A Note on the Existence and Nonexistence of Globally Bounded Classical Solutions for Nonisentropic Gas Dynamics

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Abstract

Existence of globally bounded classical solution for nonisentropic gas dynamics system has long been studied, especially in the case of polytropic gas. In [4], Liu claimed that sufficient condition has been established. However, we find that the argument he used is not true in general. In this paper, we give a counter example of his argument. Hence, his claim is not valid. We believe that it is difficult to impose general conditions on the initial data to obtain globally bounded classical solution.

It is well known that for scalar and pair of conservation laws, if the initial data satisfy certain conditions, globally bounded classical solutions exist. In some cases, those are also necessary conditions. For system with more than 2 equations, general result is difficult to obtain. In the case of nonisentropic polytropic gas dynamics system in Lagrangian coordinates,

$$\begin{aligned} v_t - u_x &= 0, \\ u_t + p(v,s)_x &= 0, \\ s_t &= 0, \end{aligned} \tag{1}$$

sufficient conditions are established (see [2],[3]), so that solutions without shocks exist globally. However, the solutions are still not classical since the derivatives are not continuous. For the system (1), if $p(v,s) = \Phi(s)\exp(-v)$, Zhu [6] proves a general result on the existence of global classical solutions. In [5], Zhao shows that globally classical solutions can exist for the Polytropic gas dynamics system, provided that the initial data satisfy some conditions. However, those conditions, as pointed out by Liu in [4], are inconsistent. In [4], Liu strengthens an argument of Zhao and claims that he obtains the global existence of bounded classical solutions. In section 1 of the present paper, by an example, we show that his change is not true in general and hence his claim is not valid. In section 2, we analyze an equation in [3]. Through our study, we believe that, in order to keep the derivatives of the solutions finite, the initial data must be delicately restricted.

1 A counter example

In [5], Zhao establishs the following maximal principle for a special type of quasilinear hyperbolic system. Namely,
Assume that $(u(x,t),\, v(x,t))$ *is a* C^0 *solution to the following Cauchy problem:*

$$\begin{array}{rcl} u_t + \lambda(x,\,t)u_x &=& a(x,\,t)(v-u), \\ v_t + \mu(x,\,t)v_x &=& b(x,\,t)(u-v), \\ u(x,0) = u_0(x), & & v(x,0) = v_0(x), \end{array} \tag{2}$$

i.e. $(u(x,\,t), v(x,\,t))$ *satisfies the corresponding integral equations along the characteristics, where* $\lambda(x,\,t)$, $\mu(x,\,t)$, $a(x,\,t)$ *and* $b(x,\,t)$ *are bounded continuous functions, and* $a(x,\,t) \geq 0,\ b(x,\,t) \geq 0,$ *then*

$$a_0 \leq u(x,\,t) \leq b_0, \qquad a_0 \leq v(x,\,t) \leq b_0, \tag{A}$$

provided that

$$a_0 \leq u_0(x) \leq b_0, \qquad a_0 \leq v_0(x) \leq b_0, \tag{C}$$

where $a_0,\, b_0$ *are constants.*

In the deduction for the global existence of classical solution (see [4] pp.31), F.G. Liu changes condition (C) of the above theorem as

$$m(x) \leq u_0(x) \leq M, \qquad m(x) \leq v_0(x) \leq M, \tag{C)$'$}$$

and asserts, without proof, that the solutions will satisfy

$$m(x) \leq u(x,\,t) \leq M, \qquad m(x) \leq v(x,\,t) \leq M. \tag{A)$'$}$$

However, this change fails even for the following simple system.

Consider a special case of (2), when $\lambda(x,\,t) \equiv -\mu(x,\,t) \equiv -1$, $a(x,\,t) \equiv b(x,\,t) \equiv 1 > 0$, that is

$$u_t - u_x \;=\; v - u, \qquad v_t + v_x \;=\; u - v, \tag{3}$$

and initial data

$$(u_0(x),\, v_0(x)) = \begin{cases} (1,0) & x > -\frac{1}{2} \\ (-(x-\frac{1}{2}),\, x+\frac{1}{2}) \quad \text{if} & |x| \leq \frac{1}{2}. \\ (0,1) & x < \frac{1}{2} \end{cases}$$

If we take $m(x) = \min\{u_0(x),\, v_0(x)\}$, it is trivial that $(C)'$ holds. Now differentiating the second equation of (3) with respect to x and t yields

$$v_{tt} + v_{xt} + v_t \;=\; u_t, \qquad v_{xt} + v_{xx} + v_x \;=\; u_x.$$

Their difference gives $v_{tt} - v_{xx} + v_t - v_x \;=\; u_t - u_x$. Using (3), v can be found to satisfy

$$v_{tt} - v_{xx} + 2v_t + v \;=\; u_t - u_x + u \;=\; v. \tag{4}$$

It is easy to check that

$$v(x,t) = \left(x + \frac{1}{2}\right)\exp(-2t), \qquad u(x,\,t) = -\left(x - \frac{1}{2}\right)\exp(-2t)$$

is a C^0 solution of (4) in B_N, where B_N is the determining region of the interval $(-1/2,\,1/2)$ on the initial line $t = 0$. However, whenever $(0,\,t) \in B_N$, we have $u(0,t) = v(0,t) = \frac{1}{2}\exp(-2t)$, but $m(0) = u_0(0) = v_0(0) = \frac{1}{2}$. That is $m(0) \ge u(0,\,t),\ m(0) \ge v(0,\,t)$. This contradicts $(A)'$.

Since Liu's proof depends essentially on the 'strengthened' conclusion $(A)'$, it follows that his proof is not valid.

2 Analysis of the problem

Now, we are going to analyze the problem. Consider the nonisentropic gas dynamics system (1) for the polytropic gas, where the pressure function is taken as $p(v,s) = \mathrm{e}^s\, v^{-\gamma}$. We consider the case $\gamma = 3$ only, for $\gamma \neq 3$, similar argument holds. With '$\cdot$' denotes $\partial/\partial t + \mu(\partial/\partial x)$, '$\acute{}$' denotes $\partial/\partial t - \mu(\partial/\partial x)$, we have (see [3] (3.20))

$$\acute{R} = -2R^2 + \frac{\mu^2}{48}s_{xx} - \frac{\mu^2}{384}s_x^2, \tag{5}$$

$$\dot{T} = -2T^2 + \frac{\mu^2}{48}s_{xx} - \frac{\mu^2}{384}s_x^2, \tag{6}$$

where $\mu = \sqrt{-p_v} > 0,\ R = (2\Phi)^{-1}(u-\Phi)_t - \frac{\mu}{48}s_x,\ T = (2\Phi)^{-1}(-u-\Phi)_t + \frac{\mu}{48}s_x,\ \Phi(v,s) = \int_v^\infty \sqrt{-p_v(y,s)}dy$. The bounds of R and T give the bounds for the derivatives of the solutions.

Since (5) and (6) are of the same type, we will analyze (6) only. Suppose that $s(x) \in C^2(\mathbf{R})$, $\int_{-\infty}^{\infty} s_x^2(x)dx = K < \infty$ and $\lim_{x\to\pm\infty} \max\{\,|s_x(x)|, |s_{xx}(x)|\} = 0$. We are going to prove that if μ is a constant, T must blow up in a finite time for sufficiently small $\xi < 0$. Thus it will be very difficult to impose a general condition on the initial data so that globally bounded classical solutions exist.

In fact, if we consider the solution T of (6) with $T(x,0) = T_0(x)$ and let $t(x;\xi) = (x-\xi)/\mu$, then $\bar{T}(x;\xi) = T(x, t(x,\xi))$ is a solution of

$$\frac{d}{dt}\bar{T}(x;\xi) = -\frac{2}{\mu}\bar{T}^2(x;\xi) + \frac{\mu}{48}s_{xx} - \frac{\mu}{384}s_x^2, \quad x > \xi, \quad \xi \in \mathbf{R}, \tag{7}$$

subject to the initial condition $\bar{T}(x;\xi)|_{x=\xi} = \bar{T}_0(\xi)$. Integrating (7) from ξ to x, we get

$$\bar{T}(x;\xi) = \bar{T}_0(\xi) - \frac{2}{\mu}\int_\xi^x \bar{T}^2(x;\xi)dx + \frac{\mu}{48}\{s_x(x) - s_x(\xi)\} - \int_\xi^x \frac{\mu}{384}s_x^2 dx, \quad x > \xi.$$

Hence

$$\bar{T} < |\bar{T}_0| + |\frac{\mu}{48}s_x(x)| + |\frac{\mu}{48}s_x(\xi)| - \frac{\mu}{384}\int_\xi^x s_x^2(x)dx. \tag{8}$$

By the boundness of the initial data, it is easy to deduce (see [3]) that $\bar{T}_0 \to 0$ as $x \to \pm\infty$. So for any $\epsilon > 0$ and sufficiently large N, we have

$$|\bar{T}_0| < \epsilon, \qquad \text{as } |x| > N. \tag{9}$$

Note that the assumptions imposed on $s(x)$ imply that for $\eta > N$

$$\int_{-\eta}^{\eta} s_x^2(x)dx > K - \epsilon, \qquad |s_x(\pm\eta)| < \epsilon, \qquad |s_{xx}(\pm\eta)| < \epsilon. \tag{10}$$

By (8),(9),(10) for sufficiently small ϵ and $\xi < -N$, $x > N$, it follows that

$$\bar{T}(x;\xi) < \Big[1 + 2\Big(\frac{\mu}{48}\Big) + \frac{\mu}{384}\Big]\epsilon - \frac{\mu}{384}K < -\Big(\frac{\mu}{384}\Big)K/2,$$

Hence

$$\frac{d}{dt}\bar{T}(x;\xi) = -\frac{2}{\mu}\bar{T}^2(x;\xi) + \frac{\mu}{48}s_{xx} - \frac{\mu}{384}s_x^2 < -\frac{1}{\mu}\bar{T}^2(x;\xi).$$

These together imply that $\bar{T}$ and so T blow up in a finite time.

By the assumptions imposed on $s(x)$, we see that the sign of s_{xx} must change. Hence the situation becomes complicated when μ is not a constant. In our case, $\mu = \sqrt{-p_v} = C\exp(-s(x)/2)(2\Phi)^2$. From the above discussion, in order that a classical solution exists globally, along every characteristic curve, the value of $\mu\, s_{xx}$, when passing through the region $\Pi_+ = \{x|s_{xx} > 0\}$ must be large enough to dominate the negative effect caused by $-\mu\, s_x^2$ and $\mu\, s_{xx}$ in the region $\Pi_- = \{x|s_{xx} < 0\}$. From [3] (pp.495, (2.9),(2.10)), we know that

$$\begin{aligned} 2\Phi(x,t) = \ & [u_0(x) + \Phi(v_0(x), s(x))]|_{x=x_2} - [u_0(x) - \Phi(v_0(x), s(x))]|_{x=x_1} \\ & + \tfrac{1}{12}\left(\textstyle\int_{\Gamma_2} \dot{s}(2\Phi)dx - \int_{\Gamma_1} \ddot{s}(2\Phi)dx\right), \end{aligned} \tag{11}$$

where Γ_i is the ith characteristics passing through (x,t), $(x_i, 0)$ is the intersection point of Γ_i and x-axis, the integration is along Γ_i, $i = 1, 2$. By (11), we know that the value of

Φ at (x,t) is determined by the value of the initial data in the interval $[x_2, x_1]$. Even for a fixed x, this interval can cover any part of the x-axis provided that t is correspondingly large. In other words, the value of the initial data at every point will affect the value of μ along a particular characteristic curve in both Π_+ and Π_- for sufficiently large t. Thus it seems that only delicately restricted initial data will admit globally bounded classical solutions and it will be very difficult to impose a general condition on the initial data so that bounded classical solutions exist globally.

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