MICRO-LOCAL STRUCTURE AND TWO KINDS OF WAVELET CHARACTERIZATIONS ABOUT THE GENERALIZED HARDY SPACES

TAO QIAN AND QI-XIANG YANG

Abstract. In this paper, we prove two kinds of wavelet characterizations of the predual spaces of the Morrey spaces through considering some micro-local quantities of the predual spaces.

1. Introduction

In chapter 5 of the celebrated book [13], Y. Meyer commented that B. Maurey did the pioneer work on the relation of the Hardy space H^1 and the L^1 unconditional convergence in [12]. L. Carleson [2] and P. Wojtaszczyk [21] also found some unconditional basis to establish such relation. According to the idea of Y. Meyer, through the wavelet characterization without a family of Borel measures, we can establish roughly the following: For function f in the Hardy space H^1, $\|f\|_{H^1}$ is equivalent to $\|f\|_{L^1} + \sum\limits_{i=1}^{n} \|R_i f\|_{L^1}$ where $R_i f$ is the Riesz transform of function f. Y. Meyer used several sections to prove such wavelet characterizations. In this paper, we consider the generalized Hardy spaces, viz., the predual spaces of the Sobolev type Morrey spaces.

We start by giving the definition of the Sobolev type Morrey spaces. For $s \in \mathbb{R}$, denote by $f_{s,Q} = |Q|^{-1} \int_Q (-\Delta)^{\frac{s}{2}} f(x) dx$ the mean value of the function $(-\Delta)^{\frac{s}{2}} f$ on the cube Q. For $\alpha \geq 0, s \in \mathbb{R}, 1 \leq p < \infty$, let

$$B_{\alpha,s,p,Q} f = |Q|^{\frac{\alpha}{n}} (|Q|^{-1} \int_Q |(-\Delta)^{\frac{s}{2}} f(x) - f_{s,Q}|^p dx)^{\frac{1}{p}}.$$

We denote also $B_{\alpha,p,Q} f = B_{\alpha,0,p,Q} f$. The Morrey spaces $M_{\alpha,s,p}$ and the vanishing Morrey spaces $M^0_{\alpha,s,p}$ are defined as follows.

Definition 1. *Let Ξ be the set of all the cubes. Define*

(i) $f \in M_{\alpha,s,p}$, *if* $\sup\limits_{Q\in\Xi} B_{\alpha,s,p,Q} f < \infty$.

(ii) $f \in M^0_{\alpha,s,p}$, *if* $\sup\limits_{Q\in\Xi} B_{\alpha,s,p,Q} f < \infty$ *and* $\lim\limits_{Q\in\Xi, |Q|\to 0 \text{ or } \infty} B_{\alpha,s,p,Q} f = 0$.

The classical Morrey spaces $M_{\alpha,p} = M_{\alpha,0,p}$ were introduced in 1938 to study certain partial differential equations. The BMO space was introduced by F. John and L. Nirenberg in 1961. See [9] and [15]. If $\alpha = s = 0$, then, for $1 \leq p < \infty$, different p produce the same space:

Professor Qian acknowledges the financial support Research Grant of University of Macau UL017/08-Y3/MAT/QT01/FST, Macau Science and Technology Fund FDCT/056/2010/A3. The corresponding author Yang Qi-Xiang is supported by NSFC No. 11271209.

1991 *Mathematics Subject Classification.* 42B20, 30H25.

Key words and phrases. Generalized Hardy spaces; micro-local structure; wavelet; characterization.

$M_{0,0,p} = BMO$ and $M^0_{0,0,p} = VMO$. We know that $(VMO)' = H^1$ and $(H^1)' = BMO$. See [7] and [9]. If $p\alpha = n$, then, modulo polynomials, they become the classical Sobolev spaces. $M_{\alpha,0,p} = L^p$. If $1 \leq p < \frac{n}{\alpha}$, then $M_{\alpha,s,p}$ are different from each other for all different triples (α, s, p). These spaces are different from the spaces discussed in the celebrated book of Triebel [20]. Further, if we replace $|Q|^{\frac{\alpha}{n}}$ by some positive function $\phi(|Q|)$, we get $M_{\phi,p}$. It is shown in [30] that there exists a large collection $M_{\phi,1}$ that have no unconditional basis.

The Q spaces $Q_\alpha = M_{\alpha,\alpha,2}$ were introduced by R. Aulaskari, J. Xiao and R. H. Zhao. The Q spaces come from the complex analysis, and were extended to $\mathbb{R}^n$ by M. Essen, S. Janson, L. Z. Peng and J. Xiao. Now many authors are interesting in studying the Q spaces, see [1, 3, 6, 8, 16, 23] and the references therein. Ignoring a difference of fractional differential, Wu-Xie [23], Xiao [24], Peng-Yang [16, 26] proved that Q spaces are in fact Morrey spaces in view of complex analysis, real analysis and wavelet theory, respectively. Q. X. Yang generalized Q spaces to $\dot{M}^{s,q}_p$ by wavelets. See §5.5 of [26]. D. C. Yang and many researchers studied systematically the properties of the Besov type Morrey spaces, the Triebel-Lizorkin type Morrey spaces and other spaces. For an overview, we refer the readers to [4, 17, 27, 28, 31, 32].

In the famous paper [7], C. Fefferman and E. M. Stein proved that the BMO space is the dual space of the Hardy space H^1. The classical Hardy spaces play an important role in harmonic analysis and PDE. See also [7, 13, 19] and the references therein. The predual spaces of Morrey spaces have a special micro-local structure. E. A. Kalita found some predual spaces by using a family of Borel measures in 1998. Afterwards, the predual of Morrey spaces were studied extensively by various skills. See [3, 5, 10, 16, 23, 25, 31, 32].

The classical Hardy spaces are special Triebel-Lizorkin spaces. The generalized Hardy spaces are neither Besov spaces nor Triebel-Lizorkin spaces. The generalized Hardy spaces have a very different micro-local structure.

(i) For the classical Hardy spaces, the micro-local structure does not play an explicit role. When we consider the micro-local structure of the generalized Hardy spaces, we found that, the high frequency part and the low frequency part make different contributions to the norm.
(ii) For the generalized Hardy spaces, D. C. Yang and W. Yuan used a group of Borel measures to get some wavelet characterizations. But for the Hardy space H^1, the L^1 unconditional convergence produces a wavelet characterization without involving a family of Borel measures. See [13] and [25].

In this paper, we obtain two kinds of wavelet characterizations through considering three kinds of micro-local quantities of the generalized Hardy spaces. To stress on the main idea, we consider only the cases $p = 2$, $s = 0$ and $0 < 2\alpha < n$ in the two last sectyions of this paper. For simplicity, we denote $M_\alpha = M_{\alpha,0,2}$ and $M^0_\alpha = M^0_{\alpha,0,2}$.

There have been many methods to determine whether a distribution f belongs to the generalized Hardy spaces. For example, G. Dafni and J. Xiao [5] used the Hausdorff capacity relative to the Careleson measure. Another method is to use the classical atomic decomposition idea like what C. Fefferman and E. M. Stein did in [7]. See [11] and [26]. Such obtained predual spaces are Banach spaces. But we did not know how to use these ideas to get the wavelet characterization of the predual spaces of the Morrey spaces. W. Yuan, W. Sickel and D. C. Yang adopted a family of Borel measure to determine whether a function belongs to the predual spaces or not. The induced norm shows that the predual spaces are only pseudo-Banach spaces, see [10, 25] and [32].

We introduce a new method to study the generalized Hardy spaces in this paper. The word '*micro-local*' appeared first in the study of PDE, we borrow this idea to study the predual spaces. Roughly speaking, we consider the functions concentrated in a compact set, whose frequencies concentrated in a band. But, the micro-local information of a distribution can reflect in its global information.

This paper contains three main results for the generalized Hardy spaces. The first one, proceeded in §3, concerns the micro-local quantities of distributions, which manifests the micro-local structure of the generalized Hardy spaces. The second result is to give a wavelet characterization by the micro-local quantities. The third result is to give a wavelet characterization of the functions in the generalized Hardy spaces through a group of L^1 functions defined by the absolute values of their wavelet coefficients. The last two results will be given in §4.

2. Wavelet preliminaries

In this section, we present some preliminaries on wavelets, functions and operators concerned in this paper.

2.1. **Wavelets and Sobolev spaces.** In this paper, we use the real-valued tensor product wavelets; which can be Meyer wavelets or Daubechies wavelets. To simplify the notations, we use also 0 to denote the zero vector $(0,\cdots,0)$ in $\mathbb{R}^n$. Let $\Phi^0(x)$ be the scale function in the wavelet terminology. Let $E_n=\{0,1\}^n\backslash\{0\}$. For $\epsilon\in E_n$, let $\Phi^\epsilon(x)$ be the wavelet functions, cf [13, 22] and [26]. Let m be a sufficiently large integer such that $m>8n$ and let M be an integer depending on m. For $\epsilon\in\{0,1\}^n$, we suppose that our Daubechies wavelets $\Phi^\epsilon(x)$ belong to $C_0^m([-2^M,2^M]^n)$; further, for $\epsilon\in E_n$, $\Phi^\epsilon(x)$ have all the vanishing moments from the order 0 to the order $m-1$. $\forall\epsilon\in\{0,1\}^n, j\in\mathbb{Z},\ k\in\mathbb{Z}^n$, we denote $Q=Q_{j,k}=\prod\limits_{s=1}^{n}[2^{-j}k_s,2^{-j}(k_s+1)]$ and $\Phi^\epsilon_Q(x)=\Phi^\epsilon_{j,k}(x)=2^{\frac{jn}{2}}\Phi^\epsilon(2^jx-k)$. Let $\Omega=\{Q_{j,k}, j\in\mathbb{Z},k\in\mathbb{Z}^n\}$ and $\Lambda_n=\{(\epsilon,j,k),\epsilon\in E_n, j\in\mathbb{Z},k\in\mathbb{Z}^n\}$.

For $1<p<\infty$, we denote by p' the conjugate index satisfying $\frac{1}{p}+\frac{1}{p'}=1$. For $1<p<\infty, r\in\mathbb{R}$, we know that the dual space of the Sobolev space $W^{r,p}$ is $W^{-r,p'}$, see [13, 20] and [26]. For any function $f(x)$, we denote, in this paper, $f^\epsilon_{j,k}=\langle f,\Phi^\epsilon_{j,k}\rangle, \forall\epsilon\in\{0,1\}^n, j\in\mathbb{Z},k\in\mathbb{Z}^n$. Let $\chi(x)$ be the character function on the unit cube $[0,1]^n$. The above wavelets can characterize the Sobolev spaces $W^{r,p}$ and the Hardy space H^1, see [13, 26] and [29]:

Lemma 2.1. *Given $1<p<\infty$ and $|r|<m$. For any function $g(x)=\sum\limits_{(\epsilon,j,k)\in\Lambda_n}g^\epsilon_{j,k}\Phi^\epsilon_{j,k}(x)$, we have*

$$g(x)=\sum_{(\epsilon,j,k)\in\Lambda_n}g^\epsilon_{j,k}\Phi^\epsilon_{j,k}(x)\in W^{r,p}\Leftrightarrow\|(\sum_{(\epsilon,j,k)\in\Lambda_n}2^{2j(r+\frac{n}{2})}|g^\epsilon_{j,k}|^2\chi(2^jx-k))^{\frac{1}{2}}\|_{L^p}<\infty.$$

$$g(x) = \sum_{(\epsilon,j,k)\in\Lambda_n} g^\epsilon_{j,k}\Phi^\epsilon_{j,k}(x) \in H^1 \Leftrightarrow \|(\sum_{(\epsilon,j,k)\in\Lambda_n} 2^{nj}|g^\epsilon_{j,k}|^2\chi(2^jx-k))^{\frac{1}{2}}\|_{L^1} < \infty.$$

2.2. **Calderón-Zygmund operators.** We introduce now some preliminaries about the Calderón-Zygmund operators, see [13] and [19]. Let $K(x,y)$ be a smooth function for $x \neq y$ such that there exists a sufficiently large $N_0 \leq m$ satisfying that

$$|\partial^\alpha_x\partial^\beta_y K(x,y)| \leq C|x-y|^{-(n+|\alpha|+|\beta|)}, \forall |\alpha|+|\beta| \leq N_0.$$

A linear operator T is said to be a Calderón-Zygmund operator if it is continuous from $C^1(\mathbb{R}^n)$ to $(C^1(\mathbb{R}^n))'$ with the kernel $K(x,y)$ where

$$Tf(x) = \int K(x,y)f(y)dy$$

such that $Tx^\alpha = T^*x^\alpha = 0, \forall\alpha \in \mathbb{N}^n$ and $|\alpha| \leq N_0$. Such Calderón-Zygmund operator was denoted by $T \in CZO(N_0)$.

Taking into account that $K(x,y)$ may have high singularities for $x=y$, the kernel $K(\cdot,\cdot)$ is only a distribution in $S'(\mathbb{R}^{2n})$. $\forall(\epsilon,j,k),(\epsilon',j',k') \in \Lambda_n$, let $a^{\epsilon,\epsilon'}_{j,k,j',k'} = \langle K(\cdot,\cdot), \Phi^\epsilon_{j,k}\Phi^{\epsilon'}_{j',k'}\rangle$. If T is a Calderón-Zygmund operator, then its kernel $K(\cdot,\cdot)$ and the related coefficients satisfy the following relations, see [13, 14] and [26]:

Lemma 2.2. *(i) If $T \in CZO(N_0)$, then the coefficients $a^{\epsilon,\epsilon'}_{j,k,j',k'}$ satisfy the following condition:*

$$|a^{\epsilon,\epsilon'}_{j,k,j',k'}| \leq C2^{-|j-j'|(\frac{n}{2}+N_0)}(\frac{2^{-j}+2^{-j'}}{2^{-j}+2^{-j'}+|k2^{-j}-k'2^{-j'}|})^{n+N_0}, \forall(\epsilon,j,k),(\epsilon',j',k') \in \Lambda_n.$$

(ii) If $a^{\epsilon,\epsilon'}_{j,k,j',k'}$ satisfy the above condition (i), then

$$K(\cdot,\cdot) = \sum_{(\epsilon,j,k),(\epsilon',j',k')\in\Lambda_n} a^{\epsilon,\epsilon'}_{j,k,j',k'}\Phi^\epsilon_{j,k}\Phi^{\epsilon'}_{j',k'}$$

in the distribution sense. Further, for any small positive real number δ, $T \in CZO(N_0-\delta)$.

To end this subsection, we recall a variant result for the continuity of the Calderón-Zygmund operators on the Sobolev spaces (also see [13, 14, 16] and [19]). For all $(\epsilon,j,k) \in \Lambda_n$, denote $\tilde{g}^\epsilon_{j,k} = \sum_{(\epsilon',j',k')\in\Lambda_n} a^{\epsilon,\epsilon'}_{j,k,j',k'}g^{\epsilon'}_{j',k'}$, we have

Lemma 2.3. *If $s > |r|, 1 < p < \infty$ and $\forall\,(\epsilon,j,k)$, $(\epsilon',j',k') \in \Lambda_n$,*

$$|a^{\epsilon,\epsilon'}_{j,k,j',k'}| \leq C2^{-|j-j'|(\frac{n}{2}+s)}(\frac{2^{-j}+2^{-j'}}{2^{-j}+2^{-j'}+|k2^{-j}-k'2^{-j'}|})^{n+s},$$

then $\int(\sum 2^{j(n+2r)}|\tilde{g}^\epsilon_{j,k}|^2\chi(2^jx-k))^{\frac{p}{2}}dx \leq C\int(\sum 2^{j(n+2r)}|g^\epsilon_{j,k}|^2\chi(2^jx-k))^{\frac{p}{2}}dx.$

2.3. **From wavelet characterization of Morrey spaces to generalized Hardy spaces.** For $|s| < m$, for any function $f(x) = \sum_{\epsilon,j,k} f^\epsilon_{j,k}\Phi^\epsilon_{j,k}(x)$ and for any dyadic cube Q, let

$$C_{\alpha,s,Q}f = |Q|^{\frac{\alpha}{n}-\frac{1}{2}}(\sum_{Q_{j,k}\subset Q} 2^{2js}|f^\epsilon_{j,k}|^2)^{\frac{1}{2}}.$$

If $s=0$, we denote $C_{\alpha,Q}f = C_{\alpha,0,Q}f$. By the wavelet characterization of the Sobolev spaces $W^{s,2}$, we get the following wavelet characterization of the Morrey spaces, cf [32]:

Proposition 1. *If $0<\alpha<\frac{n}{2}$ and $|s|<m$, then*

(i) $f(x)=\sum\limits_{(\epsilon,j,k)\in\Lambda_n} f^{\epsilon}_{j,k}\Phi^{\epsilon}_{j,k}(x)\in M_{\alpha,s,2} \quad\Leftrightarrow\quad \sup\limits_{Q\in\Omega} C_{\alpha,s,Q}f<\infty.$

(ii) $f(x)=\sum\limits_{(\epsilon,j,k)\in\Lambda_n} f^{\epsilon}_{j,k}\Phi^{\epsilon}_{j,k}(x)\in M^0_{\alpha,s,2} \quad\Leftrightarrow$

$$\sup_{Q\in\Omega} C_{\alpha,s,Q}f<\infty \text{ and } \lim_{Q\in\Omega, |Q|\to 0 \text{ or } \infty} C_{\alpha,s,Q}f=0.$$

According to the lemma 2.1 and the above proposition 1, we can identify a function $f(x)=\sum\limits_{(\epsilon,j,k)\in\Lambda_n} f^{\epsilon}_{j,k}\Phi^{\epsilon}_{j,k}(x)$ with the sequence $\{f^{\epsilon}_{j,k}\}_{(\epsilon,j,k)\in\Lambda_n}$. In this paper, we did not distinguish $f(x)$ with the sequence $\{f^{\epsilon}_{j,k}\}_{(\epsilon,j,k)\in\Lambda_n}$ sometimes.

Peng and Yang used atoms to define the predual spaces of the Q spaces in [26] like what Fefferman and Stein did for the classical Hardy spaces in [7]. Below we introduce the standard atoms, the wavelet atoms and the relative generalized Hardy space:

Definition 2. *(i) A distribution $g(x)$ is an $(\alpha,s,2)-$atom on a cube Q, if $\|(-\Delta)^{-\frac{s}{2}}g\|_{L^2}\leq |Q|^{-\frac{1}{2}+\frac{\alpha}{n}}$, $\mathrm{supp}g(x)\subset Q$, and $\int x^{\alpha}g(x)dx=0, \forall|\alpha|\leq|s|$ in the distribution sense.*

(ii) A distribution $f(x)$ belongs to the Hardy space $H^{\alpha,s,2}$, if $f(x)=\sum\limits_{u}\lambda_u g_u(x)$ where $\{\lambda_u\}\in l^1$ and $g_u(x)$ are $(\alpha,s,2)-$atoms.

Definition 3. *(i) A distribution $g(x)=\sum\limits_{\epsilon,Q_{j,k}\subset Q} g^{\epsilon}_{j,k}\Phi^{\epsilon}_{j,k}(x)$ is an $(\alpha,s,2)-$ wavelet atom on a dyadic cube Q, if $(\sum\limits_{\epsilon,j,k} 2^{-2js}|g^{\epsilon}_{j,k}|^2)^{\frac{1}{2}}\leq |Q|^{\frac{\alpha}{n}-\frac{1}{2}}$.*

(ii) A distribution $f(x)$ belongs to the Hardy space $H^{\alpha,s,2}_w$, if $f(x)=\sum\limits_{u}\lambda_u g_u(x)$, where $\{\lambda_u\}\in l^1$ and $g_u(x)$ are $(\alpha,s,2)-$wavelet atoms.

These two kinds of atomic spaces $H^{\alpha,s,2}$ and $H^{\alpha,s,2}_w$ are, in fact, identical; further Calderón-Zygmund operators are continuous from $H^{\alpha,s,2}$ to $H^{\alpha,s,2}$, see [16, 30] and [32]. In fact,

Proposition 2. *If $0<\alpha<\frac{n}{2}, |s|<N_0\leq m$, then*

(i) $H^{\alpha,s,2}=H^{\alpha,s,2}_w$.

(ii) $f(x)\in H^{\alpha,s,2}\Leftrightarrow(-\Delta)^{\frac{s}{2}}H^{\alpha,0,2}$.

(iii) Any Calderón-Zygmund operators $T\in CZO(N_0)$ is continuous from $H^{\alpha,s,2}$ to $H^{\alpha,s,2}$.

For a function space A, we denote by $(A)'$ the dual space of A. Applying the same ideas in [16, 18] and [32], we get the following duality:

Proposition 3. *If $0\leq\alpha<\frac{n}{2}$ and $|s|<m$, then*

(i) $(H^{\alpha,s,2})'=M_{\alpha,s,2}$.

(ii) $(M^0_{\alpha,s,2})'=H^{\alpha,s,2}$.

To stress on the main ideas, we assume that $s=0$ in the rest of this paper and consider only $H^{\alpha,2}=H^{\alpha,0,2}$.

3. Micro-local quantities for $H^{\alpha,2}$

If $\alpha = 0$, then $H^{0,2} = H^1$; and if $\alpha = \frac{n}{2}$, then modulo constant, $H^{\frac{n}{2},2} = L^2$. The norms of the relative function spaces depend only on the $L^p(l^2)$-norms of function series $\{f_j = Q_j f\}_{j\in\mathbb{Z}}, p = 1, 2$, where the support of the Fourier transform of f_j is contained in some ring of size 2^j. But for $0 < \alpha < \frac{n}{2}$, the relative wavelet characterization depends on both the Fourier frequency information and the local information in a self-correlation way.

The word 'micro-local' was first introduced in the PDE problems. Although our ideas in this paper is a little different from the original one, it is similar in some sense. This is the reason we borrow this word. First, we use mathematic methods to study the conditional maximum value problem for non negative sequence in §3.1. Then we will use the result obtained in §3.1 to obtain the micro-local quantities in §3.2.

3.1. **Conditional maximum value for non-negative sequence.** For $u \in \mathbb{N}$, we denote

$$\Lambda_{u,n} = \Big\{0, 1, \cdots, 2^u - 1\Big\}^n \text{ and } G_{u,n} = \Big\{(\epsilon, s, v), \epsilon \in E_n, 0 \le s \le u, v \in \Lambda_{s,n}\Big\}.$$

$\forall j \in \mathbb{Z}$, $k \in \mathbb{Z}^n$, $t \in \mathbb{N}$ and sequence $\tilde{g}^t_{j,k} = \{g^\epsilon_{j+s,2^s k+u}\}_{(\epsilon,s,u)\in G_{t,n}}$, we define

Definition 4. *$\{g^\epsilon_{j+s,2^s k+u}\}_{(\epsilon,s,u)\in G_{t,n}}$ is a non negative sequence, if*

$$g^\epsilon_{j+s,2^s k+u} \ge 0, \ \forall \ (\epsilon, s, u) \in G_{t,n}. \tag{3.1}$$

For a non-negative sequence $\tilde{g}^t_{j,k}$, we would like to find the maximum value of the following quantities:

$$\tau_{f^t_{j,k},\tilde{g}^t_{j,k}} = \sum_{(\epsilon,s,u)\in G_{t,n}} f^\epsilon_{j+s,2^s k+u} g^\epsilon_{j+s,2^s k+u}, \tag{3.2}$$

where $f^t_{j,k} = \{f^\epsilon_{j+s,2^s k+u}\}_{(\epsilon,s,u)\in G_{t,n}}$ is a non-negative sequence satisfying the following $\sum\limits_{0\le s\le t} 2^{ns}$ restrict conditions

$$\left\{\begin{array}{lll} 2^{(n-2\alpha)(j+t)} \sum\limits_{\epsilon\in E_n} (f^\epsilon_{j+t,2^t k+u})^2 & \le 1, & \forall u \in \Lambda_{t,n};\\ 2^{(n-2\alpha)(j+t-1)} \sum\limits_{(\epsilon,s,v)\in G_{1,n}} (f^\epsilon_{j+t-1+s,2^s(2^{t-1}k+u)+v})^2 & \le 1, & \forall u \in \Lambda_{t-1,n};\\ 2^{(n-2\alpha)(j+t-2)} \sum\limits_{(\epsilon,s,v)\in G_{2,n}} (f^\epsilon_{j+t-2+s,2^s(2^{t-2}k+u)+v})^2 & \le 1, & \forall u \in \Lambda_{t-2,n};\\ \cdots & \le 1, & \cdots;\\ 2^{(n-2\alpha)j} \sum\limits_{(\epsilon,s,v)\in G_{t,n}} (f^\epsilon_{j+s,2^s k+v})^2 & \le 1. & \end{array}\right. \tag{3.3}$$

There exist $(2^n - 1) \sum\limits_{0\le s\le t} 2^{ns}$ elements in $G_{t,n}$, so $f^t_{j,k}$ is a sequence of $(2^n - 1) \sum\limits_{0\le s\le t} 2^{ns}$ components.

Definition 5. *$\forall j \in \mathbb{Z}$, $k \in \mathbb{Z}^n$, $t \in \mathbb{N}$, we call $f^t_{j,k} = \{f^\epsilon_{j+s,2^s k+u}\}_{(\epsilon,s,u)\in G_{t,n}} \in F^t_{j,k}$, if $f^t_{j,k}$ is a non-negative sequence satisfying condition (3.3).*

According to the basic results in mathematical analysis, we have:

Theorem 1. *Given $0 \leq \alpha < \frac{n}{2}$ and $t \geq 0$. For $j \in \mathbb{Z}$, $k \in \mathbb{Z}^n$ and for any non-negative sequence $\tilde{g}^t_{j,k} = \{g^\epsilon_{j+s,2^sk+u}\}_{(\epsilon,s,u)\in G_{t,n}}$, there exists at least a sequence $\bar{f}^t_{j,k} = \{\tilde{f}^\epsilon_{j+s,2^sk+u}\}_{(\epsilon,s,u)\in G_{t,n}} \in F^t_{j,k}$ such that*

$$\tau_{\bar{f}^t_{j,k},\tilde{g}^t_{j,k}} = \max_{f^t_{j,k}\in F^t_{j,k}} \tau_{f^t_{j,k},\tilde{g}^t_{j,k}}.$$

Proof. The $(2^n-1)\sum_{0\leq s\leq t} 2^{ns}$ quantities $f^\epsilon_{j+s,2^sk+u}, (\epsilon,s,u) \in G_{t,n}$ of the sequence $f^t_{j,k}$ are restricted in a closed domain, so the conclusion is obvious. □

3.2. **Micro-local quantities in $H^{\alpha,2}(\mathbb{R}^n)$.** Applying the proposition 3, to determine whether a function $g(x) = \sum_{(\epsilon,j,k)\in\Lambda_n} g^\epsilon_{j,k}\Phi^\epsilon_{j,k}(x)$ belongs to a given generalized Hardy space $H^{\alpha,2}(\mathbb{R}^n)$, we consider the actions of $f \in M^0_\alpha(\mathbb{R}^n)$ on g, where $\sup_{Q\in\Omega} C_{\alpha,s,Q}f \leq 1$. The above sup of $C_{\alpha,Q}f$ is taken for all $Q \in \Omega$, we can not manifest the real structure of $g(x)$. Hence we localize $g(x)$ by restricting its wavelet coefficients $g^\epsilon_{j,k}$ such that $Q_{j,k}$ are contained in the dyadic cube Q, then we limit the range of frequencies by limiting the number of j. In fact, we consider the function

$$g_{t,Q}(x) =: \sum_{Q_{j,k}\subset Q: -\log_2|Q|\leq nj\leq nt-\log_2|Q|} g^\epsilon_{j,k}\Phi^\epsilon_{j,k}(x). \tag{3.4}$$

For such a $g_{t,Q}$, the number of (ϵ,j,k) such that $g^\epsilon_{j,k} \neq 0$ is at most $(2^n-1)\sum_{0\leq s\leq t} 2^{ns}$.

In this section, we study the micro-local functions $g_{t,Q}$ in $H^{\alpha,2}(\mathbb{R}^n)$ and obtain its three micro-local quantities. For all $t,j \in \mathbb{Z}, k \in \mathbb{Z}^n$ and $t \geq 0$, we consider the series $g^t_{j,k} = \{g^\epsilon_{j+s,2^sk+v}, \epsilon \in E_n, 0 \leq s \leq t, v \in \Lambda_{s,n}\}$. Denote

$$g^t_{j,k}(x) = \sum_{(\epsilon,s,u)\in G_{t,n}} g^\epsilon_{j+s,2^sk+u}\Phi^\epsilon_{j+s,2^sk+u}(x). \tag{3.5}$$

Because there is a one-to-one relation between the sequence $g^t_{j,k}$ and the function $g^t_{j,k}(x)$, sometimes, we do not distinguish them.

To simplify the notations, we suppose that our functions are real-valued. For two functions $f(x) = \sum_{(\epsilon,j,k)\in\Lambda_n} f^\epsilon_{j,k}\Phi^\epsilon_{j,k}(x)$ and $g(x) = \sum_{(\epsilon,j,k)\in\Lambda_n} g^\epsilon_{j,k}\Phi^\epsilon_{j,k}(x)$, if $\langle f(x), g(x)\rangle$ and $\sum_{(\epsilon,j,k)\in\Lambda_n} f^\epsilon_{j,k}g^\epsilon_{j,k}$ are well defined, then we have

$$\tau_{f,g} =: \langle f(x), g(x)\rangle = \sum_{(\epsilon,j,k)\in\Lambda_n} f^\epsilon_{j,k}g^\epsilon_{j,k}. \tag{3.6}$$

For the given function $g^t_{j,k}$, according to the equation (3.6), we can restrict f to the function $f^t_{j,k}(x) = \sum_{(\epsilon,s,u)\in G_{t,n}} f^\epsilon_{j+s,2^sk+u}\Phi^\epsilon_{j+s,2^sk+u}(x)$ with $\|f^t_{j,k}\|_{M^0_\alpha} \leq 1$. For $f^t_{j,k}(x)$, the number of (ϵ,s,u) such that $f^\epsilon_{j+s,2^sk+u} \neq 0$ is at most $(2^n-1)\sum_{0\leq s\leq t} 2^{ns}$. That is to say, applying the equation (3.6), we transfer the problem to finding out the supremum on an infinitely many restricted conditions to a maximum value problem on $\sum_{s=0}^{t} 2^{ns}$ restricted conditions on the series of quantities $\{f^\epsilon_{j+s,2^sk+u}\}_{(\epsilon,s,u)\in G_{t,n}}$.

Based on the theorem 1, we begin to consider the micro-local quantities of $g^t_{j,k}$ in $H^{\alpha,2}(\mathbb{R}^n)$.

Theorem 2. *Given $0 < \alpha < \frac{n}{2}$ and $t \geq 0$. For $g^t_{j,k}$ defined in (3.5), if $\|g^t_{j,k}\|_{H^{\alpha,2}} > 0$, then*

(i) *There exists a function* $Sf^t_{j,k}(x)=\sum\limits_{(\epsilon,s,u)\in G_{t,n}} S^t_j f^\epsilon_{j+s,2^sk+u}\Phi^\epsilon_{j+s,2^sk+u}(x)$ *with* $\|S^t_jf^t_{j,k}\|_{M^0_\alpha}$ ≤ 1 *satisfying that* $\max\limits_{\|f\|_{M^0_\alpha}\leq 1}\tau_{f,g^t_{j,k}}=\sum\limits_{(\epsilon,s,u)\in G_{t,n}} S^t_j f^\epsilon_{j+s,2^sk+u} g^\epsilon_{j+s,2^sk+u}$.

(ii) *There exists a positive number* $P^t_jg^t_{j,k}$ *which is defined by the absolute values of wavelet coefficients of* $g^t_{j,k}$ *such that*

$$P^t_jg^t_{j,k}=\|g^t_{j,k}\|_{H^{\alpha,2}}=\max_{\|f\|_{M^0_\alpha}\leq 1}\tau_{f,g^t_{j,k}}=\tau_{Sf^t_{j,k},g^t_{j,k}}.$$

(iii) *There exists a sequence* $\{Q^t_jg^\epsilon_{j,k}\}_{\epsilon\in E_n}$ *such that* $\sum\limits_{\epsilon\in E_n} Q^t_jg^\epsilon_{j,k}\Phi^\epsilon_{j,k}(x)$ *has the same* $H^{\alpha,2}$*-norm as* $g^t_{j,k}$ *does.*

Proof. For any sequence $g^t_{j,k}=\{g^\epsilon_{j+s,2^sk+u}\}_{(\epsilon,s,u)\in G_{t,n}}$, denote $\tilde{g}^t_{j,k}=\{|g^\epsilon_{j+s,2^sk+u}|\}_{(\epsilon,s,u)\in G_{t,n}}$. Denote $G^{t,j,k}_g=\{(\epsilon,s,u)\in G_{t,n}, g^\epsilon_{j+s,2^sk+u}\neq 0\}$. For $f^t_{j,k}(x)=\sum\limits_{(\epsilon,s,u)\in G_{t,n}} f^\epsilon_{j+s,2^sk+u}\Phi^\epsilon_{j+s,2^sk+u}(x)$, define

$$f^{\epsilon,g}_{j+s,2^sk+u}=\begin{cases} |f^\epsilon_{j+s,2^sk+u}|\cdot|g^\epsilon_{j+s,2^sk+u}|^{-1}\overline{g^\epsilon_{j+s,2^sk+u}}, & (\epsilon,s,u)\in G_{t,n};\\ 0, & (\epsilon,s,u)\notin G_{t,n}.\end{cases}$$

We denote by $F^{t,j,k}_g$ the set

$$\Big\{f^t_{j,k}:\ f^t_{j,k}(x)=\sum_{(\epsilon,s,u)\in G_{t,n}} f^{\epsilon,g}_{j+s,2^sk+u}\Phi^\epsilon_{j+s,2^sk+u}(x) \text{ and } \|f^t_{j,k}\|_{M^0_\alpha}\leq 1\Big\}.$$

According to the wavelet characterization of M^0_α, we have

$$\|f^t_{j,k}\|_{M^0_\alpha}\leq 1 \text{ implies } \tilde{f}^t_{j,k}\in F^t_{j,k}.$$

Hence, by the equation (3.6),

$$\max_{\|f^t_{j,k}\|_{M^0_\alpha}\leq 1}\tau_{f^t_{j,k},g^t_{j,k}}=\max_{f^t_{j,k}\in F^{t,j,k}_g}\tau_{f^t_{j,k},g^t_{j,k}}=\max_{\tilde{f}^t_{j,k}\in F^{t,j,k}_g}\tau_{\tilde{f}^t_{j,k},\tilde{g}^t_{j,k}}=\max_{\tilde{f}^t_{j,k}\in F^t_{j,k}}\tau_{\tilde{f}^t_{j,k},\tilde{g}^t_{j,k}}. \tag{3.7}$$

According to the theorem 1, there exists at least one sequence $\bar{f}^t_{j,k}=\{\tilde{f}^\epsilon_{j+s,2^sk+u}\}_{(\epsilon,s,u)\in G_{t,n}}\in F^t_{j,k}$ such that

$$\tau_{\bar{f}^t_{j,k},\tilde{g}^t_{j,k}}=\max_{f^t_{j,k}\in F^t_{j,k}}\tau_{f^t_{j,k},\tilde{g}^t_{j,k}}. \tag{3.8}$$

Let $Sf^t_{j,k}(x)=\sum\limits_{(\epsilon,s,u)\in G_{t,n}} S^t_jf^\epsilon_{j+s,2^sk+u}\Phi^\epsilon_{j+s,2^sk+u}(x)$ where

$$S^t_jf^\epsilon_{j+s,2^sk+u}=\begin{cases} \tilde{f}^\epsilon_{j+s,2^sk+u}|g^\epsilon_{j+s,2^sk+u}|^{-1}\overline{g^\epsilon_{j+s,2^sk+u}}, & \forall(\epsilon,s,u)\in G_{t,n};\\ 0, & \forall(\epsilon,s,u)\notin G_{t,n}.\end{cases}$$

According to the equations (3.6) and (3.7), $Sf^t_{j,k}(x)$ satisfies (i).

Let $P^t_jg^t_{j,k}=\tau_{\bar{f}^t_{j,k},\tilde{g}^t_{j,k}}$. According to the equations (3.7) and (3.8), $P^t_jg^t_{j,k}$ is defined by the absolute value of $g^t_{j,k}$. By applying the equations (3.6), (3.7) and (3.8), the proposition 3 implies that $P^t_jg^t_{j,k}$ satisfies (ii).

Denote

$$Q^t_jg^\epsilon_{j,k}=\begin{cases} 2^{(\frac{n}{2}-\alpha)j-\frac{n}{2}}P^t_jg^t_{j,k}, & \text{if } \sum\limits_{\epsilon\in E_n}|g^\epsilon_{j,k}|=0;\\ 2^{(\frac{n}{2}-\alpha)j}P^t_jg^t_{j,k}(\sum\limits_{\epsilon\in E_n}|g^\epsilon_{j,k}|^2)^{-\frac{1}{2}}g^\epsilon_{j,k}, & \text{if } \sum\limits_{\epsilon\in E_n}|g^\epsilon_{j,k}|\neq 0.\end{cases}$$

According to the equation (3.6) and the wavelet characterization of M^0_α, we know that $\{Q^t_j g^\epsilon_{j,k}\}_{\epsilon\in E_n}$ satisfies the condition (iii). $\square$

Remark 3.1. *For $\alpha = 0$ and $\alpha = \frac{n}{2}$, if we deal with similarly $P^t_j g^t_{j,k}$, then:*

(1) *For $\alpha = 0$, according to the wavelet characterization of the Hardy space H^1 in [13], the quantity $P^t_j g^t_{j,k}$ is equivalent to $\|(\sum\limits_{(\epsilon,s,u)\in G_{t,n}} 2^{n(j+s)}|g^\epsilon_{j+s,2^sk+u}|^2\chi(2^{j+s}x-2^sk-u))^{\frac{1}{2}}\|_{L^1}$.*

(2) *For $\alpha = \frac{n}{2}$, $P^t_j g^t_{j,k}$ can be written as $(\sum\limits_{(\epsilon,s,u)\in G_{t,n}} |g^\epsilon_{j,k}|^2)^{\frac{1}{2}}$.*

But for $0 < \alpha < \frac{n}{2}$, $P^t_j g^t_{j,k}$ can not be written in an explicit way. Luckily, the sequence $\{Q^t_j g^\epsilon_{j,k}\}_{\epsilon\in E_n}$, the function $Sf^t_{j,k} = \sum\limits_{(\epsilon,s,u)\in G_{t,n}} S^t_j f^\epsilon_{j+s,2^sk+u}\Phi^\epsilon_{j+s,2^sk+u}(x)$ and the quantity $P^t_j g^t_{j,k}$ indicate the micro-local characters. They include both the frequency structure information and the local structure information. In the last section, $\{Q^t_j g^\epsilon_{j,k}\}_{\epsilon\in E_n}$ and $P^t_j g^t_{j,k}$ will be repeatedly used to the wavelet characterization. On one hand, we show that the micro-local quantities result in the global information of functions in $H^{\alpha,2}(\mathbb{R}^n)$ in §4.1. On other hand, we prove that the functions in $H^{\alpha,2}(\mathbb{R}^n)$ can be characterized by a group of L^1 functions defined by the absolute values of their wavelet coefficients in §4.2.

4. Wavelet Characterization of Generalized Hardy spaces

In the chapter 7 of [32], W. Yuan, W. Sickel and D. C. Yang used a family of Borel measures, to determine whether or not a function belongs to the pre-dual spaces of generalized Morrey spaces. They got some wavelet characterization, but the predual spaces equipped with the induced norm are only pseudo-Banach spaces. In [16] and [26], L. Z. Peng and Q. X. Yang characterized $H^{\alpha,\alpha,2}$ with atomic decomposition like what C. Fefferman and E. M. Stein did in [7] and obtained induced Banach spaces. But their methods cannot be used to characterize these spaces with wavelet coefficients.

B. Maurey inaugurate the study about the relation between Hardy space H^1 and the L^1 unconditional convergence in [12]. L. Carleson [2] and P. Wojtaszczyk [21] make also their contributions. In chapter 5 of [13], Y. Meyer showed also the importance of a wavelet characterization without involving a family of Borel measures. It is natural to seek some wavelet characterizations without involving a family of Borel measures for generalized Hardy spaces. Comparing to the classical Hardy spaces, we have seen in the above section that the generalized Hardy spaces have a different micro-local structure. Thanks to the study in the above section, the task of this section is to establish two kinds of characterizations of functions f in the generalized Hardy spaces $H^{\alpha,2}$ by the absolute values of its wavelet coefficients. The induced spaces are Banach spaces.

4.1. **Micro-local information results in global information.** For $s \in \mathbb{Z}, N \in \mathbb{N}$, denote $\Omega_{s,N} = \{Q \in \Omega : |Q| \geq 2^{-sn}, Q \subset [-2^{N-s}, 2^{N-s}]^n\}$; for $0 \leq t \leq N$ and for $m \in \mathbb{Z}^n$, denote $\Omega^{t,N}_{s,m} = \{Q' \in \Omega : 2^{-sn} \leq |Q'| \leq 2^{(t-s)n}, Q' \subset Q_{s-N,m}\}$. $\forall s, N \in \mathbb{Z}, m \in \mathbb{Z}^n$ and $0 \leq t \leq N$, denote $g^{t,N}_{s,m}(x) = \sum\limits_{Q_{j,k}\in\Omega^{t,N}_{s,m}} g^\epsilon_{j,k}\Phi^\epsilon_{j,k}(x), g^N_{s-N,m}(x) = g^{N,N}_{s,m}(x)$ and $f^{t,N}_{s,m}(x) = \sum\limits_{Q_{j,k}\in\Omega^{t,N}_{s,m}} f^\epsilon_{j,k}\Phi^\epsilon_{j,k}(x)$.

We know that $\Omega_{s,N} = \bigcup\limits_{m\in\{-1,0\}^n} \Omega^{N,N}_{s,m}$. For $P^N_{s-N} g^N_{s-N,m}$ defined in the theorem 2, denote $P_{s,N} g = \sum\limits_{m\in\{-1,0\}^n} P^N_{s-N} g^N_{s-N,m}$. We define

Definition 6. *We call $g(x) \in P^1_\alpha$, if*

$$C_{1,g} = \sup_{s\in\mathbb{Z}, N\in\mathbb{N}} P_{s,N} g < \infty.$$

Then we have:

Theorem 3. *If $0 < \alpha < \frac{n}{2}$, then $P^1_\alpha = H^{\alpha,2}$.*

Proof. Applying the proposition 3, it is sufficient to prove $(M^0_\alpha)' = P^1_\alpha$.

(A) We transform first the problem to considering the micro-local functions. If $g(x) = \sum\limits_{\epsilon,j,k} g^\epsilon_{j,k} \Phi^\epsilon_{j,k}(x) \in (M^0_\alpha)'$, then $\forall f(x) = \sum\limits_{\epsilon,j,k} f^\epsilon_{j,k} \Phi^\epsilon_{j,k}(x) \in M^0_\alpha$, we know that $\tau_g = \sup\limits_{\|f\|_{Q_\alpha}\leq 1} |\tau_{f,g}| < \infty$. $\forall s, N \in \mathbb{Z}$ and $N \geq 0$, denote $g_{s,N}(x) = \sum\limits_{Q_{j,k}\in\Omega_{s,N}} g^\epsilon_{j,k} \Phi^\epsilon_{j,k}(x)$. Since $(M^0_\alpha)' = H^{\alpha,2}$, we know that, if $s, N \to +\infty$, then $g_{s,N}(x) \to g(x)$ in the norm of $H^{\alpha,2}$. Denote $f_{s,N}(x) = \sum\limits_{Q_{j,k}\in\Omega_{s,N}} f^\epsilon_{j,k} \Phi^\epsilon_{j,k}(x)$; by (3.6), we know that

$$\tau_{g_{s,N}} = \sup_{\|f\|_{Q_\alpha}\leq 1} |\tau_{f,g_{s,N}}| = \sup_{\|f_{s,N}\|_{Q_\alpha}\leq 1} |\tau_{f_{s,N},g_{s,N}}| \to \tau_g.$$

(B) For $g_{s,N}(x)$, we prove that its P^1_α norm is equivalent to its $H^{\alpha,2}$ norm. In fact, according to the proposition 3 and the theorem 2, we have

$$\|g^N_{s-N,m}(x)\|_{(M^0_\alpha)'} = \|g^N_{s-N,m}(x)\|_{H^{\alpha,2}} = P^N_{s-N} g^N_{s-N,m}.$$

Since we have $g_{s,N}(x) = \sum\limits_{m\in\{-1,0\}^n} g^{N,N}_{s,m}(x)$. Hence

$$\|g_{s,N}(x)\|_{H^{\alpha,2}} \leq \sum_{m\in\{-1,0\}^n} \|g^N_{s-N,m}(x)\|_{H^{\alpha,2}} \leq \|g_{s,N}\|_{P^1_\alpha}.$$

According to the wavelet characterization of M^0_α and the equation (3.6), we have

$$\|g_{s,N}\|_{(M^0_\alpha)'} = \max_{m\in\{-1,0\}^n} \|g^N_{s-N,m}(x)\|_{(M^0_\alpha)'} \geq 2^{-n} \|g_{s,N}\|_{P^1_\alpha}.$$

Hence $\|g_{s,N}\|_{P_\alpha} \sim \|g_{s,N}\|_{P^1_\alpha}$ and $(M^0_\alpha)' = P^1_\alpha$. □

4.2. **Characterization by a group of L^1 functions.** In chapter 5 of the famous book [13], Y. Meyer proved the following fact. The norm of functions in the Hardy space H^1 can be characterized by the L^1 norm of some function defined by the absolute values of wavelet coefficients. The task of this subsection is to show that the norm of each function $g(x)$ in $H^{\alpha,2}$ can characterized by a group of L^1 functions $P_{s,t,N} g(x)$.

We introduce first two lemmas.

Lemma 4.1. ([6]) *For $0 < \alpha < \frac{n}{2}$, $Q_\alpha = M_{\alpha,\alpha,2} \subset BMO$.*

By the fractional different differential and by the duality, we have

Lemma 4.2. (i) $M_\alpha = M_{\alpha,0,2} \subset BMO^{-\alpha} = \{f : (-\Delta)^{-\frac{\alpha}{2}} f \in BMO\}$.

(ii) $\dot{F}_1^{\alpha,2} = \{f : (-\Delta)^{\frac{\alpha}{2}} f \in H^1\} \subset H^{\alpha,2}$.

Now, we introduce some notations on the set of cubes. For $s \in \mathbb{Z}$ and $N \in \mathbb{N}$, denote $\Omega^{s,N} = \{Q \in \Omega : 2^{-sn} \leq |Q| \leq 2^{(N-s)n}\}$; for $0 \leq t \leq N, m \in \mathbb{Z}^n, Q = Q_{s-N,m}$, denote $\Omega_{s,t,Q} = \Omega^{t,N}_{s,m} = \{Q' \in \Omega : 2^{-sn} \leq |Q'| \leq 2^{(t-s)n}, Q' \subset Q_{s-N,m}\}$. It is easy to show that $\Omega^{s,N} = \bigcup\limits_{m\in\mathbb{Z}^n} \Omega^{N,N}_{s,m}$.

We further define $g_{s,N}(x) = \sum\limits_{m\in\mathbb{Z}^n} g^N_{s-N,m}(x)$, where

$$g^N_{s-N,m}(x) = \sum_{Q_{j,k}\in\Omega^{N,N}_{s,m}} g^{\epsilon}_{j,k}\Phi^{\epsilon}_{j,k}(x). \tag{4.1}$$

We begin with the definition of the quantities $g^{\epsilon,s,t,N}_{j,k}$.

(1) If $t = 0$ and $j > s$, then we denote $g^{\epsilon,s,t,N}_{j,k} = 0$; if $j = s$, then we denote $g^{\epsilon,s,t,N}_{j,k} = g^{\epsilon}_{j,k}$.
(2) For $t \geq 1$, if $j > s - t$, then we denote $g^{\epsilon,s,t,N}_{j,k} = 0$; if $j < s - t$, then we denote $g^{\epsilon,s,t,N}_{j,k} = g^{\epsilon}_{j,k}$; if $j = s - t$, then we denote $g^{\epsilon,s,t,N}_{j,k} = Q^t_j g^{\epsilon}_{j,k}$, where $Q^t_j g^{\epsilon}_{j,k}$ is defined in the theorem 2.

We denote $g_{s,t,N}(x) = \sum\limits_{\epsilon,j,k} g^{\epsilon,s,t,N}_{j,k}\Phi^{\epsilon}_{j,k}(x)$ and set

$$P_{s,t,N}g(x) = (\sum_{\epsilon,Q_{j,k}\in\Omega^{s,N},j\leq s-t} 2^{j(n+2\alpha)}|g^{\epsilon,s,t,N}_{j,k}|^2\chi(2^jx-k))^{\frac{1}{2}}, \tag{4.2}$$

$$Q_{s,t,N}g = \|2^{(s-t)\alpha}(\sum_{\epsilon,Q_{j,k}\in\Omega^{s,N},j=s-t} 2^{jn}|g^{\epsilon,s,t,N}_{j,k}|^2\chi(2^jx-k))^{\frac{1}{2}}\|_{L^1}. \tag{4.3}$$

We define now the second kind of spaces.

Definition 7. $P^2_\alpha = \{g : \sup\limits_{s\in\mathbb{Z},N\in\mathbb{N}} \min\limits_{0\leq t\leq N} \|P_{s,t,N}g(x)\|_{L^1} < \infty\}$.

We have

Theorem 4. *If* $0 < \alpha < \frac{n}{2}$, *then* $P^2_\alpha = H^{\alpha,2}$.

Proof. According to the part (A) in the proof for the theorem 3, for $g(x) = \sum\limits_{\epsilon,j,k} g^{\epsilon}_{j,k}\Phi^{\epsilon}_{j,k}(x) \in H^{\alpha,2}$ and for any $\delta > 0$, there exists $\tau_\delta > 0$ such that, for $s > \tau_\delta, N \geq 2s$, we have

$$\|g_{s,N}(x) - g(x)\|_{H^{\alpha,2}} + \sum_{|m|>2^n} \|g^N_{s-N,m}(x)\|_{H^{\alpha,2}} \leq \delta \tag{4.4}$$

and

$$8^{-n}\max_{|m|\leq 2^n} \|g^N_{s-N,m}(x)\|_{H^{\alpha,2}} - \delta \leq \|g_{s,N}(x)\|_{H^{\alpha,2}} \leq \sum_{|m|\leq 2^n} \|g^N_{s-N,m}(x)\|_{H^{\alpha,2}} + \delta. \tag{4.5}$$

By the construction of the above notations and using the theorem 2, we know that

$$\|g^N_{s-N,m}(x)\|_{H^{\alpha,2}} = Q_{s,N,N}g^N_{s-N,m} = \|P_{s,N,N}g^N_{s-N,m}(x)\|_{L^1}. \tag{4.6}$$

Furthermore

$$\|g_{s,t,N}(x)\|_{H^{\alpha,2}} \leq \|g_{s,t,N}(x)\|_{\dot{F}_1^{\alpha,2}} = \|P_{s,t,N}g(x)\|_{L^1}. \tag{4.7}$$

According to the above equations from (4.4) to (4.7), we finish the proof of the Theorem 4.2. □

Acknowledgements. The corresponding author would like to thank sincerely professor Hua Chen for his plentiful precious helps year after year. Both of the authors thank sincerely the precious advices of the referees.

References

[1] R. Aulaskari, J. Xiao and R. H. Zhao, On subspaces and subsets of BMOA and UBC. Analysis . 15(1995), 101-121.

[2] L. Carleson, An explicit unconditional basis in H^1, Bull. des Sciences Math., 164(1980), 405-416.

[3] L. H. Cui and Q. X. Yang, On the generalized Morrey spaces. Siberian Mathematical Journal, 46 (2005),no. 1, 133-141.

[4] L. H. Cui, Q. X. Yang and Z. X. Cheng, Notes on generalized Q spaces. (Chinese) Acta Math. Sinica (Chin. Ser.) 47 (2004), no. 2, 265-272.

[5] G. Dafni and J. Xiao, Some new tent spaces and duality theorem for fractional Carleson measures and $Q_\alpha(\mathbb{R})^n$, J. Funct. Anal. 208 (2004), 377-422.

[6] M. Essen, S. Janson, L. Z. Peng and J. Xiao, Q spaces of several real variables. Indiana University mathematics Journal, 2000, no. 2: 575-615.

[7] C. Fefferman and E. M. Stein H^p spaces of several variables, Acta Math. 129 (1972), 107-115.

[8] S. Janson, On the space Q_p and its dyadic counterpart. C. Kiselman, ed. Proc. Symposium "Complex Analysis and Differential equations", June 15-18, 1997, Uppsala, Sweden, Acta Universitatis Upsaliensis C. 64 (1999), 194-205.

[9] F. John and L. Nirenberg, On functions of bounded mean oscillation. Comm. Pure Appl. Math., 14 (1961), 415-426.

[10] E. A. Kalita, Dual Morrey spaces, Dokl. Akad. Nauk, 361 (1998), 447-449.

[11] C. Lin and Q. Yang, Semigroup characterization of Besov type Morrey spaces and well-posedness of generalized Navier-Stokes equations, J. Differential Equations, 254(2013) 804-846.

[12] B. Maurey, Isomorphismes entre espaces H^1, Acta Math., 145 (1980), 79-120.

[13] Y. Meyer, Ondelettes et opérateurs, I et II, Hermann, Paris, 1991-1992.

[14] Y. Meyer and Q. X. Yang, Continuity of Caldern-Zygmund operators on Besov or Triebel-Lizorkin spaces. Anal. Appl. (Singap.) 6 (2008), no. 1, 51-81.

[15] C. B. Morrey, On the solutions of quas-linear elliptic partial differential equations. Trans Amer Math Soc, 43 (1938), no. 1, 126-166.

[16] L. Z. Peng and Q. X. Yang, Predual spaces for Q spaces. Acta Math. Sci. Ser. B Engl. Ed. 29 (2009), no. 2, 243-250.

[17] Y. Sawano, Wavelet characterization of Besov-Morrey and Triebel-Lizorkin-Morrey spaces. Funct. Approx. Comment. Math. 38 (2008), part 1, 93-107.

[18] D. A. Stegenga, Bounded Toeplitz operators on H^1 and Applications of the duality between H^1 and the functions of bounded mean oscillation. Amer. J. of Math., 98 (1976), no. 3, 573-589.

[19] E. M. Stein, Harmonic analysis–Real variable methods, orthogonality, and integrals, Princeton University Press, 1993.

[20] H. Triebel, Theory of function spaces, Birkhauser Verlag, Basel, Boston, Stuttgart, 1983.

[21] P. Wojtaszczyk, Franklin system is an unconditional basis in H^1, Ark. Mat. 20(1982), 293-300.

[22] P. Wojtaszczyk, A Mathematical Introduction to Wavelets, London Mathematical Society Student Texts 37, Cambridge University Press, 1997.

[23] Z. J. Wu and C. P. Xie, Q spaces and Morrey spaces. J. Func Anal, 201(2003), no. 1, 282-297

[24] J. Xiao, Homothetic variant of fractional Sobolev space with application to Navier-Stokes system. Dyn. Partial Differ. Equ. 4 (2007), no. 3, 227-245

[25] D. C. Yang and W. Yuan, New Besov-type spaces and TriebelCLizorkin-type spaces including Q spaces, Math. Z. 265 (2010), 451-480

[26] Q. X. Yang, Wavelet and distribution, Beijing Science and Technology Press, 2002.

[27] Q. X. Yang, Characterization of multiplier spaces with Daubechies wavelets, Acta Math. Sci. Ser. B, 32 (2012), no. 6, 2315-2321.

[28] Q. X. Yang and F. W. Deng, Wavelet characterization of multiplier spaces, Mathematical Methods in the Applied Sciences. Math. Meth. Appl. Sci. 35 (2012), 2085-2094

[29] Q. X. Yang, Z. X. Chen and L. Z. Peng, Uniform characterization of function spaces by wavelets. Acta Math. Sci. Ser. A, 25 (2005), no. 1, 130-144.

[30] Q. X. Yang and Y. P. Zhu, Generalized Morrey spaces, multiplier spaces and stability, submited.

[31] Q. X. Yang and Y.P. Zhu, Characterization of multiplier spaces by wavelets and logarithmic Morrey spaces, Nonlinear Anal. 75 (2012), no. 13, 4920-4935.

[32] W. Yuan, W. Sickel and D. C. Yang, Morrey and Campanato Meet Besov, Lizorkin and Triebel, Lecture Notes in Mathematics 2005 Editors: J.-M. Morel, Cachan F. Takens, Groningen B. Teissier, Paris, 2010.

Tao Qian: Department of Mathematics, University of Macau, Macau, China SAR
E-mail address: `fsttq@umac.mo`

Qi-Xiang Yang: Mathematics and Statistics School, Wuhan University, Wuhan, 430072, China
E-mail address: `qxyang@whu.edu.cn`