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Windowed linear canonical transform and its applications ☆

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ABSTRACT

In this paper, we generalize the windowed Fourier transform to the windowed linear canonical transform by substituting the Fourier transform kernel with the linear canonical transform kernel in the windowed Fourier transform definition. It offers local contents, enjoys high resolution, and eliminates cross terms. Some useful properties of the windowed linear canonical transform are derived. Those include covariance property, orthogonality property and inversion formulas. As applications analogues of the Poisson summation formula, sampling formulas and series expansions are given.

1. Introduction

The linear canonical transform (LCT) has recently received much attention in signal processing and optics [1–3]. It was first introduced in 1970s [4,5] and is a four-parameter class of linear integral transform. The LCT is also known as the ABCD transform [6], the affine Fourier transform [7], and the generalized Fresnel transform [8]. Many operations, such as the Fourier transform (FT), the fractional Fourier transform (FRFT) [1,3], the Fresnel transform [9], the Lorentz transform [7] and scaling operations are its special cases. With more degrees of freedom compared to the FT and the FRFT, the LCT is more flexible but with similar computation cost as the conventional FT [10]. Due to the mentioned advantages, the LCT, as a powerful tool, has found many applications in filter design, signal synthesis, optics, radar analysis and pattern recognition, etc. [1,2]. The above-mentioned applications demonstrate the great potential of LCT in signal processing. For example, filtering in the LCT domain, as proposed in [11], can achieve better performance than in the FRFT domain owing to more degrees of freedom. Especially when multi-component chirp signals interfere with the desired signal, only one filter is used in the LCT domain usually, but several filters are required in the FRFT domain [12].

However, the LCT cannot reveal the local LCT-frequency contents due to its global kernel. The windowed Fourier transform (WFT) [1,13], with a local window function, handles this kind of problem well. The absence of undesirable cross terms and computational simplicity result in the wide-spread use of the WFT in practice, and most of the other time–frequency representations can be expressed in terms of it. Nevertheless, the WFT often performs unsatisfactorily for its low resolution. The Wigner–Ville distribution maintains high localization, however it desperately suffers from spurious values in the presence of multi-components or noise [13].

Recently, some studies have attempted to attain high localization properties using fractional Fourier transform and linear canonical transform. For example, Stankovic et al. [14] introduces the windowed fractional Fourier transform (WFRFT) which turns the time–frequency representations with an angle, and then applies the WFT to it. It costs the same number of computations as

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realizations in the time or the frequency domain. Bultheel and Martinez-Sulbaran [15] use a different approach, which generalizes the WFT by substituting the Fourier transform kernel with the linear canonical transform kernel in the WFT definition. It investigates some straightforward properties and two applications, the estimations of the time-of-arrival and pulsewidth of chirp signals, and the windowed fractional Fourier transform filtering. In the present paper we propose the windowed linear canonical transform (WLCT), as a generalization of the latter, explore its properties and applications.

Improving WFT by using the LCT is first proposed by Guven and Arikan in [16]. They, with an alternative approach, started from basic shearing operations, generalized the group of LCT. Signals with smaller time–frequency support are represented with higher resolution, and the minimum time-bandwidth product form of the signal be achieved in several ways, either rotating the support of chirp-like signals at a suitable angle or simply shearing it. Both operations are particular forms of LCT. Here they present a generalized method for improving WFT by using LCT.

In this paper, we will introduce the windowed linear canonical transform (WLCT) by substituting the Fourier transform kernel with the linear canonical transform kernel in the WFT definition. It offers local contents, enjoys high resolution, and eliminates cross terms. It is believed to be suitable for analyzing chirp signals. We derive some useful properties of WLCT such as covariance property, orthogonality property and inversion formulas. The analogues of the Poisson summation formula, sampling formulas and series expansions for the WLCT are also given.

The paper is organized as follows. In Section 2, we give the definitions and properties of LCT, translation and generalized modulation operators. Section 3 presents the WLCT and investigates their properties. The analogues of the Poisson summation formula, sampling formulas and series expansions for WLCT are studied in Section 4. Finally we conclude in Section 5.

2. Preliminary

2.1. The linear canonical transform

The *linear canonical transform* (LCT) with parameter $A=\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$ of a signal $f(t)$ is defined as in [4,2,3,17,18]

$$L_A(f)(u) := \begin{cases} \int_{-\infty}^{\infty} K_A(t,u) f(t)\, dt, & b \neq 0, \\ \sqrt{d}\, e^{j(cd/2)u^2} f(du), & b=0, \end{cases} \tag{1}$$

where a,b,c,d are real or complex numbers satisfying $ad-bc=1$, i.e., $\det(A)=1$, and the kernel of LCT is

$$K_A(t,u) := \frac{1}{\sqrt{j2\pi b}} e^{j((a/2b)t^2-(1/b)tu+(d/2b)u^2)}, \quad b \neq 0. \tag{2}$$

For typographical convenience, we shall denote matrix $A=\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$ as (a,b,c,d) in the text, but all operations have to be understood in the usual matrix sense.

By the definition (1) of LCT, the additivity property of LCT is

$$L_{A_2}(L_{A_1})(f) = L_{A_2A_1}(f), \tag{3}$$

where $A_i=(a_i,b_i,c_i,d_i)$, $i=1,2$ are real matrices and A_2A_1 is the matrix multiplication. According to the additivity property, the inverse transform of LCT can be defined as

$$L_{A^{-1}}(L_A)(f) = f, \tag{4}$$

where $A^{-1}=(d,-b,-c,a)$ is real matrix.

For $i=1,2$, if a_i,b_i,c_i,d_i are complex numbers, then the additivity property (3) holds if

$$\mathrm{Im}\left(\frac{a_2}{b_2}+\frac{d_1}{b_1}\right) > 0. \tag{5}$$

If $\mathrm{Im}(a_2/b_2+d_1/b_1)=0$, then both of b_1 and b_2 must be real. Combining with the inverse property (4), then b must be real since $A_1=(a,b,c,d)$ and $A_2=(d,-b,-c,a)$ $=A_1^{-1}$ by invoking (3).

When a,b,c,d are restricted to be real numbers, then the condition

$$\int_{\mathbf{R}} |f(t)|\, dt < \infty \tag{6}$$

guarantee the existence of LCT. When some of a,b,c,d are complex numbers, then the above integrability condition should be replaced by

$$\int_{\mathbf{R}} |f(t)e^{-\sigma t^2}|\, dt < \infty \quad \text{with } \sigma = \mathrm{Im}\left(\frac{a}{2b}\right). \tag{7}$$

The condition (7) is obviously weaker than (6) and thus defines a larger function class. To improve the flow of the paper, the proofs for (5), (6) and (7) are left to the interested readers. Nervelessly, for typographical convenience, unless otherwise stated, we shall restrict ourselves to the case of real parameters, in which the corresponding LCT is a unitary operator in $L^2(\mathbf{R})$.

An important property of LCT is Parseval's formula [1]

$$\langle f,g\rangle = \langle L_A(f), L_A(g)\rangle \tag{8}$$

for all $f,g \in L^2(\mathbf{R})$, when $\langle f,g\rangle$ denotes the inner product of f and g. It holds mainly for the reason that LCT is a unitary operator on $L^2(\mathbf{R})$. In signal analysis, it represents the fact that LCT preserves the energy of a signal.

2.2. The fundamental operations

For $x,w \in \mathbf{R}$ we define the following operators:

$$T_x f(t) := f(t-x)$$

and

$$M_\mu^{(A)} f(t) := e^{-j((a/2b)t^2-(1/b)t\mu+(d/2b)\mu^2)} f(t), \tag{9}$$

where $A=(a,b,c,d)$ with $\det(A)=1$ and $b>0$. Here T_x is a *translation* by x or a time shift and $M_\mu^{(A)}$ is a *generalized modulation* by μ. Operators of the form $T_x M_\mu^{(A)}$ or $M_\mu^{(A)} T_x$ are called *time–frequency shifts*. They are the main objects of the paper. We observe immediately the canonical *commutation relations*

$$T_x M_\mu^{(A)} f(t) = e^{-(j/b)\mu x} M_x^{(E)} M_\mu^{(A)} T_x f(t), \tag{10}$$

where $E=(0,b/a,-a/b,1)$. Here are some properties of LCT.

- *Time shift*:

$$L_A(T_\tau f)=M_\tau^{(B)}T_{a\tau}L_A(f), \tag{11}$$

where $B=(0,c^{-1},-c,a)$.

- *Modulation*:

$$L_{A_1}(M_\mu^{(A_2)}f)=M_\mu^{(D)}T_{(b_1/b_2)\mu}L_C(f), \tag{12}$$

where $C=(a_1-(a_2/b_2)b_1,b_1,c_1-(a_2/b_2)d_1,d_1)$ and $D=(0,b_2/d_1,-d_1/b_2,d_2/d_1+b_1/b_2)$.

- *Time–frequency shift*:

$$L_{A_1}(T_\tau M_\mu^{(A_2)}f)=e^{-j(a_1d_1/b_2)\mu\tau}M_\tau^{(B_1)}M_\mu^{(D)}T_{a_1\tau+(b_1/b_2)\mu}L_C(f), \tag{13}$$

where $B_1=(0,1/c_1,-c_1,a_1)$, $C=(a_1-(a_2/b_2)b_1,b_1,c_1-(a_2/b_2)d_1,d_1)$ and $D=(0,b_2/d_1,-d_1/b_2,d_2/d_1+b_1/b_2)$.

The proofs for the above properties are straightforward and are left to the interested readers.

3. The windowed linear canonical transforms

Linear canonical transform is a good tool to analyze signals in the frequency domain, but it can do nothing if we want to find out the information of the instantaneous frequency during a time interval. With a suitable window g, performing the related WLCT is an ideal choice to get the frequency contents of a function f.

The windowed linear canonical transform (WLCT) of a function $f\in L^1(\mathbf{R})$ with respect to $g\in L^\infty(\mathbf{R})$ is defined by

$$\begin{aligned}V_g^{(A)}f(x,u) &:= \int_{\mathbf{R}} f(t)\overline{T_xg(t)}K_A(t,u)\,dt\\ &= \frac{1}{\sqrt{j2\pi b}}\int_{\mathbf{R}} f(t)\overline{T_xg(t)}e^{j((a/2b)t^2-(1/b)tu+(d/2b)u^2)}\,dt, \quad b>0,\end{aligned} \tag{14}$$

where $x,u\in\mathbf{R}$ and $A=(a,b,c,d)$ with $\det(A)=1$ and $\overline{(\cdot)}$ is the complex conjugate of $(\cdot)$.

Remark 1. Note that when $A=(0,1,-1,0)$, the WLCT reduce to the WFT of a function f with respect to g. We denote it by

$$V_gf(x,u) := \frac{1}{\sqrt{2\pi j}}\int_{\mathbf{R}} f(t)\overline{T_xg(t)}e^{-jtu}dt \quad \text{for } x,u\in\mathbf{R}.$$

Clearly, if $f\in L^2(\mathbf{R})$ and window function $g\in L^2(\mathbf{R})$, the WLCT is well defined. Similarly, $f\in L^q(\mathbf{R})$ and $g\in L^p(\mathbf{R})$, $1\le p,q<\infty$ and $1/p+1/q=1$, then by Hölder's inequality [19] $fT_x\overline{g}\in L^1(\mathbf{R})$ and the WLCT is well defined.

Remark 2. When the entries of A are complex, the function space $fT_x\overline{g}\in L^1(\mathbf{R})$ should be extended to the larger class

$$fT_x\overline{g}e^{-\rho t^2}\in L^1(\mathbf{R}) \quad \text{with } \rho=\mathrm{Im}\left(\frac{a}{2b}\right).$$

Denote by $f\otimes g$ the (*tensor*) *product* $f\otimes g(x,t)=f(x)g(t)$, and $\mathcal{A}$ the *asymmetric coordinate transform* $\mathcal{A}F(x,t)=F(t,t-x)$, and $L_{A,2}$ the *partial linear canonical transform*

$$L_{A,2}(F)(x,u) := \int_{\mathbf{R}} K_A(t,u)F(x,t)\,dt$$

on a function F on $\mathbf{R}^2$. By this notation, Eq. (14) can be reformulated in terms of a factorization of the WLCT:

$$V_g^{(A)}f(x,u)=L_{A,2}(\mathcal{A}(f\otimes\overline{g})(x,\cdot))(u) \quad \text{if } f,g\in L^2(\mathbf{R}). \tag{15}$$

By (15), the domain of the WLCT can be extended even further. Note that both the operators $\mathcal{A}$ and $L_{A,2}$ are isomorphisms on the Schwartz space $S'(\mathbf{R}^2)$. If $f,g\in S'(\mathbf{R})$, then $f\otimes\overline{g}\in S'(\mathbf{R}^2)$, and consequently $V_g^{(A)}f\in S'(\mathbf{R}^2)$ as well. Thus $V_g^{(A)}f$ is a well-defined tempered distribution, whenever $f,g\in S'(\mathbf{R})$.

The next lemma indicates some useful equivalent forms of the WLCT. WLCT can be expressed in terms of FT. The proof is straightforward.

Lemma 1. *Whenever $V_g^{(A)}f$, $L_A(f)$, $\mathcal{F}(f)$ are well defined, we have the following relations*:

$$V_g^{(A)}f(x,bu)=\frac{1}{\sqrt{jb}}\mathcal{F}(f(\cdot)\overline{T_xg(\cdot)}e^{(ja/2b)(\cdot)^2})(u)e^{(jd/2b)(bu)^2}, \tag{16}$$

where

$$\mathcal{F}(f)(u)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-\infty} f(t)e^{-jut}\,dt$$

denotes the Fourier transform of f. For $A=(a,b,c,d)$ with $\det(A)=1$, $b>0$, *we have*

$$L_A(F)(bu)=e^{(jd/2b)(bu)^2}\mathcal{F}(h)(u), \tag{17}$$

where $h(t)=\sqrt{jb}F(t)e^{(ja/2b)t^2}$.

Some properties of the WLCT are given in the following.

- *Linearity*:

$$V_g^{(A)}(mf_1+nf_2)(x,u)=mV_g^{(A)}f_1(x,u)+nV_g^{(A)}f_2(x,u),$$

where m,n are arbitrary constants. This property sheds lights on the WLCT analysis of multi-component signals.

- *Time marginal constraint*: Integrating WLCT with respect to the time variable yields the LCT multiplied by a constant:

$$\int_{\mathbf{R}} V_g^{(A)}f(x,u)\,dx = CL_A(f)(u),$$

where the coefficient $C=\int_{\mathbf{R}} g(t)\,dt$.

- *Additivity of rotation*:

$$L_{A_2}(V_g^{(A_1)}f(x,u))(x,u')=V_g^{(A_2A_1)}f(x,u)(x,u'), \tag{18}$$

where $i=1,2$, $A_i=(a_i,b_i,c_i,d_i)$ with $\det(A_i)=1$ and $b_i>0$.

- *Generalized cross-ambiguity function*:

$$\begin{aligned}V_g^{(A)}f(x,u)=&\frac{1}{\sqrt{j2\pi b}}e^{(jc/2)ux}e^{-(jac/8)x^2}e^{(jd/2b)(u-(a/2)x)^2}\\ &\times\int_{\mathbf{R}} f\left(t+\frac{x}{2}\right)g\left(t-\frac{x}{2}\right)e^{-(j/b)(u-(a/2)x)}e^{(ja/2b)t^2}\,dt.\end{aligned} \tag{19}$$

If $f,g\in L^2(\mathbf{R})$, then $V_g^{(A)}f(x,u)$ is uniformly continuous on $\mathbf{R}^2$.

- *Covariance property I*:

$$V_g^{(A)}(T_\tau M_\mu^{(A)} f)(x,u) = e^{j[(d/b)\mu(u-\mu-a\tau)+c\tau(u-(a/2)\tau)]} V_g^{(A_0)} f(x-\tau,u-\mu-a\tau)$$
$$= \sqrt{\frac{1}{j2\pi b}} e^{(j/b)((d/2)u^2-(d/2)\mu^2+(d/2)\tau^2-2\tau\mu-\tau u)} V_g f\left(x-\tau,\frac{1}{b}[u-(\mu+a\tau)]\right)$$

for $x,\tau,u,\mu \in \mathbf{R}$ and $A_0=(0,b,c,d)$.

- *Covariance property II*:

$$V_g^{(A)}(M_\mu^{(A)} T_\tau f)(x,u) = \sqrt{\frac{1}{j2\pi b}} e^{-(jd/2b)\mu^2+(j/b)\tau\mu} e^{(jd/2b)u^2-(j/b)\tau u} V_g f\left(x-\tau,\frac{1}{b}(u-\mu)\right),$$

where $x,\tau,u,\mu \in \mathbf{R}$.

The proof of the first three properties are straightforward and analogous to [20]. For the proof of the generalized cross-ambiguity function, we refer the reader to Appendix A. The covariance properties follow from formulas (10)–(13).

The following theorem on inner products of WLCT's corresponds to Parseval's formula (8) for LCT.

Theorem 1 (*Orthogonality relations for WLCT*). *Let $f_1,f_2,g_1,g_2 \in L^2(\mathbf{R})$. then $V_{g_i}^{(A)} f_i \in L^2(\mathbf{R}^2)$ for $i=1,2$ and*

$$\langle V_{g_1}^{(A)} f_1, V_{g_2}^{(A)} f_2 \rangle_{L^2(\mathbf{R}^2)} = \langle f_1,f_2 \rangle \overline{\langle g_1,g_2 \rangle}. \tag{20}$$

Proof. See Appendix B.

Remark 3. Theorem 1 could be interpreted as preservation of energy by WLCT.

The following corollary follows directly from Theorem 1.

Corollary 1. *If $f,g \in L^2(\mathbf{R})$, then*

$$\|V_g^{(A)} f\|_2 = \|f\|_2 \|g\|_2.$$

In particular, if $\|g\|_2=1$, then

$$\|V_g^{(A)} f\|_2 = \|f\|_2 \quad \text{for all } f \in L^2(\mathbf{R}). \tag{21}$$

Thus WLCT is an isometry from $L^2(\mathbf{R})$ to $L^2(\mathbf{R}^2)$.

It follows from (21) that f is completely determined by $V_g^{(A)} f$. Furthermore, the condition

$$V_g^{(A)} f(x,u) = \frac{1}{\sqrt{j2\pi b}} \langle f, M_u^{(A)} T_x g \rangle = 0, \quad \forall x,u \in \mathbf{R}$$

implies $f=0$, which means that for each fixed $g \in L^2(\mathbf{R})$ the set

$$\{M_u^{(A)} T_x g : x,u \in \mathbf{R}\}$$

spans a dense subspace of $L^2(\mathbf{R})$. It is an open question how f can be recovered from $V_g^{(A)} f$. We will show that the orthogonality relations imply a remarkable inversion formula.

For the formulation of an inversion formula, we give a brief introduction on *vector-valued integrals*. If g is a function on $\mathbf{R}$ that takes values in a Banach space B, that is, $g(x) \in B$ for all $x \in \mathbf{R}$, then $f=\int_{\mathbf{R}} g(x)\,dx$ means that

$$\langle f,h \rangle = \int_{\mathbf{R}} \langle g(x),h \rangle \,dx$$

for all $h \in B^*$. If the mapping $l(h) \mapsto \int_{\mathbf{R}} \langle g(x),h \rangle\,dx$ is a bounded linear functional on B^*, then l defines a unique element $g \in B^{**}$. We work mostly with reflexive Banach spaces, $B^{**}=B$.

The most important vector-valued integrals in time–frequency analysis are superpositions of time–frequency shifts of the form

$$f(t) = \frac{1}{\sqrt{j2\pi b}} \int_{\mathbf{R}}\int_{\mathbf{R}} F(x,w) M_w^{(A)} T_x g(t)\,dx\,dw.$$

For example, if $F \in L^2(\mathbf{R}^2)$, then the bounded linear functional

$$l(h) = \frac{1}{\sqrt{j2\pi b}} \int_{\mathbf{R}}\int_{\mathbf{R}} F(x,w) \overline{\langle h, M_w^{(A)} T_x g \rangle}\,dx\,dw$$

is a bounded functional on $L^2(\mathbf{R})$. This means that l defines a unique function $f=(1/\sqrt{2\pi jb})\int_{\mathbf{R}}\int_{\mathbf{R}} F(x,w) M_w^{(A)} T_x g\,dx\,dw \in L^2(\mathbf{R})$ with norm $\|f\|_2 \le \|F\|_2 \|g\|_2$ satisfying $l(h)=\langle f,h \rangle$.

We are now ready to state a precise version of the inversion formula for the WLCT.

Corollary 2. *For $A=(a,b,c,d)$, $\det(A)=1$, suppose that $g,r \in L^2(\mathbf{R})$, and $\langle g,r \rangle \ne 0$. Then for all $f \in L^2(\mathbf{R})$*

$$f(t) = \frac{1}{\langle r,g \rangle} \int_{\mathbf{R}}\int_{\mathbf{R}} V_g^{(A)} f(x,u) \overline{K_A(t,u)} T_x r(t)\,du\,dx. \tag{22}$$

Proof. Since $V_g^{(A)} f \in L^2(\mathbf{R}^2)$, by Corollary 1, the vector-valued integral

$$\tilde{f}(t) = \frac{1}{\langle r,g \rangle} \int_{\mathbf{R}}\int_{\mathbf{R}} V_g^{(A)} f(x,u) \overline{K_A(t,u)} T_x r(t)\,du\,dx$$

is well-defined in $L^2(\mathbf{R})$. Further, using the orthogonality relations, we see that

$$\langle \tilde{f},h \rangle = \frac{1}{\langle r,g \rangle} \int_{\mathbf{R}}\int_{\mathbf{R}} V_g^{(A)} f(x,u) \overline{\frac{1}{\sqrt{j2\pi b}} \langle h, M_u^{(A)} T_x r \rangle}\,du\,dx$$
$$= \frac{1}{\langle r,g \rangle} \langle V_g^{(A)} f, V_r^{(A)} h \rangle = \langle f,h \rangle.$$

Thus $\tilde{f}=f$. The inversion formula is thus proved. □

Remark 4. The inversion formula (22) shows that f can be expressed as a continuous superposition of time–frequency shifts with the WLCT as the weight function. In this sense, (22) is similar to the inversion formula of Fourier transform. However, in the Fourier inversion formula the basic functions e^{jtu} are not in $L^2(\mathbf{R})$, whereas in Corollary 2 the basic functions $\overline{K_A(t,u)} T_x r$ are nice functions in $L^2(\mathbf{R})$.

Next we will prove a strong version of the inversion formula. Its formulation resembles an L^2-function by an approximation procedure. For the approximation it considers a nested sequence of compact sets $K_n \subseteq \mathbf{R}^2$ that exhaust $\mathbf{R}^2$. Precisely, it means that $\bigcup_{n\ge 1} K_n = \mathbf{R}^2$ and $K_n \subseteq \operatorname{int} K_{n+1}$. Then any compact set is contained in some

K_n. The cubes $[-n,n]^2$ or the balls $B(0,n)=\{x\in\mathbf{R}^2 : |x|\le n\}$ are common choices for K_n.

Theorem 2. *Fix $g,r\in L^2(\mathbf{R})$. Let $K_n\subseteq\mathbf{R}^2$ be a nested exhausting sequence of compact sets, i.e. $\bigcap K_n=\mathbf{R}^2$, and $K_n\subseteq$ int K_{n+1}. Define f_n to be*

$$f_n(t)=\frac{1}{\langle r,g\rangle}\iint_{K_n} V_g^{(A)}f(x,u)\overline{K_A(t,u)}T_x r(t)\,du\,dx.$$

Then $\lim_{n\to\infty}\|f-f_n\|_2=0$.

Proof. See Appendix C.

Here we would like to mention the work done by Feichtinger and Weisz [21,22], which generalizes the inversion formula of WFT to norm convergence for the Wiener amalgam space by considering the θ-means of the WFT of f. The authors treat it as a generalization of θ-summation of inversion FT. Under some conditions on θ, they prove that the summation of the WFT of f converges to f in the Wiener amalgam norm, hence also in the L_p norm, and pointwise almost everywhere. The WLCT have the same conclusions, since the generalized modulation operator in the present paper is also unitary.

As shown above, the WLCT is not only a linear transform, which will not suffer from cross-terms, nor distort the time–frequency structure of a signal after de-chirping, but have similar properties as WFT, a basic quadratic time–frequency representation.

Remark 5. The WLCT in presented paper

$$V_a^{(A)}f(x,u)=\exp\left(j\frac{d}{2b}u^2\right)V_g(f(t)e^{j(a/2b)t^2})\left(x,\frac{1}{b}u\right)$$

is different with the time–frequency distribution proposed by Guven and Arikan [16]. They proposed $V_g(L_A(f)(t))(x,u)$ as the time–frequency distribution. By decomposing the LCT into chirp multiplications and convolutions, $V_g(L_A(f)(t))(x,u)$ is proved to be equivalent with a single WFT. Only when its special case, the chirp modulation case, the proposed $V_g(L_A(f)(t))(x,u)$ in [16] is analogue with the WLCT, but the WLCT has more degree of freedom.

4. Applications

This section presents the analogue of the Poisson summation formula, sampling formulas and series expansions for the WLCT.

4.1. Poisson summation formula

The Poisson summation formula demonstrates that the sum of infinite samples in time domain of a signal f is equivalent to the sum of infinite samples of $\mathcal{F}(f)$ in the Fourier domain [13]. Mathematically, the Poisson sum formula can be represented as

$$\sum_{k=-\infty}^{+\infty} f(t+k\tau)=\frac{1}{\tau}\sum_{k=-\infty}^{+\infty}\mathcal{F}(f)\left(\frac{k}{\tau}\right)e^{j(k/\tau)t}, \tag{23}$$

and thus

$$\sum_{k=-\infty}^{+\infty} f(k\tau)=\frac{1}{\tau}\sum_{k=-\infty}^{+\infty}\mathcal{F}(f)\left(\frac{k}{\tau}\right), \tag{24}$$

where $\tau\in\mathbf{R}$, $\mathcal{F}(f)$ is the Fourier transform of signal f.

The following is the analogue of the Poisson summation formula under WLCT. It derives the relationship of infinite time samples and the infinite WLCT domain samples.

Theorem 3. *The Poisson summation formula of a signal $f(t)$ in the WLCT domain with parameter $A=(a,b,c,d)$, $\det(A)=1$ and $b>0$ is*

$$\sum_{k=-\infty}^{+\infty} f(t+k\tau)\overline{g(t+k\tau-x)}e^{(ja/2b)(2kt\tau+k^2\tau^2)} = \frac{1}{\tau}e^{-(jd/2b)\tau^2}\sum_{k=-\infty}^{+\infty} e^{-(jd/2b)(bk/\tau)^2}V_g^{(A)}f\left(x,\frac{bk}{\tau}\right)e^{j(k/\tau)t}.$$

In particular,

$$\sum_{k=-\infty}^{+\infty} f(k\tau)\overline{g(k\tau-x)}e^{(ja/2b)k^2\tau^2} = \frac{1}{\tau}e^{-(jd/2b)\tau^2}\sum_{k=-\infty}^{+\infty} e^{-(jd/2b)(bk/\tau)^2}V_g^{(A)}f\left(x,\frac{bk}{\tau}\right).$$

Proof. Applying the Poisson summation formula (23) to the signal $h(t)=F(t)e^{(ja/2b)t^2}$ in the Fourier domain. By (17), we obtain

$$\begin{aligned}\sqrt{jb}\sum_{k=-\infty}^{+\infty} F(t+k\tau)e^{(ja/2b)(2kt\tau+k^2\tau^2)} &= \frac{1}{\tau}e^{-(jd/2b)\tau^2}\sum_{k=-\infty}^{+\infty}\mathcal{F}(h)\left(\frac{k}{\tau}\right)e^{j(k/\tau)t}\\ &= \sqrt{jb}\frac{1}{\tau}e^{-(jd/2b)\tau^2}\sum_{k=-\infty}^{+\infty} e^{-(jd/2b)(bk/\tau)^2}L_A(F)\left(\frac{bk}{\tau}\right)e^{j(k/\tau)t}.\end{aligned}$$

Let $F(t)=fT_x\overline{g}(t)$, then the result follows. □

4.2. Sampling formulas

Sampling is the basis process for converting the continuous signals to discrete signals, and it is central in almost any domain since it provides the link between the continuous physical signals and the discrete ones.

A signal f is said to be Ω-bandlimited under a linear transform $\mathcal{T}$, if $\mathcal{T}(f)(u)=0$ for $|u|>\Omega$.

The classical Shannon sampling theorem of Ω-bandlimited signal $h(t)$ reads as

$$h(t)=\sum_{k=-\infty}^{+\infty} h(kT)\,\mathrm{sinc}\left(\frac{\Omega(t-kT)}{\pi}\right), \tag{25}$$

where $T=\pi/\Omega$ is the sampling period, and Ω/π is called as the Nyquist rate of sampling associated with the Fourier transform.

It is easy to derive the following lemma.

Lemma 2. *Whenever $V_g^{(A)}f$, $L_A(f)$ and $\mathcal{F}(f)$ are well defined, the following three statements are equivalent:*

(a) *f is Ω-bandlimited under the WLCT;*

(b) *Fix* $x \in \mathbf{R}$, $F(t)=f(t)T_x\overline{g(t)}$ *is* Ω*-bandlimited under LCT*;
(c) $h(t)=\sqrt{jb}F(t)e^{(ja/2b)t^2}=\sqrt{jb}f(t)T_x\overline{g(t)}e^{(ja/2b)t^2}$ *is* Ω/b*-bandlimited under the FT.*

In this section, the sampling theorems for bandlimited signals associated with WLCT is to be deduced. First, the sampling theorem for bandlimited signals associated with WLCT from the samples taken at the Nyquist rate is derived. Then, based on the relationship between Fourier transform and WLCT, other two new sampling formulaes using samples taken at half of the Nyquist rate from the signal with its first derivative and the signal with its generalized Hilbert transform [23]

$$H_A(f)(v)=\frac{e^{-(jd/2b)v^2}}{\pi}\int_{-\infty}^{\infty}\frac{f(t)e^{(ja/2b)t^2}}{v-t}\,dt$$

are obtained.

Theorem 4. *Suppose that f is* Ω_A*-bandlimited under WLCT with parameter* $A=(a,b,c,d)$, $\det(A)=1$ *and* $b>0$. *Then*

$$f(t)T_x\overline{g(t)}=e^{-(jd/2b)t^2}\sum_{k=-\infty}^{+\infty}e^{(ja/2b)(nT)^2}f(kT)\overline{g(kT-x)}\times\mathrm{sinc}\left(\frac{\Omega_A(t-kT)}{b\pi}\right),\tag{26}$$

where $T=\pi b/\Omega_A$ *is the sampling period, and* $\Omega_A/\pi b$ *is the Nyquist rate of the sampling associated with WLCT.*

Proof. By Lemma 2, since f is Ω_A-bandlimited under WLCT, then signal $F(t)=f(t)\overline{g(t-x)}$, $x\in\mathbf{R}$, is Ω_A-bandlimited under LCT domain.

Applying (25) to $h(t)$, we get the following equation associated with $F(t)=(1/\sqrt{jb})e^{-(ja/2b)t^2}h(t)$:

$$F(t)=e^{-(ja/2b)t^2}\sum_{k=-\infty}^{+\infty}F(kT)e^{(ja/2b)(kT)^2}\,\mathrm{sinc}\left(\frac{\Omega_A(t-kT)}{b\pi}\right).$$

Then the result follows:

$$f(t)T_x\overline{g(t)}=e^{-(jd/2b)t^2}\sum_{k=-\infty}^{+\infty}e^{(ja/2b)(nT)^2}f(kT)\overline{g(kT-x)}\times\mathrm{sinc}\left(\frac{\Omega_A(t-kT)}{b\pi}\right).\qquad\square$$

Remark 6.

(1) By exchanging the role of the signal and its LCT, the sampling theorem of time limited signal in LCT domain can be easily obtained from Theorem 1:

$$V_g^{(A)}f(x,u)=\sqrt{\frac{1}{-j2\pi b}}e^{(jd/2b)u^2}\times\sum_{k=-\infty}^{+\infty}e^{(ja/2b)(kT)^2}f(kT)\overline{g(kT-x)}e^{jukT/b}.$$

(2) The resulting equation is a little complicated, but more flexible than LCT, since there is a undetermined windowed function g.

(3) When LCT parameter reduces to $A=(\cos y,\sin y,\sin y,\cos y)$, the above results reduce to the sampling theorem of short time fractional Fourier domain defined in [3], which in the specific case $y=\pi/2$ becomes the classical sampling theorem of short time Fourier transform.

Though the sampling theorem introduced above works well, it is interesting and worthwhile to derive new sampling formulae that make the sampling rate as low as possible. In the following sections other two new sampling formulae involving the signal and its first derivative or its generalized Hilbert transform associated with WLCT are derived.

Theorem 5. *Assume signals f and g are continuously differentiable, f is* Ω_A*-bandlimited under the WLCT. Then the following sampling formula for* fT_xg *holds*

$$f(t)T_x\overline{g(t)}=e^{-(ja/2b)t^2}\sum_{k=-\infty}^{+\infty}e^{(ja/2b)(kT_1)^2}\left\{\left[1+jkT_1(t-kT_1)\frac{a}{b}\right]f(kT_1)\overline{g(kT_1-x)}+(t-kT_1)[f'(kT_1)\overline{g(kT_1-x)}+f(kT_1)\overline{g'(kT_1-x)}]\right\}\times\left[\mathrm{sinc}\left(\frac{t}{T_1}-k\right)\right]^2$$

where $T_1=2T=2\pi b/\Omega_A$ *and* $f'(kT_1)=f'(t)|_{t=kT_1}$.

Proof. Applying the classic sampling results to the signal $h(t)=\sqrt{jb}e^{(ja/2b)t^2}F(t)$ yields

$$h(t)=\sum_{k=-\infty}^{+\infty}\left\{h\left(\frac{2k\pi}{\Omega_A}\right)+(t-\frac{2k\pi}{\Omega_A})h'\left(\frac{2k\pi}{\Omega_A}\right)\right\}\times\left[\mathrm{sinc}\,\frac{1}{2}\left(\frac{\Omega_A t}{\pi}-2k\right)\right]^2\tag{27}$$

and

$$h'(t)=\sqrt{jb}e^{(ja/2b)t^2}\left[jtF(t)\frac{a}{b}+F'(t)\right].$$

Substituting the above equations in (27), we obtain

$$F(t)=e^{-(ja/2b)t^2}\sum_{k=-\infty}^{+\infty}e^{(ja/2b)(kT_1)^2}\left\{\left[1+j(kT_1)(t-kT_1)\frac{a}{b}\right]F(kT_1)+(t-kT_1)F'(kT_1)\right\}\left[\mathrm{sinc}\left(\frac{t}{T_1}-k\right)\right]^2.$$

Replacing $F(x)$ with $f(t)T_x\overline{g(t)}$, the above becomes

$$f(t)T_x\overline{g(t)}=e^{-(ja/2b)t^2}\sum_{k=-\infty}^{+\infty}e^{(ja/2b)(kT_1)^2}\left\{\left[1+j(kT_1)(t-kT_1)\frac{a}{b}\right]f(kT_1)\times\overline{g(kT_1-x)}+(t-kT_1)[f'(kT_1)\overline{g(kT_1-x)}+f(kT_1)\overline{g'(kT_1-x)}]\right\}\times\left[\mathrm{sinc}\left(\frac{t}{T_1}-k\right)\right]^2.\qquad\square$$

The above theorem provides alternative sampling formulae for reconstruction of signals from the samples of the signal and its first derivative, each at half of the Nyquist rate. There are potential applications of this sampling theorem in radar and sonar signal processing, as well as digital flight control, where the velocity and the position are recorded, and the velocity can be seen as the first derivative of position.

Theorem 6. *Assume signal f is Ω_A-bandlimited under the WLCT. Then fT_xg can be represented by the samples of f and its generalized Hilbert transform $H_A[f]$ at half Nyquist rate as follows*:

$$f(t)T_x\overline{g(t)} = e^{-(ja/2b)t^2}\sum_{k=-\infty}^{+\infty}\left\{e^{(ja/2b)(kT_2)^2}f(kT_2)\overline{g(kT_2-x)}\times\cos\left(\frac{\pi}{T_2}t-2k\pi\right)-e^{(jd/2b)(kT_2)^2}H_A[fT_x\overline{g}](kT_2)\times\sin\left(\frac{\pi}{T_2}t-2k\pi\right)\right\}\mathrm{sinc}\left(\frac{\pi}{T_2}t-2k\pi\right),$$

where $T_2 = 2T = 2\pi b/\Omega_A$.

Proof. From Lemma 2, $h(t) = \sqrt{jb}F(t)e^{(ja/2b)t^2}$ is Ω_A/b-bandlimited in the conventional Fourier domain. Applying the well-known formulae associated with the traditional Hilbert transform in the FT domain, we get

$$h(t) = \sum_{k=-\infty}^{+\infty}\left\{h\left(\frac{2k\pi b}{\Omega_A}\right)\cos\left(\frac{1}{2}\left(\frac{\Omega_A}{b}t-2k\pi\right)\right)-\tilde{h}\left(\frac{2k\pi b}{\Omega_A}\right)\sin\left(\frac{1}{2}\left(\frac{\Omega_A}{b}t-2k\pi\right)\right)\right\}\times\mathrm{sinc}\left(\frac{1}{2\pi}\left(\frac{\Omega_A}{b}t-2k\pi\right)\right)\Bigg\}, \tag{28}$$

where $\tilde{h}(t) = (1/\pi)p.v.\int_{-\infty}^{\infty}(h(v)/(t-v))\,dv$ is the traditional Hilbert transform of signal h.

From the definition of the traditional Hilbert transform we have

$$\tilde{h}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{h(v)}{t-v}dv = \frac{1}{\pi}\sqrt{jb}\int_{-\infty}^{\infty}\frac{F(v)e^{(ja/2b)v^2}}{t-v}dv = \sqrt{jb}e^{(jd/2b)t^2}H_A(F)(t).$$

Substituting $\tilde{h}(t)$ into (28) and replacing $F(t)$ with $f(t)\overline{g(t-x)}$ yield

$$f(t)\overline{g(t-x)}e^{(ja/2b)t^2} = \sum_{k=-\infty}^{+\infty}\left\{e^{(ja/2b)(kT_2)^2}f(kT_2)\overline{g(kT_2-x)}\cos\left(\frac{\pi}{T_2}t-2k\pi\right)-e^{(jd/2b)(kT_2)^2}H_A[fT_x\overline{g}](kT_2)\sin\left(\frac{\pi}{T_2}t-2k\pi\right)\right\}\times\mathrm{sinc}\left(\frac{\pi}{T_2}t-2k\pi\right). \qquad \square$$

If the parameter of LCT reduces to $A = (\cos y, \sin y, \sin y, \cos y)$, then the sampling theorem related to short time fractional Fourier transform domain can be obtained.

4.3. Series expansions

This section aims at the series expansion of the WLCT. Recall Shannon's interpolation theorem for the traditional Fourier transform, which expresses a bandlimited function in terms of its time domain samples. For the timelimited functions (A signal f is said to be T-timelimited if $f(t) = 0$ for $|t| > T$.), the corresponding dual theorem says that if f is $T/2$-timelimited, then the FT of h can be expressed as

$$\mathcal{F}(h)(w) = \sum_n \mathcal{F}(h)(nW)\mathrm{sinc}\left(\frac{w}{W}-n\right), \tag{29}$$

where $W = 2\pi/T$.

To derive the sampling theorem for WLCT, we should first get the sampling theorem for LCT. If $h(t)$ is $T/2$-timelimited, so is $F(t) = \sqrt{1/jb}e^{-(ja/2b)t^2}h(t)$. By (17) in Lemma 1, (29) becomes

$$L_A(F)(u) = e^{j(d/2b)u^2}\mathcal{F}(h)\left(\frac{u}{b}\right) = e^{j(d/2b)u^2}\sum_n \mathcal{F}(h)(nW)\mathrm{sinc}\left(\frac{u}{bW}-n\right).$$

To eliminate $\mathcal{F}(h)(nW)$, we evaluate the expression (1) with respect to signal $h(t)$ at $u = nbW$ (m is an arbitrary integer). Upon this evaluation, we obtain the relation for $\mathcal{F}(h)(nW)$

$$\mathcal{F}(h)(nW) = L_{A_0}(F)(nbW)e^{-j(d/2b)(nbW)^2}.$$

Then we get

$$L_A(F)(u) = e^{j(d/2b)u^2}\sum_n L_{A_0}(F)(nbW)e^{-j(d/2b)(nbW)^2}\mathrm{sinc}\left(\frac{u}{bW}-n\right) \tag{30}$$

as the interpolation theorem of LCT for the domain limited functions. This relation implies that a function limited at a LCT domain can be represented by its samples at any other LCT domain.

If $A_0 = A$, by applying the inverse LCT $L_{A^{-1}}$ to the both sides of (30) with respect to u, we immediately get the equivalent form of the classical Fourier series for LCT:

$$F(t) = \sqrt{\frac{1}{2\pi jb}}bWe^{-j(a/2b)u^2}\sum_n L_A(F)(nbW)e^{-j(d/2b)(nbW)^2+jnWt}.$$

Replacing $F(t)$ with $f(t)g(t-x)$, we get the series for WLCT:

$$f(t) = \sqrt{\frac{1}{2\pi jb}}\frac{bW}{g(t-x)}\sum_n V_g^{(A)}(f)(x,nbW)e^{-j(a/2b)u^2-j(d/2b)(nbW)^2+jnWt}.$$

4.4. Potential applications and future work

For applications of the WLCT, we have

$$V_g^{(A)}f(x,u) = \int_{\mathbf{R}} K_A(t,u)\overline{T_xg(t)}f(t)\,dt,$$

where $K_A(t,u) = (1/\sqrt{j2\pi b})e^{j((a/2b)t^2-(1/b)tu+(d/2b)u^2}$ and $A = (a,b,c,d)$ is the parameter matrix. There are two kinds of free parameters: matrix A and an unrestricted windowed function $g(t) \in S'(\mathbf{R})$. Accordingly potential extension may be the WLCT with specifically chosen matrix A or windows $g(t)$.

Consider that $g(t)$ is a suitably chosen lowpass unit-energy window function centered around the origin, which suppresses $f(t)$ outside an interval centered at x. Take the parameter matrix A to be $A(x)$, a function of the position x at which the window is centered. To this end, the WLCT will also be referred to as the time- or space-dependent LCT. The WLCT at a given value of x is essentially LCT of the function $f(t)$ over an interval around x. The order of $A(x)$ may be different for each value of x.

The WLCT can be expressed in terms of FRFT [24]. It is useful as we decide the choice for the parameters in WLCT. Owing to

$$V_g^{(A)}f(x,u) = e^{-j\pi q u^2}\sqrt{\frac{1}{M}}F_\alpha(fT_x\overline{g})\left(\frac{u}{M}\right), \tag{31}$$

where $F_\alpha(f)$ denotes the FRFT of f

$$F_\alpha(f) := \sqrt{\frac{1-j\cot\alpha}{2\pi}}\int_{-\infty}^{\infty} e^{j(((t^2+u^2)/2)\cot\alpha - jtu/\sin\alpha)}f(t)\,dt$$

and

$$T=\begin{pmatrix} A & B\\ C & D\end{pmatrix}=\begin{pmatrix} 1 & 0\\ -q & 1\end{pmatrix}\begin{pmatrix} M & 0\\ 0 & \frac{1}{M}\end{pmatrix}\begin{pmatrix} \cos\varphi & \sin\varphi\\ -\sin\varphi & \cos\varphi\end{pmatrix}$$

with $\varphi = \dfrac{a\pi}{2}$.

We can determine $A=(a,b,c,d)$ by choosing α, M, q by order in (31). We first rotate angle α, scale M and then shear q. As proposed in [25], the optimal fractional order α_0 could be estimated by determining the orientation of the signal in the time–frequency plane. For example, we search for the peaks in the FRFT magnitudes or compute fractional moments at various fractional orders. In the comparison of WLCT with WFRFT $F_a(fT_x\overline{g})(u)$ given in [20], WLCT takes its advantage in choosing a suitable scale M for better performance [26].

The WLCT has many potential applications, such as noise removal, multiplexing, correlation, detection and pattern recognition. We start by first considering noise removal. If different regions of a signal or image are corrupted with noise or distortions of different characteristics, it will clearly be advantageous to adaptively choose the order A for those different regions, rather than working with a single order for the whole signal or image. Likewise, when multiplexing signals, it would be advantageous to be able to adapt the order to the local characteristics of the class of signals to be multiplexed. In correlation and detection applications, it may be desirable to have different degrees of shift invariance for different parts of the signal or image. For instance, we may be looking for objects characterized by a number of defining characteristics such that some of those are expected to be always in the same position, while others may appear in different positions. Being able to spatially adjust the degree of shift invariance can increase our discrimination capability in such circumstances.

The possibilities and potential room for improvement of the adaptive approach are far from being exhausted. First, it is worthy to investigating more on how to systematically determine or estimate the optimal partitioning into regions and the optimal transform orders for a given class of objects. Another issue is to develop the corresponding frame theory of WLCT by taking advantage of Gabor frame theory and in the light of [24,16,27]. Developments along these lines is expected to offer further improvements. By [28] realizing that many calculations in present paper would become more natural if we used the machinery of Heisenberg operators to formulate the results. It would be easy to convert these results in terms of the modulation and time-shift operators conventionally used in time–frequency analysis. In a recent paper [29] we prove the Paley-Wiener theorems and the uncertainty principles for the (inverse) windowed linear canonical transform. They are new in literature and has some consequences that are now under investigation.

5. Conclusion

In this paper, we introduce a new kind of integral transform, which we name as windowed linear canonical transform (WLCT). WLCT is not only a linear transform, which will not suffer from cross-terms and distort the time–frequency structure of a signal after de-chirping. It has similar properties as windowed Fourier transform, including covariance property, orthogonality property and inversion formulas, provide local information and high resolution. Finally, we generalize the analogues of Poisson summation theorem, sampling formulas and series expansions for the WLCT.

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Appendix A. Proof of the generalized cross-ambiguity function

Proof. The symmetric form (19) can be obtained by simple change of variable on the definition. The term

$$\int_{\mathbf{R}} f\left(t+\frac{x}{2}\right)g\left(t-\frac{x}{2}\right)e^{-(j/b)(u-(a/2)x)}e^{(ja/2b)t^2}\,dt$$

could be treated as a *generalized cross-ambiguity function*. It is believed to be useful in radar and optics, referring to [1].

The uniform continuity of $V_g^{(A)}f$ on $\mathbf{R}^2$ is shown as follows. For any $x,u\in\mathbf{R}$, $A=(a,b,c,d)$ real, h and k sufficient small

$$\begin{aligned}
&\left|V_g^{(A)}f(x+k,u+h)-V_g^{(A)}f(x,u)\right|\\
&\quad=\left|\int_{\mathbf{R}} f(t)g(t-x-k)e^{j((a/2b)t^2-(1/b)t(u+h)+(d/2b)(u+h)^2)}\,dt\right.\\
&\qquad\left.-\int_{\mathbf{R}} f(t)g(t-x)e^{j((a/2b)t^2-(1/b)tu+(d/2b)u^2)}\,dt\right|\\
&\quad\le\left|\int_{\mathbf{R}} f(t)g(t-x-k)e^{j((a/2b)t^2-(1/b)t(u+h)+(d/2b)(u+h)^2)}\,dt\right.\\
&\qquad\left.-\int_{\mathbf{R}} f(t)g(t-x)e^{j((a/2b)t^2-(1/b)t(u+h)+(d/2b)(u+h)^2)}\,dt\right|\\
&\qquad+\left|\int_{\mathbf{R}} f(t)g(t-x)e^{j((a/2b)t^2-(1/b)t(u+h)+(d/2b)(u+h)^2)}\,dt\right.\\
&\qquad\left.-\int_{\mathbf{R}} f(t)g(t-x)e^{j((a/2b)t^2-(1/b)tu+(d/2b)u^2)}\,dt\right|\\
&\quad\le\int_{\mathbf{R}}|f(t)|\,|g(t-x-k)-g(t-x)|\,dt\\
&\qquad+\int_{\mathbf{R}}|f(t)|\,|g(t-x)|\,\left|e^{-\frac{j}{b}(t-ud)h}-1\right|\,dt.
\end{aligned}\tag{A.1}$$

For the first integral, applying Hölder's Inequality, we have

$$\int_{\mathbf{R}} |f(t)|\,|g(t-x-k)-g(t-x)|\,dt \le \|f\|_2 \|g(\cdot-x-k)-g(\cdot-x)\|_2.$$

Since C_0^∞ is dense in L^2, for any $\varepsilon > 0$ and any $g \in L^2$, there exists $h \in C_0^\infty$, such that $\|g-h\|_2 < \varepsilon/3$. Thus

$$\begin{aligned}&\|g(\cdot-x-k)-g(\cdot-x)\|_2\\ &\quad\le \|g(\cdot-x-k)-h(\cdot-x-k)\|_2\\ &\quad+\|h(\cdot-x-k)-h(\cdot-x)\|_2+\|h(\cdot-x)-g(\cdot-x)\|_2\\ &\quad< \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon \quad \text{as } k\to 0.\end{aligned}$$

Therefore, for $f \in L^2$, the first integral (A.2) of (A.1) approaches to zero, as k tends to zero.

For the second integral in (A.1), for any $t,u \in \mathbf{R}$, we have

$$\lim_{h\to 0}|e^{-(j/b)(t-ud)h}-1| = 0$$

and for $f,g \in L^2(\mathbf{R})$

$$|f(t)|\,|g(t-x)|\,|e^{-(j/b)(t-ud)h}-1| \le 2|f(t)|\,|g(t-x)|.$$

Therefore applying the dominated convergence Theorem, the second integral of (A.1) tends to zero, as h tends to zero.

Hence, for sufficiently small h and k, $|V_g^{(A)}f(x+k,u+h) -V_g^{(A)}f(x,u)|$ tends to zero. □

Appendix B. Proof of Theorem 1

Proof. We first assume that the windows g_i are in $L^1 \bigcap L^\infty(\mathbf{R}) \subseteq L^2(\mathbf{R})$. So that $f_i T_x \overline{g_i} \in L^2(\mathbf{R})$ for all $x \in \mathbf{R}$. Therefore Parseval's formula of LCT (8) applies to the u-integral and yields

$$\begin{aligned}&\int_{\mathbf{R}}\int_{\mathbf{R}} V_{g_1}^{(A)}f_1(x,u)\overline{V_{g_2}^{(A)}f_2(x,u)}\,du\,dx\\ &\quad=\int_{\mathbf{R}}\left(\int_{\mathbf{R}} L_A(f_1T_xg_1)(u)\overline{L_A(f_2T_xg_2)(u)}\,du\right)dx\\ &\quad=\int_{\mathbf{R}}\left(\int_{\mathbf{R}} f_1(t)\overline{f_2(t)}\,\overline{g_1(t-x)}g_2(t-x)e^{-\mathrm{Im}(a/b)t^2}\,dt\right)dx.\end{aligned}$$

Here $f_1\overline{f_2} \in L^1(\mathbf{R},dt)$ and $g_1\overline{g_2} \in L^1(\mathbf{R},dx)$, therefore Fubini's theorem [19] allows us to interchange the order of integration. We continue as follows:

$$\begin{aligned}\langle V_{g_1}^{(A)}f_1, V_{g_2}^{(A)}f_2\rangle_{L^2(\mathbf{R}^2)} &= \int_{\mathbf{R}} f_1(t)\overline{f_2(t)}e^{-\mathrm{Im}(a/b)t^2}\left(\int_{\mathbf{R}}\overline{g_1(t-x)}g_2(t-x)\,dx\right)dt\\ &= \langle f_1,f_2\rangle_\sigma \overline{\langle g_1,g_2\rangle}.\end{aligned}$$

The extension to general $g_j \in L^2(\mathbf{R})$ is done by a standard density argument. With $g_1 \in L^1 \bigcap L^\infty(\mathbf{R})$ fixed, the mapping $g_2 \to \langle V_{g_1}^{(A)}f_1, V_{g_2}^{(A)}f_2\rangle_{L^2(\mathbf{R}^2)}$ is a linear functional that coincides with $\langle f_1,f_2\rangle\overline{\langle g_2,g_1\rangle}$ on the dense subspace $L^1 \bigcap L^\infty$. It is therefore bounded and extends to all $g_2 \in L^2(\mathbf{R})$. In the same way, for arbitrary f_1,f_2 and $g_2 \in L^2(\mathbf{R})$, the conjugate linear functional $g_1 \to \langle V_{g_1}^{(A)}f_1, V_{g_2}^{(A)}f_2\rangle_{L^2(\mathbf{R}^2)}$ equals to $\langle f_1,f_2\rangle_\sigma\overline{\langle g_1,g_2\rangle}$ on $L^1 \bigcap L^\infty$ and extends to all L^2.

The orthogonality relations are therefore established for all $f_j,g_j \in L^2(\mathbf{R})$. □

Appendix C. Proof of Theorem 2

Proof. Using the Cauchy–Schwartz inequality and Corollary 1, we estimate for $h \in L^2(\mathbf{R})$ that

$$\begin{aligned}|\langle f_n,h\rangle|_\sigma &= \frac{1}{|\langle r,g\rangle|}\left|\iint_{K_n} V_g^{(A)}f(x,u)\overline{V_r^{(A)}h(x,u)}\,du\,dx\right|\\ &\le \frac{1}{|\langle r,g\rangle|}\|V_g^{(A)}f\|_2\|V_r^{(A)}h\|_2\\ &= \frac{1}{|\langle r,g\rangle|}\|f\|_{\sigma,2}\|g\|_2\|r\|_2\|h\|_{\sigma,2}.\end{aligned}$$

Therefore for each n, f_n is a well-defined element of $L^2(\mathbf{R})$, and furthermore, by Corollary 1, $\|f_n\|_2 \le |\langle r,g\rangle|^{-1}\|g\|_2 \|r\|_2\|f\|_{\sigma,2}$.

Thus

$$\begin{aligned}|\langle f-f_n,h\rangle| &= \frac{1}{|\langle r,g\rangle|}\left|\left(\int_{\mathbf{R}}\int_{\mathbf{R}}-\iint_{K_n}\right)V_g^{(A)}f(x,u)\overline{V_r^{(A)}h(x,u)}\,dx\,du\right|\\ &= \frac{1}{|\langle r,g\rangle|}\left|\iint_{K_n^C} V_g^{(A)}f(x,u)\overline{V_r^{(A)}h(x,u)}\,dx\,du\right|\\ &\le \frac{1}{|\langle r,g\rangle|}\|V_r^{(A)}h\|_2\left(\iint_{K_n^C}|V_g^{(A)}f(x,u)|^2\,dx\,du\right)^{1/2}\\ &= \frac{1}{|\langle r,g\rangle|}\|r\|_2\|h\|_{\sigma,2}\left(\iint_{K_n^C}|V_g^{(A)}f(x,u)|^2\,dx\,du\right)^{1/2}.\end{aligned}$$

Since this is true for all $h \in L^2(\mathbf{R},)$, we therefore have

$$\begin{aligned}\|f-f_n\|_2 &= \sup_{\|h\|_{\sigma,2}\le 1}|\langle f-f_n,h\rangle_\sigma| \le \frac{1}{|\langle r,g\rangle|}\|r\|_2\\ &\quad\times\left(\iint_{K_n^C}|V_g^{(A)}f(x,u)|^2\,dx\,du\right)^{1/2}.\end{aligned}$$

Since $V_g^{(A)}f \in L^2(\mathbf{R}^2)$, and K_n is exhausting, the right-hand side becomes arbitrarily small as n increases. □

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