Generalized holomorphic Szegö kernel in 3D spheroids

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Abstract

Monogenic orthogonal polynomials over 3D prolate spheroids were previously introduced and shown to have some remarkable properties. In particular, the underlying functions take values in the quaternions (identified with $\mathbb{R}^4$), and are generally assumed to be nullsolutions of the well known Moisil-Théodoresco system. In this paper, we show that these polynomial functions play an important role in defining the Szegö kernel function over the surface of 3D (prolate) spheroids. As a concrete application, we prove an explicit expression of the monogenic Szegö kernel function over 3D (prolate) spheroids and present two numerical examples.

Keywords: Quaternion analysis, Ferrer's associated Legendre functions, Chebyshev polynomials, hyperbolic functions, prolate spheroidal harmonics, Moisil-Théodoresco system, prolate spheroidal monogenics, Szegö kernel function.

1. Introduction

Quaternion analysis arises as an attempt to generalize adequately the theory of holomorphic functions of one complex variable, and also provides the foundations to refine the theory of harmonic functions to higher dimensions. The rich and powerful structure of this function theory involves the analysis of quaternion functions that are defined in 3D open subsets and that are null-solutions of higher-dimensional Cauchy-Riemann or Dirac systems. They are often called generalized holomorphic functions; for historical reasons they are also called monogenic functions. In the meantime quaternion analysis has become proverbial in mathematics and extremely successful in many different directions. For a fairly account of this multidimensional function theory and related topics we refer the reader to [28, 29, 35, 36, 54, 55, 58].

Monogenic polynomials constitute today a well-developed and very active area of research in quaternion analysis. Actually the general problem of approximating a monogenic function by monogenic polynomials did not really come into its own until the appearance of R. Fueter's work [23, 24] in 1932, which was done by means of the hypercomplex variables. This study culminated in the works of F. Brackx, R. Delanghe and F. Sommen in [4] and H. Malonek [41] whose work has led to the development of a monogenic function by a local approximation (Taylor series) in terms of the Fueter polynomials. A survey on this work and an extensive list of references can be found in [30]. In our view much of the older theory has been given a new interpretation, and a new light has been shed upon the study of monogenic polynomial bases in the context of quaternion and Clifford analyses. For later developments the reader is referred to [1, 2, 5, 6, 7, 8, 9, 16, 17, 18, 19, 37, 38, 40, 45, 46, 47, 49, 62] and elsewhere. The standard domain, in which the previous contributions are developed, is the ball. Therefore it is of great interest to progress with a function theory on a different type (and more general) domains.

Most relevant to our study are the intimate connections between monogenic functions and spheroidal structures, and the potential flexibility afforded by a spheroid's nonspherical canonical geometry. Developments are described in the sequence of papers by H. Malonek et al. in [42, 43] and J. Morais et al. in [31, 51]. In light of this, in [48, 50] a very recent approach has been developed to discuss complete sets of monogenic orthogonal polynomials over prolate spheroids in $\mathbb{R}^3$ of which could be expressed in terms of products of Ferrer's associated Legendre functions multiplied by Chebyshev polynomial factors (see Theorem 3.1 below). This new method generalizes the harmonic polynomial systems exploited by P. Garabedian in [25] and shows connections with the classical works of S. Zaremba [63], K. Friedrichs [21, 22], and G. Szegö [59, 60]. There are several features of particular relevance here. A spheroid is a common tool in computer graphics applications including geometric modelling, 3D metamorphosis and collision detection. Particularly, the importance of constructing the underlying spheroidal monogenics stems from the role which they play in the calculation of the Bergman and Szegö kernels, and Green's monogenic functions in a spheroid. Once these functions are known, it is possible to solve both Dirichlet boundary value and conformal mapping problems in spheroidal structures arising in hydrodynamics, elasticity and electromagnetism. For example, magnetometric problems for prolate spheroids are often applied to magnetometry in some volcanic and post volcanic regions [33]. For the solvability of boundary value problems of radiation, scattering and propagation of acoustic signals and electromagnetism waves in spheroidal domains, spheroidal functions are commonly encountered. These applications have stimulated a recent surge of new techniques and have reawakened interest in approximation theory, potential theory, and the theory of partial differential equations of elliptic type for spheroidal domains [3, 10].

The article is organized as follows. Section 2 gives a brief introduction to some general definitions and basic properties of quaternion analysis. Section 3 reviews a special complete set of monogenic polynomials over prolate spheroids in $\mathbb{R}^3$. Some properties of the system are briefly discussed. The examples presented at the end of Section 3 illustrate approximations up to degree 10 for the image of a (prolate) spheroid under certain spheroidal monogenic mappings, and lead to qualitatively very good numerical results. With the help of this system, in Section 4 we define the monogenic Szegö kernel function of this class. This will be done in the spaces of square integrable functions over the quaternions. We finish the section by presenting two examples showing the applicability of our approach. Some conclusions are drawn in Section 5.

In this note, the authors confine themselves exclusively to studies on prolate spheroids in the 3D Euclidean space. Interestingly, the used methods can be either extended to oblate spheroids or even to arbitrary ellipsoids. But such a procedure would make the computations somewhat laborious, and for this reason we shall not discuss these cases here. Further investigations and extensions of this topic will be reported in a forthcoming paper. The results in this article are new in the literature and have some consequences that are now under investigation.

2. Preliminaries

2.1. Prolate spheroidal harmonics

A prolate spheroid is generated by rotating an ellipse about its major axis. For the prolate spheroidal coordinate system (μ, θ, ϕ) the coordinate surfaces are two families of orthogonal surfaces of revolution. The surfaces of constant μ are a family of confocal prolate spheroids, and the surfaces of constant θ are a family of confocal hyperboloids of revolution.

In prolate spheroidal coordinates (see e.g. E.W. Hobson [32], N.N. Lebedev [39]), the Cartesian coordinates may be parameterized by $x = x(\mu, \theta, \phi)$, $\mu \in [0, \infty)$, $\theta \in [0, \pi)$, and $\phi \in [0, 2\pi)$, such that

$$x_0 = ca\cos\theta, \quad x_1 = cb\sin\theta\cos\phi, \quad x_2 = cb\sin\theta\sin\phi,$$

where c is the prolatness parameter, and $a = \cosh\mu$, $b = \sinh\mu$, are respectively, the semimajor and semiminor axis of the generating ellipse. Using these transformation relations the surfaces of revolution for which μ is the parameter consist of the confocal prolate spheroids:

$$\mathcal{S} : \frac{x_0^2}{c^2\cosh^2\mu} + \frac{x_1^2 + x_2^2}{c^2\sinh^2\mu} = 1. \tag{1}$$

Accordingly, the surface of $\mathcal{S}$ is matched with the surface of the supporting spheroid $\mu = \alpha$ if we put $c^2\cosh^2\alpha = a^2$, and $c^2\sinh^2\alpha = b^2$. Then we obtain the prolatness parameter $c = \sqrt{a^2 - b^2} \in (0, 1)$, which means that c is the eccentricity of the ellipse with foci on the x_0-axis: $(-c, 0, 0)$, $(+c, 0, 0)$.

The particular solutions of the Laplace equation in prolate spheroidal coordinates are the well known *prolate spheroidal harmonics* [20, 32, 39]. The prolate spheroidal harmonics form a complete and orthogomal set of functions for the space interior of (1), and are a combination of products of spherical functions:

$$\Theta(\theta)\ \Xi(\mu) \begin{cases} T_l(\cos\phi) \\ \sin\phi\, U_{l-1}(\cos\phi) \end{cases}, \quad 0 \leq l \leq n, \tag{2}$$

where n is a constant, and $\Theta(\theta) := P_n^l(\cos\theta)$ and $\Xi(\mu) := P_n^l(\cosh\mu)$ satisfy the differential equations

$$\frac{d^2\Theta(\theta)}{d\theta^2} + \cot\theta\frac{d\Theta(\theta)}{d\theta} + \Big[n(n+1) - \frac{l^2}{\sin^2\theta}\Big]\Theta(\theta)\sin\theta = 0,$$

$$\frac{d^2\Xi(\mu)}{d\mu^2} + \coth\mu\frac{d\Xi(\mu)}{d\mu} - \Big[\frac{l^2}{\sinh^2\mu} + n(n+1)\Big]\Xi(\mu)\sinh\mu = 0.$$

Here P_n^l are the *Ferrer's associated Legendre functions* of the first kind of n-th degree and l-th order, T_l and U_l are, respectively, the *Chebyshev polynomials* of the first and second kinds. We remark that whenever $l = 0$, the corresponding associated Legendre function P_n^0 coincides with the Legendre polynomial P_n. In this assignment, the sign convention of including the Condon-Shortley phase is adopted:

$$P_n^l(\cosh\mu) := (-1)^l(\sinh\mu)^l\ \frac{d^l}{dt^l}[P_n(t)]\bigg|_{t=\cosh\mu}.$$

Originally, the spheroidal prolate functions (C. Flammer [20], E.W. Hobson [32], N.N. Lebedev [39]) were introduced by C. Niven in 1880 while studying the conduction of heat in an ellipsoid of revolution, which lead to a Helmholtz equation in spheroidal coordinates.

Higher dimensional extensions of the prolate spheroidal functions were first studied by D. Slepian in [57], which provided many of their analytical properties, as well as properties that support the construction of numerical schemes (see also A.I. Zayed [61]). Very recently, K.I. Kou et al. [34] introduced the continuous Clifford prolate spheroidal functions in the finite Clifford Fourier transform setting. These generalized spheroidal functions (for offset Clifford linear canonical transform) were successfully applied for the analysis of the energy concentration problem introduced in the early-sixties by D. Slepian and H.O. Pollak [56].

2.2. Quaternion analysis

In the present section we review some definitions and basic properties of quaternion analysis. For each element $\mathbf{z} \in \mathbb{H}$, it can be uniquely written in the form

$$\mathbf{z} = z_0 + z_1\mathbf{i} + z_2\mathbf{j} + z_3\mathbf{k}, \quad z_l \in \mathbb{R}\ \ (l = 0, 1, 2, 3)$$

where the imaginary units $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ stand for the elements of the basis of $\mathbb{H}$, subject to the rules

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1;$$
$$\mathbf{ij} = \mathbf{k} = -\mathbf{ji}, \qquad \mathbf{jk} = \mathbf{i} = -\mathbf{kj}, \qquad \mathbf{ki} = \mathbf{j} = -\mathbf{ik}.$$

The scalar and vector parts of $\mathbf{z}$, $\mathrm{Sc}(\mathbf{z})$ and $\mathrm{Vec}(\mathbf{z})$, are defined as the z_0 and $z_1\mathbf{i} + z_2\mathbf{j} + z_3\mathbf{k} := \underline{\mathbf{z}}$ terms, respectively. Like in the complex case, the conjugate of $\mathbf{z}$ is the element $\overline{\mathbf{z}} = z_0 - z_1\mathbf{i} - z_2\mathbf{j} - z_3\mathbf{k}$, and the norm $|\mathbf{z}|$ of $\mathbf{z}$ is defined by

$$|\mathbf{z}| = \sqrt{\mathbf{z}\overline{\mathbf{z}}} = \sqrt{\overline{\mathbf{z}}\mathbf{z}} = \sqrt{z_0^2 + z_1^2 + z_2^2 + z_3^2}.$$

Let Ω be an open subset of $\mathbb{R}^3$ with a piecewise smooth boundary, and $x := (x_0, x_1, x_2) \in \Omega$. We say that $\mathbf{f} : \Omega \longrightarrow \mathbb{H}$ such that

$$\mathbf{f}(x) = [\mathbf{f}(x)]_0 + [\mathbf{f}(x)]_1\mathbf{i} + [\mathbf{f}(x)]_2\mathbf{j} + [\mathbf{f}(x)]_3\mathbf{k} =: [\mathbf{f}(x)]_0 + \underline{\mathbf{f}}(x) \tag{3}$$

is a quaternion-valued function or, briefly, an $\mathbb{H}$-valued function, where $[\mathbf{f}]_l$ $(l = 0, 1, 2, 3)$ are real-valued functions defined in Ω. Continuity, differentiability, integrability, and so on, which are ascribed to $\mathbf{f}$ are defined componentwise.

A possibility to generalize complex holomorphy is offered by following the Riemann approach, which is introduced by means of the classical Dirac operator (also known as Moisil-Théodoresco operator [23, 44])

$$\partial = \mathbf{i}\,\frac{\partial}{\partial x_0} + \mathbf{j}\,\frac{\partial}{\partial x_1} + \mathbf{k}\,\frac{\partial}{\partial x_2}. \tag{4}$$

Nullsolutions to this operator provide us with the class of the so-called monogenic functions.

Definition 2.1 (Monogenicity)**.** *An $\mathbb{H}$-valued function $\mathbf{f}$ is called left (resp. right) monogenic in Ω if $\mathbf{f}$ is in $C^1(\Omega; \mathbb{H})$ and satisfies $\partial\mathbf{f} = 0$ (resp. $\mathbf{f}\partial = 0$) in Ω.*

Remark 2.1. *Throughout the text, and unless otherwise stated, we only use left $\mathbb{H}$-valued monogenic functions that, for simplicity, we call monogenic. Nevertheless, all results accomplished to left $\mathbb{H}$-valued monogenic functions can be easily adapted to right $\mathbb{H}$-valued monogenic functions.*

It is worthy of note that the equation $\partial \mathbf{f} = 0$ is equivalent to the system

$$
(\mathrm{MT}) \qquad \begin{cases} \dfrac{\partial [\mathbf{f}]_1}{\partial x_0} + \dfrac{\partial [\mathbf{f}]_2}{\partial x_1} + \dfrac{\partial [\mathbf{f}]_3}{\partial x_2} & = & 0 \\ \dfrac{\partial [\mathbf{f}]_0}{\partial x_0} - \dfrac{\partial [\mathbf{f}]_2}{\partial x_2} + \dfrac{\partial [\mathbf{f}]_3}{\partial x_1} & = & 0 \\ \dfrac{\partial [\mathbf{f}]_0}{\partial x_1} + \dfrac{\partial [\mathbf{f}]_1}{\partial x_2} - \dfrac{\partial [\mathbf{f}]_3}{\partial x_0} & = & 0 \\ \dfrac{\partial [\mathbf{f}]_0}{\partial x_2} - \dfrac{\partial [\mathbf{f}]_1}{\partial x_1} + \dfrac{\partial [\mathbf{f}]_2}{\partial x_0} & = & 0 \end{cases}
$$

or, in a more compact form:

$$
\begin{cases} \operatorname{div} \underline{\mathbf{f}} & = & 0 \\ \operatorname{grad} [\mathbf{f}]_0 + \operatorname{rot} \underline{\mathbf{f}} & = & 0. \end{cases}
$$

We may point out that system (MT) is called the Moisil-Théodoresco system [44] (cf. [35]), and as shown in [53] it has the form of the classical Cauchy-Riemann equations for $\mathbb{H}$-valued functions $\mathbf{f}$ so that $\partial \mathbf{f} = 0$. For the interpretation of the (MT) system in viewpoint of $\mathbb{H} \cong \mathcal{C}\ell^+_{0,3}$ we also refer to [17]. The solutions of the (MT)-system will be called (MT)-solutions, and the subspace of polynomial (MT)-solutions of degree n is denoted by $\mathcal{M}^+(\Omega; \mathbb{H}; n)$. In [58] (see also [17, Remark 1]), it is proved that $\dim \mathcal{M}^+(\Omega; \mathbb{H}; n) = n+1$. We further introduce the right linear Hilbert space of square integrable $\mathbb{H}$-valued functions in Ω that we denote by $L^2(\Omega; \mathbb{H}; \mathbb{H})$, and the space of square integrable $\mathbb{H}$-valued monogenic functions in Ω by $\mathcal{M}^+(\Omega; \mathbb{H}) := L^2(\Omega; \mathbb{H}; \mathbb{H}) \cap \ker \partial$.

In the sequel, let $\partial \mathcal{S}$ denote a prolate spheroidal boundary surface in $\mathbb{R}^3$. In the next section we review an explicit set of special monogenic polynomials, which forms a complete orthogonal system in $\mathcal{M}^+(\mathcal{S}; \mathbb{H})$ in the sense of the quaternion inner product

$$
< \mathbf{f}, \mathbf{g} >_{L^2(\mathcal{S}; \mathbb{H}; \mathbb{H})} = \int_{\partial \mathcal{S}} \overline{\mathbf{f}}\, \mathbf{g}\, \boldsymbol{\omega}\, d\sigma\,. \tag{5}
$$

Here $\boldsymbol{\omega}$ corresponds to an arbitrary positive weight function on $\partial \mathcal{S}$ that is chosen so that the indicated integral exists, and $d\sigma$ is the Lebesgue measure on $\partial \mathcal{S}$.

3. A complete orthogonal system of monogenic polynomials over 3D prolate spheroids

3.1. Prolate spheroidal monogenics

In [50] a special complete orthogonal system of (MT)-polynomial solutions is obtained over the interior of prolate spheroids in $\mathbb{R}^3$. The principal point of interest is that the orthogonality of the polynomials in question does not depend on the shape of the spheroids, but only on the location of the foci of the ellipse generating the spheroid. It is shown a corresponding orthogonality over the surface of these spheroids with respect to a suitable weight function. The explicit expressions of the mentioned (prolate) spheroidal monogenics are given by

Theorem 3.1 (see [50]). *Monogenic polynomials of the form*

$$\begin{aligned}\boldsymbol{\mathcal{S}}_{n,l}(\mu,\theta,\phi) &:= \frac{1}{2}(n+2+l)(n+1+l)(n-l+1)\,A_{n,l}(\mu,\theta)\,T_l(\cos\phi)\\ &+ \frac{1}{2}(n+2+l)\,A_{n,l+1}(\mu,\theta)\,T_{l+1}(\cos\phi)\,\mathbf{i}\\ &+ \frac{1}{2}(n+2+l)\,A_{n,l+1}(\mu,\theta)\,\sin\phi\,U_l(\cos\phi)\,\mathbf{j}\\ &- \frac{1}{2}(n+2+l)(n+1+l)(n-l+1)\,A_{n,l}(\mu,\theta)\,\sin\phi\,U_{l-1}(\cos\phi)\,\mathbf{k}\end{aligned}$$

for $l = 0, \ldots, n$, with the notation

$$A_{n,l}(\mu,\theta) := \sum_{k=0}^{\lceil \frac{n-l}{2} \rceil} \frac{(2n+1-2k)\,(n+l)_{2k}}{(n+1-l)_{2k+1}}\, P^l_{n-2k}(\cosh\mu)\, P^l_{n-2k}(\cos\theta)$$

form a complete orthogonal system over the surface of a prolate spheroid in the sense of the product (5) with weight function

$$\boldsymbol{\omega} := |c^2 - (ca\cos\theta + \mathbf{i}\,cb\sin\theta)^2|^{1/2}(\sin^2\theta + \sinh^2\mu) \quad (a > b)$$

and their norms are given by

$$
\begin{aligned}
\|\boldsymbol{\mathcal{S}}_{n,l}\|^2_{L^2(\mathcal{S};\mathbb{H};\mathbb{H})} &= \pi\,(n+2+l)\,\frac{(n+2+l)!}{(n-l)!} \\
&\times \Big[(n+1+l)(n-l+1)\,P^l_n(\cosh\alpha)\,\sinh\alpha\,\cosh\alpha\,P^l_{n+1}(\cosh\alpha) \\
&\qquad - \frac{(n+1+l)^2(n-l+1)}{2n+3}\,[P^l_n(\cosh\alpha)]^2\,\sinh\alpha \\
&\qquad + P^{l+1}_n(\cosh\alpha)\,\sinh\alpha\,\cosh\alpha\,P^{l+1}_{n+1}(\cosh\alpha) \\
&\qquad - \frac{(n+2+l)}{2n+3}\,[P^{l+1}_n(\cosh\alpha)]^2\,\sinh\alpha\Big].
\end{aligned}
$$

Remark 3.1. *For the usual applications we define these $n+1$ polynomials in a spheroid which has an infinite boundary, because $P^l_n(\cosh\mu)$ becomes infinite with μ. Of course, the results can be extended to the case of the region outside a spheroid as well. One has merely to replace the Ferrer's associated Legendre functions by the Legendre functions of second kind [32].*

In the considerations to follow we will often omit the argument and write simply $\boldsymbol{\mathcal{S}}_{n,l}$ instead of $\boldsymbol{\mathcal{S}}_{n,l}(\mu,\theta,\phi)$. These $n+1$ system constituents satisfy the first order partial differential equation

$$
\begin{aligned}
0 &= c\,\partial\boldsymbol{\mathcal{S}}_{n,l} \\
&= \mathbf{i}\left(\frac{\cos\theta\sinh\mu}{\sin^2\theta+\sinh^2\mu}\frac{\partial\boldsymbol{\mathcal{S}}_{n,l}}{\partial\mu} - \frac{\sin\theta\cosh\mu}{\sin^2\theta+\sinh^2\mu}\frac{\partial\boldsymbol{\mathcal{S}}_{n,l}}{\partial\theta}\right) \\
&+ \mathbf{j}\left(\frac{\sin\theta\cosh\mu\cos\phi}{\sin^2\theta+\sinh^2\mu}\frac{\partial\boldsymbol{\mathcal{S}}_{n,l}}{\partial\mu} + \frac{\cos\theta\sinh\mu\cos\phi}{\sin^2\theta+\sinh^2\mu}\frac{\partial\boldsymbol{\mathcal{S}}_{n,l}}{\partial\theta} - \frac{\sin\phi}{\sin\theta\sinh\mu}\frac{\partial\boldsymbol{\mathcal{S}}_{n,l}}{\partial\phi}\right) \\
&+ \mathbf{k}\left(\frac{\sin\theta\cosh\mu\sin\phi}{\sin^2\theta+\sinh^2\mu}\frac{\partial\boldsymbol{\mathcal{S}}_{n,l}}{\partial\mu} + \frac{\cos\theta\sinh\mu\sin\phi}{\sin^2\theta+\sinh^2\mu}\frac{\partial\boldsymbol{\mathcal{S}}_{n,l}}{\partial\theta} + \frac{\cos\phi}{\sin\theta\sinh\mu}\frac{\partial\boldsymbol{\mathcal{S}}_{n,l}}{\partial\phi}\right).
\end{aligned}
$$

We further assume the reader to be familiar with the fact that ∂ is a square root of the Laplace operator in $\mathbb{R}^3$ in the sense that

$$
\begin{aligned}
\Delta_3\boldsymbol{\mathcal{S}}_{n,l} &= -\partial^2\boldsymbol{\mathcal{S}}_{n,l} \\
&= \frac{1}{c^2(\sin^2\theta+\sinh^2\mu)}\left(\frac{\partial^2\boldsymbol{\mathcal{S}}_{n,l}}{\partial\mu^2} + \frac{\partial^2\boldsymbol{\mathcal{S}}_{n,l}}{\partial\theta^2} + \coth\mu\frac{\partial\boldsymbol{\mathcal{S}}_{n,l}}{\partial\mu} + \cot\theta\frac{\partial\boldsymbol{\mathcal{S}}_{n,l}}{\partial\theta}\right) \\
&+ \frac{1}{c^2\sin^2\theta\sinh^2\mu}\frac{\partial^2\boldsymbol{\mathcal{S}}_{n,l}}{\partial\phi^2}.
\end{aligned}
$$

For, the monogenic polynomials $\boldsymbol{\mathcal{S}}_{n,l}$ can be seen as a refinement of the harmonic polynomials exploited by Garabedian in [25].

It is of interest to remark at this point that the Laplacian in (prolate) spheroidal coordinates reduces to the classical Laplacian in spherical coordinates if $a = b$, which occurs as μ appoaches infinity, and in which case the two foci coincide at the origin.

3.2. Properties

This subsection summarizes some basic properties of the prolate spheroidal monogenics.

Proposition 3.1. *The monogenic polynomials $\boldsymbol{\mathcal{S}}_{n,l}$ $(l = 0, \dots, n)$ satisfy the following properties:*

1. $\boldsymbol{\mathcal{S}}_{n,l}(0,0,0) = \begin{cases} \frac{(n+2)(n+1)}{2} \sum_{k=0}^{\lceil \frac{n}{2} \rceil} \frac{(2n+1-2k)\,(n)_{2k}}{(n+1)_{2k+1}} & , \quad l = 0 \\ 0 & , \quad l > 0 \end{cases}$ *;*
2. $\boldsymbol{\mathcal{S}}_{n,l}(\mu, \theta, \pi) = \frac{(n+2+l)}{2} (-1)^l \Big[(n+1+l) A_{n,l}(\mu,\theta) - A_{n,l+1}(\mu,\theta)\mathbf{i} \Big]$*;*
3. $\boldsymbol{\mathcal{S}}_{n,1}(\mu, \theta, \phi) = \frac{(n+3)}{2} \Big[(n+2) A_{n,1}(\mu,\theta) + A_{n,2}(\mu,\theta)\mathbf{i} e^{-\mathbf{k}\phi} \Big] e^{-\mathbf{k}\phi}$*;*
4. *The polynomials $\boldsymbol{\mathcal{S}}_{n,l}$ are 2π-periodic with respect to the variable ϕ.*

Proof. For the proof of Statement 1 we simply note that

$$\boldsymbol{\mathcal{S}}_{n,l}(0,0,0) = \frac{(n+2+l)(n+1+l)}{2} A_{n,l}(0,0) + \frac{(n+2+l)}{2} A_{n,l+1}(0,0)\mathbf{i}\,,$$

where

$$A_{n,l}(0,0) = \sum_{k=0}^{\lceil \frac{n-l}{2} \rceil} \frac{(2n+1-2k)\,(n+l)_{2k}}{(n+1-l)_{2k+1}} \left[P^l_{n-2k}(1) \right]^2 \; = \; 0, \quad l > 0,$$

and $A_{n,l+1}(0,0) = 0$ for $l \geq 0$. For Statement 2 a straightforward computation shows that

$$\begin{aligned} \boldsymbol{\mathcal{S}}_{n,l}(\mu,\theta,\pi) &= \frac{(n+2+l)(n+1+l)}{2} A_{n,l}(\mu,\theta)\, T_l(-1) \\ &+ \frac{(n+2+l)}{2} A_{n,l+1}(\mu,\theta)\, T_{l+1}(-1)\mathbf{i} \\ &= \frac{(n+2+l)}{2} (-1)^l \Big[(n+1+l) A_{n,l}(\mu,\theta) - A_{n,l+1}(\mu,\theta)\mathbf{i} \Big]. \end{aligned}$$

For the proof of Statement 3 we note that

$$\begin{aligned} S_{n,1}(\mu,\theta,\phi) &= \frac{(n+3)(n+2)}{2} A_{n,1}(\mu,\theta)\left(\cos\phi - \sin\phi\mathbf{k}\right) \\ &+ \frac{(n+3)}{2} A_{n,2}(\mu,\theta)\Big((2\cos^2\phi - 1)\mathbf{i} + 2\sin\phi\cos\phi\mathbf{j}\Big) \\ &= \frac{(n+3)}{2}\Big[(n+2)A_{n,1}(\mu,\theta) + A_{n,2}(\mu,\theta)\mathbf{i}e^{-\mathbf{k}\phi}\Big]e^{-\mathbf{k}\phi}. \end{aligned}$$

The proof of Statement 4 can be performed using the periodic properties of the Chebyshev polynomials of the first and second kinds. We have then that $\boldsymbol{\mathcal{S}}_{n,l}$ are periodic with period 2π with respect to the variable ϕ. It means that for all ϕ it holds $\boldsymbol{\mathcal{S}}_{n,l}(\mu,\theta,\phi+2\pi) = \boldsymbol{\mathcal{S}}_{n,l}(\mu,\theta,\phi)$ $(l = 0,\ldots,n)$. We show now that a smaller period is not possible. Taking into account a previous representation it is clear that the period of $T_l(\cos\phi)$ and $T_{l+1}(\cos\phi)$ is $\frac{2\pi}{d}$, where d is the greatest common divisor of l and $l+1$, respectively. If $l \geq 0$ it follows that $d = 1$. □

3.3. Numerical examples

This subsection presents some numerical examples showing approximations up to degree 10 for the image of a prolate spheroid under a special (prolate) spheroidal monogenic mapping.

From the geometrical and practical point of view, we are interested to map a prolate spheroid to a domain in $\mathbb{R}^3$ (not necessarily a prolate spheroid). To begin with, a direct observation shows that for each degree $n \in \mathbb{N}_0$ the polynomial $\boldsymbol{\mathcal{S}}_{n,0}$ is monogenic from both sides ($\partial\boldsymbol{\mathcal{S}}_{n,0} = \boldsymbol{\mathcal{S}}_{n,0}\partial = 0$) and is such that $[\boldsymbol{\mathcal{S}}_{n,0}]_3 = 0$, i.e. $\boldsymbol{\mathcal{S}}_{n,0} : \mathcal{S} \longrightarrow \mathcal{A} := \text{span}_{\mathbb{R}}\{1,\mathbf{i},\mathbf{j}\} \cong \mathbb{R}^3$. We use this insight to motivate our numerical procedures for computing the image of a 3D prolate spheroid under $\boldsymbol{\mathcal{S}}_{n,0}$. We did not go further than $n = 10$, as our program becomes very time-consuming.

Figures 1, 2 and 3 visualize approximations of degrees 3, 7 and 10 for the image of a prolate spheroid with semi-axes $a = 4$ and $b = \sqrt{15}$, and centered at the origin.

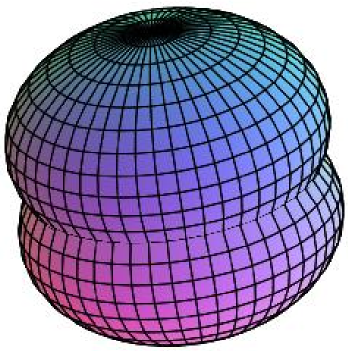

Fig. 1:

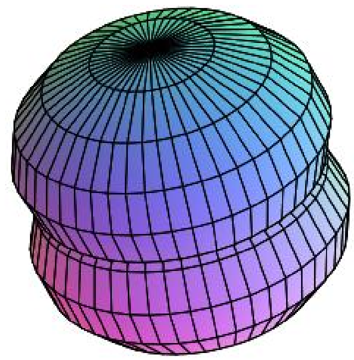

Fig. 2:

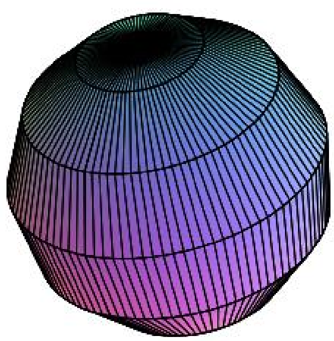

Fig. 3:

Figures 4, 5 and 6 visualize the image of a prolate spheroid with semi-axes $a = 4$ and $b = \sqrt{15}$, and centered at $(1/2, 10, 10)$.

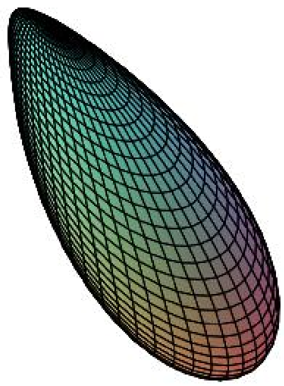

Fig. 4:

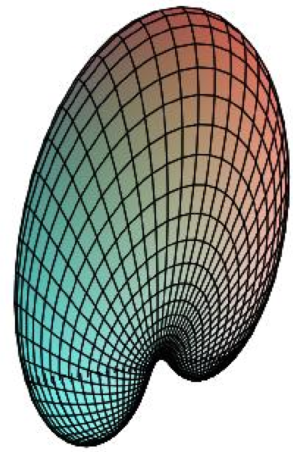

Fig. 5:

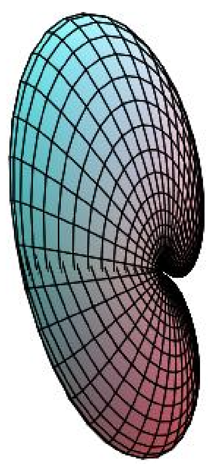

Fig. 6:

3.4. A special Fourier expansion by means of spheroidal monogenics

This subsection discusses a suitable Fourier expansion for monogenic functions over 3D prolate spheroids in terms of orthogonal monogenic polynomials. To begin with, note that for each degree $n \in \mathbb{N}_0$ the set

$$\{\boldsymbol{\mathcal{S}}_{n,l} : l = 0, \dots, n\} \tag{6}$$

is formed by $n+1 = \dim \mathcal{M}^+(\mathcal{S};\mathbb{H};n)$ monogenic polynomials, and therefore, it is complete in $\mathcal{M}^+(\mathcal{S};\mathbb{H};n)$. Furthermore, based on the orthogonal decomposition

$$\mathcal{M}^+(\mathcal{S};\mathbb{H}) = \oplus_{n=0}^{\infty} \mathcal{M}^+(\mathcal{S};\mathbb{H};n),$$

and the completeness of the system in each subspace $\mathcal{M}^+(\mathcal{S};\mathbb{H};n)$, it follows the result.

Theorem 3.2. *For each n, the set (6) forms an orthogonal basis in the subspace $\mathcal{M}^+(\mathcal{S};\mathbb{H};n)$ in the sense of the product (5) with weight function*

$$\boldsymbol{\omega} := |c^2 - (ca\cos\theta + \mathbf{i}\, cb\sin\theta)^2|^{1/2}(\sin^2\theta + \sinh^2\mu)$$

such that $a > b$. Consequently,

$$\{\boldsymbol{\mathcal{S}}_{n,l} : l = 0,\ldots,n;\ n = 0,1,\ldots\} \tag{7}$$

is an orthogonal basis in $\mathcal{M}^+(\mathcal{S};\mathbb{H})$.

Using the L^2-norms of the constructed polynomials, we can normalize them in order to get a complete orthonormal system in $L^2(\mathcal{S};\mathbb{H};\mathbb{H})$. For, from now on we shall denote by $\boldsymbol{\mathcal{S}}^*_{n,l}$ $(l = 0,\ldots,n)$ the new normalized basis functions $\boldsymbol{\mathcal{S}}_{n,l}$ in $L^2(\mathcal{S};\mathbb{H};\mathbb{H})$ endowed with the inner product (5).

Yet clearly we can easily write down the Fourier expansion of a square integrable $\mathbb{H}$-valued monogenic function over 3D prolate spheroids. Next we formulate the result.

Lemma 3.1 (Fourier Expansion in $\mathcal{S}$)**.** *Let $\mathbf{f} \in \mathcal{M}^+(\mathcal{S};\mathbb{H})$. The function $\mathbf{f}$ can be uniquely represented with the orthogonal system (7):*

$$\mathbf{f}(x) = \sum_{n=0}^{\infty}\sum_{l=0}^{n} \boldsymbol{\mathcal{S}}^*_{n,l}\, a^*_{n,l}\,, \tag{8}$$

where for each $n \in \mathbb{N}_0$, the associated (quaternion-valued) Fourier coefficients are given by

$$a^*_{n,l} \;= < \boldsymbol{\mathcal{S}}^*_{n,l}, \mathbf{f} >_{L^2(\mathcal{S};\mathbb{H};\mathbb{H})} \qquad (l = 0,\ldots,n).$$

4. Applications

4.1. Monogenic Szegö kernel function over 3D prolate spheroids

Due to the absence of a direct analogue of the famous Riemann mapping theorem for higher dimensions, at first glance it seems extremely difficult to get closed formulae for the Szegö kernel on monogenic functions. However, in 2002 D. Constales and R. Kraußhar [11] provided an important breakthrough in this research direction. As far as we know, before their work explicit formulae for the Bergman kernels were only known for very special domains, such as for instance the unit ball and the half-space. In several papers [12, 13, 14, 15], the authors were able to give explicit representation formulae for the monogenic Bergman kernel for block domains, wedge shaped domains, cylinders, triangular channels and hyperbolic polyhedron domains which are bounded by parts of spheres and hyperplanes. Recently Kraußhar et al. also managed to set up explicit formulae for the Bergman kernel of polynomial Dirac equations, including the Maxwell-, Helmholtz- and Klein-Gordon equations as special subcases, for spheres and annular shaped domains.

With the help of the above-mentioned polynomials we can now obtain an explicit representation for the monogenic Szegö kernel function over 3D prolate spheroids. Now, since the right linear set $\mathcal{M}^+(\mathcal{S};\mathbb{H})$ is a subspace of $L^2(\mathcal{S};\mathbb{H};\mathbb{H})$, to each $\xi \in \mathcal{S}$, if $\mathbf{K}_{\mathcal{S}}(x,\xi)$ is a positive definite Hermitian quaternion element in $\mathcal{M}^+(\mathcal{S};\mathbb{H})$, then it can be easily shown that there exists a uniquely determined Hilbert space of functions on $\mathcal{S}$ admitting the reproducing kernel $\mathbf{K}_{\mathcal{S}}(x,\xi)$, and such that

$$\mathbf{f}(\xi) = < \mathbf{f}, \mathbf{K}_{\mathcal{S}}(.,\xi) >_{L^2(\mathcal{S};\mathbb{H};\mathbb{H})}, \tag{9}$$

or equivalently,

$$\mathbf{f}(\xi) = \int_{\partial \mathcal{S}} \overline{\mathbf{f}}\, \mathbf{K}_{\mathcal{S}}(x,\xi)\, \boldsymbol{\omega}\, d\sigma(x),$$

for any $\mathbf{f} \in \mathcal{M}^+(\mathcal{S};\mathbb{H})$. The function $\mathbf{K}_{\mathcal{S}}(x,\xi)$, with $(x,\xi) \in \mathcal{S} \times \mathcal{S}$, is called the monogenic Szegö kernel function of $\mathcal{S}$ with respect to ξ, and is given by

$$\mathbf{K}_{\mathcal{S}}(x,\xi) = \sum_{n=0}^{\infty} \sum_{l=0}^{n} \boldsymbol{\mathcal{S}}^*_{n,l} < \boldsymbol{\mathcal{S}}^*_{n,l}, \mathbf{K}_{\mathcal{S}}(x,\xi) >_{L^2(\mathcal{S};\mathbb{H};\mathbb{H})} .$$

Next we formulate our main result.

Theorem 4.1. *The monogenic Szegö kernel of* $\mathcal{S}$

$$\mathbf{K}_{\mathcal{S}} : \mathcal{S} \times \mathcal{S} \longrightarrow \mathbb{H}$$

is given explicitly by the formula

$$\mathbf{K}_{\mathcal{S}}\Big((\mu,\theta,\phi),(\eta,\beta,\varphi)\Big) = \frac{1}{4}\sum_{n=0}^{\infty}\sum_{l=0}^{n}\frac{(n+2+l)^2(n+1+l)(n-l+1)}{\|\boldsymbol{\mathcal{S}}_{n,l}\|^2_{L^2(\mathcal{S};\mathbb{H};\mathbb{H})}}(A+B+C+D),$$

where the L^2*-norms of* $\boldsymbol{\mathcal{S}}_{n,l}$ *are explicitly known, and with the subscript coefficient functions*

$$\begin{aligned}
A &= (n+1+l)(n-l+1)A_{n,l}(\mu,\theta)\,A_{n,l}(\eta,\beta)\Big\{\cos[l(\phi-\varphi)] - \sin[l(\phi+\varphi)]\mathbf{k}\Big\},\\
B &= -\frac{A_{n,l+1}(\mu,\theta)\,A_{n,l+1}(\eta,\beta)}{(n+1+l)(n-l+1)}\Big\{\cos[(l+1)(\phi-\varphi)] + \sin[(l+1)(\phi-\varphi)]\mathbf{k}\Big\},\\
C &= A_{n,l}(\mu,\theta)\,A_{n,l+1}(\eta,\beta)\Big\{\cos[l\phi-(l+1)\varphi]\mathbf{i} - \sin[l\phi-(l+1)\varphi]\mathbf{j}\Big\},\\
D &= A_{n,l+1}(\mu,\theta)\,A_{n,l}(\eta,\beta)\Big\{\cos[(l+1)\phi+l\varphi]\mathbf{i} + \sin[(l+1)\phi+l\varphi]\mathbf{j}\Big\}.
\end{aligned}$$

for $l = 0,\ldots,n$.

Proof. Let $\mathbf{f} \in \mathcal{M}^+(\mathcal{S};\mathbb{H})$, and $(x,\xi) \in \mathcal{S}\times\mathcal{S}$ be fixed. To begin with, if we take the Fourier series expansion (8)

$$\mathbf{f}(x) = \sum_{n=0}^{\infty}\sum_{l=0}^{n}\boldsymbol{\mathcal{S}}^*_{n,l} \; < \boldsymbol{\mathcal{S}}^*_{n,l}, \mathbf{f} >_{L^2(\mathcal{S};\mathbb{H};\mathbb{H})},$$

and using the reproducing property (9), it follows that

$$\begin{aligned}
\mathbf{K}_{\mathcal{S}}(x,\xi) &= \sum_{n=0}^{\infty}\sum_{l=0}^{n}\boldsymbol{\mathcal{S}}^*_{n,l} < \boldsymbol{\mathcal{S}}^*_{n,l}, \mathbf{K}_{\mathcal{S}}(x,\xi) >_{L^2(\mathcal{S};\mathbb{H};\mathbb{H})}\\
&= \sum_{n=0}^{\infty}\sum_{l=0}^{n}\boldsymbol{\mathcal{S}}^*_{n,l}(x)\,\overline{\boldsymbol{\mathcal{S}}^*_{n,l}(\xi)}.
\end{aligned}$$

By setting $x = (\mu,\theta,\phi)$ and $\xi = (\eta,\beta,\varphi)$ we have

$$\begin{aligned}
\boldsymbol{\mathcal{S}}_{n,l}(x) &:= \frac{(n+2+l)(n+1+l)(n-l+1)}{2}A_{n,l}(\mu,\theta)\,[T_l(\cos\phi) - \sin\phi U_{l-1}(\cos\phi)\mathbf{k}]\\
&+ \frac{(n+2+l)}{2}A_{n,l+1}(\mu,\theta)\,[T_{l+1}(\cos\phi)\mathbf{i} + \sin\phi U_l(\cos\phi)\mathbf{j}]
\end{aligned}$$

and

$$\begin{aligned}
\overline{\boldsymbol{\mathcal{S}}_{n,l}(\xi)} &:= \frac{(n+2+l)(n+1+l)(n-l+1)}{2} A_{n,l}(\eta,\beta)\left[T_l(\cos\varphi)+\sin\varphi U_{l-1}(\cos\varphi)\mathbf{k}\right] \\
&- \frac{(n+2+l)}{2} A_{n,l+1}(\eta,\beta)\left[T_{l+1}(\cos\varphi)\mathbf{i}+\sin\varphi U_l(\cos\varphi)\mathbf{j}\right].
\end{aligned}$$

The quaternion multiplication of $\boldsymbol{\mathcal{S}}_{n,l}(x)$ and $\overline{\boldsymbol{\mathcal{S}}_{n,l}(\xi)}$ may be performed straightforwardly. □

4.2. Properties

This subsection describes some of the basic properties of $\mathbf{K}_{\mathcal{S}}$.

Proposition 4.1. *The monogenic Szegö kernel function $\mathbf{K}_{\mathcal{S}}$ satisfies the following properties:*

1. $\mathbf{K}_{\mathcal{S}}\Big((0,0,0),(0,0,0)\Big) =$
$$= \begin{cases} \frac{1}{2}\frac{(n+2)^2(n+1)^4}{\|\boldsymbol{\mathcal{S}}_{n,0}\|^2_{L^2(\mathcal{S};\mathbb{H};\mathbb{H})}}\left|\sum_{k=0}^{\lceil \frac{n}{2}\rceil}\frac{(2n+1-2k)\,(n)_{2k}}{(n+1)_{2k+1}}\right|^2 & , \quad l=0 \\ 0 & , \quad l>0 \end{cases};$$
2. *The function $\mathbf{K}_{\mathcal{S}}$ is 2π-periodic with respect to the variables ϕ and φ;*
3. $\mathbf{K}_{\mathcal{S}}((\mu,\theta,\phi),(\mu,\theta,\phi)) = \sum_{n=0}^{\infty}\sum_{l=0}^{n}\frac{1}{4}\frac{(n+2+l)^2}{\|\boldsymbol{\mathcal{S}}_{n,l}\|^2_{L^2(\mathcal{S};\mathbb{H};\mathbb{H})}}\Big[(n+1+l)^2(n-l+1)^2(A_{n,l}(\mu,\theta))^2+(A_{n,l+1}(\mu,\theta))^2\Big]$.

Proof. The proofs of these statements are a consequence of Statements 1 and 4 in Proposition 3.1. □

4.3. Geometric properties

The numerical results presented in Subsection 3.3. give us a motivation for computing the image of a prolate spheroid under the kernel $\mathbf{K}_{\mathcal{S}}\Big(x,(0,0,0)\Big)$ where $x=(\mu,\theta,\phi)$. A direct observation shows that

$$\mathbf{K}_{\mathcal{S}} : \mathcal{S} \longrightarrow \mathcal{A}$$

where

$$\begin{aligned}\mathbf{K}_{\mathcal{S}}((\mu,\theta,\phi),(0,0,0)) &= \sum_{n=0}^{\infty}\sum_{l=0}^{n}\boldsymbol{\mathcal{S}}^{*}_{n,l}(\mu,\theta,\phi)\,\overline{\boldsymbol{\mathcal{S}}^{*}_{n,l}(0,0,0)}\\ &= \sum_{n=0}^{\infty}\frac{(n+2)(n+1)}{2}\boldsymbol{\mathcal{S}}^{*}_{n,0}(\mu,\theta,\phi)\sum_{k=0}^{\left\lceil\frac{n}{2}\right\rceil}\frac{(2n+1-2k)\,(n)_{2k}}{(n+1)_{2k+1}}.\end{aligned}$$

Figures 7, 8 and 9 visualize approximations of degrees 3, 7 and 10 for the image of a prolate spheroid with semi-axes $a = 4$ and $b = \sqrt{15}$, and centered at the origin.

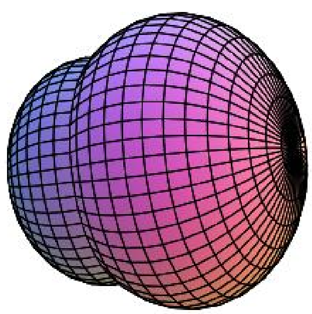

Fig. 7:

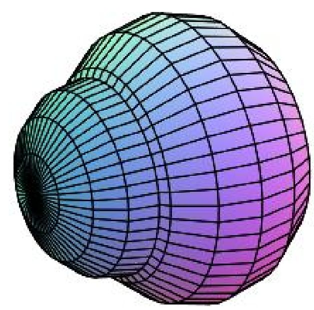

Fig. 8:

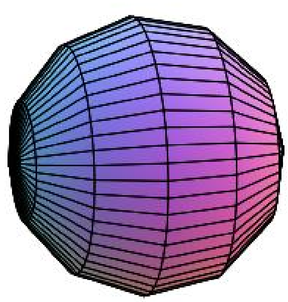

Fig. 9:

Figures 10, 11 and 12 visualize the image of a prolate spheroid with semi-axes $a = 4$ and $b = \sqrt{15}$, and centered at $(1/2, 10, 10)$.

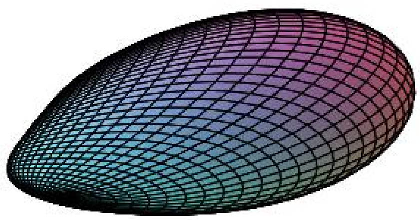

Fig. 10:

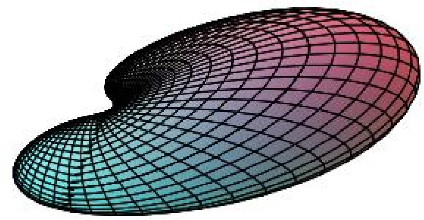

Fig. 11:

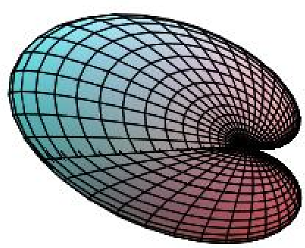

Fig. 12:

5. Conclusion

In summary, we have defined and investigated a Szegö kernel monogenic function in 3D prolate spheroids. The importance of constructing this function stems from the role which it plays when solving both basic boundary value and conformal mapping problems. It may also be useful when revisiting the so-called Szegö's kernel approach for solving 3D numerical conformal mapping problems. With the help of the properties of this function, we hope to contribute to the existence of extremal problems through this investigation and theorems for partial differential equations of elasticity. Further investigations on this topic are now under investigation and will be reported in a forthcoming paper.

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References

[1] M. Abul-Ez and D. Constales. *Basic sets of polynomials in Clifford analysis.* Complex Variables, Vol. 14, No. 1-4, 177-185 (1990).

[2] S. Bock and K. Gürlebeck. *On a generalized Appell system and monogenic power series.* Mathematical Methods in the Applied Sciences, Vol. 33, No. 4, 394-411 (2009).

[3] J.P. Boyd. *Prolate spheroidal wavefunctions as an alternative to Chebyshev and Legendre polynomials for spectral element and pseudospectral algorithms.* J. Comput. Phys., 199(2): 688716 (2004).

[4] F. Brackx, R. Delanghe and F. Sommen. *Clifford Analysis.* Pitman Publishing, Boston-London-Melbourne, 1982.

[5] I. Cação. *Constructive Approximation by Monogenic polynomials*. Ph.D. diss., Universidade de Aveiro, 2004.

[6] I. Cação, K. Gürlebeck and S. Bock. *Complete Orthonormal Systems of Spherical Monogenics - A Constructive Approach*. L. H. Son et al. (ed.), Methods of Complex and Clifford Analysis, SAS International Publications, Delhi, 241-260 (2005).

[7] I. Cação, K. Gürlebeck and S. Bock. *On Derivatives of Spherical Monogenics*. Complex Variables and Elliptic Equations, Vol. 51, No. 8-11, 847-869 (2006).

[8] I. Cação and H. Malonek. *Remarks on some properties of monogenic polynomials*. In: T.E. Simos, G.Psihoyios, Ch. Tsitouras, Special Volume of Wiley-VCH, ICNAAM, 596-599 (2006).

[9] I. Cação. *Complete orthonormal sets of polynomial solutions of the Riesz and Moisil-Teodorescu systems in* $\mathbb{R}^3$. Numer. Algor., Vol. 55, No. 2-3, 191-203 (2010).

[10] Q.Y. Chen, D. Gottlieb and J. S. Hesthaven. *Spectral methods based on prolate spheroidal wave functions for hyperbolic PDEs*. SIAM J. Numer. Anal., Vol. 43, No. 5, 1912-1933 (2005).

[11] D. Constales and R. Kraußhar. *Bergman Kernels for rectangular domains and Multiperiodic Functions in Clifford Analysis*. Mathematical Methods in the Applied Sciences, Vol. 25, No. 16-18, 1509-1526 (2002).

[12] D. Constales and R. Kraußhar. *The Bergman Kernels for the half-ball and for fractional wedge-shaped domains in Clifford Analysis*. Forum Mathematicum, Vol. 17, No. 5, 809-821 (2005).

[13] D. Constales and R. Kraußhar. *Hilbert Spaces of Solutions to Polynomial Dirac equations, Fourier Transforms and Reproducing Kernel Functions for Cylindrical Domains*. Zeitschrift fur Analysis und Ihre Anwendungen, Vol. 24, No. 3, 611-636 (2005).

[14] D. Constales and R. Kraußhar. *Bergman Spaces of higher dimensional hyperbolic polyhedron type domains I*. Mathematical Methods in the Applied Sciences, Vol. 29, No. 1, 85-98 (2006).

[15] D. Constales and R. Kraußhar. *On the Navier-Stokes equation with Free Convection in three dimensional unbounded triangular channels*. Mathematical Methods in the Applied Sciences Vol. 31, No. 6, 735-751 (2008).

[16] R. Delanghe. *On homogeneous polynomial solutions of the Riesz system and their harmonic potentials*. Complex Variables and Elliptic Equations, Vol. 52, No. 10-11, 1047-1062 (2007).

[17] R. Delanghe. *On homogeneous polynomial solutions of the Moisil-Théodoresco system in* $\mathbb{R}^3$. Computational Methods and Function Theory, Vol. 9, No. 1, 199-212 (2009).

[18] R. Delanghe, R. Lávička and V. Soucek. *On polynomial solutions of generalized Moisil-Théodoresco systems and Hodge systems*. Adv. appl. Clifford alg., Vol. 21, No. 3, 521-530 (2011).

[19] M. Falcão, J. Cruz and H. Malonek. *Remarks on the generation of monogenic functions*. K. Gürlebeck and C. Könke (Editors): Proceedings 17th International Conference on the Applications of Computer Science and Mathematics in Architecture and Civil Engineering, Weimar (2006).

[20] C. Flammer. *Spheroidal Wave Functions*. Stanford University Press, Stanford, CA, 1957.

[21] K. Friedrichs. *On certain inequalities and characteristic value problems for analytic functions and for functions of two variables*. Trans. Amer. Math. Soc., Vol. 41, No. 3, 321-364 (1937).

[22] K. Friedrichs. *An inequality for potential functions*. Amer. J. Math., Vol. 68, 581-592 (1946).

[23] R. Fueter. *Analytische Funktionen einer Quaternionenvariablen*. Comment. Math. Helv., Vol. 4, 9-20 (1932).

[24] R. Fueter. *Functions of a Hyper Complex Variable*. Lecture notes written and supplemented by E. Bareiss, Math. Inst. Univ. Zürich, Fall Semester, 1949.

[25] P. Garabedian. *Orthogonal harmonic polynomials*. Pacific J. Math. Vol. 3, No. 3, 585-603 (1953).

[26] S. Georgiev and J. Morais. *On convergence aspects of Spheroidal monogenics*. AIP Conf. Proc., Vol. 1389, 276-279 (2011).

[27] S. Georgiev and J. Morais. *An explicit formula for the monogenic Szegö kernel function on 3D spheroids*. AIP Conf. Proc., Vol. 1479, 292-295 (2012).

[28] K. Gürlebeck and W. Sprössig. *Quaternionic Analysis and Elliptic Boundary Value Problems*. Akademie Verlag, Berlin, 1989.

[29] K. Gürlebeck and W. Sprössig. *Quaternionic Calculus for Engineers and Physicists*. John Wiley and Sons, Chichester, 1997.

[30] K. Gürlebeck, K. Habetha and W. Sprößig. *Holomorphic Functions in the Plane and n-Dimensional Space*. Birkhäuser: Basel, 2008.

[31] K. Gürlebeck and J. Morais. *Geometric characterization of M-conformal mappings*. Geometric Algebra Computing: in Engineering and Computer Science, Bayro-Corrochano, Eduardo; Scheuermann, Gerik (Eds.), Springer, 1st Edition, 327-342 (2010).

[32] E.W. Hobson. *The theory of spherical and ellipsoidal harmonics*. Cambridge, 1931.

[33] M. Hvozdara and J. Vozár. *Anomalies of geomagnetic field due to a vertical prolate rotational ellipsoid*. In Contributions to Geophysics and Geodesy. Vol. 40, No. 3, 185-205 (2010).

[34] K.I. Kou, J. Morais and Y. Zhang. *Generalized prolate spheroidal wave functions for offset linear canonical transform in Clifford analysis*. Mathematical Methods in the Applied Sciences (2012), doi: 10.1002/mma.2657.

[35] V. Kravchenko and M. Shapiro. *Integral Representations for Spatial Models of Mathematical Physics*. Research Notes in Mathematics, Pitman Advanced Publishing Program, London, 1996.

[36] V. Kravchenko. *Applied quaternionic analysis. Research and Exposition in Mathematics*. Lemgo: Heldermann Verlag, Vol. 28, 2003.

[37] R. Lávicka. *Generalized Appell property for the Riesz system in dimension* 3. AIP Conference Proceedings, Vol. 1389, 291-294 (2011).

[38] R. Lávicka. *Complete orthogonal Appell systems for spherical monogenics*. Complex Analysis and Operator Theory, Vol. 6, No. 2, 477-489 (2012).

[39] N.N. Lebedev. *Special Functions and their Applications*. Dover, New York, 1972.

[40] H. Leutwiler. *Quaternionic analysis in $\mathbb{R}^3$ versus its hyperbolic modification*. Brackx, F., Chisholm, J.S.R. and Soucek, V. (ed.). NATO Science Series II. Mathematics, Physics and Chemistry, Vol. 25, Kluwer Academic Publishers, Dordrecht, Boston, London (2001).

[41] H. Malonek. *Power series representation for monogenic functions in $\mathbb{R}^{m+1}$ based on a permutational product*. Complex Variables and Elliptic Equations, Vol. 15, No. 3, 181-191 (1990).

[42] H. Malonek and M.I Falcão. *3D-mappings by means of monogenic functions and their approximation*. Mathematical Methods in the Applied Sciences, Vol. 33, No. 4, 423-430 (2010).

[43] H. Malonek and R. Almeida. *A note on a generalized Joukowski transformation*. Applied Mathematics Letters, Vol. 23, No. 10, 1174-1178 (2010).

[44] G. Moisil and N. Théodoresco. *Fonctions holomorphes dans l'espace*. Matematica (Cluj) 5, 142–159 (1931).

[45] J. Morais. *Approximation by homogeneous polynomial solutions of the Riesz system in $\mathbb{R}^3$*. Ph.D. diss., Bauhaus-Universität Weimar, 2009.

[46] J. Morais, and H.T. Le. *Orthogonal Appell systems of monogenic functions in the cylinder*. Mathematical Methods in the Applied Sciences, Vol. 34, No. 12, 1472-1486 (2011).

[47] J. Morais, H.T. Le., and W. Sprößig. *On some constructive aspects of monogenic function theory in $\mathbb{R}^4$*. Mathematical Methods in the Applied Sciences, Vol. 34, No. 14, 1685-1706 (2011).

[48] J. Morais. *A complete orthogonal system of spheroidal monogenics*. Journal of Numerical Analysis, Industrial and Applied Mathematics, Vol. 6, No. 3-4, 105-119 (2011).

[49] J. Morais and K. Gürlebeck. *Real-Part Estimates for Solutions of the Riesz System in* $\mathbb{R}^3$. Complex Variables and Elliptic Equations, Vol. 57, No. 5, 505-522 (2012).

[50] J. Morais. *An orthogonal system of monogenic polynomials over prolate spheroids in* $\mathbb{R}^3$. Mathematical and Computer Modelling (2012), doi: 10.1016/j.mcm.2012.06.020.

[51] J. Morais and M. Ferreira. *Quasiconformal mappings in 3D by means of monogenic functions*. Mathematical Methods in the Applied Sciences (2012), doi: 10.1002/mma.2625.

[52] J. Morais, S. Georgiev and K.I. Kou. *On convergence properties of 3D spheroidal monogenics*. To appear in International Journal of Wavelets, Multiresolution and Information Processing (2012).

[53] J.B. Reyes and M. Shapiro. *Clifford analysis versus its quaternionic counterparts*. Mathematical Methods in the Applied Sciences, Vol. 33, No. 9, 1089-1101 (2010).

[54] M. Shapiro and N. L. Vasilevski. *Quaternionic ψ-hyperholomorphic functions, singular operators and boundary value problems I. ψ-*hyperholomorphic function theory. Complex Variables, Vol. 27, 17-46 (1995).

[55] M. Shapiro and N. L. Vasilevski. *Quaternionic ψ-hyperholomorphic functions, singular operators and boundary value problems II*. Algebras of singular integral operators and Riemann type boundary value problems. Complex Variables, Vol. 27, 67-96 (1995).

[56] D. Slepian and H. O. Pollak. *Prolate Spheroidal Wave Functions, Fourier Analysis and Uncertainty, I.* Bell System Tech. J., Vol. 40, 43-64 (1961).

[57] D. Slepian. *Prolate spheroidal wave functions, Fourier analysis and uncertainty. IV. Extensions to many dimensions; generalized prolate spheroidal functions.* Bell System Tech. J., Vol. 43, 3009-3057, 1964.

[58] A. Sudbery. *Quaternionic analysis*. Math. Proc. Cambridge Phil. Soc., Vol. 85, 199-225 (1979).

[59] G. Szegö. *A problem concerning orthogonal polynomials*. Trans. Amer. Math. Soc., Vol. 37, 196-206 (1935).

[60] G. Szegö. *Orthogonal Polynomials*. Amer. Math. Soc. Colloq. Publ., Vol. 23, New York (1939).

[61] A.I. Zayed. *A generalization of the prolate spheroidal wave functions*. Proc. Amer. Math. Soc., Vol. 135, No. 7, 2193-2203 (2007).

[62] M. Zayed, M. Abul-Ez and J. Morais. *Generalized derivative and primitive of Cliffordian bases of polynomials constructed through Appell monomials*. Computational Methods and Function Theory, Vol. 12, No. 2, 501-515 (2012).

[63] S. Zaremba. *L'équation biharmonique et une classe remarquable de functions fondamentales harmoniques*. Bull. Int. Acad. Polon. Sci Lett. Cl Sci. Math. Nat. Sér. A. Sci. Math., 147-196 (1907).