathcal{P}$. For $1\le p<\infty$, we say that $f\in G_m^p(\mathcal{D}_d^n)$ if f satisfies

$$\begin{aligned}&\int_{-d}^{d}\cdots\int_{-d}^{d} ds_{i_1}\cdots ds_{i_l}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}\frac{|f(r_1+\mathbf{i}s_1,\cdots,r_n+\mathbf{i}s_n)|}{|r_{j_1}+\mathbf{i}s_{j_1}-x_{j_1}|\cdots|r_{j_m}+\mathbf{i}s_{j_m}-x_{j_m}|}dr_{j_1}\cdots dr_{j_m}\\ \le\ & C_{\mathcal{P}}|r_{i_1}|^{\alpha_{i_1}}\cdots|r_{i_l}|^{\alpha_{i_l}},\end{aligned}\tag{10}$$

where $\alpha_{i_k}\in(0,1)$, $r_{i_k}=\pm(N_{i_k}+1/2)h$ for $k=1,\ldots,l$; $s_{j_k}=\pm d_{j_k}$ for $k=1,\ldots,m$; $s_{q_k}=0$, for $k=1,\cdots,h$, and $C_{\mathcal{P}}$ depends on $\mathcal{P}$ but not on r_{q_k}, $k=1,\ldots,h$, nor on $x_k, k=1,...,n$;

and, if $\{i_1, \cdots, i_l\} = \emptyset$, then

$$N_m^p(f, \mathcal{D}_d^n) = \left[\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \sum_{s_{j_1}=-d,d} \cdots \sum_{s_{j_m}=-d,d} |f(r_1 + \mathbf{i}s_1, \cdots, r_n + \mathbf{i}s_n)|^p dr_{j_1} \cdots dr_{j_m}\right]^{1/p} < \infty. \quad (11)$$

Let N_j be as in Definition 3.1. Define the boundaries of the truncated strips

$$L_{N_j} = \left\{r_j + \mathbf{i}s_j : |r_j| \leq \left(N_j + \frac{1}{2}\right) h \text{ and } s_j = \pm d_j; \text{ or } r_j = \pm \left(N_j + \frac{1}{2}\right) h \text{ and } |s_j| \leq d_j\right\},$$

L_{N_j} encloses the points $r_j = kh$, $k = 0, \pm 1, \cdots, \pm N_j$, and $L_{N_j} \to \partial\mathcal{D}_d$, the boundary of $\mathcal{D}_d$, as $N_j \to \infty$.

Before we state the main theorem of the section, we give some notation. Denote

$$a(x_j) = \frac{\sin(\pi x_j/h)}{2\pi\mathbf{i}}$$

and

$$b(z_j) = \frac{1}{(z_j - x_j)\sin(\pi z_j/h)}, \quad j = 1, ..., n.$$

If g is a function of n variables, then denote

$$A_k(g; x_{l_1}, \cdots, x_{l_k}) = \prod_{t=1}^{k} a(x_{l_t}) \int_{L_{N_{l_1}} \times \cdots \times L_{N_{l_k}}} \prod_{t=1}^{k} b(z_{l_t}) g(\cdots, z_{l_1}, \cdots, z_{l_k}, \cdots) dz_{l_1} \cdots dz_{l_k},$$

and

$$B_k(g; x_{l_1}, \cdots, x_{l_k}) = \prod_{t=1}^{k} a(x_{l_t}) \int_{\partial\mathcal{D}_d^k} \prod_{t=1}^{k} b(z_{l_t}) g(\cdots, z_{l_1}, \cdots, z_{l_k}, \cdots) dz_{l_1} \cdots dz_{l_k},$$

where $k = 1, \cdots, n$, and A_k, B_k, once $x_{l_1}, ..., x_{l_k}$ are fixed, are functions in $n - k$ variables.

Theorem 3.1 *Let f be holomorphic in $\mathcal{D}_d^n$, $h > 0$,*

(i) For

$$\varepsilon_{N_1,...,N_n}(f; \underline{x}) = f(\underline{x}) - C_{N_1,...,N_n}(f; \underline{x}),$$

we have

$$\begin{aligned} \varepsilon_{N_1,...,N_n}(f; \underline{x}) \quad = \quad & \sum_{j=1}^{n} A_1(f(\cdots, x_{j-1}, \cdot, x_{j+1}, \cdots); x_j) \\ & - \sum_{q<r} A_2(f(\cdots, x_{q-1}, \cdot, x_{q+1}, \cdots, x_{r-1}, \cdot, x_{r+1}, \cdots); x_q, x_r) \\ & + \cdots + (-1)^{n+1} A_n(f; \underline{x}). \end{aligned} \quad (12)$$

(ii) If additionally $f \in G_m^p(D_d^n)$, $1 \le p < \infty$, *with*

$$\varepsilon(f;\underline{x}) = f(\underline{x}) - C(f;\underline{x}),$$

we have

$$\begin{aligned}
\varepsilon(f;\underline{x}) &= \sum_{j=1}^{n} B_1(f(\cdots,x_{j-1},\cdot,x_{j+1},\cdots);x_j) \\
&\quad - \sum_{q<r} B_2(f(\cdots,x_{q-1},\cdot,x_{q+1},\cdots,x_{r-1},\cdot,x_{r+1},\cdots);x_q,x_r) \\
&\quad + \cdots + (-1)^{n+1} B_n(f;\underline{x}).
\end{aligned} \tag{13}$$

(iii) For $1 \le p < \infty$, *we have*

$$\|f - C(f;\cdot)\|_\infty = \|\varepsilon(f;\underline{\cdot})\|_\infty \le C \sum_{j=1}^{n} \begin{pmatrix} n \\ j \end{pmatrix} \frac{N_j^p(f,\mathcal{D}_d^n)}{[d^{1/p}\sinh(\pi d/h)]^j} = \mathcal{O}(e^{-\pi d/h}). \tag{14}$$

Proof (i) While essentially the formula is a consequence of the Residue Theorem of several complex variables, we choose to prove the assertion step by step based on the Residue Theorem in one complex variable.

To describe the main idea and avoid the complication of the notation, we first deal with the cases $n = 1, 2, 3$, and indicate how to proceed with the proof for general $n > 3$.

Applying the Residue Theorem to the integral of $a(x_1)b(z_1)f(z_1)$ along the contour L_{N_1}, always in the positive orientation with respect to the finite strip area, we have

$$\begin{aligned}
a(x_1)\int_{L_{N_1}} b(z_1)f(z_1)dz_1 &= \frac{\sin(\pi x_1/h)}{2\pi\mathbf{i}} \int_{L_{N_1}} \frac{f(z_1)}{(z_1 - x_1)\sin(\pi z_1/h)} dz_1 \\
&= f(x_1) - C_{N_1}(f;x_1) \\
&= \epsilon_{N_1}(f;x_1).
\end{aligned} \tag{15}$$

So (12) is true for $n = 1$.

For $n = 2$, we have

$$\begin{aligned}
\varepsilon_{N_1,N_2}(f;x_1,x_2) &= f(x_1,x_2) - C_{N_1,N_2}(f;x_1,x_2) \\
&= [f(x_1,x_2) - C_{N_1}(f(\cdot,x_2);x_1)] + [C_{N_1}(f(\cdot,x_2);x_1) - C_{N_1,N_2}(f;x_1,x_2)] \\
&= I_1 + I_2.
\end{aligned}$$

Using the one dimensional error formula in (15), we have

$$I_1 = f(x_1, x_2) - C_{N_1}(f(\cdot, x_2); x_1) = A_1(f(\cdot, x_2); x_1),$$

which is one term on the right-hand-side of (12). Taking out the common factor of the two terms in I_2, we have

$$\begin{aligned} I_2 &= \sum_{|k_1|\le N_1} \operatorname{sinc}\left(\frac{x_1 - hk_1}{h}\right) [f(hk_1, x_2) - C_{N_2}(f(hk_1, \cdot); x_2)] \\ &= \sum_{|k_1|\le N_1} \operatorname{sinc}\left(\frac{x_1 - hk_1}{h}\right) A_1(f(hk_1, \cdot); x_2) \\ &= \sum_{|k_1|\le N_1} \operatorname{sinc}\left(\frac{x_1 - hk_1}{h}\right) a(x_2) \int_{L_{N_2}} b(z_2) f(hk_1, z_2) dz_2. \end{aligned}$$

Exchanging the order of taking summation and integration, and by the definition of $C_{N_1}(f(\cdot, z_2); x_1)$, we obtain

$$I_2 = a(x_2) \int_{L_{N_2}} b(z_2) C_{N_1}(f(\cdot, z_2); x_1) dz_2.$$

Owing to the relations

$$C_{N_1}(f(\cdot, z_2); x_1) = f(x_1, z_2) - A_1(f(\cdot, z_2); x_1),$$

$$a(x_2) \int_{L_{N_2}} b(z_2) f(x_1, z_2) dz_2 = A_1(f(x_1, \cdot); x_2)$$

and

$$a(x_2) \int_{L_{N_2}} b(z_2) A_1(f(\cdot, z_2); x_1) dz_2 = A_2(f; x_1, x_2),$$

we have

$$\varepsilon_{N_1,N_2}(f; x_1, x_2) = I_1 + I_2 = A_1(f(\cdot, x_2); x_1) + A_1(f(x_1, \cdot); x_2) - A_2(f; x_1, x_2), \tag{16}$$

as desired for $n = 2$.

For $n = 3$, we have

$$\begin{aligned} & \varepsilon_{N_1,N_2,N_3}(f; x_1, x_2, x_3) \\ =\ & f(x_1, x_2, x_3) - C_{N_1,N_2,N_3}(f; x_1, x_2, x_3) \\ =\ & [f(x_1, x_2, x_3) - C_{N_1}(f(\cdot, x_2, x_3); x_1)] \\ & + [C_{N_1}(f(\cdot, x_2, x_3); x_1) - C_{N_1,N_2,N_3}(f; x_1, x_2, x_3)] \\ =\ & I_1 + I_2. \end{aligned}$$

Using the one dimensional error formula, we have

$$I_1 = f(x_1, x_2, x_3) - C_{N_1}(f(\cdot, x_2, x_3); x_1) = A_1(f(\cdot, x_2, x_3); x_1).$$

Taking out the common factor $\operatorname{sinc}((x_1 - h\underline{k})/h)$ in I_2, we have

$$I_2 = \sum_{|k_1| \le N_1} \operatorname{sinc}\left(\frac{x_1 - hk_1}{h}\right) \left[f(hk, x_2, x_3) - C_{N_2,N_3}(f(hk_1, \cdot, \cdot); x_2, x_3)\right].$$

Applying the two dimensional error formula given by (16), we have

$$\begin{aligned} & f(hk_1, x_2, x_3) - C_{N_2,N_3}(f(hk_1, \cdot, \cdot); x_2, x_3) \\ = \; & A_1(f(hk_1, \cdot, x_3); x_2) + A_1(f(hk_1, x_2, \cdot); x_3) - A_2(f(hk_1, \cdot, \cdot); x_2, x_3) \\ = \; & a(x_2) \int_{L_{N_2}} b(z_2) f(hk_1, z_2, x_3) dz_2 + a(x_3) \int_{L_{N_3}} b(z_3) f(hk_1, x_2, z_3) dz_3 \\ & - a(x_2) a(x_3) \int_{L_{N_2} \times L_{N_3}} b(z_2) b(z_3) f(hk_1, z_2, z_3) dz_2 dz_3. \end{aligned}$$

Inserting the last expression into the summation expansion of I_2, by exchanging the order of summation and integration, we have

$$\begin{aligned} I_2 = \; & a(x_2) \int_{L_{N_2}} b(z_2) C_{N_1}(f(\cdot, z_2, x_3); x_1) dz_2 + a(x_3) \int_{L_{N_3}} b(z_3) C_{N_1}(f(\cdot, x_2, z_3); x_1) dz_3 \\ & - a(x_2) a(x_3) \int_{L_{N_2} \times L_{N_3}} b(z_2) b(z_3) C_{N_1}(f(\cdot, z_2, z_3); x_1) dz_2 dz_3. \end{aligned}$$

By the one dimensional error formula (15) for $C_{N_1}(f(\cdot, z_2, x_3); x_1)$, $C_{N_1}(f(\cdot, x_2, z_3); x_1)$ and $C_{N_1}(f(\cdot, z_2, z_3); x_1)$, one has

$$\begin{aligned} I_2 = \; & a(x_2) \int_{L_{N_2}} b(z_2) \left[f(x_1, z_2, x_3) - A_1(f(\cdot, z_2, x_3); x_1)\right] dz_2 \\ & + a(x_3) \int_{L_{N_3}} b(z_3) \left[f(x_1, x_2, z_3) - A_1(f(\cdot, x_2, z_3); x_1)\right] dz_3 \\ & - a(x_2) a(x_3) \int_{L_{N_2} \times L_{N_3}} b(z_2) b(z_3) \left[f(x_1, z_2, z_3) - A_1(f(\cdot, z_2, z_3); x_1)\right] dz_2 dz_3. \end{aligned}$$

In the given notation, we have, for example,

$$a(x_2) \int_{L_{N_2}} b(z_2) f(x_1, z_2, x_3) dz_2 = A_1(f(x_1, \cdot, x_3); x_2),$$

$$a(x_2) \int_{L_{N_2}} b(z_2) A_1(f(\cdot, z_2, x_3); x_1) dz_2 = A_2(f(\cdot, \cdot, x_3); x_1, x_2),$$

$$a(x_2) a(x_3) \int_{L_{N_2} \times L_{N_3}} b(z_2) b(z_3) f(x_1, z_2, z_3) dz_2 dz_3 = A_2(f(x_1, \cdot, \cdot); x_2, x_3)$$

and

$$a(x_2)a(x_3)\int_{L_{N_2}\times L_{N_3}} b(z_2)b(z_3)A_1(f(\cdot,z_2,z_3);x_1)dz_2dz_3 = A_3(f;x_1,x_2,x_3).$$

Likewise, integration of the above kind in k variables always raises A_i to A_{i+k}. We thus finally arrive

$$\begin{aligned}\varepsilon_{N_1,N_2,N_3}(f;x_1,x_2,x_3) &= A_1(f(\cdot,x_2,x_3);x_1)+A_1(f(x_1,\cdot,x_3);x_2)+A_1(f(x_1,x_2,\cdot);x_3)\\ &\quad -A_2(f(\cdot,\cdot,x_3);x_1,x_2)-A_2(f(\cdot,x_2,\cdot);x_1,x_3)-A_2(f(x_1,\cdot,\cdot);x_2,x_3)\\ &\quad +A_3(f;x_1,x_2,x_3).\end{aligned}$$

This shows that (12) is true for $n=3$.

Inductively, to deal with a general positive integer n, we first write

$$\begin{aligned}&\varepsilon_{N_1,\dots,N_n}(f;x_1,\cdots,x_n)\\ =\;& f(\underline{x})-C_{N_1,\cdots,N_n}(f;\underline{x})\\ =\;& [f(\underline{x})-C_{N_1}(f(\cdot,x_2,\cdots,x_n);x_1)]+[C_{N_1}(f(\cdot,x_2,\cdots,x_n);x_1)-C_{N_1,\cdots,N_n}(f;\underline{x})]\\ =\;& I_1+I_2.\end{aligned}$$

To I_1 we use the proved formula for $\varepsilon_{N_1}(f;\underline{x})$ to get $I_1 = A_1(f(\cdot,x_2,\cdots,x_n);x_1)$.

To I_2 we first factorize out the common factor $\operatorname{sinc}((x_1-hk_1)/h)$, and obtain

$$\begin{aligned}I_2 &= \sum_{|k_1|\le N_1}\operatorname{sinc}\left(\frac{x_1-hk_1}{h}\right)[f(hk_1,x_2,\cdots,x_n)-C_{N_2,\cdots,N_n}(f(hk_1,\cdots);x_2,\cdots,x_n))]\\ &= \sum_{|k_1|\le N_1}\operatorname{sinc}\left(\frac{x_1-hk_1}{h}\right)\varepsilon_{N_2,\dots,N_n}(f(hk_1,\cdots);x_2,\cdots,x_n).\end{aligned}$$

Then using the induction hypothesis on $\varepsilon_{N_2,\dots,N_n}(f(hk_1,\cdots);x_2,\cdots,x_n)$.

By exchanging summation with integration, then combining $\operatorname{sinc}((x_1-hk_1)/h)$ with the numerator part in each term of $\varepsilon_{N_2,\dots,N_n}(f(hk_1,\cdots);x_2,\cdots,x_n)$ in the integral to form the cardinal function of the first variable x_1, that are subsequently replaced by $f(x_1,\cdots)-A_1(f(\cdot,\cdots);x_1)$. Using the definition of A_j, we finally get the desired (12) for general n.

The proof of (i) is complete.

(ii) To prove (13), study the integral

$$\begin{aligned} & A_k(f; x_{l_1}, \cdots, x_{l_k}) \\ = & \prod_{t=1}^{k} a(x_{l_t}) \int_{L_{N_{l_1}} \times \cdots \times L_{N_{l_k}}} \prod_{i=1}^{k} b(z_{l_i}) f(\cdots, z_{l_1}, \cdots, z_{l_k}, \cdots) dz_{l_1} \cdots dz_{l_k}. \end{aligned} \tag{17}$$

Decompose the integral with respect to $dz_{l_j}, j = 1, ..., k$, into the sum

$$\int_{L_{N_{l_j}}} \cdots dz_{l_j} = \int_{-d_{l_j}}^{d_{l_j}} \cdots dz_{l_j} + \int_{-(N_{l_j}+1/2)/h}^{(N_{l_j}+1/2)/h} \cdots dz_{l_j}.$$

By the distribution law, the multiple integral (17) is a sum of some multiple integrals of which each is associated with a partition of the index set $\{l_1, \ldots, l_k\}$, viz. $\{l_1, \ldots, l_k\} = \{i_1, \ldots, i_p\} \cup \{j_1, \ldots, j_m\}$, and the corresponding multiple integral is

$$\begin{aligned} & \prod_{i=1}^{k} a(x_{l_i}) \int_{-d_{i_1}}^{d_{i_1}} \cdots \int_{-d_{i_p}}^{d_{i_p}} ds_{i_1} \cdots ds_{i_p} \\ & \int_{-(N_{j_1}+1/2)h}^{(N_{j_1}+1/2)h} \cdots \int_{-(N_{j_m}+1/2)h}^{(N_{j_m}+1/2)h} \prod_{i=1}^{k} b(z_{l_i}) f(\cdots, z_{l_1}, \cdots, z_{l_k}, \cdots) dz_{l_1} \cdots dz_{l_k}. \end{aligned}$$

We divide all the integrals, corresponding to all different partitions of $\{l_1, \ldots, l_k\}$, into two groups.

Integrals in the first group correspond to the partitions with $p > 0$ of which each involves some vertical segments of $L_{N_{l_1}} \times \cdots \times L_{N_{l_k}}$. Using the condition (10) with $\alpha_{i_k} \in (0, 1)$, and the fact that $|\sin(\pi(r_i + \mathbf{i}s_i)/h)| \geq \cosh(\pi s_1/h) \geq 1$ and $|r_{i_k} + \mathbf{i}s_{i_k} - x_{i_k}| \geq |r_{i_k} - x_{i_k}| = |(N_{i_k} + 1/2)h - x_{i_k}|$, $k = 1, \ldots, p$, the module of each in the first kind is not greater than

$$\begin{aligned} & \frac{C_k}{|(N_{i_1} + 1/2)h - x_{i_1}| \cdots |(N_{i_p} + 1/2)h - x_{i_p}|} \int_{-d_{i_1}}^{d_{i_1}} \cdots \int_{-d_{i_p}}^{d_{i_p}} ds_{i_1} \cdots ds_{i_p} \\ & \int_{-(N_{j_1}+1/2)h}^{(N_{j_1}+1/2)h} \cdots \int_{-(N_{j_m}+1/2)h}^{(N_{j_m}+1/2)h} \frac{|f(r_1 + \mathbf{i}s_1, \cdots, r_n + \mathbf{i}s_n)|}{|r_{j_1} + \mathbf{i}s_{i_1} - x_{i_1}| \cdots |r_{j_1 m} + \mathbf{i}s_{i_m} - x_{i_m}|} dr_{j_1} \cdots dr_{j_m} \\ \leq \; & \frac{C_k |r_{i_1}|^{\alpha_{i_1}} \cdots |r_{i_p}|^{\alpha_{i_p}}}{|(N_{i_1} + 1/2)h - x_{i_1}| \cdots |(N_{i_p} + 1/2)h - x_{i_p}|}, \end{aligned}$$

and thus goes to zero as $N_{l_t} \to \infty$, $t = 1, \ldots, k$.

The second group contains only one multiple integral corresponding to $p = 0$ and $\{j_1, \ldots, j_m\} = \{l_1, \ldots, l_k\}$. The integral is

$$\prod_{t=1}^{k} a(x_{l_t}) \int_{-(N_{l_1}+1/2)h}^{(N_{l_1}+1/2)h} \cdots \int_{-(N_{l_k}+1/2)h}^{(N_{l_k}+1/2)h} \prod_{i=1}^{k} b(r_{l_i} + \mathbf{i}s_{l_i}) f(r_1 + \mathbf{i}s_1, \cdots, r_n + \mathbf{i}s_n) dr_{j_{l_1}} \cdots dr_{j_{l_k}},$$

where $s_{l_i} = \pm d_i$, $i = 1, \ldots, k$. This is $B_k(f; x_{l_1}, \ldots, x_{l_k})$ as $N_{l_i} \to \infty$, $i = 1, \ldots, k$.

So

$$\lim_{\substack{N_{l_t} \to \infty \\ t=1,\cdots,k}} A_k(f, ; x_{l_1}, \ldots, x_{l_k}) = B_k(f; x_{l_1}, \ldots, x_{l_k}),$$

where $k = 1, \ldots, n$.

This implies

$$\lim_{\substack{N_j \to \infty \\ j=1,\cdots,n}} \varepsilon_{N_1,\ldots,N_n}(f; \underline{x}) = \varepsilon(f; \underline{x}).$$

The result (13) follows.

(iii) To prove (14), we will show that

$$\|B_k\|_\infty \leq \frac{C N_k^p(f, D_d^n)}{[d^{1/p} \sinh(\pi d/h)]^k}, \tag{18}$$

for $k = 1, \cdots, n$.

Using the fact that $|\sin(\pi(r_1 \pm \mathbf{i} d_1)/h)| \geq \sinh(\pi d_1/h)$, we have

$$|b(r_{l_i} \pm \mathbf{i} s_{l_i})| \leq \frac{1}{\sinh(\pi d/h) |r_{l_i} \pm \mathbf{i} s_{l_i} - x_{l_i}|}.$$

This implies

$$\begin{aligned} &\Big| B_k(f; x_{l_1}, \ldots, x_{l_j}) \Big| \\ \leq\ & \frac{C}{[\sinh(\pi d/h)]^k} \int_{\mathbb{R}^k} \prod_{i=1}^{k} \frac{1}{|r_{l_i} \pm \mathbf{i} s_{l_i} - x_{l_i}|} |f(r_1 + \mathbf{i} s_1, \cdots, r_n + \mathbf{i} s_n)| dr_{j_{l_1}} \cdots dr_{j_{l_k}}, \end{aligned} \tag{19}$$

where $s_{l_i} = \pm d$, $i = 1, \ldots, k$.

For $p = 1$, since $|r_{l_i} \pm \mathbf{i} s_{l_i} - x_{l_i}| \geq |s_{l_i}| = d$, we have

$$\begin{aligned} \Big| B_k(f; x_{l_1}, \ldots, x_{l_j}) \Big| &\leq \frac{C}{[d \sinh(\pi d/h)]^k} \int_{\mathbb{R}^k} |f(r_1 + \mathbf{i} s_1, \cdots, r_n + \mathbf{i} s_n)| dr_{j_{l_1}} \cdots dr_{j_{l_k}} \\ &= \frac{C_p N_k^1(f, \mathcal{D}_d^n)}{[d \sinh(\pi d/h)]^k}. \end{aligned}$$

For $1 < p < \infty$, using the Hölder inequality in the integral in (19) with $1/p' + 1/p = 1$, we have

$$\int_{\mathbb{R}^k} \prod_{i=1}^{k} \frac{1}{|r_{l_i} \pm \mathbf{i} s_{l_i} - x_{l_i}|} |f(r_1 + \mathbf{i} s_1, \cdots, r_n + \mathbf{i} s_n)| dr_{j_{l_1}} \cdots dr_{j_{l_k}}$$

$$\leq \left\{\prod_{i=1}^{k}\int_{\mathbb{R}}\frac{1}{|r_{l_i}\pm \mathbf{i}s_{l_i}-x_{l_i}|^{p'}}dr_{l_i}\right\}^{1/p'}\left\{\int_{\mathbb{R}^k}|f(r_1+\mathbf{i}s_1,\cdots,r_n+\mathbf{i}s_n)|^p dr_{j_{l_1}}\cdots dr_{j_{l_k}}\right\}^{1/p}$$
$$\leq \frac{CN_k^p(f,\mathcal{D}_d^n)}{d^{k/p}}.$$

Therefore (18) holds for $1\leq p<\infty$.

Hence using the formula in (13), there exists a constant C, depending on p, such that

$$\|\varepsilon(f;\underline{.})\|_\infty \leq C\sum_{k=1}^{n}\begin{pmatrix} n \\ k\end{pmatrix}\frac{N_k^p(f,D_d^n)}{[d^{1/p}\sinh(\pi d/h)]^k}. \tag{20}$$

For $h\leq 2\pi d/\ln(2)$, one has $1/\sinh(\pi d/h)\leq 4e^{-\pi d/h}$ and therefore for any $h>0$, $1/\sinh(\pi d/h)\leq Ce^{-\pi d/h}$. Since $N_k^p(f,D_d^n)$ are bounded for all k, applying the binomial theorem, (20) becomes

$$\|\varepsilon(f;\underline{.})\|_\infty \leq C\left[\left(1+\frac{1}{d^{1/p}\sinh(\pi d/h)}\right)^n-1\right]\leq Cne^{-\pi d/h}=\mathcal{O}(e^{-\pi d/h}).$$

The bound (14) follows.□

The estimate in (iii) of the above theorem is expected, as the sinc function in $\mathbb{C}^n$ is a tensor product of those in $\mathbb{C}$.

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