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Paley-Wiener Theorems and Uncertainty Principles for the Windowed Linear Canonical Transform†

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In a recent paper the authors have introduced the windowed linear canonical transform and shown its good properties together with some applications such as Poisson summation formulas, sampling interpolation and series expansion. In this paper we prove the Paley-Wiener theorems and the uncertainty principles for the (inverse) windowed linear canonical transform. They are new in literature and has some consequences that are now under investigation. Copyright © 2009 John Wiley & Sons, Ltd.

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1. Introduction

The linear canonical transform (LCT) is a generalization of the traditional Fourier transform (FT), which has found many applications to signal processing and optics [1, 2, 3]. It was first introduced in 1970s [4, 5] and is a four-parameter class linear integral transform. The LCT is also known as the ABCD transform [6], the affine Fourier transform [7], and the generalized Fresnel transform [8]. Many operations, such as the FT, the fractional Fourier transform (FRFT) [1, 3], the Fresnel transform (FRT) [9], the Lorentz transform [7] and scaling operations are its special cases. With more degrees of freedom compared to the FT and the FRFT, the LCT is more flexible but with similar computation cost as the conventional FT does [10]. Due to its advantages discussed above, the LCT, as a powerful tool, has found many applications in filter design, signal synthesis, optics, radar analysis and pattern recognition, etc [1, 2]. The above mentioned applications demonstrate the great potential of LCT in signal processing. For example, filtering in the LCT domain, proposed as in [11], can achieve better performance than in the FRFT domain because of more degrees of freedom. Especially when multi-component chirp signals interfere with the desired signal, only one filter is used in the LCT domain usually, but several filters are required in the FRFT domain [12].

However, the LCT cannot reveal the local LCT-frequency contents due to its global kernel. The windowed Fourier transform (WFT) [1, 13], with a local window function, handles this kind of problem well. The absence of undesirable cross terms and computational simplicity result in the wide-spread use of the WFT in practice, and most other time-frequency representations can be expressed in terms of it. Nevertheless, the WFT often performs unsatisfactorily for its low resolution. The Wigner-Ville distribution maintains high localization, however it desperately suffers from spurious values in the presence of multi-components or noise [13].

Recently, some studies have attempted to attain high localization properties using fractional Fourier transform and linear canonical transform. For example, L. Stankovic et al. [14] introduces the windowed fractional Fourier transform (WFRFT) which turnes the time-frequency representations with an angle α, and then applies the WFT to it. It costs the same number of computations as realizations in the time or the frequency domain. A. Bultheel and H. Martinez-Sulbaran [15] uses a different approach, which generalizes the WFT by substituting the Fourier transform kernel with the linear canonical transform kernel in

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the WFT definition. It investigates some straightforward properties and two applications, the estimations of the time-of-arrival and pulsewidth of chirp signals, and the windowed fractional Fourier transform filtering.

Improving WFT by using the LCT is first proposed by H. E. Guven and O. Arikan [16]. They started from basic shearing operations, generalized the group of the LCT. Signals with smaller time-frequency support are represented with higher resolution, and the minimum time-bandwidth product form of the signal be achieved in several ways, either rotating the support of chirp-like signals at a suitable angle, or simply shearing it. Both operations are particular form of the LCT. Here they present a generalized method for improving WFT by using LCT.

In [17] we introduced the WLCT by substituting the FT kernel with the linear canonical transform kernel in the WFT definition. It offers local contents, enjoys high resolution, and eliminates cross terms. It is believed to be suitable for analyzing chirp signals. We derived some useful properties of the WLCT such as covariance property, orthogonality property and inversion formulas. The analogue of the Poisson summation formula, series expansions and sampling formulas for the WLCT are also given. It is well known that the classical Paley-Wiener theorem has a close relation with the sampling interpolations. In the Shannon sampling and interpolation, the sampling scale is determined by the constant Ω (see Section 3) appearing in the exponential part of the estimate for the holomorphic function under study [18]. Owing to the analogous complex structure in real variable induced by the WLCT, the Paley-Wiener theorem for WLCT proved in this note offers the same applications to topics in real variable.

As another properties of the WLCT, the uncertainty principles for the WLCT are considered. The uncertainty principle means that a function can not be concentrated on small sets in the time-frequency plane. There are various versions of uncertainty principles according to various notions and time-frequency representations [19, 13]. Recently some versions for the linear canonical transform (LCT) are appeared [20, 21, 22], and we generalized the analogue of Lieb's uncertainty principle [23] for the WLCT, which takes the LCT version as one of its special cases.

The paper is organized as follows. In Section 2 we give the knowledge of the fundamental operations and WLCT. Section 3 presents the analogues of Paley-Wiener theorem for WLCT. Our proof is guided by the one for the classical Paley-Wiener Theorem cited in [24]. The analogues of the uncertainty principle for WLCT are studied in Section 4. Finally we conclude in Section 5.

2. Preliminary

2.1. The Fundamental operations

Translation and *modulation* operator plays an important role in time-frequency analysis, they are respectively defined by

$$T_x f(t) := f(t-x),$$

and

$$M_w f(t) := e^{2\pi j w t} f(t), \tag{1}$$

for $x, w \in \mathbf{R}$ [13]. Here T_x is a *time shift*, M_w is a modulation by w, which is called *frequency shift*. Obviously, they are both unitary operators. Operators of the form $T_x M_w$ or $M_w T_x$ are called *time-frequency shifts*.

The generalized modulation operator $M_\mu^{(A)}$, characterized by a complex matrix $A = (a, b, c, d)$, $det(A) = 1$, with respect to variable t and effects on f is defined by

$$M_\mu^{(A)} f(t) := e^{-j\left(\frac{a}{2b}t^2 - \frac{1}{b}t\mu + \frac{d}{2b}\mu^2\right)} f(t).$$

It was first introduced in [17], and is motivated by the kernel of the WLCT (see (2)). The authors in [17] explored some of its good properties which seems useful and convenient. Notice that when a, b, c and d are real, it is a unitary operator, and when $A = (0, 1, -1, 0)$ it reduces to the classical modulation operator in (1), that is $M_\mu^{(A)} f(t) = M_{\frac{\mu}{2\pi}} f(t) = e^{j\mu t} f(t)$. Operators of the form $T_x M_w^{(A)}$ or $M_w^{(A)} T_x$ are called *generalized time-frequency shifts*. Now we are ready to introduce the WLCT.

For typographical convenience, we shall denote matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ as (a, b, c, d) in the text, but all operations have to be understood in the usual matrix sense.

2.2. WLCT

Linear canonical transform is a good tool to analyze signal in the frequency domain, but it can do nothing if we want to find out the information of the instantaneous frequency during a time interval. With a suitable window g, performing the WLCT is a good choose to get the frequency contents of a function f.

The windowed linear canonical transform (WLCT) [17] of a function $f \in L^1(\mathbf{R})$ with respect to $g \in L^\infty(\mathbf{R})$ is defined by

$$V_g^{(A)} f(x, u) := \int_{\mathbf{R}} f(t) \overline{T_x g(t)} K_A(t, u) dt, \tag{2}$$

$$K_A(t, u) := \frac{1}{\sqrt{j2\pi b}} e^{j\left(\frac{a}{2b}t^2 - \frac{1}{b}tu + \frac{d}{2b}u^2\right)}, \qquad b > 0, \tag{3}$$

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where $x, u \in \mathbf{R}$ and $A = (a, b, c, d)$ with $\det(A) = 1$ and $\overline{(\cdot)}$ is the complex conjugates of $(\cdot)$.

Clearly, if $f \in L^2(\mathbf{R})$ and window function $g \in L^2(\mathbf{R})$, the WLCT is well defined. Similarly, $f \in L^q(\mathbf{R})$ and $g \in L^p(\mathbf{R})$, $1 \leq p, q < \infty$ and $1/p + 1/q = 1$, then by Hölder's inequality [25] $fT_x\overline{g} \in L^1(\mathbf{R})$ and the WLCT is defined pointwise.

Remark 1 *Note that when $A = (0, 1, -1, 0)$ it reduce to the windowed Fourier transform (WFT) of a function f with respect to g, we denote by*

$$V_g f(x, w) := \frac{1}{\sqrt{2\pi j}} \int_{\mathbf{R}} f(t)\overline{T_x g(t)} e^{-jtu} dt, \qquad \text{for } x, w \in \mathbf{R}.$$

Remark 2 *When the entries of A are complex, the function space $fT_x\overline{g} \in L^1(\mathbf{R})$ would be extended to the larger class*

$$fT_x\overline{g}\, e^{-\rho t^2} \in L^1(\mathbf{R}), \qquad \text{with } \rho = Im\left(\frac{a}{2b}\right).$$

Nervelessly, for typographical convenience, unless otherwise stated, we shall restrict ourselves to the case of real parameters, in which case the corresponding LCT is a unitary operator in $L^2(\mathbf{R})$.

Remark 3 *Let $f \otimes g$ be the (tensor) product $f \otimes g(x, t) = f(x)g(t)$, and $\mathcal{A}$ be the asymmetric coordinate transform $\mathcal{A}F(x, t) = F(t, t - x)$, and $L_{A,2}$ be the partial linear canonical transform*

$$L_{A,2}(F)(x, u) := \int_{\mathbf{R}} K_A(t, u) F(x, t) dt$$

on a function F on $\mathbf{R}^2$. By this notation, equation (2) can be reformulated in terms of a factorization of the WLCT.

$$V_g^{(A)} f(x, u) = L_{A,2}\left(\mathcal{A}(f \otimes \overline{g})(x, \cdot)\right)(u) \quad \text{if } f, g \in L^2(\mathbf{R}). \tag{4}$$

By (4), the domain of the WLCT can be extended even further. Note that both operators $\mathcal{A}$ and $L_{A,2}$ are isomorphisms on the Schwartz space $S'(\mathbf{R}^2)$. If $f, g \in S'(\mathbf{R})$, then $f \otimes \overline{g} \in S'(\mathbf{R}^2)$, and consequently $V_g^{(A)} f \in S'(\mathbf{R}^2)$ as well. Thus $V_g^{(A)} f$ is a well-defined tempered distribution, whenever $f, g \in S'(\mathbf{R})$.

The following theorem on inner products of the WLCT's corresponds to Parseval's formula for the LCT, and will be used frequently [17].

Theorem 1 *[17] (Orthogonality relations for the WLCT)*
Let $f_1, f_2, g_1, g_2 \in L^2(\mathbf{R})$, then $V_{g_i}^{(A)} f_i \in L^2(\mathbf{R}^2)$ for $i = 1, 2$, and

$$\left\langle V_{g_1}^{(A)} f_1, V_{g_2}^{(A)} f_2 \right\rangle_{L^2(\mathbf{R}^2)} = \langle f_1, f_2 \rangle \overline{\langle g_1, g_2 \rangle}. \tag{5}$$

Theorem 1 could be interpreted as a statement that WLCT preserve the energy of a signal. The following corollary follows directly from Theorem 1.

Corollary 1 *[17] If $f, g \in L^2(\mathbf{R})$, then*

$$\left\| V_g^{(A)} f \right\|_2 = \|f\|_2 \|g\|_2 .$$

In particular, if $\|g\|_2 = 1$, then

$$\left\| V_g^{(A)} f \right\|_2 = \|f\|_2 \quad \text{for all } f \in L^2(\mathbf{R}). \tag{6}$$

Thus, in this case the WLCT is an isometry from $L^2(\mathbf{R})$ to $L^2(\mathbf{R}^2)$.

The following corollary states a precise version of the inversion formula for the WLCT.

Corollary 2 *[17] For $A = (a, b, c, d)$, $det(A) = 1$, suppose that $g, r \in L^2(\mathbf{R})$, and $\langle g, r \rangle \neq 0$. Then for all $f \in L^2(\mathbf{R})$*

$$f(t) = \frac{1}{\langle r, g \rangle} \int_{\mathbf{R}} \int_{\mathbf{R}} V_g^{(A)} f(x, u) \overline{K_A(t, u)} T_x r(t) du dx. \tag{7}$$

Remark 4 *The inversion formula (7) shows that f can be expressed as a continuous superposition of time-frequency shifts with the WLCT as weight function. In this sense, (7) is similar to the inversion formula for the Fourier transform, that is, $f(x) = \int_{\mathbf{R}} \hat{f}(w) e^{2\pi j x w} dw$. However, in Fourier inversion the elementary functions $e^{2\pi j x w}$ are not in $L^2(\mathbf{R})$, whereas in Corollary 2 the elementary functions $\overline{K_A(t, u)} T_x r$ are particularly nice functions in $L^2(\mathbf{R})$.*

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Remark 5 *The inversion formula (7) shows that f can be expressed as a continuous superposition of time-frequency shifts with the WLCT as weight function. In this sense, (7) is similar to the inversion formula for the Fourier transform, that is, $f(x) = \int_{\mathbb{R}} \hat{f}(w) e^{2\pi j x w} dw$. However, in Fourier inversion the elementary functions $e^{2\pi j x w}$ are not in $L^2(\mathbf{R})$, whereas in Corollary 2 the elementary functions $\overline{K_A(t,u)} T_x r$ are particularly nice functions in $L^2(\mathbf{R})$.*

A strong version of the inversion formula is then given. Its formulation resembles an L^2-function by an approximation procedure. For the approximation it considers a nested sequence of compact sets $K_n \subseteq \mathbf{R}^2$ that exhaust $\mathbf{R}^2$. This means that $\cup_{n\geq 1} K_n = \mathbf{R}^2$ and $K_n \subseteq \text{int} K_{n+1}$. Then any compact set is contained in some K_n. The cubes $[-n, n]^2$ or the balls $B(0, n) = \{x \in \mathbf{R}^2 : | x | \leq n\}$ are common choices for K_n.

Theorem 2 *[17] Fix $g, r \in L^2(\mathbf{R})$, and let $K_n \subseteq \mathbf{R}^2$ for $n \geq 1$ be a nested exhausting sequence of compact sets, i.e. $\bigcap_{n\geq 1} K_n = \mathbf{R}^2$, and $K_n \subseteq K_{n+1}$. Define f_n to be*

$$f_n(t) = \frac{1}{\langle r, g\rangle} \int \int_{K_n} V_g^{(A)} f(x,u) \overline{K_A(t,u)} T_x r(t) du dx.$$

Then $\lim_{n\to\infty} \|f - f_n\|_2 = 0$.

Remark 6 *Hans G.Feichtinger and Ferenc Weisz in [26] [27] generalized the inversion formula of WFT to norm convergence for the Wiener amalgam space by considering the θ-means of the WFT of f. In fact, the authors treat it as a generalization of θ-summation of inversion FT. Under some conditions on θ, they prove that the summation of the WFT of f converges to f in Wiener amalgam norm, hence also in the L_p sense for L_p functions, and pointwise almost everywhere. The WLCT may have the same conclusions, since the generalized modulation operator in this paper is also unitary.*

Remark 7 *The WLCT in presented paper*

$$V_g^{(A)} f(x,u) = exp\left(j\frac{d}{2b}u^2\right) V_g\left(f(t)e^{j(a/2b)t^2}\right)\left(x, \frac{1}{b}u\right)$$

is different with the time-frequency distribution proposed by Guven and Arikan [16]. They proposed $V_g(L_A(f)(t))(x,u)$ as the time-frequency distribution, where the Linear canonical transform $L_A(f)$ of f is given by

$$L_A(f)(u) := \begin{cases} \int_{-\infty}^{\infty} K_A(t,u) f(t) dt, & b \neq 0; \\ \sqrt{d} e^{j\frac{cd}{2}u^2} f(du), & b = 0. \end{cases} \tag{8}$$

with parameter $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and a, b, c, d are real or complex numbers satisfying $ad - bc = 1$, i.e., det(A)=1 [4, 2]. The kernel function K_A is given in (3).

By decomposing the LCT into chirp multiplications and convolutions, $V_g(L_A(f)(t))(x,u)$ is proved to be equivalent with a single WFT. Only when its special case, the chirp modulation case, the proposed $V_g(L_A(f)(t))(x,u)$ in [16] is analogue with WLCT, but WLCT has more degree of freedom.

3. The Paley-Wiener theorem for WLCT

The Paley-Wiener Theorem states that a sufficient and necessary condition for a square-integrable function f, being extendable to become an entire function in the complex plane with the exponential type bounds

$$|f(z)| \leq Ce^{\Omega|z|}$$

is that supp $\hat{f} \subset [-\Omega, \Omega]$. People interested on the higher dimensional extensions of the Paley-Wiener Theorem, for example, in [28, 29, 30, 31]. In [28] a corresponding extension is obtained by imbedding of $\mathbf{R}^n$ into the real-Clifford algebra. The imbedding provides $\mathbf{R}^n$ with a global complex structure in analogy with the imbedding of $\mathbf{R}$ into the complex plane. In [29] a set of results in several complex available is obtained using Clifford algebra. In [30] results of the non-commutative and non-associative octonions are studied. In [31] the authors investigate the case in Co-dimension-p.

Due to the fundamental role of the theorem in harmonic analysis and complex analysis [25, 32, 18], people have been seeking for generalizations of the results with the other integral transforms [33]. While in this section, we give the analogue of theorem in the WLCT sense. In below the constant C in the proof may vary from line to line.

Theorem 3 *Suppose that $f, r \in L^2(\mathbf{R})$, $g \in L^\infty \bigcap L^2(\mathbf{R})$, $\langle g, r\rangle \neq 0$, f, r are entire functions and Ω is a positive real number. Then the following two conditions are equivalent:*

(i) There exists a constant $C > 0$ such that

$$|f(z)| \leq Ce^{\frac{\Omega}{b}|y|+\frac{a}{b}xy}, \quad \forall z = x + jy \in \mathbf{C}; \tag{9}$$

(ii) supp $V_g^{(A)}(f|_{\mathrm{R}})(s,\cdot) \subset [-\Omega,\Omega], \quad \forall s \in \mathbf{R}$.

Moreover, if one of the above conditions holds, then we have

$$f(z) = \frac{1}{\langle r, g\rangle}\int_{\mathrm{R}}\int_{[-\Omega,\Omega]} V_g^{(A)}(f|_{\mathrm{R}})(s,u)\overline{K_A(t,u)}T_s r(z)\,du\,ds \tag{10}$$

for all $z \in \mathbf{C}$, where $A = (a, b, c, d)$.

Proof. (ii) $\Longrightarrow$ (i). Assume that (ii) holds. Let

$$F(z) = \frac{1}{\langle r, g\rangle}\int_{\mathrm{R}}\int_{[-\Omega,\Omega]} V_g^{(A)}(f|_{\mathrm{R}})(s,u)\overline{K_A(t,u)}T_s r(z)\,du\,ds,$$

where $z = x + jy \in \mathbf{C}$. Since $\overline{K_A(t,u)}T_s r(z) = \frac{1}{\sqrt{j2\pi b}}\, e^{-\frac{jd}{2b}u^2+\frac{j}{b}uz-\frac{ja}{2b}z^2}T_s r(z)$, $z = x + jy$, we have $\left|M_u^{(A)}T_s r(z)\right| = Ce^{-\frac{1}{b}uy+\frac{a}{b}xy}\,|T_s r(z)|$. Therefore

$$|F(z)| \leq Ce^{\frac{\Omega}{b}|y|+\frac{a}{b}xy}\int_{\mathrm{R}}\left|\int_{[-\Omega,\Omega]} V_g^{(A)}(f|_{\mathrm{R}})(s,u)\,du\right| |T_s r(z)|\,ds. \tag{11}$$

The Cauchy-Schwarz inequality then implies

$$\left|\int_{[-\Omega,\Omega]} V_g^{(A)}(f|_{\mathrm{R}})(s,u)\,du\right| \leq C\|V_g^{(A)}(f|_{\mathrm{R}})(s,\cdot)\|_2$$

and the estimate (11) becomes

$$|F(z)| \leq Ce^{\frac{\Omega}{b}|y|+\frac{a}{b}xy}\int_{\mathrm{R}}\left\|V_g^{(A)}(f|_{\mathrm{R}})(s,\cdot)\right\|_2 |T_s r(z)|\,ds.$$

Applying the Hölder inequality again and by corollary 1, we have

$$\int_{\mathrm{R}}\left\|V_g^{(A)}(f|_{\mathrm{R}})(s,\cdot)\right\|_2 |T_s r(z)|\,ds \leq \|V_g^{(A)}f\|_2\|r\|_2 = \|f\|_2\|g\|_2\|r\|_2.$$

Hence

$$|F(z)| \leq Ce^{\frac{\Omega}{b}|y|+\frac{a}{b}xy}.$$

Since f, r are entire functions, by inversion formulas of WLCT (7), we have $f(x) = F(x)$ in $\mathbf{R}$ and both are analytic in $\mathbf{C}$, we conclude that $f(z) = F(z)$. Thus $f(z)$ is of the desired estimate.

(i) $\Longrightarrow$ (ii). Let $b, d > 0$. For $\epsilon > 0$, and fixed $x \in \mathbf{R}$, define the function

$$f_{\epsilon,x}(t) := f(t)h_{\epsilon,x}(t)$$

where

$$h_{\epsilon,x}(t) = e^{-j\frac{d}{2b}\epsilon^2+\frac{d}{b}\,\mathrm{sgn}(t)x\epsilon-\frac{1}{b}\epsilon|t|}, \qquad t \in \mathbf{R}.$$

It is easy to see that $f_{\epsilon,x} \in L^1$. We shall show that

$$\lim_{\epsilon\to 0}\int_{\mathrm{R}}\overline{M_x^{(A)}T_s g(t)}f_{\epsilon,x}(t)\,dt = 0, \qquad s \in \mathbf{R},\ x \notin [-\Omega,\Omega]. \tag{12}$$

Then by the direct computation,

$$\hat{h}_{\epsilon,x}(w) = 2\pi e^{-j\frac{d}{2b}\epsilon^2}\left[P_{\frac{\epsilon}{b}}(2\pi w)\cosh\left(\frac{d}{b}x\epsilon\right) + jQ_{\frac{\epsilon}{b}}(2\pi w)\sinh\left(\frac{d}{b}x\epsilon\right)\right],$$

where $P_\epsilon(t) := \frac{1}{\pi}\frac{\epsilon}{\epsilon^2+t^2}$ the Poisson kernel and $Q_\epsilon(t) := \frac{1}{\pi}\frac{t}{\epsilon^2+t^2}$ the conjugate Poisson kernel.

The Plancherel theorem implies that

$$\|f_{\epsilon,x} - f\|_2 = \|\hat{f}_{\epsilon,x} - \hat{f}\|_2.$$

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In fact,

$$\begin{aligned}
\|\hat{f}_{\epsilon,x}-\hat{f}\|_2 &= \|\hat{f}*\hat{h}_{\epsilon,x}-\hat{f}\|_2 \\
&= 2\pi\left\|\hat{f}*\left(P_{\frac{\epsilon}{b}}(2\pi w)\cosh\left(\frac{d}{b}x\epsilon\right)+jQ_{\frac{\epsilon}{b}}(2\pi w)\sinh\left(\frac{d}{b}x\epsilon\right)\right)-\hat{f}\right\|_2 \\
&\le \left\|\hat{f}*P_{\frac{\epsilon}{b}}(2\pi w)-\hat{f}\right\|_2+\left|\sinh\left(\frac{d}{b}x\epsilon\right)\right|\left\|\hat{f}*Q_{\frac{\epsilon}{b}}(2\pi w)\right\|_2 \\
&\quad +\left|\cosh\left(\frac{d}{b}x\epsilon\right)-1\right|\left\|\hat{f}*P_{\frac{\epsilon}{b}}(2\pi w)\right\|_2.
\end{aligned}$$

Since $\left\|\hat{f}*P_{\frac{\epsilon}{b}}(2\pi w)-\hat{f}\right\|_2\to 0$, as $\epsilon\to 0$, $\left\|\hat{f}*Q_{\frac{\epsilon}{b}}(2\pi w)\right\|_2\le C_1\|\hat{f}\|_2$ and $\left\|\hat{f}*P_{\frac{\epsilon}{b}}(2\pi w)\right\|_2\le C_2\|\hat{f}\|_2$, for sufficient small ϵ, $\|\hat{f}_{\epsilon,x}-\hat{f}\|_2\to 0$.

Therefore the Plancherel theorem of the WLCT (6) implies that

$$\|V_g^{(A)}f_{\epsilon,x}-V_g^{(A)}f\|_2=\|f_{\epsilon,x}-f\|_2\|g\|_2\to 0,\qquad \text{as } \epsilon\to 0,$$

for any $g\in L^2$. Hence (12) will imply that $V_g^{(A)}f(s,\cdot)$ vanishes outside $[-\Omega,\Omega]$, for almost every real s. Then (10) holds for almost every real variable. Since f and r are entire functions, each side of (10) is an entire function, it follows that (10) holds for every complex z. Thus (12) implies the theorem.

For each real θ, let Γ_θ be the path defined by

$$\Gamma_\theta(t)=te^{j\theta},\qquad 0\le t<\infty,$$

put $\Pi_\theta=\left\{\mu:\ \mathrm{Re}(\mu e^{j\theta})>\Omega\right\}$ and if $\mu\in\Pi_\theta$, define

$$\begin{aligned}
\Phi_{\theta,s}(\mu) &:= \int_{\Gamma_\theta}K_A(z,j\mu)T_s\overline{g}(z)f(z)dz \\
&= \frac{1}{\sqrt{j2\pi b}}\int_{\Gamma_\theta}e^{-\frac{jd}{2b}\mu^2-\frac{1}{b}\mu z+\frac{ja}{2b}z^2}T_s\overline{g}(z)f(z)dz.
\end{aligned}$$

Substituting the path Γ_θ in the integral,

$$\Phi_{\theta,s}(\mu)=\frac{e^{j\theta}e^{-j\frac{d}{2b}\mu^2}}{\sqrt{j2\pi b}}\int_0^\infty e^{-\frac{1}{b}\mu te^{j\theta}+j\frac{a}{2b}t^2e^{2j\theta}}T_s\overline{g}(te^{j\theta})f(te^{j\theta})dt. \tag{13}$$

Then the absolute value of the integrand (13) is

$$e^{-\frac{1}{b}\mu t\cos\theta-\frac{a}{2b}t^2\sin 2\theta}|T_s\overline{g}(te^{j\theta})||f(te^{j\theta})|.$$

By (9), $|f(te^{j\theta})|\le Ce^{\frac{a}{2b}t^2\sin(2\theta)+\frac{\Omega}{b}t}$ and $g\in L^\infty(\mathbf{R})$, the absolute value of the integrand (13) is at most

$$C\exp\left\{-\left[\ \mathrm{Re}(\mu e^{j\theta})-\Omega\right]\frac{t}{b}\right\} \tag{14}$$

and it follows that $\Phi_{\theta,s}$ is holomorphic in the half plane Π_θ.

However, more is true if $\theta=0$ and if $\theta=\pi$. We have

$$\Phi_{0,s}(\mu)=\frac{1}{\sqrt{j2\pi b}}\int_0^\infty e^{-\frac{jd}{2b}\mu^2-\frac{1}{b}\mu t+\frac{ja}{2b}t^2}\overline{g}(t-s)f(t)dt,\qquad \mathrm{Re}(\mu)>\Omega,$$

$$\Phi_{\pi,s}(\mu)=-\frac{1}{\sqrt{j2\pi b}}\int_{-\infty}^0 e^{-\frac{jd}{2b}\mu^2-\frac{1}{b}\mu t+\frac{ja}{2b}t^2}\overline{g}(t-s)f(t)dt,\qquad \mathrm{Re}(\mu)<-\Omega.$$

The functions $\Phi_{0,\tau}$ and $\Phi_{\pi,\tau}$ are holomorphic in the indicated half planes.

The significance of the functions $\Phi_{\theta,\tau}$ to (12) lies in the following relation

$$\frac{1}{\sqrt{j2\pi b}}\int_{\mathrm{R}}\overline{M_x^{(A)}T_sg(t)}f_{\epsilon,x}(t)dt=\Phi_{0,s}(\epsilon+jx)-\Phi_{\pi,s}(-\epsilon+jx),\qquad x,s\in\mathbf{R}. \tag{15}$$

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In fact,

$$\begin{aligned}
& \Phi_{0,s}(\epsilon+jx)-\Phi_{\pi,s}(-\epsilon+jx) \\
= & \frac{1}{\sqrt{j2\pi b}}\int_0^{\infty} e^{-\frac{jd}{2b}(\epsilon+jx)^2-\frac{1}{b}(\epsilon+jx)t+\frac{ja}{2b}t^2}\overline{g}(t-s)f(t)dt \\
& +\frac{1}{\sqrt{j2\pi b}}\int_{-\infty}^{0} e^{-\frac{jd}{2b}(-\epsilon+jx)^2-\frac{1}{b}(-\epsilon+jx)t+\frac{ja}{2b}t^2}\overline{g}(t-s)f(t)dt \\
= & \frac{1}{\sqrt{j2\pi b}}\int_0^{\infty} \overline{M_x^{(A)}T_s g(t)}e^{-\frac{jd}{2b}\epsilon^2+\frac{d}{b}x\epsilon-\frac{1}{b}t\epsilon}f(t)dt \\
& +\frac{1}{\sqrt{j2\pi b}}\int_{-\infty}^{0} \overline{M_x^{(A)}T_s g(t)}e^{-\frac{jd}{2b}\epsilon^2-\frac{d}{b}x\epsilon+\frac{1}{b}t\epsilon}f(t)dt \\
= & \frac{1}{\sqrt{j2\pi b}}\int_{-\infty}^{\infty} \overline{M_x^{(A)}T_s g(t)}\left[e^{-\frac{jd}{2b}\epsilon^2+\frac{d}{b}\ sgn(t)\ x\epsilon+\frac{1}{b}|t|\epsilon}f(t)\right]dt \\
= & \frac{1}{\sqrt{j2\pi b}}\int_{\mathrm{R}} \overline{M_x^{(A)}T_s g(t)}f_{\epsilon,x}(t)dt.
\end{aligned}$$

Hence we have to prove that the right side of (15) tends to 0 as $\epsilon \to 0$, if $x \notin [-\Omega, \Omega]$.

We claim that any two of our functions $\Phi_{\theta,s}$ agree in the intersection of their domains of definition, i.e., that they are analytic continuations of each other.

Once this is done, we can replace $\Phi_{0,s}$ and $\Phi_{\pi,s}$ by $\Phi_{\frac{\pi}{2},s}$ in (15) if $\mathrm{Re}(\mu) < -\Omega$, and by $\Phi_{-\frac{\pi}{2},s}$ if $\mathrm{Re}(\mu) > \Omega$, and it is then obvious that the difference tends to 0 as $\epsilon \to 0$.

Suppose $0 < \beta - \alpha < \pi$. Put

$$\gamma = \frac{\alpha+\beta}{2}, \quad \eta = \cos\left(\frac{\beta-\alpha}{2}\right) > 0.$$

If $\mu = |\mu|\, e^{-j\gamma}$, then

$$\mathrm{Re}(\mu e^{j\alpha}) = \eta\,|\mu| = \mathrm{Re}(\mu e^{j\beta})$$

so that $\mu \in \Pi_\alpha \bigcap \Pi_\beta$ as soon as $|\mu| > \Omega/\eta$. Consider the integral

$$\int_\Gamma K_A(z, j\mu)T_s\overline{g}(z)f(z)dz \tag{16}$$

over the circular arc Γ given by $\Gamma(t) = re^{jt}$, $\alpha \le t \le \beta, 0 \le r < \infty$. Similar with 14, the absolute value of exponential part of the integrand in (16) does not exceed

$$C\exp\left\{\frac{1}{b}(\Omega - |\mu|\eta)r\right\}.$$

If $|\mu| > \Omega/\eta$, it follows that (16) tends to 0 as $r \to \infty$.

We now apply the Cauchy theorem. The integral of $K_A(z, -j\mu)T_s\overline{g}(z)f(z)$ over the interval $[0, re^{j\beta}]$ is equal to the sum of (16) and the integral over $[0, re^{j\alpha}]$. Since (16) tends to 0 as $r \to \infty$, we conclude that $\Phi_{\alpha,s}(\mu) = \Phi_{\beta,s}(\mu)$ if $\mu = |\mu|\, e^{-j\gamma}$ and $|\mu| > \Omega/\eta$, and then $\Phi_{\alpha,s}$ and $\Phi_{\beta,s}$ coincide in the intersection of the half planes in which they were originally defined.

This completes the proof.□

The following corollary follows directly from Theorem 3. It gives the Paley-Wiener formulas for the linear canonical transform $L_A(f)(u)$ of a signal f [4, 2].

Corollary 3 *Suppose that $f \in L^2(\mathbf{R})$, f is entire function and Ω is a positive real number. Then the following two conditions are equivalent:*

(i) There exists a constant $C > 0$ such that

$$|f(z)| \le Ce^{\frac{\Omega}{b}|y|+\frac{a}{b}xy},$$

for all $z = x + jy \in \mathbf{C}$;

(ii) supp $L_A(f|_{\mathrm{R}}) \subset [-\Omega, \Omega]$, $\quad \forall s \in \mathbf{R}$.

Moreover, if one of the above conditions holds, then we have

$$f(z) = \int_{-\Omega}^{\Omega} K_{A^{-1}}(t, z)L_A(f|_{\mathrm{R}})(t)dt,$$

for all $z \in \mathbf{C}$, where $A = (a, b, c, d)$.

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4. Uncertainty principle for WLCT

In the discussion of the inversion formula for the WLCT we saw how the time-frequency resolution of the WLCT depends on the choice of the window function g. In particular, the time-frequency resolution of $V_g^{(A)}f$ is limited by the size of the essential supports of g and $\hat{g}$. The classical uncertainty principle for g thus implies an uncertainty principle for $V_g^{(A)}f$. By contrast, in this section we present uncertainty principles that apply directly to the WLCT.

Proposition 1 *Suppose that $f, g \in L^2(\mathbf{R})$ and $\|f\|_2 = \|g\|_2 = 1$ and that $U \subseteq \mathbf{R}^2$ and $\epsilon \geq 0$ and such that*

$$\int\int_U \left|V_g^{(A)}f(x,u)\right|^2 dxdu \geq 1-\epsilon.$$

Then $|U| \geq 1-\epsilon$.

Proof. The Cauchy-Schwartz inequity implies that

$$\left|V_g^{(A)}f(x,u)\right| = \left|\left\langle f, M_u^{(A)}T_x g\right\rangle\right| \leq \|f\|_2 \|g\|_2 = 1 \quad \text{for all } (x,u) \in \mathbf{R}^2.$$

Therefore

$$1-\epsilon \leq \int\int_U \left|V_g^{(A)}f(x,u)\right|^2 dxdu \leq \left\|V_g^{(A)}f\right\|_\infty^2 |U| \leq |U|. \qquad \square$$

Before we give a much deeper and stronger inequality for the WLCT. We need a Hausdorff–Young inequalities associated with the LCT [22].

Lemma 1 *Let $2 \leq p < \infty$ and q be such that $1/p + 1/q = 1$. Then*

$$\| L_{A_1} f \|_p \leq E_{A_1,A_2} C_q \| L_{A_2} f \|_q \tag{17}$$

holds for all $f \in L^p(\mathbf{R}), 2 \leq p < \infty$. Here $C_q = (q^{1/q}p^{-1/p})^{1/2}$ and $E_{A_1,A_2} = (2\pi)^{1/p-1/2} \mid a_1b_2 - a_2b_1 \mid^{1/2-1/p}$.

We are now ready to state a precise version of the uncertainty inequality for the WLCT. It is analogous to the one associated with the windowed Fourier transform by E. Lieb [23].

Theorem 4 *If $f, g \in L^2(\mathbf{R})$ and $2 \leq p < \infty$, then*

$$\int\int_{\mathrm{R}^2} \mid V_g^{(A)}f(x,u) \mid^p dxdu \leq \frac{2}{p}(E_A)^p(\| f \|_2 \| g \|_2)^p. \tag{18}$$

where $E_A = (2\pi)^{1/p-1/2} |b|^{1/2-1/p}$.

Proof. Let q be the conjugate index defined by $1/p + 1/q = 1$. Since $2 \leq p < \infty$, we have $1 < q \leq 2$. We note that $fT_x\overline{g} \in L^1(\mathbf{R})$ by the Cauchy-Schwartz inequality. Moreover, since $V_g^{(A)}f(x,u) = L_A(fT_x\overline{g})(u) \in L^2(\mathbf{R}^2)$ by Corollary 1. We conclude from Fubini's theorem that $L_A(fT_x\overline{g}) \in L^2(\mathbf{R})$ for almost all $x \in \mathbf{R}$. Thus $fT_x\overline{g} \in L^1 \cap L^2(\mathbf{R})$ for almost all x, which implies $fT_x\overline{g} \in L^q(\mathbf{R})$ for almost all x.

The Hausdorff–Young inequalities (17) implies

$$\begin{aligned}\left(\int_{\mathrm{R}} \left|V_g^{(A)}f(x,u)\right|^p du\right)^{1/p} &= \|L_A(fT_x\overline{g})\|_p \\ &\leq E_{A_1,A_2}C_q\|L_{A_2}(fT_x\overline{g})\|_q.\end{aligned}$$

Taking $A_1 = A = (a,b,c,d)$ and $A_2 = (1,0,0,1)$, then $E_{A_1,A_2} = (2\pi)^{1/p-1/2}|b|^{1/2-1/p}$ denoted by E_A, we have

$$\begin{aligned}\left(\int_{\mathrm{R}} \left|V_g^{(A)}f(x,u)\right|^p du\right)^{1/p} &\leq E_AC_q\left(\int_R |(fT_x\overline{g})(y)|^q dy\right)^{1/q} \\ &= E_AC_q\left(|f|^q * |g^*|^q(x)\right)^{1/q}\end{aligned}$$

where $C_q = (q^{1/q}p^{-1/p})^{1/2}$ and $g^*(x) = \overline{g(-x)}$. Hence

$$\begin{aligned}\left\|V_g^{(A)}f\right\|_p &= \left(\int_{\mathrm{R}}\left(\int_{\mathrm{R}} \left|V_g^{(A)}f(x,u)\right|^p du\right)dx\right)^{1/p} \\ &\leq E_AC_q\left(\int_{\mathrm{R}} \left(|f|^q * |g^*|^q(x)\right)^{p/q} dx\right)^{1/p} \\ &= E_AC_q\left\||f|^q * |g^*|^q\right\|_{p/q}^{1/q}.\end{aligned}$$

Applying the Young's Inequality [13][23] to the functions $|f|^q$ and $|g^*|^q$, which are elements of $L^{2/q}(\mathbf{R})$, with triple $(s,s,t) = (\frac{2}{q}, \frac{2}{q}, \frac{p}{q})(\frac{1}{s}+\frac{1}{s} = 1+\frac{1}{t})$, we obtain

$$\left\| |f|^q * |g^*|^q \right\|_t \leq C_s C_s C_{t\prime} \left\| |f|^q \right\|_s \left\| |g^*|^q \right\|_s = C_s C_s C_t \, \|f\|_2^q \, \|g\|_2^q$$

where $C_s = (s^{\frac{1}{s}}(s')^{-\frac{1}{s'}})^{\frac{1}{2}}$, $C_{t\prime} = ((t')^{\frac{1}{t'}} t^{-\frac{1}{t}})^{\frac{1}{2}}$, and $\frac{1}{s}+\frac{1}{s'} = 1$, $\frac{1}{t}+\frac{1}{t'} = 1$.

Therefore

$$\begin{aligned} \left\| V_g^{(A)} f \right\|_p &\leq E_A C_q \left(C_s C_s C_{t\prime} \, \|f\|_2^q \, \|g\|_2^q\right)^{1/q} \\ &= E_A C_p (C_s C_s C_{t\prime})^{\frac{1}{q}} \parallel f \parallel_2 \parallel g \parallel_2 \\ &= \left(\frac{2}{p}\right)^{\frac{1}{p}} E_A \parallel f \parallel_2 \parallel g \parallel_2 . \end{aligned}$$

Then

$$\int\int_{R^2} \mid V_g^{(A)} f(x,u) \mid^p dx du = \left\| V_g^{(A)} f \right\|_p^p \leq \frac{2}{p} (E_A)^p (\parallel f \parallel_2 \parallel g \parallel_2)^p .$$

□

Remark 8 *(1) A careful analysis of the minimizing functions in the sharp version of the inequalities of Young and Hausdorff-Young shows that equality in (18) is obtained if and only if f and g are time-frequency shifts of Gaussians [23].*

(2) Lieb's paper [23] contains a number of other important inequalities about the WFT. The analogous one of Theorem 4 is that: for $f, g \in L^2(R)$, if $2 \leq p < \infty$, then $\int\int_{R^2} \mid V_g^{(A)} f(x,u) \mid^p dx du \leq \frac{2}{p} \parallel f \parallel_2 \parallel g \parallel_2$. Its counterpart is for the case $1 \leq p \leq 2$. If $f, g \in L^2(R)$ and $1 \leq p < 2$, then $\int\int_{R^2} \mid V_g^{(A)} f(x,u) \mid^p dx du \geq \frac{2}{p} \parallel f \parallel_2 \parallel g \parallel_2$. Equality holds if and only if $p > 1$ and f, g are certain Gaussians. We have homologous counterpart for Theorem 4, with the proof more technical.

The next theorem shows that uncertainty principle improves Proposition 1 and yields a sharper estimate for the essential support of $V_g^A f$.

Theorem 5 *Suppose that $f, g \in L^2(\mathbf{R})$ and $\|f\|_2 = \|g\|_2 = 1$. If $U \subseteq \mathbf{R}^2$, $\epsilon \geq 0$ such that*

$$\int\int_U \left| V_g^{(A)} f(x,u) \right|^2 dx du \geq 1 - \epsilon.$$

Then

$$|U| \geq (2\pi) \mid b \mid^{-1} (1-\epsilon)^{\frac{p}{p-2}} \left(\frac{p}{2}\right)^{\frac{2}{p-2}}, \qquad \text{for all } p > 2.$$

Proof. We first apply Hölder inequality with exponents $\frac{p}{2}$ and $\frac{p}{p-2}$, and then by (18)

$$\begin{aligned} 1-\epsilon &\leq \int\int_U \left| V_g^{(A)} f(x,w) \right|^2 dx dw \\ &\leq \left(\int\int_{R^2} \left| V_g^{(A)} f(x,w) \right|^{2 \cdot \frac{p}{2}} dx dw \right)^{\frac{2}{p}} \left(\int\int_{R^2} |\chi_U(x,w)|^{\frac{p}{p-2}} dx dw \right)^{\frac{p-2}{p}} \\ &\leq \left(\frac{2}{p}\right)^{\frac{2}{p}} (E_A)^2 \left(\|f\|_2 \|g\|_2\right)^2 |U|^{\frac{p-2}{p}} . \end{aligned}$$

where $E_A = (2\pi)^{\frac{1}{p}-\frac{1}{2}} |b|^{\frac{1}{2}-\frac{1}{p}}$, $\chi_U(x,w)$ denotes the two-dimensional characteristic function.

Thus for $p > 2$

$$|U| \geq (1-\epsilon)^{\frac{p}{p-2}} \left(\frac{p}{2}\right)^{\frac{2}{p-2}} (E_A)^{\frac{2p}{2-p}} = (2\pi) \mid b \mid^{-1} (1-\epsilon)^{\frac{p}{p-2}} \left(\frac{p}{2}\right)^{\frac{2}{p-2}}, \qquad \text{for all } p > 2.$$

□

Remark 9 *Taking $\epsilon = 0$ in Theorem 5 yields the following lower bound for the support of $V_g^{(A)} f$:*

$$\mid supp \, V_g^{(A)} f \mid \geq (2\pi) \mid b \mid^{-1} \lim_{p \to 2+} \left(\frac{p}{2}\right)^{\frac{2}{p-2}} = e.$$

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5. Conclusion and future directions

The WLCT is not only a linear transform, which won't suffer from cross-terms and distort the time-frequency structure of a signal after de-chirping, but also have similar properties as the windowed fourier transform, provide local information and high concentration. As some of the important applications, we generalize the Paley-Wiener Theorem from fourier sense to the WLCT sense. The uncertainty principles which cannot be avoided and entail its own peculiar version for each time-frequency representation, are also considered for WLCT. They are new in literature. The higher dimensional extension of the Paley-Wiener theorem for the WLCT are now under investigation. Notice that the corresponding extension by imbedding $\mathbf{R}^n$ to $\mathbf{C}^n$ actually is the Bargmann transform [34]. The imbedding of $\mathbf{R}^n$ into the real-Clifford algebra involves more skill to formulate it. The latter imbedding provides $\mathbf{R}^n$ with a global complex structure in analogy with the imbedding of $\mathbf{R}$ into the complex plane. The study is under investigations.

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