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New sampling formulae for non-bandlimited signals associated with linear canonical transform and nonlinear Fourier atoms

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ABSTRACT

The sampling theory is basic and crucial in engineering sciences. On the other hand, the linear canonical transform (LCT) is also of great power in optics, filter design, radar system analysis and pattern recognition, etc. The Fourier transform (FT), the fractional Fourier transform (FRFT), Fresnel transform (FRT) and scaling operations are considered as special cases of the LCT. In this paper, we structure certain types of non-bandlimited signals based on two ladder-shape filters designed in the LCT domain. Subsequently, these non-bandlimited signals are reconstructed from their samples together with the generalized sinc function, their parameter *M*-Hilbert transforms or their first derivatives and other information provided by the phase function of the nonlinear Fourier atom which is the boundary value of the *Möbius* transform, respectively. Simultaneously, mathematical characterizations for these non-bandlimited signals are given. Experimental results presented also offer a foundation for the sampling theorems established.

1. Introduction

The linear canonical transform (LCT) [1–6] is a four-parameter (a,b,c,d) class of linear integral transforms. It was first introduced in the 1970s and is also known as the ABCD transform [3], the affine Fourier transform [8] and the generalized Fresnel transform [9]. The Fourier transform (FT), the fractional Fourier transform (FRFT), Fresnel transform (FRT) and scaling operations are special cases of the LCT [1,10]. With more degrees of freedom compared to the FT and the FRFT, the LCT is more flexible but with similar computation cost as the conventional FT [12]. Due to its advantages discussed above, the LCT, as a powerful tool, has found many applications in filter design, signal synthesis, optics, radar analysis and pattern recognition, etc. [1,5].

On the other hand, the sampling theory, as one of the basic and fascinating topics in engineering sciences, is crucial for reconstructing the continuous signals from the information collected at a series of discrete points without aliasing because it bridges the continuous physical signals and the discrete domain. After the celebrated Whittaker–Shannon–Kotel'nikov (WSK) sampling theorem established, sampling expansions for bandlimited signals associated with the FRFT, the LCT and offset LCT (OLCT) have been derived [7,11,13,14,16,29]. However, in the real world, many analog signals arising in engineering applications are non-bandlimited, e.g. causal signals, time-limited signals such as images, most outputs of finite order systems, etc. In view of this, some recent researches have attempted to employ new methods to reconstruct certain non-bandlimited signals. For example, Vetterli et al. [17] perfectly reconstructed periodic and finite-length streams of Diracs, non-uniform splines and piecewise polynomials by using sinc and Gaussian kernels. Walter and Shen [18] gave perfect reconstruction for a class of non-bandlimited signals based on the chromatic

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Nomenclature

Unusual symbols

$L_M[f](\cdot)$	linear canonical transform with parameter M of f
$\mathcal{A}f(\cdot)$	analytic signal of f
$\mathcal{H}[f](\cdot)$	Hilbert transform of f
$\tilde{\mathcal{A}}f(\cdot)$	generalized analytic signal of f
$\mathcal{H}_M[f](\cdot)$	parameter M-Hilbert transform of f
$e^{j\theta_h(\cdot)}$	nonlinear Fourier atom associated with h
$\theta_h(\cdot)$	phase function of the nonlinear Fourier atom associated with h
$\mathrm{sinc}(\cdot)$	the sinc function
$\mathrm{Sinc}_h(\cdot)$	generalized sinc function associated with h
$\mathrm{sgn}(\cdot)$	the sign function
$\chi_E(\cdot)$	the indicator function with a set E

derivative filters and the synthesis filter banks. Vaidyanathan and Vrcelj [19] obtained derivative sampling theorems and pulse sampling theorems for splines, and discrete time versions of these results were derived as well. Chen and Qian [20] investigated certain type of non-bandlimited signals in the conventional FT sense by introducing a ladder-shape filter with one parameter, and also provided a Shannon-type sampling theorem. After that, Chen et al. [22] generalized the above results with a two-parameter ladder shaped filter.

Nevertheless, to the best of our knowledge, there was no investigation into the reconstructions of non-bandlimited signals in the LCT sense before. In this paper, we structure certain types of non-bandlimited signals in the LCT sense based on two ladder-shape filters designed in the LCT domain enlightened by [20,22]. Subsequently, sampling expansions are established for such non-bandlimited signals. The paper is organized as follows: Section 2 is devoted to necessary preliminaries. In Section 3, on the basis of the filtering processes designed in the LCT domain, we originate some spaces determined by six parameters a,b,c,d,h,Ω_M. In particular, when $h=0$, they are actually the same space of Ω_M-bandlimited signals in the LCT sense. If $h\neq 0$, certain types of non-bandlimited signals will be given. In Section 4, sampling formulae for the non-bandlimited signals defined in Section 3 are derived, respectively. Furthermore, the sampling formulae are shown related to the generalized sinc function, the parameter M-Hilbert transforms or the first derivatives of the signals, and the phase function $\theta_h(t)$ of the nonlinear Fourier atom. Characterizations for these non-bandlimited signals are provided as well. Experimental results are presented in Section 5. Finally, conclusions are given in Section 6.

2. Preliminary

2.1. The linear canonical transform

The *linear canonical transform* (*LCT*) with parameter $M=\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$ of a signal $f(t)$ is defined as [2,5]

$$L_M[f](u)=L_M[f(t)](u)=\begin{cases}\sqrt{\dfrac{1}{j2\pi b}}\int_{-\infty}^{\infty} f(t)e^{j(1/2)((a/b)t^2-(2/b)ut+(d/b)u^2)}\,dt, & b\neq 0,\\ \sqrt{d}e^{j(1/2)cdu^2}f(du), & b=0,\end{cases} \tag{2.1}$$

where a,b,c,d are real numbers satisfying $ad-bc=1$, i.e. $\det(M)=1$.

The inverse transform of the LCT with parameter M (ILCT) is given by

$$f(t)=L_{M^{-1}}[L_M[f](u)](t)$$
$$=\begin{cases}\sqrt{\dfrac{j}{2\pi b}}\int_{-\infty}^{\infty} L_M[f](u)e^{j(1/2)(-(d/b)u^2+(2/b)tu-(a/b)t^2)}\,du, & b\neq 0,\\ \sqrt{a}e^{-j(1/2)cat^2}f(at), & b=0.\end{cases} \tag{2.2}$$

Note that when $b=0$, the LCT of a signal is essentially a chirp multiplication and it is of no particular interest for our objective in this work. Hence, without loss of generality, we set $b>0$ in the following sections.

A signal $f(t)$ is said to be *Ω_M-bandlimited in the LCT sense*, if

$$L_M[f](u)=0 \quad \text{for } |u|>\Omega_M.$$

Otherwise, the signal $f(t)$ is non-bandlimited in the LCT sense.

2.2. The parameter M-Hilbert transform

Hilbert transform (*HT*) is another important tool in signal processing and optical analysis. It is used to introduce the *analytic signal* $\mathcal{A}f(t)$, which is a complex extension of a real signal $f(t)$ as [27]

$$\mathcal{A}f(t)=f(t)+j\mathcal{H}[f](t),$$

where $\mathcal{H}[f](t)$ represents the Hilbert transform of $f(t)$ given by [23]

$$\mathcal{H}[f](t)=\frac{1}{\pi}\lim_{\varepsilon\to 0+}\int_{|t-x|>\varepsilon}\frac{f(x)}{t-x}\,dx.$$

The analytic signal contains no negative frequency components in the FT domain [24]. And its counterparts in the FRFT domain and the LCT domain have the same property [15,25]. Associated with the LCT and *parameter M-Hilbert transform*, the *generalized analytic signal* is defined as [25]

$$\tilde{\mathcal{A}}f(t)=f(t)+j\mathcal{H}_M[f](t),$$

where $\mathcal{H}_M[f](t)$ is the parameter M-Hilbert transform (the generalized Hilbert transform [7]) given by

$$\mathcal{H}_M[f](t)=\frac{e^{-j(d/2b)t^2}}{\pi}\lim_{\varepsilon\to 0+}\int_{|t-x|>\varepsilon}\frac{f(x)}{t-x}e^{j(a/2b)x^2}\,dx. \tag{2.3}$$

2.3. *The nonlinear Fourier atom and generalized sinc function*

The definition of the *sinc function* sinc(t) is as [28]

$$\operatorname{sinc}(t)=\begin{cases}\dfrac{\sin(\pi t)}{\pi t}, & t\neq 0,\\ 1, & t=0.\end{cases}\tag{2.4}$$

On the other hand, the phase function $\theta_h(t)$ is defined [21,26] by the boundary value

$$e^{j\theta_h(t)}=\frac{e^{jt}-h}{1-\bar{h}e^{jt}},\quad t\in\mathbb{R}\tag{2.5}$$

of the Möbius transform

$$\tau_h(z)=\frac{z-h}{1-\bar{h}z},\quad |h|<1,\ |z|<1,\tag{2.6}$$

where $h=|h|e^{jt_h}$. The boundary value $e^{j\theta_h(t)}$ is called the *nonlinear Fourier atom associated with* h. In fact,

$$\theta_h(t)=t+2\arctan\frac{|h|\sin(t-t_h)}{1-|h|\cos(t-t_h)},\quad t\in\mathbb{R},$$

$$\theta_h'(t)=\frac{1-|h|^2}{1-2|h|\cos(t-t_h)+|h|^2},\quad t\in\mathbb{R}.$$

Actually, $\theta_h'(t)$ is the instantaneous frequency of the nonlinear Fourier atom and the phase function $\theta_h(t)$ satisfies

$$\mathcal{H}\cos\theta_h(t)=\sin\theta_h(t),\quad t\in\mathbb{R},$$

which implies that the nonlinear Fourier atoms are unit analytic signals with periodic and positive instantaneous frequency.

The *generalized sinc function* Sinc$_h(t)$ (see Fig. 1) is defined as

$$\operatorname{Sinc}_h(t)=\begin{cases}\dfrac{\sin\theta_h(t)}{t}, & t\neq 0,\\ \dfrac{1+h}{1-h}, & t=0.\end{cases}\tag{2.7}$$

Some useful identities with real number h ($|h|<1$) for the phase function $\theta_h(t)$ of the nonlinear Fourier atom are presented in [21,26] as follows:

$$\sin\theta_h(t)=(1-h^2)\sum_{l=1}^{\infty}h^{l-1}\sin(lt)=\frac{(1-h^2)\sin t}{1-2h\cos t+h^2}\tag{2.8}$$

and

$$\cos\theta_h(t)=\frac{(1+h^2)\cos t-2h}{1-2h\cos t+h^2}.\tag{2.9}$$

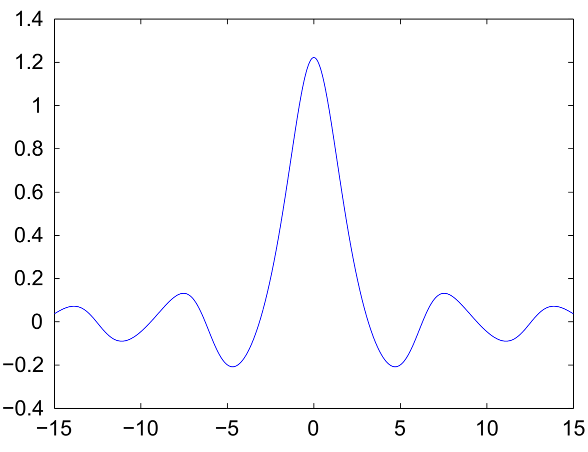

Fig. 1. The generalized sinc function Sinc$_h(t)$ with $h=0.1$.

3. Spaces of non-bandlimited signals in the LCT sense related to ladder-shape filters

3.1. *A new perspective on the Ω_M-bandlimited signals*

To start this section, the existed sampling theorems for the bandlimited signals in the LCT sense are recalled at first. If $f(t)$ is an Ω_M-bandlimited signal in the LCT sense, and its first derivative is continuous, then it can be reconstructed from the equivalent sampling expansions [7] stated in the following:

$$f(t)=e^{-j(a/2b)t^2}\sum_{k\in\mathbb{Z}}e^{j(a/2b)\tilde{t}_k^2}f(\tilde{t}_k)\operatorname{sinc}\left(\frac{\tilde{T}}{\pi}\right),\tag{3.1}$$

$$f(t)=e^{-j(a/2b)t^2}\sum_{k\in\mathbb{Z}}[e^{j(a/2b)t_k^2}f(t_k)\cos T-e^{j(d/2b)t_k^2}\mathcal{H}_M[f](t_k)\sin T]\operatorname{sinc}\left(\frac{T}{\pi}\right),\tag{3.2}$$

and

$$f(t)=e^{-j(a/2b)t^2}\sum_{k\in\mathbb{Z}}e^{j(a/2b)t_k^2}\left\{\left[1+jt_k(t-t_k)\frac{a}{b}\right]f(t_k)+(t-t_k)f'(t_k)\right\}\operatorname{sinc}^2\left(\frac{T}{\pi}\right),\tag{3.3}$$

where

$$\tilde{t}_k=k\frac{\pi}{\Omega_M}b,\quad \tilde{T}=\frac{\Omega_M}{b}(t-\tilde{t}_k),\quad t_k=k\frac{2\pi}{\Omega_M}b,$$
$$T=\frac{\Omega_M}{2b}(t-t_k),\quad k\in\mathbb{Z}.$$

Subsequently, it would be demonstrated that each Ω_M-bandlimited signal is associated with $Q^m_{M,f,\Omega_M}(u)$ ($m=1,2,3$) defined in Eqs. (3.4)–(3.8).

$$Q^1_{M,f,\Omega_M}(u)=\frac{1}{2\Omega_M}\sqrt{-2\pi bj}e^{j(d/2b)u^2}\sum_{k\in\mathbb{Z}}e^{j(a/2b)\tilde{t}_k^2}f(\tilde{t}_k)e^{-j(1/b)\tilde{t}_k u},\tag{3.4}$$

$$Q^2_{M,f,\Omega_M}(u)=\frac{1}{2\Omega_M}\sqrt{-2\pi bj}e^{j(d/2b)u^2}\sum_{k\in\mathbb{Z}}[e^{j(a/2b)t_k^2}f(t_k)+je^{j(d/2b)t_k^2}\operatorname{sgn}(u)\mathcal{H}_M[f](t_k)]e^{-j(1/b)t_k u},\tag{3.5}$$

$$Q^3_{M,f,\Omega_M}(u)=\sqrt{-2\pi bj}e^{j(d/2b)u^2}\sum_{k\in\mathbb{Z}}e^{j(a/2b)t_k^2}\left[-\frac{1}{\Omega_M}f(t_k)+\frac{2}{\Omega_M}f(t_k)\cos\left(\frac{1}{2b}t_k u\right)e^{-j(1/2b)t_k u}+g(u)e^{-j(1/b)t_k u}\right],\tag{3.6}$$

where

$$g(u) = -\frac{1}{\Omega_M^2}\operatorname{sgn}(u)f(t_k)u + \frac{1}{\Omega_M^2}\operatorname{sgn}(u)\left[\frac{a}{b}t_k f(t_k) - jf'(t_k)\right]b, \tag{3.7}$$

$\tilde{t}_k$, t_k are defined as above, and $\operatorname{sgn}(u)$ is the sign function as

$$\operatorname{sgn}(u) = \begin{cases} 1, & u>0, \\ 0, & u=0, \\ -1, & u<0. \end{cases} \tag{3.8}$$

Naturally, $Q^m_{M,f,\Omega_M}(u)$ $(m=1,2,3)$ are well-defined as long as $f(\tilde{t}_k)$, $\mathcal{H}_M[f](t_k)$ and $f'(t_k)$ decay at sufficient rates, respectively, when $k \longrightarrow \infty$.

Consider the spaces

$$D^m_{M,\Omega_M} = \{f \in L^2(\mathbb{R}) : L_M[f](u) = Q^m_{M,f,\Omega_M}(u)\chi_{[-\Omega_M,\Omega_M]}(u)\}, \quad m=1,2,3. \tag{3.9}$$

Then the following four statements are equivalent: (1) f is Ω_M-bandlimited in the LCT sense, (2) $f \in D^1_{M,\Omega_M}$, (3) $f \in D^2_{M,\Omega_M}$, (4) $f \in D^3_{M,\Omega_M}$. To illustrate, if $f \in D^1_{M,\Omega_M}$, obviously, f is Ω_M-bandlimited in the LCT sense. Reversely, if f is Ω_M-bandlimited in the LCT sense, according to Eq. (3.1), and by straightforward computation

$$L_{M^{-1}}[Q^1_{M,f,\Omega_M}(u)\chi_{[-\Omega_M,\Omega_M]}(u)](t) = e^{-j(a/2b)t^2}\sum_{k\in\mathbb{Z}} e^{j(a/2b)\tilde{t}_k^2} f(\tilde{t}_k)\operatorname{sinc}\left(\frac{\tilde{T}}{\pi}\right),$$

hence, $f \in D^1_{M,\Omega_M}$. Similarly, we can easily prove that (1)$\Leftrightarrow$(3)$\Leftrightarrow$(4). Conclude from the above, in fact, D^m_{M,Ω_M} $(m=1,2,3)$ stand for the same space of all Ω_M-bandlimited signals in the LCT sense.

3.2. Non-bandlimited signals in the LCT sense with ladder-shape filters

Next, we move further to extend the above spaces using two ladder-shape filters designed in the LCT domain in order to introduce the non-bandlimited signals. The extended spaces are defined as

$$B^{h,m}_{M,\Omega_M} = \{f \in L^2(\mathbb{R}) : L_M[f](u) = G^{h,m}_{M,f,\Omega_M}(u)\}, \quad m = 1,2,3, \tag{3.10}$$

where $G^{h,m}_{M,f,\Omega_M}(u)$ $(m=1,2)$ are given by

$$G^{h,m}_{M,f,\Omega_M}(u) = \begin{cases} (1-h)Q^m_{M,f,\Omega_M}(u), & |u| \in [0,\Omega_M), \\ h(1-h)Q^m_{M,f,\Omega_M}(u), & |u| \in [\Omega_M, 2\Omega_M), \\ \vdots & \vdots \\ h^n(1-h)Q^m_{M,f,\Omega_M}(u), & |u| \in [n\Omega_M, (n+1)\Omega_M), \\ \vdots & \vdots \end{cases} \tag{3.11}$$

Particularly, $G^{h,3}_{M,f,\Omega_M}(u)$ is defined as

$$G^{h,3}_{M,f,\Omega_M}(u) = \begin{cases} \frac{(1-h)^2}{1-3h}Q^3_{M,f,\Omega_M}(u), & |u| \in [0,\Omega_M), \\ h\frac{(1-h)^2}{1-3h}Q^3_{M,f,\Omega_M}(u), & |u| \in [\Omega_M, 2\Omega_M), \\ \vdots & \vdots \\ h^n\frac{(1-h)^2}{1-3h}Q^3_{M,f,\Omega_M}(u), & |u| \in [n\Omega_M, (n+1)\Omega_M), \\ \vdots & \vdots \end{cases} \tag{3.12}$$

More specifically, two ladder-shape filters are designed corresponding to $G^{h,m}_{M,f,\Omega_M}(u)$ $(m=1,2)$ and $G^{h,3}_{M,f,\Omega_M}(u)$, respectively. The ladder-shape filters $\phi_1(u)$ (see Fig. 2) for $G^{h,m}_{M,f,\Omega_M}(u)$ $(m=1,2)$ and $\phi_2(u)$ (see Fig. 3) for $G^{h,3}_{M,f,\Omega_M}(u)$ are in the forms of

$$\phi_1(u) = \begin{cases} (1-h), & |u| \in [0,\Omega_M), \\ h(1-h), & |u| \in [\Omega_M, 2\Omega_M), \\ \vdots & \vdots \\ h^n(1-h), & |u| \in [n\Omega_M, (n+1)\Omega_M), \\ \vdots & \vdots \end{cases} \tag{3.13}$$

and

$$\phi_2(u) = \begin{cases} (1-h)^2/(1-3h), & |u| \in [0,\Omega_M), \\ h(1-h)^2/(1-3h), & |u| \in [\Omega_M, 2\Omega_M), \\ \vdots & \vdots \\ h^n(1-h)^2/(1-3h), & |u| \in [n\Omega_M, (n+1)\Omega_M), \\ \vdots & \vdots \end{cases} \tag{3.14}$$

in which n is a non-negative integer, and h is a real number with $|h|<1$ ensuring that $\lim_{u\to\infty} G^{h,m}_{M,f,\Omega_M}(u) = 0$. Furthermore, $h \neq \frac{1}{3}$ in the definition of $G^{h,3}_{M,f,\Omega_M}(u)$.

The definitions of $G^{h,m}_{M,f,\Omega_M}(u)$ $(m=1,2,3)$ are considered as filtering processes in the LCT domain. The spectra in the LCT domain of the signals belonging to $B^{h,m}_{M,\Omega_M}$ $(m=1,2,3)$ are obtained by scaling $Q^m_{M,f,\Omega_M}(u)$ $(m=1,2,3)$ in different frequency bands with corresponding ladder-shape filter

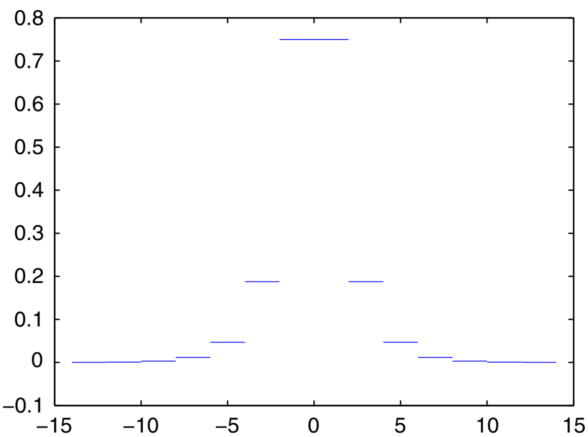

Fig. 2. Ladder-shape filter $\phi_1(u)$ with parameters $h=0.25$, $\Omega_M=2$.

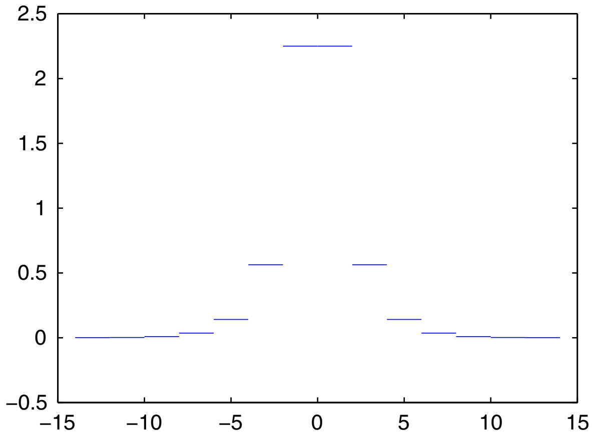

Fig. 3. Ladder-shape filter $\phi_2(u)$ with parameters $h=0.25$, $\Omega_M=2$.

defined in Eqs. (3.13) and (3.14). In addition, we can see that the spaces $B^{h,m}_{M,\Omega_M}$ $(m=1,2,3)$ are determined by six parameters a, b, c, d, h, Ω_M. Especially, when $h=0$,

$$G^{0,m}_{M,f,\Omega_M}(u)=Q^m_{M,f,\Omega_M}(u)\chi_{(-\Omega_M,\Omega_M)}(u),\quad u\in\mathbb{R}.$$

Therefore, $B^{0,m}_{M,\Omega_M}$ $(m=1,2,3)$ represent the same space of bandlimited signals whose supports of the LCTs belong to $[-\Omega_M,\Omega_M]$, i.e. $f\in B^{0,m}_{M,\Omega_M}$ $(m=1,2,3)$ if and only if f is Ω_M-bandlimited in the LCT sense. When $h\neq 0$ for $B^{h,m}_{M,\Omega_M}$ $(m=1,2)$ and $h\neq 0,\frac{1}{3}$ for $B^{h,3}_{M,\Omega_M}$, $B^{h,m}_{M,\Omega_M}$ $(m=1,2,3)$ are spaces of certain non-bandlimited signals, however, they are not necessarily the same.

Apparently, the non-bandlimited signals defined above are with generally decaying spectra in the LCT domain. In fact, the idea stems from three points stated below. Firstly, on the basis of the above discussions, we know sampling expansions for the signals in the extended spaces must be compatible with the existed results for bandlimited signals, that is why $Q^m_{M,f,\Omega_M}(u)$ $(m=1,2,3)$ are defined as in Eqs. (3.4)–(3.8) and the factor h^n is placed in the definitions of $B^{h,m}_{M,\Omega_M}$ $(m=1,2,3)$. Secondly, the factors $1-h$ in $G^{h,m}_{M,f,\Omega_M}(u)$ $(m=1,2)$ and $(1-h)^2/(1-3h)$ in $G^{h,3}_{M,f,\Omega_M}(u)$ cannot be discarded or distributed with other values, or the extended spaces would actually be the null set. The last and also the most fascinating thing is that we find that the phase function $\theta_h(t)$ defined by the boundary value (nonlinear Fourier atom) of the Möbius transform is associated with the filtering processes designed above in the LCT domain. The reasons for the latter two points will be presented in the next section.

4. Sampling and characterization for non-bandlimited signals in the LCT sense

4.1. Sampling theorems for non-bandlimited signals in the LCT sense

This part serves as the main results of the paper. In this section, the sufficient and necessary conditions for a non-bandlimited signal f belonging to $B^{h,m}_{M,\Omega_M}$ $(m=1,2)$ with $h\neq 0$ or $B^{h,3}_{M,\Omega_M}$ with $h\neq 0,\frac{1}{3}$ are given below in terms of sampling theorems, respectively. That means, theoretically, corresponding non-bandlimited signals in the LCT sense can be perfectly reconstructed.

Theorem 1. *A signal $f(t)$ is non-bandlimited in the LCT sense belonging to $B^{h,1}_{M,\Omega_M}$ $(h\neq 0)$ if and only if*

$$f(t)=\frac{1-h}{1+h}e^{-j(a/2b)t^2}\sum_{k\in\mathbb{Z}}e^{j(a/2b)\tilde{t}_k^2}f(\tilde{t}_k)\mathrm{Sinc}_h(\tilde{T}),\quad t\in\mathbb{R},$$

where

$$\tilde{T}=\frac{\Omega_M}{b}(t-\tilde{t}_k),\quad \tilde{t}_k=k\frac{\pi}{\Omega_M}b,\ \ k\in\mathbb{Z}.$$

Proof. We firstly consider the necessity. If f is a non-bandlimited signal in $B^{h,1}_{M,\Omega_M}$ $(h\neq 0)$, then

$$\begin{aligned}L_M[f](u)&=G^{h,1}_{M,f,\Omega_M}(u)\\&=\sum_{l=0}^{\infty}G^{h,1}_{M,f,\Omega_M}(u)[\chi_{(-(l+1)\Omega_M,-l\Omega_M]}(u)+\chi_{[l\Omega_M,(l+1)\Omega_M)}(u)].\end{aligned}$$

By the definitions of $Q^1_{M,f,\Omega_M}(u)$ in Eq. (3.4) and $G^{h,1}_{M,f,\Omega_M}(u)$ in Eq. (3.11),

$$\begin{aligned}L_M[f](u)=&\frac{1-h}{1+h}\frac{\sqrt{-2\pi bj}}{2\Omega_M}e^{j(d/2b)u^2}\sum_{k\in\mathbb{Z}}e^{j(a/2b)\tilde{t}_k^2}f(\tilde{t}_k)e\\&{}^{-j(1/b)\tilde{t}_ku}\sum_{l=0}^{\infty}h^l(1+h)[\chi_{(-(l+1)\Omega_M,-l\Omega_M]}(u)\\&+\chi_{[l\Omega_M,(l+1)\Omega_M)}(u)],\end{aligned}$$

in which $\tilde{t}_k=k(\pi/\Omega_M)b,\ k\in\mathbb{Z}$. Note that

$$\begin{aligned}&\sum_{l=0}^{\infty}h^l(1+h)[\chi_{(-(l+1)\Omega_M,-l\Omega_M]}(u)+\chi_{[l\Omega_M,(l+1)\Omega_M)}(u)]\\&\quad=(1-h^2)\sum_{l=1}^{\infty}h^{l-1}\chi_{[-l\Omega_M,l\Omega_M]}(u).\end{aligned}\tag{4.1}$$

Hence

$$\begin{aligned}L_M[f](u)=&\frac{1-h}{1+h}\frac{1}{2\Omega_M}\sqrt{-2\pi bj}\sum_{k\in\mathbb{Z}}e^{j(a/2b)\tilde{t}_k^2}f(\tilde{t}_k)(1-h^2)\\&\times\sum_{l=1}^{\infty}h^{l-1}e^{j(d/2b)u^2}e^{-j(1/b)\tilde{t}_ku}\chi_{[-l\Omega_M,l\Omega_M]}(u).\end{aligned}$$

Applying the ILCT to the both sides, with

$$\begin{aligned}&\sqrt{\frac{j}{2\pi b}}\int_{-\infty}^{+\infty}e^{j(1/2)(-(d/b)u^2+(2/b)tu-(a/b)t^2)}e^{j(d/2b)u^2}e^{-j(1/b)\tilde{t}_ku}\chi_{[-l\Omega_M,l\Omega_M]}(u)\,du\\&\quad=\sqrt{\frac{j}{2\pi b}}e^{-j(a/2b)t^2}\int_{-l\Omega_M}^{l\Omega_M}e^{j(1/b)(t-\tilde{t}_k)u}\,du\\&\quad=2\Omega_M\sqrt{\frac{j}{2\pi b}}e^{-j(a/2b)t^2}\frac{\sin\left[l\frac{\Omega_M}{b}(t-\tilde{t}_k)\right]}{\frac{\Omega_M}{b}(t-\tilde{t}_k)}.\end{aligned}$$

We have

$$\begin{aligned}f(t)=&\frac{1-h}{1+h}e^{-j(a/2b)t^2}\sum_{k\in\mathbb{Z}}e^{j(a/2b)\tilde{t}_k^2}f(\tilde{t}_k)(1-h^2)\\&\times\sum_{l=1}^{\infty}h^{l-1}\frac{\sin\left[l\frac{\Omega_M}{b}(t-\tilde{t}_k)\right]}{\frac{\Omega_M}{b}(t-\tilde{t}_k)}.\end{aligned}$$

By Eq. (2.8), it follows that

$$f(t)=\frac{1-h}{1+h}e^{-j(a/2b)t^2}\sum_{k\in\mathbb{Z}}e^{j(a/2b)\tilde{t}_k^2}f(\tilde{t}_k)\mathrm{Sinc}_h(\tilde{T}),\quad t\in\mathbb{R},$$

where $\tilde{T}=(\Omega_M/b)(t-\tilde{t}_k),\ k\in\mathbb{Z}$.

For the sufficiency, it can be proved by reversing the above steps directly. □

We can see from Theorem 1 that the non-bandlimited signals in $B^{h,1}_{M,\Omega_M}$ $(h\neq 0)$ can be exactly reconstructed with the sampling interval $T_1=(\pi/\Omega_M)b$, and the generalized sinc function is included in the sampling formula obtained. When $h\longrightarrow 0$, the sampling formula in Theorem 1 reduces to Eq. (3.1).

Remark 1. Theorem 1 is a generalization of the sampling theorem with classical Fourier spectrum derived in [20] i.e. the sampling theorem in [20] is just a special case of Theorem 1 with parameters $(a,b,c,d)=(0,1,-1,0)$. That means some signals cannot be perfectly reconstructed by sampling theorem in [20] are able to be exactly reconstructed by the sampling formula in Theorem 1.

Next, the sampling expansion related to the parameter M-Hilbert transform for the non-bandlimited signals belonging to $B^{h,2}_{M,\Omega_M}$ $(h\neq 0)$ is stated below.

Theorem 2. *A signal $f(t)$ is non-bandlimited in the LCT sense belonging to $B^{h,2}_{M,\Omega_M}$ $(h\neq 0)$ if and only if*

$$f(t)=\frac{1-h}{1+h}e^{-j(a/2b)t^2}\sum_{k\in\mathbb{Z}}\Big\{e^{j(a/2b)t_k^2}f(t_k)\mathrm{Sinc}_h(2T)-\frac{1}{2T}e^{j(d/2b)t_k^2}\mathcal{H}_M[f](t_k)[1-\cos\theta_h(2T)]\Big\},\quad t\in\mathbb{R},$$

where

$$T=\frac{\Omega_M}{2b}(t-t_k),\quad t_k=k\frac{2\pi}{\Omega_M}b,\ k\in\mathbb{Z}.$$

Proof. If $f\in B^{h,2}_{M,\Omega_M}$ $(h\neq 0)$, recall the definitions of $Q^2_{M,f,\Omega_M}(u)$ in Eq. (3.5), $G^{h,2}_{M,f,\Omega_M}(u)$ in Eq. (3.11), and employ the identity (4.1), then

$$L_M[f](u)=G^{h,2}_{M,f,\Omega_M}(u)=\frac{1-h}{1+h}\sum_{k\in\mathbb{Z}}(1-h^2)\sum_{l=1}^{\infty}h^{l-1}p(u),$$

with

$$p(u)=\frac{1}{2\Omega_M}\sqrt{-2\pi bj}e^{j(d/2b)u^2}[e^{j(a/2b)t_k^2}f(t_k)+je^{j(d/2b)t_k^2}\operatorname{sgn}(u)\mathcal{H}_M[f](t_k)]e^{-j(1/b)t_ku}\chi_{[-l\Omega_M,l\Omega_M]}(u),$$

and $t_k=k(2\pi/\Omega_M)b,\ k\in\mathbb{Z}$.

Applying the ILCT to the both sides, since

$$\begin{aligned}&\sqrt{\frac{j}{2\pi b}}\int_{-\infty}^{+\infty}e^{j(1/2)(-(d/b)u^2+(2/b)tu-(a/b)t^2)}p(u)\,du\\&=\frac{1}{2\Omega_M}e^{-j(a/2b)t^2}\int_{-l\Omega_M}^{l\Omega_M}[e^{j(a/2b)t_k^2}f(t_k)+je^{j(d/2b)t_k^2}\operatorname{sgn}(u)\mathcal{H}_M[f](t_k)]e^{j(1/b)(t-t_k)u}\,du\\&=e^{-j(a/2b)t^2}\left[e^{j(a/2b)t_k^2}f(t_k)\frac{\sin(2lT)}{2T}-e^{j(d/2b)t_k^2}\mathcal{H}_M[f](t_k)\frac{\sin^2(lT)}{T}\right],\end{aligned}$$

where $T=(\Omega_M/2b)(t-t_k),\ k\in\mathbb{Z}$.

Therefore

$$f(t)=\frac{1-h}{1+h}\sum_{k\in\mathbb{Z}}(1-h^2)\sum_{l=1}^{\infty}h^{l-1}e^{-j(a/2b)t^2}\times\left[e^{j(a/2b)t_k^2}f(t_k)\frac{\sin(2lT)}{2T}-e^{j(d/2b)t_k^2}\mathcal{H}_M[f](t_k)\frac{\sin^2(lT)}{T}\right].$$

Recall $|h|<1$, consider

$$\sum_{l=1}^{\infty}h^{l-1}\cos(2lT)=\frac{1}{h}\mathrm{Re}\left(\sum_{l=1}^{\infty}h^le^{j2lT}\right)=\frac{\cos 2T-h}{1-2h\cos 2T+h^2},\tag{4.2}$$

and by Eq. (2.9),

$$\begin{aligned}&(1-h^2)\sum_{l=1}^{\infty}h^{l-1}\sin^2(lT)\\&=\frac{1-h^2}{2}\sum_{l=1}^{\infty}h^{l-1}-\frac{1-h^2}{2}\sum_{l=1}^{\infty}h^{l-1}\cos(2lT)\\&=\frac{1+h}{2}-\frac{1-h^2}{2}\frac{\cos(2T)-h}{1-2h\cos(2T)+h^2}\\&=\frac{1}{2}\left[1-\frac{(1+h^2)\cos(2T)-2h}{1-2h\cos(2T)+h^2}\right]\\&=\frac{1}{2}[1-\cos\theta_h(2T)].\end{aligned}\tag{4.3}$$

With Eq. (2.8), it follows that

$$f(t)=\frac{1-h}{1+h}e^{-j(a/2b)t^2}\sum_{k\in\mathbb{Z}}\Big\{e^{j(a/2b)t_k^2}f(t_k)\mathrm{Sinc}_h(2T)-\frac{1}{2T}e^{j(d/2b)t_k^2}\mathcal{H}_M[f](t_k)[1-\cos\theta_h(2T)]\Big\},\quad t\in\mathbb{R}.$$

Additionally, we can get the proof of the sufficiency by reversing the steps above directly. □

It indicates that the sampling expansion of the non-bandlimited signals in $B^{h,2}_{M,\Omega_M}$ $(h\neq 0)$ is related to the parameter M-Hilbert transform and the information carried by the phase function $\theta_h(t)$ of the nonlinear Fourier atom with the sampling interval $T_2=(2\pi/\Omega_M)b$, which is actually twice the length of that in Theorem 1. When $h\longrightarrow 0$, the sampling formula in Theorem 2 will become Eq. (3.2).

On the other hand, derivative sampling would be useful in some cases, for example, when we want to measure the position and velocity of a moving target. The sampling theorem for the non-bandlimited signals in $B^{h,3}_{M,\Omega_M}$ $(h\neq 0,\frac{1}{3})$ is in the derivative form with the sampling interval $T_3=(2\pi/\Omega_M)b$ as well.

Theorem 3. *A signal $f(t)$ is non-bandlimited in the LCT sense belonging to $B^{h,3}_{M,\Omega_M}$ $(h\neq 0,\frac{1}{3})$ if and only if*

$$f(t)=\frac{(1-h)^2}{(1+h)(1-3h)}e^{-j(a/2b)t^2}\sum_{k\in\mathbb{Z}}e^{j(a/2b)t_k^2}\times\Big\{\frac{1}{2T^2}\Big[\Big(1+jt_k(t-t_k)\frac{a}{b}\Big)f(t_k)+(t-t_k)f'(t_k)\Big][1-\cos\theta_h(2T)]+\frac{1}{T}f(t_k)[\sin\theta_h(2T)-\sin^2\theta_h(2T)\csc(2T)]\Big\},\quad t\in\mathbb{R},$$

where

$$T=\frac{\Omega_M}{2b}(t-t_k),\quad t_k=k\frac{2\pi}{\Omega_M}b,\ k\in\mathbb{Z}.$$

Proof. The idea is similar to that in Theorems 1 and 2 but with much more complicated calculation. Thus, for the purpose of clear presentation, the proof is attached in the Appendix.

It is easy to see that when $h\longrightarrow 0$, the sampling formula in Theorem 3 will be simplified to Eq. (3.3).

Remark 2. From the sampling formulae derived above, we know that if the factors $1-h$ in $G^{h,m}_{M,f,\Omega_M}(u)$ $(m=1,2)$ and $(1-h)^2/(1-3h)$ in $G^{h,3}_{M,f,\Omega_M}(u)$ are excluded or distributed with other values, then the right hand sides of the sampling formulae cannot converge to the signals themselves, when $t=\tilde{t}_N$ $(N\in\mathbb{Z})$ in Theorem 1 or $t=t_N$ $(N\in\mathbb{Z})$ in Theorems 2 and 3, which implies that essentially the non-bandlimited signals contained in the extended spaces $B^{h,m}_{M,\Omega_M}$ $(m=1,2)$ with $h\neq 0$ and $B^{h,3}_{M,\Omega_M}$ with $h\neq 0,\frac{1}{3}$ do not exist. That means our definitions for $B^{h,m}_{M,\Omega_M}$ $(m=1,2,3)$ are natural. Furthermore, interestingly, the phase function $\theta_h(t)$ defined by the boundary value (nonlinear Fourier atom) of the *Möbious* transform is deduced in the time domain by the filtering processes designed in the LCT domain. Also, Theorems 1–3 show that such kinds of non-bandlimited signals can be reconstructed from their samples together with the generalized sinc function, their parameter M-Hilbert transforms or their first derivatives and the information provided by the phase function$\theta_h(t)$. At the same time, the sampling rates in Theorems 2 and 3 are half of that in Theorem 1.

4.2. Characterization for non-bandlimited signals in the LCT sense

Sampling formulae for certain non-bandlimited signals in the LCT sense are presented above, however, for practical applications, one may concern more about how can one know a signal f belongs to $B^{h,m}_{M,\Omega_M}$ $(m=1,2,3)$ or not and if it does, how h and Ω_M should be chosen? Here we provide some necessary conditions for $f\in B^{h,m}_{M,\Omega_M}$ $(m=1,2,3)$, respectively.

The following theorem asserts that if $f\in B^{h,1}_{M,\Omega_M}$, $0<|h|<1$, then f may be holomorphically extended to a strip containing the real axis. This is in contrast with bandlimited functions in the LCT sense.

Theorem 4. *If a signal f belongs to $B^{h,1}_{M,\Omega_M}$, then f may be holomorphically extended to the strip*

$$\left\{z=x+jy\left|\frac{b\log|h|}{\Omega_M}<y<-\frac{b\log|h|}{\Omega_M},-\infty<x<\infty\right.\right\},$$

and, inside the strip, the extended function satisfies the estimate

$$|f(x+jy)|\le\frac{C_{\Omega_M,h}}{1-e^{2(\log|h|+(\Omega_M/b)|y|)}},$$

where $C_{\Omega_M,h}$ is a constant depending on Ω_M and h.

Proof. Consider the possible complex numbers z that make the following two integrals $f^+(z)$ and $f^-(z)$ both well defined:

$$f^+(z)=\sqrt{\frac{j}{2\pi b}}\int_0^{+\infty}G^{h,1}_{M,f,\Omega_M}(u)e^{j(1/2)(-(d/b)u^2+(2/b)zu-(a/b)z^2)}\,du$$

and

$$f^-(z)=\sqrt{\frac{j}{2\pi b}}\int_{-\infty}^{0}G^{h,1}_{M,f,\Omega_M}(u)e^{j(1/2)(-(d/b)u^2+(2/b)zu-(a/b)z^2)}\,du.$$

By the definition of $G^{h,1}_{M,f,\Omega_M}$, for $u\in[0,2\Omega_M)$, $G^{h,1}_{M,f,\Omega_M}(u+2n\Omega_M)=h^{2n}e^{j(2d/b)n\Omega_M(u+n\Omega_M)}G^{h,1}_{M,f,\Omega_M}(u)$, $n=0,1,2,3,\ldots$. Writing $\log h=\log|h|+j\arg h$, we have

$$\begin{aligned}f^+(z)&=\sqrt{\frac{j}{2\pi b}}\int_0^{2\Omega_M}\sum_{n=0}^{\infty}h^{2n}e^{j(2d/b)n\Omega_M(u+n\Omega_M)}e^{j(1/2)[-(d/b)(u+2n\Omega_M)^2+(2/b)z(u+2n\Omega_M)-(a/b)z^2]}G^{h,1}_{M,f,\Omega_M}(u)\,du\\&=\sqrt{\frac{j}{2\pi b}}\int_0^{2\Omega_M}e^{j(1/2)(-(d/b)u^2+(2/b)zu-(a/b)z^2)}A^{h,1}_{z,\Omega_M}G^{h,1}_{M,f,\Omega_M}(u)\,du,\end{aligned}$$

where $A^{h,1}_{z,\Omega_M}$ represents the geometric series in the integral, namely,

$$A^{h,1}_{z,\Omega_M}=\sum_{n=0}^{\infty}e^{2n(\log|h|+j\arg h+j(\Omega_M/b)z)}.$$

For $z=x+jy$, due to the relation

$$|e^{2(\log|h|+i\arg h+j(\Omega_M/b)z)}|\le e^{2(\log|h|-(\Omega_M/b)y)},$$

for $y>b\log|h|/\Omega_M$, the geometric series is absolutely convergent to

$$A^{h,1}_{z,\Omega_M}=\frac{1}{1-e^{2(\log|h|+i\arg h+j(\Omega_M/b)z)}}.$$

The function $A^{h,1}_{z,\Omega_M}$ is bounded by

$$\frac{1}{1-e^{2(\log|h|-(\Omega_M/b)y)}}.$$

The factor $A^{h,1}_{z,\Omega_M}$ may be moved out of the integral while $G^{h,1}_{M,f,\Omega_M}$ is integrable and $e^{j(1/2)(-(d/b)u^2+(2/b)zu-(a/b)z^2)}$ is bounded in the domain of integration. Therefore f^+ is well defined through the integral in the half-plane $y>b\log|h|/\Omega_M$ with

$$|f^+(z)|\le\frac{C_{\Omega_M,h}}{1-e^{2(\log|h|-(\Omega_M/b)y)}}.$$

Using the direct computation and the dominated convergence theorem, one can show that f^+ is holomorphic in the half-plane. We therefore conclude that the function f^+ is holomorphic and bounded above any line $\{t+iy|-\infty<t<\infty\}$ for $y>b\log|h|/\Omega_M$. Similarly, we can find that

$$|f^-(z)| \le \frac{C_{\Omega_M,h}}{1-e^{2(\log|h|+(\Omega_M/b)y)}},$$

for $y < -b\log|h|/\Omega_M$, and the function f^- is holomorphic and bounded below any line $\{t+jy| -\infty < t < \infty\}$ for $y > b\log|h|/\Omega_M$. It follows that restricted in the strip

$$\frac{b\log|h|}{\Omega_M} < y < -\frac{b\log|h|}{\Omega_M},$$

the function $f(z) = f^+(z) + f^-(z)$ is holomorphic and satisfies the estimate

$$|f(z)| \le \frac{C_{\Omega_M,h}}{1-e^{2(\log|h|+(\Omega_M/b)|y|)}}.$$

The proof is complete. □

Theorem 4 infers that the necessary criterion for judging a signal f belonging to $B^{h,1}_{M,\Omega_M}$ is that f must be bounded by

$$\frac{C_{\Omega_M,h}}{1-e^{2(\log|h|+(\Omega_M/b)|y|)}}.$$

Remark 3. Using Theorem 4, one can extend the sampling formula in Theorem 1 to the strip where $f \in B^{h,1}_{M,\Omega_M}$ is holomorphic.

Likewise, we have similar necessary criteria for $f \in B^{h,m}_{M,\Omega_M}$ $(m = 2,3)$ as follows.

Theorem 5. *If a signal f belongs to $B^{h,2}_{M,\Omega_M}$, then f may be holomorphically extended to the strip*

$$\left\{z = x+jy \middle| \frac{b\log|h|}{\Omega_M} < y < -\frac{b\log|h|}{\Omega_M}, -\infty < x < \infty\right\},$$

and, inside the strip, the extended function satisfies the estimate

$$|f(x+jy)| \le \frac{C_{\Omega_M,h}}{1-e^{\log|h|+(\Omega_M/b)|y|}},$$

where $C_{\Omega_M,h}$ is a constant depending on Ω_M and h.

Proof. See the proof in the Appendix.

Theorem 6. *If a signal f belongs to $B^{h,3}_{M,\Omega_M}$, then f may be holomorphically extended to the strip*

$$\left\{z = x+jy \middle| \frac{b\log|h|}{\Omega_M} < y < -\frac{b\log|h|}{\Omega_M}, -\infty < x < \infty\right\},$$

and, inside the strip, the extended function satisfies the estimate

$$|f(x+jy)| \le \frac{C_{\Omega_M,h}}{1-e^{2(\log|h|+(\Omega_M/b)|y|)}},$$

where $C_{\Omega_M,h}$ is a constant depending on Ω_M and h.

Proof. See the proof in the Appendix.

Though Theorems 4–6 are not exact criteria for determining a signal f belonging to $B^{h,m}_{M,\Omega_M}$ $(m = 1,2,3)$, they do give us some intuitions on the relationship between the signal and the parameters (a,b,c,d,Ω_M,h), and extends the sampling formulae in Theorem 1–3 to the strip where $f \in B^{h,m}_{M,\Omega_M}$, $m = 1,2,3$ is holomorphic.

5. Experimental results

Certain types of non-bandlimited signals are structured in Section 3, and it is demonstrated that these non-bandlimited signals can be perfectly reconstructed by Theorems 1–3, respectively. At the same time, necessary conditions for signals belonging to the spaces $B^{h,m}_{M,\Omega_M}$ $(m = 1,2,3)$ are given in Theorems 4–6. For the time being, though it is rather difficult to give a strict explicit mathematical characterization for the non-bandlimited signals, from our intuitions, it is believed that the non-bandlimited signals satisfying Theorems 4–6 should be well approximated by the sampling formulae derived in Theorems 1–3.

In order to investigate the practical applications of the sampling formulae derived in Theorems 1–3, for instance, we observe a signal given by

$$f(t) = \frac{\cos t}{1+|t|^{3/2}}, \quad t \in \mathbb{R},$$

with a LCT pair $(a,b,c,d) = (1,16,0,1)$, $h = 0.1$, $\Omega_M = 64$. It is not difficult to check f satisfies the conditions in Theorems 4–6. The original signal $f(t)$ (see Fig. 4) is non-bandlimited and its LCT (see Fig. 5) is in the desired shape. We try to reconstruct the original signal $f(t)$ by Theorems 1–3 with sampling intervals $T_1 = (\pi/\Omega_M)b = \pi/4$ in Theorem 1 and $T_2 = T_3 = (2\pi/\Omega_M)b = \pi/2$ in Theorems 2 and 3. The real and imaginary parts of the original and reconstructed signals for Theorems 1–3 are plotted in Figs. 6–11, respectively. The original signals are in blue lines and the reconstructed ones are in red dotted lines.

As shown in Figs. 4–11, the experimental results illustrate that the sampling formulae derived in Theorems 1–3 really apply to the reconstruction of the original signal $f(t)$. Specifically, the sampling formulae in Theorems 1 and 2 almost perfectly reconstruct the original signal $f(t)$. Though the reconstruction by Theorem 3 is poorer than those of Theorems 1 and 2, the original signal is also well approximated by the formula in Theorem 3. Not surprisingly, that may result from the truncation error

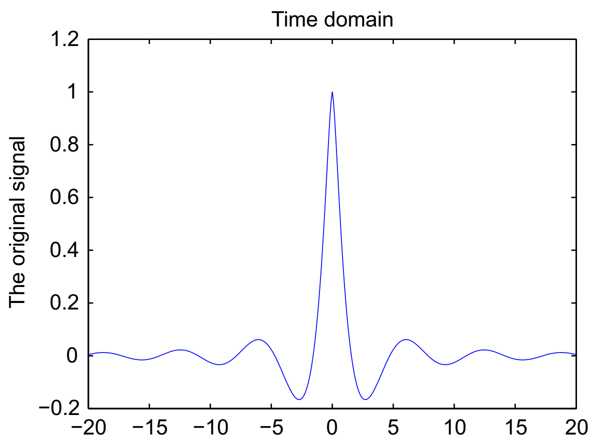

Fig. 4. The original signal $f(t)$.

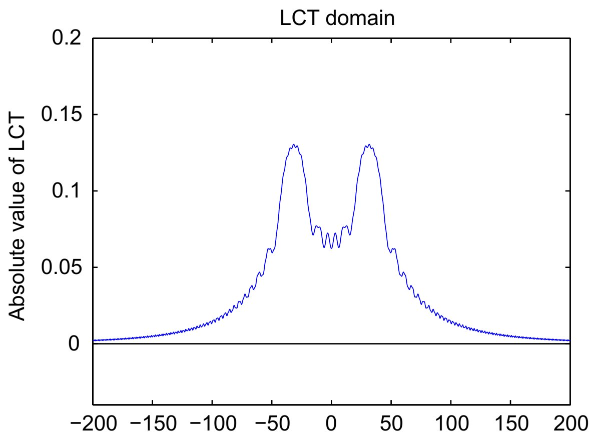

Fig. 5. Absolute value of the LCT of the original signal.

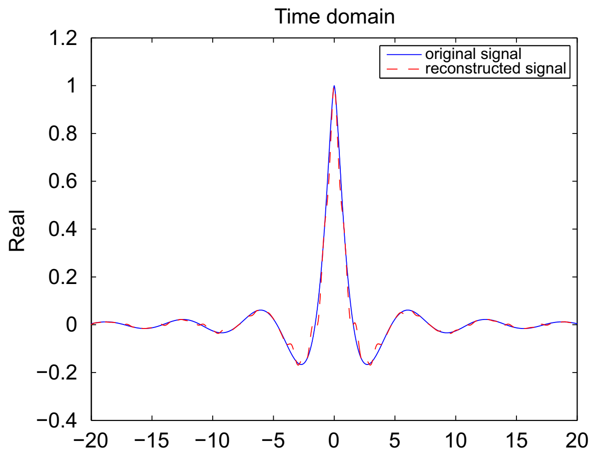

Fig. 6. Real parts of the original and reconstructed signals for Theorem 1.

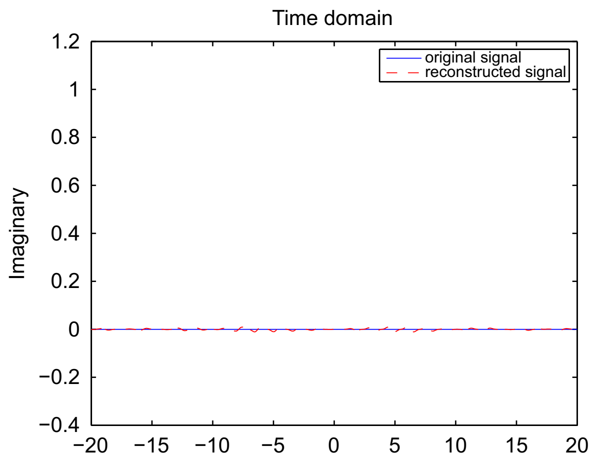

Fig. 7. Imaginary parts of the original and reconstructed signals for Theorem 1.

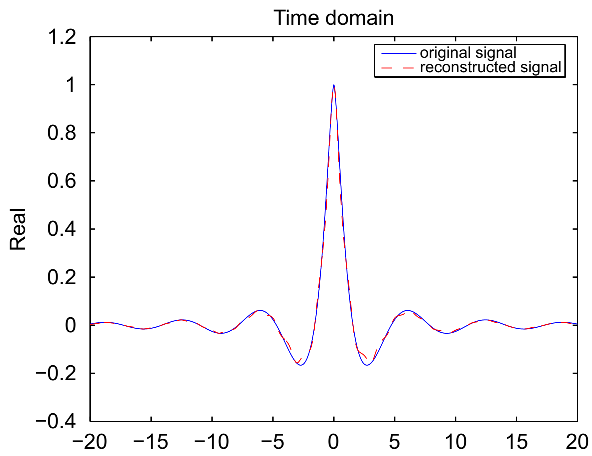

Fig. 8. Real parts of the original and reconstructed signals for Theorem 2.

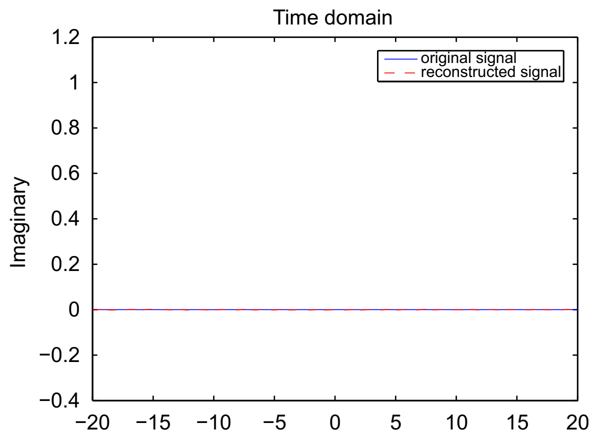

Fig. 9. Imaginary parts of the original and reconstructed signals for Theorem 2.

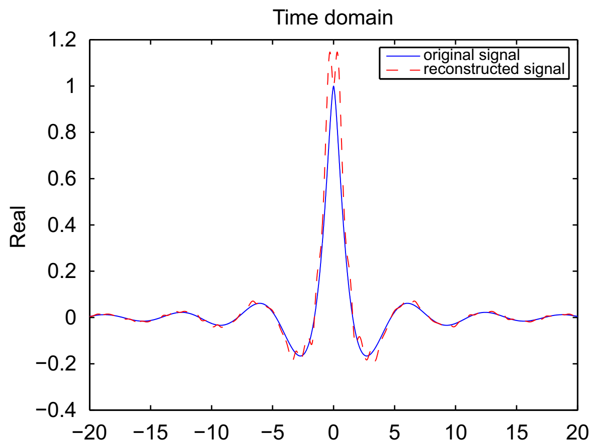

Fig. 10. Real parts of the original and reconstructed signals for Theorem 3.

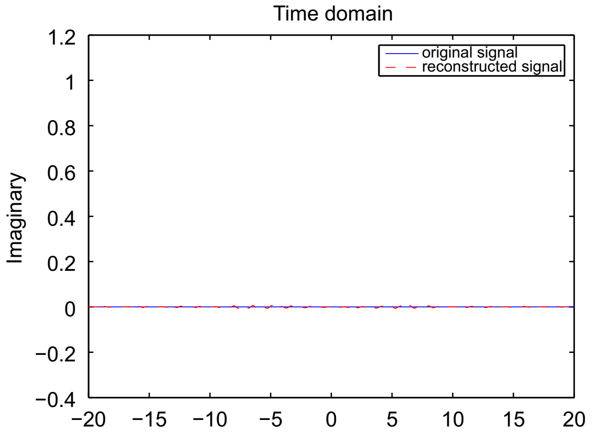

Fig. 11. Imaginary parts of the original and reconstructed signals for Theorem 3.

and other numerical errors derived from computing the extremely complicated terms. At the same time, the sampling formulae in Theorems 2 and 3 are with lower, actually half, sampling rates compared to that in Theorem 1.

6. Conclusions

In this paper, based on the filtering processes designed in the LCT domain, certain types of non-bandlimited signals in the LCT sense are studied. For such kinds of non-bandlimited signals, their spectra in the LCT domain are obtained by scaling $Q^m_{M,f,\Omega_M}(u)$ $(m=1,2,3)$ in different frequency bands with corresponding ladder-shape filter. Simultaneously, we characterize the spaces of the investigated non-bandlimited signals in terms of sampling theorems. It is shown that these non-bandlimited signals can be reconstructed from their samples together with the generalized sinc function, their parameter M-Hilbert transforms or their first derivatives and the information carried by the phase function $\theta_h(t)$ defined by the nonlinear Fourier atom which is the boundary value of the *Möbius* transform. Necessary conditions for judging a signal belonging to $B^{h,m}_{M,\Omega_M}$ $(m=1,2,3)$ are given in Theorems 4–6.

Our experimental results also provide a foundation for the application prospects of the sampling formulae stated in Theorems 1–3. From the simulation, it seems that the signals satisfying the conditions stated in Theorems 4–6 can be well approximated by the sampling theorems derived in Theorems 1–3. Nevertheless, we have to concede that there are no normative rules for determining the parameters a,b,c,d,h,Ω_M in practical implementations at present.

For more practical applications, we would present a new type of spectrum for sequences in the LCT sense, referred as *multi-scale LCT spectrum*, in the future paper. Different from the traditional Fourier spectrum for sequences, which is essentially suitable for only bandlimited signals, multi-scale LCT spectrum is specifically designed for signals in $B^{h,m}_{M,\Omega_M}$ $(m=1,2,3)$. We give a simple explanation about the difference between Fourier spectrum and multi-scale LCT spectrum. Suppose that we are investigating a continuous signal $f(t)$, $t\in\mathbf{R}$. For computer implementation, we actually deal with digital signal of length N, namely, $(d_1,d_2,\ldots,d_N)$. With more degrees of freedom compared to the FT and the FRFT, the LCT is more flexible but with similar computation cost as the conventional FT [12]. By LCT algorithm, one get the LCT spectrum data $(\hat{d_1},\hat{d_2},\ldots,\hat{d_N})$ of $f(t)$ with the same length as the time domain data. Essentially, in LCT algorithm, considering the signal $f(t)$ are both time-limited and bandlimited. But the bandwidth of a time limited signal is infinite. Thus N LCT spectrum data are not enough to represent the signal $f(t)$ since one cannot get the higher frequency information of $f(t)$ when $f(t)$ has small duration. One can obtain the higher frequency data unless one extend the length of the time domain data of the signal $f(t)$ under study. But by multi-scale LCT spectrum algorithm, one can easily obtain high frequency information at any band from N data in the time domain.

For more future study, it is worth to characterize the spaces of non-bandlimited signals explicitly, then one can perfectly reconstruct these non-bandlimited signals. The other is to give an effective algorithm for the practical implementation, i.e. to find the optimal related parameters to approximate the original signal.

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Appendix

Proof of Theorem 3. We follow the idea in the proofs of Theorems 1 and 2. For the necessity, if $f\in B^{h,3}_{M,\Omega_M}$ $(h\neq 0,\frac{1}{3})$, by the definitions of $Q^3_{M,f,\Omega_M}(u)$ in Eq. (3.6), $G^{h,3}_{M,f,\Omega_M}(u)$ in Eq. (3.12) and the identity (4.1), then

$$L_M[f](u)=G^{h,3}_{M,f,\Omega_M}(u)$$
$$=\frac{(1-h)^2}{(1+h)(1-3h)}\sum_{k\in\mathbb{Z}}e^{j(a/2b)t_k^2}(1-h^2)\sum_{l=1}^{\infty}h^{l-1}s(u), \qquad (*)$$

where $t_k = k(2\pi/\Omega_M)b,\ k \in \mathbb{Z}$,

$$s(u) = \sqrt{-2\pi bj}e^{j(d/2b)u^2} \times \left[-\frac{1}{\Omega_M}f(t_k) + \frac{2}{\Omega_M}f(t_k)\cos\left(\frac{1}{2b}t_k u\right)e^{-j(1/2b)t_k u} + g(u)e^{-j(1/b)t_k u}\right]\chi_{[-l\Omega_M, l\Omega_M]}(u),$$

and with Eq. (3.7)

$$g(u) = -\frac{1}{\Omega_M^2}\operatorname{sgn}(u)f(t_k)u + \frac{1}{\Omega_M^2}\operatorname{sgn}(u)\left[\frac{a}{b}t_k f(t_k) - jf'(t_k)\right]b.$$

By direct computation,

$$\sqrt{\frac{j}{2\pi b}}\int_{-\infty}^{+\infty} s(u)e^{j(1/2)(-(d/b)u^2+(2/b)tu-(a/b)t^2)}\,du$$
$$= e^{-j(a/2b)t^2}\int_{-l\Omega_M}^{l\Omega_M}\left[-\frac{1}{\Omega_M}f(t_k) + \frac{2}{\Omega_M}f(t_k)\cos\left(\frac{1}{2b}t_k u\right)e^{-j(1/2b)t_k u} + g(u)e^{-j(1/b)t_k u}\right]e^{j(1/b)tu}\,du = e^{-j(a/2b)t^2}(I_1+I_2+I_3),$$

with

$$I_1 = \int_{-l\Omega_M}^{l\Omega_M} -\frac{1}{\Omega_M}f(t_k)e^{j(1/b)tu}\,du = -\frac{2b}{\Omega_M}f(t_k)\frac{\sin\left(l\frac{1}{b}\Omega_M t\right)}{t},$$

$$I_2 = \int_{-l\Omega_M}^{l\Omega_M} \frac{2}{\Omega_M}f(t_k)\cos\left(\frac{1}{2b}t_k u\right)e^{j(1/2b)(2t-t_k)u}\,du$$
$$= 2\int_0^{l\Omega_M}\frac{2}{\Omega_M}f(t_k)\cos\left(\frac{1}{2b}t_k u\right)\cos\left(\frac{1}{2b}(2t-t_k)u\right)du$$
$$= \frac{2}{\Omega_M}f(t_k)\int_0^{l\Omega_M}\left[\cos\left(\frac{1}{b}tu\right) + \cos\left(\frac{1}{b}(t-t_k)u\right)\right]du$$
$$= \frac{2b}{\Omega_M}f(t_k)\frac{\sin\left(l\frac{1}{b}\Omega_M t\right)}{t} + f(t_k)\frac{\sin(2lT)}{T},$$

and

$$I_3 = \int_{-l\Omega_M}^{l\Omega_M} g(u)e^{j(1/b)(t-t_k)u}\,du = \int_{-l\Omega_M}^{l\Omega_M}\left[-\frac{1}{\Omega_M^2}\operatorname{sgn}(u)f(t_k)u + \frac{1}{\Omega_M^2}\operatorname{sgn}(u)\left(t_k f(t_k)\frac{a}{b} - jf'(t_k)\right)b\right]e^{j(1/b)(t-t_k)u}\,du$$
$$= \int_{-l\Omega_M}^{l\Omega_M} -\frac{1}{\Omega_M^2}\operatorname{sgn}(u)f(t_k)u\cos\left(\frac{1}{b}(t-t_k)u\right) + \frac{b}{\Omega_M^2}\operatorname{sgn}(u)\left(jt_k f(t_k)\frac{a}{b} + f'(t_k)\right)\sin\left(\frac{1}{b}(t-t_k)u\right)du$$
$$= -\frac{2}{\Omega_M^2}f(t_k)\int_0^{l\Omega_M}u\cos\left(\frac{1}{b}(t-t_k)u\right)du + \frac{2b}{\Omega_M^2}\left(jt_k f(t_k)\frac{a}{b} + f'(t_k)\right)\int_0^{l\Omega_M}\sin\left(\frac{1}{b}(t-t_k)u\right)du$$
$$= -lf(t_k)\frac{\sin(2lT)}{T} + f(t_k)\frac{\sin^2(lT)}{T^2} + \left(jt_k f(t_k)\frac{a}{b} + f'(t_k)\right)(t-t_k)\frac{\sin^2(lT)}{T^2},$$

where $T = (\Omega_M/2b)(t-t_k),\ k \in \mathbb{Z}$. It follows that

$$\sqrt{\frac{j}{2\pi b}}\int_{-\infty}^{+\infty} s(u)e^{j(1/2)(-(d/b)u^2+(2/b)tu-(a/b)t^2)}\,du$$
$$= e^{-j(a/2b)t^2}\left\{\left[\left(1 + jt_k(t-t_k)\frac{a}{b}\right)f(t_k) + (t-t_k)f'(t_k)\right] \times \frac{\sin^2(lT)}{T^2} + (1-l)f(t_k)\frac{\sin(2lT)}{T}\right\}.$$

Applying the ILCT to the both sides of Eq. $(*)$, we get

$$f(t) = \frac{(1-h)^2}{(1+h)(1-3h)}e^{-j(a/2b)t^2}\sum_{k\in\mathbb{Z}}e^{j(a/2b)t_k^2}(1-h^2)\sum_{l=1}^{\infty}h^{l-1} \times \left\{(1-l)f(t_k)\frac{\sin(2lT)}{T} + \left[\left(1 + jt_k(t-t_k)\frac{a}{b}\right)f(t_k) + (t-t_k)f'(t_k)\right]\frac{\sin^2(lT)}{T^2}\right\}.$$

Note that $|h| < 1$, so $\sum_{l=1}^{\infty}h^{l-1}\cos(2lx)$ and $\sum_{l=1}^{\infty}h^{l-1}(1-l)\sin(2lx)$ are absolutely convergent for all $x \in \mathbb{R}$. By Eqs. (2.8) and (4.2), thus

$$(1-h^2)\sum_{l=1}^{\infty}h^{l-1}(1-l)\sin(2lT)$$
$$= \sin\theta_h(2T) + \frac{1}{2}\left[(1-h^2)\sum_{l=1}^{\infty}h^{l-1}\cos(2lx)\right]'\Bigg|_{x=T}$$
$$= \sin\theta_h(2T) - \sin^2\theta_h(2T)\csc(2T).$$

With Eq. (4.3), it gives

$$f(t) = \frac{(1-h)^2}{(1+h)(1-3h)}e^{-j(a/2b)t^2}\sum_{k\in\mathbb{Z}}e^{j(a/2b)t_k^2} \times \left\{\frac{1}{2T^2}\left[\left(1 + jt_k(t-t_k)\frac{a}{b}\right)f(t_k) + (t-t_k)f'(t_k)\right][1-\cos\theta_h(2T)] + \frac{1}{T}f(t_k)[\sin\theta_h(2T) - \sin^2\theta_h(2T)\csc(2T)]\right\},\quad t \in \mathbb{R}.$$

As for the sufficiency, it can be easily proved by reversing the above steps directly. □

Proof of Theorem 5. Consider the possible complex numbers z that make the following two integrals $f^+(z)$ and $f^-(z)$ both well defined:

$$f^+(z) = \sqrt{\frac{j}{2\pi b}}\int_0^{+\infty}G_{M,f,\Omega_M}^{h,2}(u)e^{j(1/2)\left(-(d/b)u^2+(2/b)zu-(a/b)z^2\right)}\,du$$

and

$$f^-(z) = \sqrt{\frac{j}{2\pi b}}\int_{-\infty}^{0}G_{M,f,\Omega_M}^{h,2}(u)e^{j(1/2)(-(d/b)u^2+(2/b)zu-(a/b)z^2)}\,du.$$

By the definition of $G_{M,f,\Omega_M}^{h,2}$, for $u \in [0,\Omega_M)$, $G_{M,f,\Omega_M}^{h,2}(u+n\Omega_M) = h^n e^{j(d/2b)n\Omega_M(2u+n\Omega_M)}G_{M,f,\Omega_M}^{h,2}(u),\ n = 0,1,2,3,\ldots$. Writing $\log h = \log|h| + j\arg h$, we have

$$f^+(z) = \sqrt{\frac{j}{2\pi b}}\int_0^{\Omega_M}\sum_{n=0}^{\infty}h^n e^{j(d/2b)n\Omega_M(2u+n\Omega_M)}e^{j(1/2)[-(d/b)(u+n\Omega_M)^2 + (2/b)z(u+n\Omega_M) - (a/b)z^2]}G_{M,f,\Omega_M}^{h,2}(u)\,du$$
$$= \sqrt{\frac{j}{2\pi b}}\int_0^{2\Omega_M}e^{j(1/2)(-(d/b)u^2+(2/b)zu-(a/b)z^2)}A_{z,\Omega_M}^{h,2}G_{M,f,\Omega_M}^{h,2}(u)\,du,$$

where $A^{h,2}_{z,\Omega_M}$ represents the geometric series in the integral, namely,

$$A^{h,2}_{z,\Omega_M}=\sum_{n=0}^{\infty}e^{n(\log|h|+j\arg h+j(\Omega_M/b)z)}.$$

For $z=x+jy$, due to the relation

$$|e^{\log|h|+i\arg h+j(\Omega_M/b)z}|\le e^{\log|h|-(\Omega_M/b)y},$$

for $y>b\log|h|/\Omega_M$, the geometric series is absolutely convergent to

$$A^{h,2}_{z,\Omega_M}=\frac{1}{1-e^{\log|h|+i\arg h+j(\Omega_M/b)z}}.$$

The function $A^{h,2}_{z,\Omega_M}$ is bounded by

$$\frac{1}{1-e^{\log|h|-(\Omega_M/b)y}}.$$

The factor $A^{h,2}_{z,\Omega_M}$ may be moved out of the integral while $G^{h,2}_{M,f,\Omega_M}$ is integrable and $e^{j(1/2)(-(d/b)u^2+(2/b)zu-(a/b)z^2)}$ is bounded in the domain of integration. Therefore f^+ is well defined through the integral in the half-plane $y>b\log|h|/\Omega_M$ with

$$|f^+(z)|\le\frac{C_{\Omega_M,h}}{1-e^{\log|h|-(\Omega_M/b)y}}.$$

Using the direct computation and the dominated convergence theorem, one can show that f^+ is holomorphic in the half-plane. We therefore conclude that the function f^+ is holomorphic and bounded above any line $\{t+iy|-\infty<t<\infty\}$ for $y>b\log|h|/\Omega_M$. Similarly, we can find that

$$|f^-(z)|\le\frac{C_{\Omega_M,h}}{1-e^{\log|h|+(\Omega_M/b)y}},$$

for $y<-b\log|h|/\Omega_M$, and the function f^- is holomorphic and bounded below any line $\{t+jy|-\infty<t<\infty\}$ for $y>b\log|h|/\Omega_M$. It follows that restricted in the strip

$$\frac{b\log|h|}{\Omega_M}<y<-\frac{b\log|h|}{\Omega_M},$$

the function $f(z)=f^+(z)+f^-(z)$ is holomorphic and satisfies the estimate

$$|f(z)|\le\frac{C_{\Omega_M,h}}{1-e^{\log|h|+(\Omega_M/b)|y|}}.$$

The proof is complete. □

Proof of Theorem 6. Consider the possible complex numbers z that make the following two integrals $f^+(z)$ and $f^-(z)$ both well defined:

$$f^+(z)=\sqrt{\frac{j}{2\pi b}}\int_0^{+\infty}G^{h,3}_{M,f,\Omega_M}(u)e^{j(1/2)(-(d/b)u^2+(2/b)zu-(a/b)z^2)}\,du$$

and

$$f^-(z)=\sqrt{\frac{j}{2\pi b}}\int_{-\infty}^{0}G^{h,3}_{M,f,\Omega_M}(u)e^{j(1/2)(-(d/b)u^2+(2/b)zu-(a/b)z^2)}\,du.$$

By the definition of $G^{h,3}_{M,f,\Omega_M}$, for $u\in[0,2\Omega_M)$, $n=0,1,2,3,\ldots,$

$$G^{h,3}_{M,f,\Omega_M}(u+2n\Omega_M)$$
$$=h^{2n}e^{j(2d/b)n\Omega_M(u+n\Omega_M)}\left[G^{h,3}_{M,f,\Omega_M}(u)-\sqrt{-2\pi bj}\frac{2n}{\Omega_M}e^{j(d/2b)u^2}\sum_{k\in\mathbb{Z}}e^{j(a/2b)t_k^2}f(t_k)e^{-j(1/b)t_ku}\right].$$

Writing $\log h=\log|h|+j\arg h$, we have

$$f^+(z)=\sqrt{\frac{j}{2\pi b}}\int_0^{2\Omega_M}\sum_{n=0}^{\infty}h^{2n}e^{j(2d/b)n\Omega_M(u+n\Omega_M)}e^{j(1/2)[-(d/b)(u+2n\Omega_M)^2+(2/b)z(u+2n\Omega_M)-(a/b)z^2]}\times\left[G^{h,3}_{M,f,\Omega_M}(u)-\sqrt{-2\pi bj}\frac{2n}{\Omega_M}e^{j(d/2b)u^2}\sum_{k\in\mathbb{Z}}e^{j(a/2b)t_k^2}f(t_k)e^{-j(1/b)t_ku}\right]du$$
$$=\sqrt{\frac{j}{2\pi b}}\int_0^{2\Omega_M}e^{j(1/2)(-(d/b)u^2+(2/b)zu-(a/b)z^2)}A^{h,3}_{z,\Omega_M}\times\left[G^{h,3}_{M,f,\Omega_M}(u)-\sqrt{-2\pi bj}\frac{2n}{\Omega_M}e^{j(d/2b)u^2}\sum_{k\in\mathbb{Z}}e^{j(a/2b)t_k^2}f(t_k)e^{-j(1/b)t_ku}\right]du,$$

where $A^{h,3}_{z,\Omega_M}$ represents the geometric series in the integral, namely,

$$A^{h,3}_{z,\Omega_M}=\sum_{n=0}^{\infty}e^{2n(\log|h|+j\arg h+j(\Omega_M/b)z)}.$$

For $z=x+jy$, due to the relation

$$|e^{2(\log|h|+i\arg h+j(\Omega_M/b)z)}|\le e^{2(\log|h|-(\Omega_M/b)y)},$$

for $y>b\log|h|/\Omega_M$, the geometric series is absolutely convergent to

$$A^{h,3}_{z,\Omega_M}=\frac{1}{1-e^{2(\log|h|+i\arg h+j(\Omega_M/b)z)}}.$$

The function $A^{h,3}_{z,\Omega_M}$ is bounded by

$$\frac{1}{1-e^{2(\log|h|-(\Omega_M/b)y)}}.$$

The factor $A^{h,3}_{z,\Omega_M}$ may be moved out of the integral while $G^{h,3}_{M,f,\Omega_M}(u)-\sqrt{-2\pi bj}(2n/\Omega_M)e^{j(d/2b)u^2}\sum_{k\in\mathbb{Z}}e^{j(a/2b)t_k^2}f(t_k)$ $e^{-j(1/b)t_ku}$ is integrable and $e^{j(1/2)(-(d/b)u^2+(2/b)zu-(a/b)z^2)}$ is bounded in the domain of integration. Therefore f^+ is well defined through the integral in the half-plane $y>b\log|h|/\Omega_M$ with

$$|f^+(z)|\le\frac{C_{\Omega_M,h}}{1-e^{2(\log|h|-(\Omega_M/b)y)}}.$$

Using the direct computation and the dominated convergence theorem, one can show that f^+ is holomorphic in the half-plane. We therefore conclude that the function f^+ is holomorphic and bounded above any line $\{t+iy|-\infty<t<\infty\}$ for $y>b\log|h|/\Omega_M$. Similarly, we can find that

$$|f^-(z)|\le\frac{C_{\Omega_M,h}}{1-e^{2(\log|h|+(\Omega_M/b)y)}},$$

for $y<-b\log|h|/\Omega_M$, and the function f^- is holomorphic and bounded below any line $\{t+jy|-\infty<t<\infty\}$ for $y>b\log|h|/\Omega_M$. It follows that restricted in the strip

$$\frac{b\log|h|}{\Omega_M}<y<-\frac{b\log|h|}{\Omega_M},$$

the function $f(z) = f^+(z) + f^-(z)$ is holomorphic and satisfies the estimate

$$|f(z)| \le \frac{C_{\Omega_M,h}}{1 - e^{2(h\log|h| + (\Omega_M/b)|y|)}}.$$

The proof is complete. $\square$

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