



ON THE LOWER BOUND FOR A CLASS OF HARMONIC FUNCTIONS IN THE HALF SPACE*

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Abstract The main objective is to derive a lower bound from an upper one for harmonic functions in the half space, which extends a result of B. Y. Levin from dimension 2 to dimension $n \geq 2$. To this end, we first generalize the Carleman's formula for harmonic functions in the half plane to higher dimensional half space, and then establish a Nevanlinna's representation for harmonic functions in the half sphere by using Hörmander's theorem.

Key words harmonic function; Carleman's formula; Nevanlinna's representation for half sphere; lower bound

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1 Introduction and Main Results

A version of the well-know Liouville theorem states that if a real harmonic function $u(x)$ ($x \in \mathbf{R}^n$) has a finite upper bound, then $u(x)$ is a constant in \mathbf{R}^n . In this paper, we derive an estimate of lower bound of harmonic functions from its upper bound in the upper half space. This estimate is useful and important for studying harmonic functions and the growth properties because its associated assumption is weaker than that of the Maximum Principle.

The method to derive an estimate of a lower bound for a harmonic function from its upper bound in a half plane is using Carleman's formula and Nevanlinna's representation in the half-disk. This motivates that to obtain a similar estimate in a higher dimension, one might need to

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first generalize Carleman's formula from the half plane to the half space in \mathbf{R}^n and Nevanlinna's representation from the half disk to the half sphere in \mathbf{R}^n for a harmonic function. We know that Carleman's formula for harmonic functions in half plane is derived by taking the real part of contour integral for analytic functions. Taking the real part of the contour integral is impossible in half space of \mathbf{R}^n , but it is possible for us to use the contour integral to construct Green's function of half sphere in \mathbf{R}^n by Hörmander's method. In this paper, Carleman's formula for a harmonic function in the half space is established by Green's function, Nevanlinna's representation in the half sphere is obtained by using Hörmander's theorem, and the estimate of lower bound from an upper one in the half space is obtained by using Carleman's formula and Nevanlinna's representation in the half sphere. For the proof, we let \mathbf{R}^n denote the n -dimensional Euclidean space and $|x|$ denote the Euclidean norm. Set $x = (x_1, \dots, x_{n-1}, x_n) = (x', x_n)$ with $x' = (x_1, \dots, x_{n-1})$. Then $|x'|^2 = x_1^2 + \dots + x_{n-1}^2$, $|x|^2 = |x'|^2 + x_n^2$. For simplicity, a point x' in \mathbf{R}^{n-1} is often identified with $(x', 0)$ in \mathbf{R}^n . The boundary and closure of a subset Ω of \mathbf{R}^n are denoted by $\partial\Omega$ and $\bar{\Omega}$ respectively. Let $\Omega^+ = \{x = (x', x_n) : x_n > 0, x \in \Omega\}$. We also write $B(r)$ and $S(r)$ for the open ball and the sphere of radius $r (> 0)$ and centered at the origin, and $B(r, R) = B(R) \setminus \bar{B}(r)$ ($R > r > 0$).

In the sense of Lebesgue measure $dx' = dx_1 \cdots dx_{n-1}$, $dx = dx' dx_n$. The mirror image of $x = (x', x_n)$ in the hyperplane $\mathbf{R}^{n-1} \times \{0\} = \mathbf{R}^{n-1}$ is denoted by $x^* = (x', -x_n)$.

A twice continuously differentiable function $u(x)$ defined on Ω^+ is harmonic if $\Delta u \equiv 0$, where $\Delta = \partial_1^2 + \partial_2^2 + \dots + \partial_n^2$ with ∂_j^2 denote the second partial derivative with respect to the j th coordinate variable ($j = 1, \dots, n$).

The following theorem is well-known [1–4]:

Theorem A Let $u(z)$ be a harmonic function in the upper half-plane $\mathbf{C}_+ = \{z = x + iy = re^{i\varphi}, y > 0\}$ with continuous boundary values on the real axis. Suppose that

$$u(z) \leq Kr^\rho, \quad z \in \mathbf{C}_+, \quad r = |z| \geq 1, \quad \rho > 1,$$

and

$$|u(z)| \leq K, \quad |z| \leq 1, \quad \text{Im}z \geq 0.$$

Then

$$u(re^{i\varphi}) \geq -cK \frac{(1+r^\rho)}{\sin \varphi}, \quad re^{i\varphi} \in \mathbf{C}_+,$$

where c does not depend on K, r, φ and the function $u(z)$.

Our objective is to establish the following theorem.

Theorem 1 Let $u(x)$ be a harmonic function in the half space $\mathbf{R}_+^n = \{x \in \mathbf{R}^n, x_n > 0\}$ with continuous boundary values on the boundary $\partial\mathbf{R}_+^n$. Suppose that

$$u(x) \leq Kr^\rho, \quad x \in \mathbf{R}_+^n, \quad r = |x| \geq 1, \quad \rho > n - 1, \quad (1)$$

and

$$|u(x)| \leq K, \quad |x| \leq 1, \quad x_n \geq 0. \quad (2)$$

Then

$$u(x) \geq -cK \frac{(1+r^\rho)}{\sin^{n-1} \varphi}, \quad (3)$$

where c is a constant independent of K, r, φ and the function $u(x)$.

Remark When $n = 2$, the above theorem reduces to Theorem A.

2 Two Preliminary Lemmas

The following Lemma 1 is a generalization of the Carleman’s formula for harmonic functions in the half plane to the higher dimensional Euclidean half space.

Lemma 1 (Carleman’s formula of harmonic functions in the half space) Let $u(x)$ be a harmonic function in \mathbf{R}_+^n and continuous on $\overline{\mathbf{R}}_+^n$, and $d\sigma(x)$ be the surface element of sphere in \mathbf{R}_+^n . Then, for $R > r > 0$, we have

$$\int_{\{x: |x|=R, x_n>0\}} u(x) \frac{nx_n}{R^{n+1}} d\sigma(x) + \int_{\{x: r<|x'|<R, x_n=0\}} u(x') \left(\frac{1}{|x'|^n} - \frac{1}{R^n} \right) dx' = A_u(r, R),$$

where $A_u(r, R) = R^{-n}c_1(r) + c_2(r)$ is a function depending on r and R . Here, functions $c_1(r), c_2(r)$ are defined respectively by

$$c_1(r) = \int_{\{x: |x|=R, x_n>0\}} \left(\frac{x_n}{u(x)} - x_n \frac{\partial u(x)}{\partial r} \right) d\sigma(x),$$

$$c_2(r) = - \int_{\{x: |x|=r, x_n>0\}} \left(\frac{(n-1)x_n}{r^{n+1}} - x_n \frac{\partial u(x)}{\partial r} \right) d\sigma(x)$$

with $\frac{\partial u(x)}{\partial r} = \sum_{i=1}^n \frac{x_i}{r} \frac{\partial u}{\partial x_i}$.

Proof Applying the second Green’s formula to the harmonic functions $u(x)$ and $v(x) = \frac{x_n}{|x|^n} - \frac{x_n}{R^n}$ in the resulting sphere $B^+(r, R) = B(r, R) \cap \mathbf{R}_+^n$, we obtain

$$\int_{\partial B^+(r,R)} \left(u(x) \frac{\partial v(x)}{\partial n} - v(x) \frac{\partial u(x)}{\partial n} \right) d\sigma(x) = 0,$$

where $\partial/\partial n$ is the differentiation along the inward normal onto $B^+(r, R)$.

Since the function $v(x)$ is harmonic in $\mathbf{R}_+^n \setminus \{0\}$, the equations

$$v(x) = 0, \quad \frac{\partial v(x)}{\partial n} = \frac{nx_n}{R^{n+1}}, \quad x = (x', x_n)$$

hold on the half sphere $\{x : |x| = R, x_n \geq 0\}$, and the equations

$$v(x) = 0, \quad \frac{\partial v(x)}{\partial n} = -\frac{|x|^n}{r} \left(\frac{n-1}{r^n} + \frac{1}{R^n} \right)$$

hold on $\{x : |x| = r, x_n > 0\}$. Moreover, the equations

$$v(x) = 0, \quad \frac{\partial v(x)}{\partial n} = \frac{1}{|x'|^n} - \frac{1}{R^n}$$

are true on $\{x = (x', 0) : r < |x'| < R\}$. Thus

$$\begin{aligned} 0 &= \int_{\partial B_{r,R}^+} \left(u(x) \frac{\partial v(x)}{\partial n} - v(x) \frac{\partial u(x)}{\partial n} \right) d\sigma(x) \\ &= \int_{\{x \in \partial B_{r,R}^+, x_n > 0\}} \left(u(x) \frac{\partial v(x)}{\partial n} - v(x) \frac{\partial u(x)}{\partial n} \right) d\sigma(x) + \int_{\{x \in \partial B_{r,R}^+, x_n = 0\}} u(x') \frac{\partial v(x')}{\partial n} dx' \\ &= \int_{\{x: |x|=R; x_n>0\}} u(x) \frac{nx_n}{R^{n+1}} dx' + \int_{\{x: r<|x'|<R, x_n=0\}} u(x') \left(\frac{1}{|x'|^n} - \frac{1}{R^n} \right) dx' \\ &\quad - \int_{\{x: |x|=r; x_n>0\}} u(x) \frac{nx_n}{r^{n+1}} dx' - \int_{\{x: |x'|=r, x_n=0\}} \left(\frac{x_n}{|x|^n} - \frac{x_n}{R^n} \right) \frac{\partial u(x)}{\partial r} d\sigma(x). \end{aligned}$$

The remainder term

$$\begin{aligned}
 & - \int_{\{x: |x|=r; x_n>0\}} \frac{nx_n}{r^{n+1}} u(x) d\sigma(x) - \int_{\{x: r<|x|<R, x_n=0\}} \left(\frac{x_n}{|x|^n} - \frac{x_n}{R^n} \right) \frac{\partial u(x)}{\partial r} d\sigma(x) \\
 & = \frac{c_1(r)}{R^{n+1}} - c_2(r)
 \end{aligned}$$

is denoted by $A_u(r, R)$, a function depending on r and R . This completes the proof of Lemma 1. □

Next, we will give a precise Nevanlinna’s representation of harmonic functions in the half sphere by using Hörmander’s theorem [5].

Lemma 2 (Nevanlinna’s representation of harmonic functions in the half sphere) Let $u(x)$ be a harmonic function in \mathbf{R}_+^n . Then, on the closed half sphere $\overline{B}_R^+ = \overline{B}_R \cap \mathbf{R}_+^n$ with $\overline{B}_R = \{x : |x| = R\}$, we have

$$\begin{aligned}
 u(x) & = \int_{\{y \in \mathbf{R}_+^n : |y|=R, y_n>0\}} \frac{R^2 - |x|^2}{\omega_n R} \left(\frac{1}{|y - x|^n} - \frac{1}{|y - x^*|^n} \right) u(y) d\sigma(y) \\
 & \quad + \frac{2x_n}{\omega_n} \int_{\{y \in \mathbf{R}_+^n : r<|y'|<R, y_n=0\}} \left(\frac{1}{|y' - x|^n} - \frac{R^n}{|x|^n} \frac{1}{|y' - \tilde{x}|^n} \right) u(y') dy',
 \end{aligned}$$

where the reflection \tilde{x} of x in ∂B_R is defined by $\tilde{x} = R^2 x / |x|^2$, its direction corresponds with x , and $|x| |\tilde{x}| = R^2$.

Proof Note that $x \in \mathbf{R}^n \setminus \{0\}$, the inversion $x \rightarrow \tilde{x}$ is the identity on ∂B_R . It is also an involution, that is, $(\tilde{\tilde{x}}) = x$ for every $x \neq 0$. If x, x' and y, y' are two pairs of corresponding points, then the equation $|x| |x'| = |y| |y'|$ shows that the triangles with vertices $0, x, y$ and $0, x', y'$ are similar, so that we may interchange x and x' to obtain

$$|y|/|x'| = |x|/|y'| = |x - y|/|x' - y'|, \tag{4}$$

$$|y|/|x| = |x'|/|y'| = |x' - y|/|x - y'|. \tag{5}$$

If $|y| = R$, then $y' = y$ and (4) gives $|x - y|/|x' - y| = \sqrt{|x|/|x'|} = |x|/R$, which means that the sphere is harmonic with respect to x and x' .

Let

$$E(x) = -\frac{|x|^{2-n}}{(n-2)\omega_n}, \quad n > 2,$$

where ω_n is the area of the unit sphere in \mathbf{R}^n . We know E is locally integrable in \mathbf{R}^n . Now we define the following Green’s function

$$G_R(x, y) = E(x - y) - E\left(\tilde{x} - y \frac{|x|}{R}\right) = E(x - y) - E\left((x - \tilde{y}) \frac{|y|}{R}\right)$$

for $x, y \in \overline{B}_R$ and $x \neq y$, where the second equality follows from (5). The first expression shows that G_R is a harmonic function of y for fixed $x \neq y$, while the second expression shows that G_R is a harmonic function of x for fixed $y \neq x$. Clearly, we have

$$G_R(x, y) = 0 \quad \text{if } |x| = R \quad \text{or} \quad |y| = R, \quad \text{and} \quad G_R(x, y) \leq 0.$$

For fixed x with $|x| < R$ the inequality $|x - y| < |x||x' - y|/R$ is satisfied for all y in B_R . Define a Green's function by

$$G_R^+(x, y) = G_R(x, y^*) - G_R(x, y) = G_R(x, y) - G_R(x^*, y), \quad x, y \in B_R^+, \quad x \neq y$$

for the half ball $B_R^+ = B_R \cap \mathbf{R}_+^n$. Recall that $*$ denotes the reflection operation on the boundary plane $\partial\mathbf{R}_+^n$. It is clear that $G_R^+(x, y) = 0$ if x or y is in $\partial\mathbf{R}_+^n$, and $G_R^+(x, y) - E(x - y)$ is harmonic in x and y . We see that the Poisson kernel

$$P_R^+(x, y) = \partial G_R^+(x, y)/\partial n, \quad x \in B_R^+, \quad y \in \partial B_R^+$$

is positive, because $G_R^+(x, y) \rightarrow -\infty$ as $x \rightarrow y$, so that by the maximum principle $G_R^+(x, y) < 0$ in $B_R^+ \times B_R^+$. If $u(x)$ is harmonic near $\overline{B_R^+}$, Hörmander [2] proved that

$$u(x) = \int_{\partial B_R^+} P_R^+(x, y)u(y)d\sigma(y).$$

Below we provide an explicit calculation for $P_R^+(x, y)$.

When $|y| = R$,

$$\begin{aligned} \frac{\partial G_R(x, y)}{\partial n} &= \frac{\partial G_R(x, y)}{\partial y} = \frac{1}{\omega_n R} \frac{R^2 - |x|^2}{|y - x|^n}, \quad |y| = R, \\ \frac{\partial G_R(x, y^*)}{\partial n} &= \frac{\partial G_R(x, y^*)}{\partial y^*} = \frac{1}{\omega_n R} \frac{R^2 - |x|^2}{|y^* - x|^n}, \quad |y^*| = R. \end{aligned}$$

So

$$\begin{aligned} P_R^+(x, y) &= \frac{\partial G_R^+(x, y)}{\partial n} = \frac{\partial G_R(x, y)}{\partial n} - \frac{\partial G_R(x^*, y)}{\partial n} \\ &= \frac{R^2 - |x|^2}{\omega_n R} \left(\frac{1}{|y - x|^n} - \frac{1}{|y - x^*|^n} \right). \end{aligned}$$

When $|y| < R, \quad y_n = 0$,

$$\begin{aligned} \frac{\partial G_R(x, y)}{\partial n} &= -\frac{\partial G_R(x, y)}{\partial y_n} \Big|_{y_n=0} = \frac{x_n}{\omega_n} \left[\frac{1}{|y - x|^n} - \frac{1}{|y - \tilde{x}|^n} \frac{R^n}{|x|^n} \right], \\ \frac{\partial G_R(x^*, y)}{\partial n} &= -\frac{\partial G_R(x^*, y)}{\partial y_n} \Big|_{y_n=0} = -\frac{x_n}{\omega_n} \left[\frac{1}{|y - x^*|^n} - \frac{1}{|y - \tilde{x}^*|^n} \frac{R^n}{|x|^n} \right]. \end{aligned}$$

Hence,

$$\begin{aligned} P_R^+(x, y) &= \frac{\partial G_R^+(x, y)}{\partial n} = \frac{\partial G_R(x, y)}{\partial n} - \frac{\partial G_R(x^*, y)}{\partial n} \\ &= \frac{2x_n}{\omega_n} \left(\frac{1}{|y - x|^n} - \frac{1}{|y - \tilde{x}|^n} \frac{R^n}{|x|^n} \right), \end{aligned}$$

$$\begin{aligned} u(x) &= \int_{\{y \in \mathbf{R}_+^n: |y|=R, y_n>0\}} \frac{R^2 - |x|^2}{\omega_n R} \left(\frac{1}{|y - x|^n} - \frac{1}{|y - x^*|^n} \right) u(y)d\sigma(y) \\ &\quad + \frac{2x_n}{\omega_n} \int_{\{y \in \mathbf{R}_+^n: r<|y'|<R, y_n=0\}} \left(\frac{1}{|y' - x|^n} - \frac{R^n}{|x|^n} \frac{1}{|y' - \tilde{x}|^n} \right) u(y')dy'. \end{aligned}$$

The proof of Lemma 2 is completed. □

2 Proof of Theorem 1

We apply Lemma 1 to the harmonic function $u(x)$ to obtain

$$\begin{aligned} & \frac{n}{R^{n+1}} \int_{\{x: |x|=R, x_n>0\}} x_n u^-(x) d\sigma(x) \\ & + \int_{\{x: r<|x'|<R, x_n=0\}} u^-(x') \left(\frac{1}{|x'|^n} - \frac{1}{R^n} \right) dx' + A_u(r, R) \\ = & \frac{n}{R^{n+1}} \int_{\{x: |x|=R, x_n>0\}} x_n u^+(x) d\sigma(x) + \int_{\{x: r<|x'|<R, x_n=0\}} u^+(x') \left(\frac{1}{|x'|^n} - \frac{1}{R^n} \right) dx'. \end{aligned} \tag{6}$$

Here and in the sequel, $u^- = (-u)^+$, i.e., $u = u^+ - u^-$, and the remainder term is

$$A_u(r, R) = \frac{c_1(r)}{R^{n+1}} - c_2(r)$$

in which c_1, c_2 are functions depending only on r .

The terms on the right-hand side of (6) can be estimated by using (1):

$$\begin{aligned} & \frac{n}{R^{n+1}} \int_{\{x: |x|=R, x_n>0\}} x_n u^+(x) d\sigma(x) \leq n\omega_n K R^{\rho-1}, \\ & \int_{\{x: r<|x'|<R, x_n=0\}} u^+(x') \left(\frac{1}{|x'|^n} - \frac{1}{R^n} \right) dx' \leq \frac{n\omega_{n-1} K R^{\rho-1}}{(\rho-1)(\rho+n-1)}. \end{aligned}$$

Thus, for $R \geq 2$ and (6), we can obtain

$$\frac{n}{R^{n+1}} \int_{\{x: |x|=R, x_n>0\}} x_n u^-(x) d\sigma(x) \leq cR^{\rho-1}, \tag{7}$$

$$\begin{aligned} & \int_{\{x: r<|x'|<R, x_n=0\}} \frac{u^-(x')}{|x'|^n} dx' \\ & \leq \frac{2^n}{2^n-1} \int_{\{x: r<|x'|<R, x_n=0\}} u^-(x') \left(\frac{1}{|x'|^n} - \frac{1}{(2R)^n} \right) dx' \leq cR^{\rho-1}, \end{aligned} \tag{8}$$

where c is a constant independent of the variables.

Lemma 2 and estimates (7) and (8) allow us to find a lower bound for the function $u(x)$.

We have

$$\begin{aligned} -u(x) = & \frac{R^2 - |x|^2}{\omega_n R} \int_{\{y: |y|=R, y_n>0\}} \left(\frac{1}{|y-x|^n} - \frac{1}{|y-x^*|^n} \right) (-u(y)) d\sigma(y) \\ & + \frac{2x_n}{\omega_n} \int_{\{y: r<|y'|<R, y_n=0\}} \left(\frac{1}{|y'-x|^n} - \frac{R^n}{|x|^n} \frac{1}{|y'-\tilde{x}|^n} \right) (-u(y')) dy'. \end{aligned} \tag{9}$$

Both integral kernels are positive, since they are derivatives of the Green function with respect to the outward normal. This permits us to replace $-u$ by u^- in the integrals and simultaneously to transform identity (9) into the inequality. In the following we will estimate the kernels appearing in the integrals.

Let $r = |x| \geq 2$, $R = 2r$. Then, by the mean value theorem for derivatives, we obtain the estimate of the kernel in the first integral

$$\begin{aligned} & \frac{R^2 - |x|^2}{\omega_n R} \left(\frac{1}{|y - x|^n} - \frac{1}{|y - x^*|^n} \right) \\ &= \frac{R^2 - r^2}{\omega_n R} \left(\frac{1}{|R^2 - 2Rr \cos(\theta - \varphi) + r^2|^{n/2}} - \frac{1}{|R^2 - 2Rr \cos(\theta + \varphi) + r^2|^{n/2}} \right) \\ &\leq \frac{R^2 - r^2}{\omega_n R} \frac{2Rr \sin \theta \sin \varphi}{(R - r)^{n+2}} \leq \frac{c \sin \theta}{R^{n-1}}, \end{aligned} \tag{10}$$

where $x_n = |x| \cos \theta$, $y_n = |y| \cos \varphi$, and c is a constant depending only on n . So

$$\begin{aligned} & \int_{\{y: |y|=R, y_n > 0\}} \frac{R^2 - |x|^2}{\omega_n R} \left(\frac{1}{|y - x|^n} - \frac{1}{|y - x^*|^n} \right) (-u(y)) d\sigma(y) \\ &\leq \int_{\{y: |y|=R, y_n > 0\}} \frac{R^2 - |x|^2}{\omega_n R} \leq \left(\frac{1}{|y - x|^n} - \frac{1}{|y - x^*|^n} \right) u^-(y) d\sigma(y) \\ &\leq \frac{c}{R^n} \int_{\{y: |y|=R, y_n > 0\}} u^-(y) y_n d\sigma(y) \leq cR^\rho, \end{aligned}$$

where c is a constant independent of the variables.

Using the inequalities

$$|y - x|^n \geq \begin{cases} |y|^n \sin^n \varphi, & y \in \mathbf{R}_+^n, \quad y_n = 0, \quad |y| \geq 1, \\ \frac{|x|^n}{2^n}, & y \in \mathbf{R}_+^n, \quad y_n = 0, \quad |y| < 1, \end{cases}$$

and setting $r = |x| \geq 2$, $R = 2r$, we obtain the estimate of the kernel in the second integral:

$$\frac{x_n}{\omega_n} \left(\frac{1}{|y' - x|^n} - \frac{1}{|y' - \tilde{x}|^n} \frac{R^n}{|x|^n} \right) \leq \begin{cases} \frac{x_n}{\omega_n} \frac{1}{|y|^n \sin^n \varphi}, & y \in \mathbf{R}_+^n, \quad y_n = 0, \quad |y| \geq 1, \\ \frac{x_n}{\omega_n} \frac{2^n}{|x|^n}, & y \in \mathbf{R}_+^n, \quad y_n = 0, \quad |y| < 1. \end{cases}$$

For the second integral, we have, for $|y| \geq 1$,

$$\begin{aligned} & \int_{\{y: r < |y'| < R, y_n = 0\}} \frac{2x_n}{\omega_n} \left(\frac{1}{|y' - x|^n} - \frac{1}{|y' - \tilde{x}|^n} \frac{R^n}{|x|^n} \right) (-u(y')) dy' \\ &= \int_{\{y: r < |y'| < R, y_n = 0\}} \frac{2x_n}{\omega_n} \left(\frac{1}{|y' - x|^n} - \frac{1}{|y' - \tilde{x}|^n} \frac{R^n}{|x|^n} \right) u^-(y') dy' \\ &\leq cK \frac{1 + R^\rho}{\sin^{n-1} \varphi}. \end{aligned} \tag{11}$$

Substituting inequalities (10) and (11) into (9), we obtain

$$-u(x) \leq AR^\rho + cK \frac{1 + r^\rho}{\sin^{n-1} \varphi} + \frac{2^n}{\omega_n r^{n-1}} \int_{-1}^1 (u^-(t) + u^-(-t)) dt.$$

By using (2), the third integral is bounded uniformly with respect to x' .

Therefore, for $|x| \geq 2$, we get

$$u(x) \geq -cK \frac{1 + r^\rho}{\sin^{n-1} \varphi}.$$

Thus Theorem 1 is proved.

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