A Tighter Uncertainty Principle For Linear Canonical Transform in Terms of Phase Derivative

Pei Dang*, Guan-Tie Deng, Tao Qian

Abstract

This study devotes to uncertainty principles under the linear canonical transform (LCT) of a complex signal. A lower-bound for the uncertainty product of a signal in the two LCT domains is proposed that is sharper than those in the existing literature. We also deduce the conditions that give rise to the equal relation of the new uncertainty principle. The uncertainty principle for the fractional Fourier transform is a particular case of the general result for LCT. Examples, including simulations, are provided to show that the new uncertainty principle is truly sharper than the latest one in the literature, and illustrate when the new and old lower bounds are the same and when different.

Index Terms

Complex signal, Fractional Fourier transform, linear canonical transform (LCT), uncertainty principle.

EDICS: DSP-TFSR

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I. INTRODUCTION

Uncertainty principle is important in signal analysis ([1], [2], [3], [4], [7], [8], [9], [10], [11], [12], [13], [15]). The classical Heisenberg's uncertainty principle provides a lower-bound on the product of spreads of a signal energy in the time and Fourier frequency domains as specified by the inequality

$$\sigma_{t,s}^2\sigma_{\omega,s}^2 \geq \frac{1}{4}, \tag{1.1}$$

where $\sigma_{t,s}$ and $\sigma_{\omega,s}$ are defined in Definition 2.1. The inequality (1.1) is the most general version of uncertainty principle but not the best. For a specific signal $s(t) = \rho(t)e^{i\varphi(t)}$, a stronger result is available ([2], [3]):

$$\sigma_{t,s}^2\sigma_{\omega,s}^2 \geq \frac{1}{4} + \mathrm{Cov}_s^2, \tag{1.2}$$

where

$$\mathrm{Cov}_s = \int_{-\infty}^{\infty} t\varphi'(t)|s(t)|^2 dt - \langle t\rangle_s\langle\omega\rangle_s$$

is the covariance of the signal s, $\langle t\rangle_s$ and $\langle\omega\rangle_s$ are defined in Definition 2.1. Due to [8], the Covariance can also be given by

$$\mathrm{Cov}_s = \int_{-\infty}^{\infty}(t - \langle t\rangle_s)(\varphi'(t) - \langle\omega\rangle_s)\rho^2(t)dt.$$

The recent paper [8] improves the result (1.2) through proving a larger lower-bound:

$$\sigma_{t,s}^2\sigma_{\omega,s}^2 \geq \frac{1}{4} + \mathrm{COV}_s^2, \tag{1.3}$$

where COV_s is the *absolute covariance* of the signal $s(t) = \rho(t)e^{i\varphi(t)}$, defined by

$$\mathrm{COV}_s = \int_{-\infty}^{\infty}|(t - \langle t\rangle_s)(\varphi'(t) - \langle\omega\rangle_s)|\rho^2(t)dt. \tag{1.4}$$

Owing to the basic integral inequality

$$\int_{-\infty}^{\infty}|f(t)|dt \geq |\int_{-\infty}^{\infty} f(t)dt|$$

for any integrable function $f(t)$, the uncertainty principle given by (1.3) is stronger than that given by (1.2). In this communication concrete examples (see Example 3.7) are provided to show that for certain classes of signals the right-hand-side of (1.3) is strictly larger than that of (1.2). If a signal gives rise to the equal relation of (1.2), then it also gives rise to the equal relation of (1.3). We also provide numerical examples for this case (Example 3.8).

The linear Canonical transform (LCT) is a generalized form of the classical Fourier Transform (FT), the Fractional Fourier transform (FrFT), as well as Fresnel transform (FRT) ([14], [22]). Recently, there is an ample amount of literature devoting to studies on various uncertainty principles for FrFT ([18], [23]) and LCT ([16], [17], [19], [20], [21], [22], [24], [25]). Uncertainty principles for LCT or FrFT always concern the products of the spreads of signals in the two LCT or the two FrFT domains. LCT and FrFT are generalized forms of the classical Fourier transform. When $M = (a, b, c, d)$ in (2.5) or α in (2.6) are assigned to some special values, uncertainty principles in the two LCT or FrFT domains will reduce to uncertainty principles in the time and Fourier frequency domains. [18] gives a lower-bound about the product of the spreads of a real signal in the two FrFT domains, that is different from what is for complex-valued signals given in [23]. However, both [18] and [23] can be reduced to the uncertainty principle (1.1). Although the lower-bounds given in [16] and [21] are not as the same as that for the complex signals case given in [24], all those correspond to the uncertainty principle (1.1). [22] and [25], with different conditions and proofs, provide a lower-bound for uncertainty principle for LCT corresponding to the sharper uncertainty principle (1.2).

Some concerns have been devoted to proofs of uncertainty principles such as (1.2) that involve phase and amplitude derivatives. The same concerns are also expressed to uncertainty principles for LCT and FrFT that involve such derivatives. The concerns are related to the effort to establish a fundamental theory of signal analysis on Lebesgue square integrable functions, but not smooth functions. In fact, $s(t) = \rho(t)e^{i\varphi(t)} \in L^2(\mathbb{R})$ can not guarantee the differentiability property of $s(t), \varphi(t)$ and $\rho(t)$. To solve this problem, [6], [7] and [8] work with different types of derivatives, viz., Hardy-Sobolev derivative, derivatives as non-tangential boundary limits and Fourier transform derivative, etc., through Hardy decomposition and Fourier transformation. We showed that under certain conditions the various types of derivatives can be unified. In this study, we will use Fourier transform derivatives.

The main result of this paper is a new uncertainty principle for LCT corresponding to the uncertainty principle (1.3). It is not only a truly stronger uncertainty inequality for LCT but also under the weakest possible conditions on the signals that satisfy the new uncertainty principle. Indeed, the conditions are based on Lebesgue measure and integral theory and signals of finite energy. The necessary and sufficient conditions for a signal to satisfy the equal relation in the uncertainty principle are specified. Concrete examples of signals with simulations are given to illustrate all the possible relations of the new and old lower bounds of the LCT uncertainty principle.

II. Preliminary

A. Review of the LCT

The LCT of a signal $s(t)$ with parameter $M=(a,b,c,d)$ is defined by [14]

$$S_M(u)=L_{(a,b,c,d)}(s(t))(u)=\begin{cases}\sqrt{\frac{1}{i2\pi b}}e^{i\frac{d}{2b}u^2}\int_{-\infty}^{\infty}s(t)e^{i\frac{a}{2b}t^2}e^{-i\frac{1}{b}ut}dt, & \text{if } b\neq 0,\\ \sqrt{d}e^{i\frac{cd}{2}u^2}s(du), & \text{if } b=0,\end{cases} \tag{2.5}$$

where $a,b,c,d\in\mathbb{R}$ and $ad-bc=1$.

When $(a,b,c,d)=(\cos\alpha,\sin\alpha,-\sin\alpha,\cos\alpha)$, the LCT reduces to the Fractional Fourier transform, that is,

$$L_{(\cos\alpha,\ \sin\alpha,\ -\sin\alpha,\ \cos\alpha)}(s(t))(u)=e^{-i\frac{\alpha}{2}}S_\alpha(u),$$

where $S_\alpha(u)$, the FrFT of the signal $s(t)$, is defined by

$$S_\alpha(u)=\int_{-\infty}^{\infty}s(t)K_\alpha(t,u)dt, \tag{2.6}$$

where

$$K_\alpha(t,u)=\begin{cases}\sqrt{\frac{1-i\cot\alpha}{2\pi}}e^{i\frac{t^2+u^2}{2}\cot\alpha-iut\csc\alpha}, & \text{if } \alpha \text{ is not a multiple of } \pi,\\ \delta(t-u), & \text{if } \alpha \text{ is a multiple of } 2\pi,\\ \delta(t+u), & \text{if } \alpha+\pi \text{ is a multiple of } 2\pi.\end{cases}$$

When $(a,b,c,d)=(0,1,-1,0)$, the LCT reduces to the classical Fourier transform (FT), that is,

$$L_{(0,1,-1,0)}(s(t))(\omega)=e^{-i\frac{\pi}{4}}\hat{s}(\omega),$$

where $\hat{s}(\omega)$ is the Fourier transform of $s(t)$.

The Fourier transform of $s\in L^1(\mathbb{R})$ is defined by

$$\hat{s}(\omega)\triangleq\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-it\omega}s(t)dt. \tag{2.7}$$

If $\hat{s}$ is also in $L^1(\mathbb{R})$, then the inversion Fourier transform formula holds, that is

$$s(t)=(\hat{s})^\vee(t)\triangleq\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{it\omega}\hat{s}(\omega)d\omega, \quad \text{a.e.} \tag{2.8}$$

It is standard knowledge that through a density argument the restricted Plancherel Theorem

$$\|\hat{s}\|_2^2=\|s\|_2^2, \qquad s\in L^1(\mathbb{R})\cap L^2(\mathbb{R})$$

may be extended to $L^2(\mathbb{R})$. Below, when we use the formulas (2.7) and (2.8) for $L^2(\mathbb{R})$ functions, we keep in mind that the convergence of the integrals is in the L^2-sense. Throughout the paper we assume that s is of unit energy.

When $(a,b,c,d)=(1,0,0,1)$, the LCT reduces to the signal s itself, that is,

$$L_{(1,0,0,1)}(s(x))(t)=s(t).$$

B. Properties of signal moments in the LCT domain and other related knowledge

Definition 2.1: Let $s(t)$ be a square-integral signal, then we can define

(i) $\langle \omega \rangle_s \triangleq \int_{-\infty}^{\infty} \omega |\hat{s}(\omega)|^2 d\omega$,

(ii) $\langle t \rangle_s \triangleq \int_{-\infty}^{\infty} t |s(t)|^2 dt$,

(iii) $\langle u \rangle_{\alpha,s} \triangleq \int_{-\infty}^{\infty} u |S_\alpha(u)|^2 du$,

(iv) $\langle u \rangle_{M,s} \triangleq \int_{-\infty}^{\infty} u |S_M(u)|^2 du$,

(v) $\langle t^2 \rangle_s \triangleq \int_{-\infty}^{\infty} t^2 |s(t)|^2 dt$,

(vi) $\sigma_{t,s}^2 \triangleq \int_{-\infty}^{\infty} (t-\langle t \rangle_s)^2 |s(t)|^2 dt$,

(vii) $\sigma_{\omega,s}^2 \triangleq \int_{-\infty}^{\infty} (\omega-\langle \omega \rangle_s)^2 |\hat{s}(\omega)|^2 d\omega$,

(viii) $\sigma_{\alpha,s}^2 \triangleq \int_{-\infty}^{\infty} (u-\langle u \rangle_{\alpha,s})^2 |S_\alpha(u)|^2 du$,

(ix) $\sigma_{M,u,s}^2 \triangleq \int_{-\infty}^{\infty} (u-\langle u \rangle_{M,s})^2 |S_M(u)|^2 du$,

provided that the right-hand sides of the above formulas are well defined integrals, where $\hat{s}(\omega), S_\alpha(u)$ and $S_M(u)$ are, respectively, the Fourier transform, the fractional Fourier transform and the linear canonical transform of $s(t)$.

Because the classical derivatives for signals of finite energy may not exist ([5], [7]), we adopt the Fourier transform derivatives as follows (also see [8]):

Definition 2.2: Assume $s(t) = \rho(t)e^{i\varphi(t)} \in L^2(\mathbb{R})$ and $\omega\hat{s}(\omega) \in L^2(\mathbb{R})$. We can define the Fourier transform derivative for $s(t),\ \rho(t)$ and $\varphi(t)$ as

$$(Ds)(t) = [i\omega\hat{s}(\omega)]^{\vee}(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}[i\omega\hat{s}(\omega)]e^{it\omega}d\omega,$$

$$(D\rho)(t) = \begin{cases} \rho(t)\mathrm{Re}\frac{(Ds)(t)}{s(t)}, & \text{if } s(t) \neq 0, \\ 0, & \text{if } s(t) = 0, \end{cases}$$

and

$$(D\varphi)(t) = \begin{cases} \mathrm{Im}\frac{(Ds)(t)}{s(t)}, & \text{if } s(t) \neq 0, \\ 0, & \text{if } s(t) = 0. \end{cases}$$

Suppose that we have an amplitude-phase representation of the signal s, given by

$$s(t) = \rho(t)e^{i\varphi(t)}.$$

suppose that both functions $\rho(t)$ and $\varphi(t)$ have the classical derivatives $\rho'(t)$ and $\varphi'(t)$. By taking the derivative with respect to t on both sides of the above equation, and then dividing by $s(t)$, if it is non-zero, we obtain

$$\rho'(t) = \rho(t)\mathrm{Re}\{\frac{s'(t)}{s(t)}\}, \quad \varphi'(t) = \mathrm{Im}\{\frac{s'(t)}{s(t)}\}.$$

These formulas have the same forms as those given in the definition. In view of this, when we deal with general signals in Sobolev spaces, since the classical derivatives may not exist, a reasonable replacement of the classical derivative s' is the Fourier type derivative. In general, the Sobolev space condition implies the existence of an L^2-convergence sense derivative. The assumption of the zero value of the new derivatives at the points $s(t) = 0$ is conventional that makes the proofs going. A detailed analysis and comprehensive development of relations of several types of derivatives with applications in signal analysis are given in [7], [5] and [6].

Notice that in Lebesgue theory if two functions are equal except points in a null set, then the two functions are considered to be the same. Secondly, if $\rho'(t), \varphi'(t)$ and $s'(t)$ all exist in the classical derivative sense and as Lebesgue measurable functions, and $s'(t)$ is in $L^2(\mathbb{R})$, then $(Ds)(t) = s'(t),\ (D\rho)(t) = \rho'(t)$ and $(D\varphi)(t) = \varphi'(t)$ almost everywhere. With a little abuse of the notation we have

Definition 2.3: Let $s(t) = \rho(t)e^{i\varphi(t)},\ \omega\hat{s}(\omega) \in L^2(\mathbb{R})$. The covariance for $s(t)$ is defined by

$$\mathrm{Cov}_s \triangleq \langle t(D\varphi)(t)\rangle - \langle t\rangle_s\langle\omega\rangle_s = \int_{-\infty}^{\infty} t(D\varphi)(t)|s(t)|^2dt - \langle t\rangle_s\langle\omega\rangle_s,$$

and the absolute covariance is given by

$$\mathrm{COV}_s \triangleq \int_{-\infty}^{\infty} |(t - \langle t \rangle_s)((D\varphi)(t) - \langle \omega \rangle_s)| s(t)|^2 dt.$$

Lemma 2.4: [8] Let $s(t) = \rho(t)e^{i\varphi(t)}$ and $\omega\hat{s}(\omega) \in L^2(\mathbb{R})$. Then

$$\langle \omega \rangle_s = \int_{-\infty}^{\infty} (D\varphi)(t)|s(t)|^2 dt, \tag{2.9}$$

and

$$\sigma_{\omega,s}^2 = \int_{-\infty}^{\infty} (D\rho)^2(t)dt + \int_{-\infty}^{\infty} [(D\varphi)(t) - \langle \omega \rangle_s]^2 |s(t)|^2 dt. \tag{2.10}$$

Lemma 2.5: [8], [26] Assume that $1 \le p_1 \le 2, 1 \le p_2 \le 2$, $s(t) \in L^{p_1}(\mathbb{R})$, $h(\omega) = i\omega\hat{s}(\omega) \in L^{p_2}(\mathbb{R})$. Let

$$g(t) = \int_a^t (Ds)(u)du + s(a),$$

where a is a Lebesgue point of s. Then $s(t)$ is identical almost everywhere with the absolutely continuous function $g(t)$, and

$$(Ds)(t) = g'(t) \text{ for almost all } t \in \mathbb{R}. \tag{2.11}$$

Lemma 2.6: Let $s(t) = \rho(t)e^{i\varphi(t)}$, $\omega\hat{s}(\omega)$ and $ts(t) \in L^2(\mathbb{R})$. Then

$$\langle u \rangle_{M,s} = a\langle t \rangle_s + b\int_{-\infty}^{\infty} (D\varphi)(t)|s(t)|^2 dt \tag{2.12}$$

and

$$\sigma_{M,u,s}^2 = a^2\sigma_{t,s}^2 + 2ab\mathrm{Cov}_s + b^2\sigma_{\omega,s}^2. \tag{2.13}$$

Proof of Lemma 2.6 The proof of Lemma 2.6 is given in Appendix A.

Under weaker assumptions, Lemma 2.6 proves the same equality relations as those proved in Lemma 1 and Lemma 2 of [22]. In [22] differentiability of the phase and amplitude functions together with their respective and relevant integrability are implicitly assumed, while in our setting only the Sobolev space condition, being equivalent to the square- integrability of the signal itself and a weaker type derivative, is assumed.

III. Uncertainty Principle For LCT

A. Uncertainty Principle for LCT

The reference [8] derives a strong form of uncertainty principle in the time and Fourier frequency domains as cited below that plays an important role in the proof of the uncertainty principle in the LCT domains in this study.

Lemma 3.1: Assume $s(t)=\rho(t)e^{i\varphi(t)}, ts(t)$ and $\omega\hat{s}(\omega)\in L^2(\mathbb{R})$. Then

$$\sigma_{t,s}^2\sigma_{\omega,s}^2 \geq \frac{1}{4}+\mathrm{COV}_s^2. \tag{3.14}$$

Under the extra assumptions that $s(t)=\rho(t)e^{i\varphi(t)}$ has the classical derivatives $s'(t),\varphi'(t),\rho'(t)$, where $\varphi'(t)$ is continuous and ρ is almost everywhere non-zero, the equality holds if and only if $s(t)$ has one of the following four forms

$$s(t)=e^{-\frac{1}{\zeta}(t-\langle t\rangle_s)^2+\gamma_1}e^{i[\frac{1}{2\varepsilon}(t-\langle t\rangle_s)^2+\langle\omega\rangle_s t+\gamma_2]},$$

$$s(t)=e^{-\frac{1}{\zeta}(t-\langle t\rangle_s)^2+\gamma_1}e^{i[-\frac{1}{2\varepsilon}(t-\langle t\rangle_s)^2+\langle\omega\rangle_s t+\gamma_3]},$$

$$s(t)=\begin{cases} e^{-\frac{1}{\zeta}(t-\langle t\rangle_s)^2+\gamma_1}e^{i[\frac{1}{2\varepsilon}(t-\langle t\rangle_s)^2+\langle\omega\rangle_s t+\gamma_4]} & \text{if } \ t\geq\langle t\rangle_s,\\ e^{-\frac{1}{\zeta}(t-\langle t\rangle_s)^2+\gamma_1}e^{i[-\frac{1}{2\varepsilon}(t-\langle t\rangle_s)^2+\langle\omega\rangle_s t+\gamma_5]} & \text{if } \ t<\langle t\rangle_s,\end{cases}$$

or

$$s(t)=\begin{cases} e^{-\frac{1}{\zeta}(t-\langle t\rangle_s)^2+\gamma_1}e^{i[-\frac{1}{2\varepsilon}(t-\langle t\rangle_s)^2+\langle\omega\rangle_s t+\gamma_6]} & \text{if } \ t\geq\langle t\rangle_s,\\ e^{-\frac{1}{\zeta}(t-\langle t\rangle_s)^2+\gamma_1}e^{i[\frac{1}{2\varepsilon}(t-\langle t\rangle_s)^2+\langle\omega\rangle_s t+\gamma_7]} & \text{if } \ t<\langle t\rangle_s,\end{cases}$$

for some $\gamma_1,\gamma_2,\gamma_3,\gamma_4,\gamma_5,\gamma_6,\gamma_7,\zeta,\varepsilon\in\mathbb{R},\zeta,\varepsilon>0$, and $e^{2\gamma_1}\sqrt{\frac{\zeta\pi}{2}}=1$.

Note that the proof of the Lemma 3.1 is referred to [8]. The proof itself shows the physical reason why the above four cases of signals can make the equality in (3.14) hold. We will show (see Example 3.8), in fact, the first two classes of signals can make the equality in (1.2) hold, therefore, the equality in (3.14) holds, too. The last two classes of signals can really make the strict inequality hold (see Example 3.7), that is,

$$\mathrm{COV}_s^2>\mathrm{Cov}_s^2.$$

Theorem 3.2: Let $s(t)=\rho(t)e^{i\varphi(t)}$, $\omega\hat{s}(\omega)$ and $ts(t)\in L^2(\mathbb{R})$. $M_1=[a_1,b_1,c_1,d_1]$ and $M_2=[a_2,b_2,c_2,d_2]$. Then

$$\begin{aligned}\sigma_{M_1,u,s}^2\sigma_{M_2,u,s}^2 \geq\ & (a_1b_2-a_2b_1)^2(\frac{1}{4}+\mathrm{COV}_s^2-\mathrm{Cov}_s^2)\\ &+[a_1a_2\sigma_{t,s}^2+b_1b_2\sigma_{\omega,s}^2+(a_1b_2+a_2b_1)\mathrm{Cov}_s]^2.\end{aligned} \tag{3.15}$$

Under the extra assumptions that $s(t)=\rho(t)e^{i\varphi(t)}$ has the classical derivatives $s'(t),\varphi'(t),\rho'(t)$, where $\varphi'(t)$ is continuous and ρ is almost everywhere non-zero, the equality holds if and only if $s(t)$ has one of the following four forms

$$s(t)=e^{-\frac{1}{\zeta}(t-\langle t\rangle_s)^2+\gamma_1}e^{i[\frac{1}{2\varepsilon}(t-\langle t\rangle_s)^2+\langle\omega\rangle_s t+\gamma_2]}, \tag{3.16}$$

$$s(t)=e^{-\frac{1}{\zeta}(t-\langle t\rangle_s)^2+\gamma_1}e^{i[-\frac{1}{2\varepsilon}(t-\langle t\rangle_s)^2+\langle\omega\rangle_s t+\gamma_3]}, \tag{3.17}$$

$$s(t)=\begin{cases} e^{-\frac{1}{\zeta}(t-\langle t\rangle_s)^2+\gamma_1}e^{i[\frac{1}{2\varepsilon}(t-\langle t\rangle_s)^2+\langle\omega\rangle_s t+\gamma_4]} & \text{if} \quad t\geq\langle t\rangle_s,\\ e^{-\frac{1}{\zeta}(t-\langle t\rangle_s)^2+\gamma_1}e^{i[-\frac{1}{2\varepsilon}(t-\langle t\rangle_s)^2+\langle\omega\rangle_s t+\gamma_5]} & \text{if} \quad t<\langle t\rangle_s,\end{cases} \tag{3.18}$$

or

$$s(t)=\begin{cases} e^{-\frac{1}{\zeta}(t-\langle t\rangle_s)^2+\gamma_1}e^{i[-\frac{1}{2\varepsilon}(t-\langle t\rangle_s)^2+\langle\omega\rangle_s t+\gamma_6]} & \text{if} \quad t\geq\langle t\rangle_s,\\ e^{-\frac{1}{\zeta}(t-\langle t\rangle_s)^2+\gamma_1}e^{i[\frac{1}{2\varepsilon}(t-\langle t\rangle_s)^2+\langle\omega\rangle_s t+\gamma_7]} & \text{if} \quad t<\langle t\rangle_s,\end{cases} \tag{3.19}$$

for some $\gamma_1,\gamma_2,\gamma_3,\gamma_4,\gamma_5,\gamma_6,\gamma_7,\zeta,\varepsilon\in\mathbb{R},\zeta,\varepsilon>0$, and $e^{2\gamma_1}\sqrt{\frac{\zeta\pi}{2}}=1$.

Proof of Theorem 3.2 Thanks to the equal relations given by Lemma 2.6, we have

$$\sigma^2_{M_1,u,s}=a_1^2\sigma^2_{t,s}+2a_1b_1\mathrm{Cov}_s+b_1^2\sigma^2_{\omega,s},$$

and

$$\sigma^2_{M_2,u,s}=a_2^2\sigma^2_{t,s}+2a_2b_2\mathrm{Cov}_s+b_2^2\sigma^2_{\omega,s}.$$

By invoking the sharper uncertainty principle of the classical setting given in Lemma 3.1, viz., the inequality

$$\sigma^2_{t,s}\sigma^2_{\omega,s} \quad\geq\quad \frac{1}{4}+\mathrm{COV}^2_s, \tag{3.20}$$

we have

$$\begin{aligned}
&\sigma^2_{M_1,u,s}\sigma^2_{M_2,u,s}\\
=\;&(a_1^2\sigma^2_{t,s}+2a_1b_1\mathrm{Cov}_s+b_1^2\sigma^2_{\omega,s})(a_2^2\sigma^2_{t,s}+2a_2b_2\mathrm{Cov}_s+b_2^2\sigma^2_{\omega,s})\\
=\;&a_1^2a_2^2\sigma^4_{t,s}+2a_1^2a_2b_2\sigma^2_{t,s}\mathrm{Cov}_s+a_1^2b_2^2\sigma^2_{t,s}\sigma^2_{\omega,s}+2a_2^2a_1b_1\sigma^2_{t,s}\mathrm{Cov}_s+4a_1b_1a_2b_2\mathrm{Cov}^2_s\\
&+2b_2^2a_1b_1\sigma^2_{\omega,s}\mathrm{Cov}_s+b_1^2a_2^2\sigma^2_{\omega,s}\sigma^2_{t,s}+2b_1^2a_2b_2\sigma^2_{\omega,s}\mathrm{Cov}_s+b_1^2b_2^2\sigma^4_{\omega,s}\\
=\;&[a_1a_2\sigma^2_{t,s}+b_1b_2\sigma^2_{\omega,s}+(a_1b_2+a_2b_1)\mathrm{Cov}_s]^2+(a_1b_2-a_2b_1)^2(\sigma^2_{t,s}\sigma^2_{\omega,s}-\mathrm{Cov}^2_s)\\
\geq\;&[a_1a_2\sigma^2_{t,s}+b_1b_2\sigma^2_{\omega,s}+(a_1b_2+a_2b_1)\mathrm{Cov}_s]^2+(a_1b_2-a_2b_1)^2(\frac{1}{4}+\mathrm{COV}^2_s-\mathrm{Cov}^2_s), \qquad (3.21)
\end{aligned}$$

TABLE III.1

COMPARISONS BETWEEN THE LOWER-BOUNDS OF UNCERTAINTY PRINCIPLE IN LCT DOMAINS

Uncertainty Product	Type of signals	Lower-bound of Uncertainty Product	References
$\sigma^2_{M_1,u,s}\sigma^2_{M_2,u,s}$	real signals	$\frac{1}{4}(a_1b_2-a_2b_1)^2+[a_1a_2\sigma^2_{t,s}+\frac{b_1b_2}{4\sigma^2_{t,s}}]^2$	[16], [21], [24]
$\sigma^2_{M_1,u,s}\sigma^2_{M_2,u,s}$	complex signals	$\frac{1}{4}(a_1b_2-a_2b_1)^2$	[20], [24]
$\sigma^2_{M_1,u,s}\sigma^2_{M_2,u,s}$	complex signals	$\frac{1}{4}(a_1b_2-a_2b_1)^2$ $+[a_1a_2\sigma^2_{t,s}+b_1b_2\sigma^2_{\omega,s}+(a_1b_2+a_2b_1)\mathrm{Cov}_s]^2$	[22], [25]
$\sigma^2_{M_1,u,s}\sigma^2_{M_2,u,s}$	complex signals	$(\frac{1}{4}+\mathrm{COV}^2_s-\mathrm{Cov}^2_s)(a_1b_2-a_2b_1)^2$ $+[a_1a_2\sigma^2_{t,s}+b_1b_2\sigma^2_{\omega,s}+(a_1b_2+a_2b_1)\mathrm{Cov}_s]^2$	Present Study

as desired. We note that the equality in (3.21) holds if and only if the equality in (3.20) holds.

We provide the Table III.1 in order to compare the existing and proposed results.

Remark 3.3: One of the purposes of the series of studies in signal analysis given in [7], [5] and [8], etc., is to establish a theoretical foundation of signal analysis/processing for signals of finite energy, viz., of Lebesgue square integrable functions, those, in particular, are not necessary to be continuous, nor of particular forms. The essence of using Lebesgue integration in the proof is that the integrals eliminate the effect of the possible infinite jumps of the phase derivative induced by discontinuity of signals. Moreover, the use of Hölder inequality in Lemma 3.1 (see [8]) justifies first taking the absolute value on the integrand and validates the role of the larger quantity, the absolute covariance COV_s, in the uncertainty product estimation.

B. Some Special Cases of Uncertainty Principle for LCT

Corollary 3.4: Let $s(t)=\rho(t)e^{i\varphi(t)},\ \omega\hat{s}(\omega)$ and $ts(t)\in L^2(\mathbb{R}),\ M=(a,b,c,d)$. Then

$$\sigma^2_{t,s}\sigma^2_{M,u,s}\geq b^2(\frac{1}{4}+\mathrm{COV}^2_s-\mathrm{Cov}^2_s)+(a\sigma^2_{t,s}+b\mathrm{Cov}_s)^2. \tag{3.22}$$

The equality in (3.22) holds if the signal under study is of one of the forms (3.16), (3.17), (3.18) and (3.19).

When $M=(\cos\alpha,\sin\alpha,-\sin\alpha,\cos\alpha)$, we can obtain the following uncertainty relation in two FrFT domains.

Corollary 3.5: Let $s(t)=\rho(t)e^{i\varphi(t)},\omega\hat{s}(\omega),ts(t)\in L^2(\mathbb{R}),M_1=(\cos\alpha,\sin\alpha,-\sin\alpha,\cos\alpha),M_2=(\cos\beta,\sin\beta,\ -\sin\beta,\cos\beta)$. Then

$$\begin{aligned}\sigma^2_{M_1,u,s}\sigma^2_{M_2,u,s} &= \sigma^2_{\alpha,s}\sigma^2_{\beta,s}\\ &\geq (\cos\alpha\sin\beta-\cos\beta\sin\alpha)^2(\frac{1}{4}+\mathrm{COV}_s^2-\mathrm{Cov}_s^2)\\ &\quad+[\cos\alpha\cos\beta\sigma^2_{t,s}+\sin\alpha\sin\beta\sigma^2_{\omega,s}+(\cos\alpha\sin\beta+\cos\beta\sin\alpha)\mathrm{Cov}_s]^2. \quad (3.23)\end{aligned}$$

In particular,

$$\sigma^2_{t,s}\sigma^2_{\alpha,s} \geq \sin^2\alpha(\frac{1}{4}+\mathrm{COV}_s^2-\mathrm{Cov}_s^2)+(\cos\alpha\sigma^2_{t,s}+\sin\alpha\mathrm{Cov}_s)^2. \quad (3.24)$$

The equalities in (3.23) and (3.24) hold if the signal under study is of one of the forms (3.16), (3.17), (3.18) and (3.19).

The uncertainty principle in the LCT domains in [22] and [25] is essentially

$$\sigma^2_{M_1,u,s}\sigma^2_{M_2,u,s} \geq \frac{(a_1b_2-a_2b_1)^2}{4}+[a_1a_2\sigma^2_{t,s}+b_1b_2\sigma^2_{\omega,s}+(a_1b_2+a_2b_1)\mathrm{Cov}_s]^2. \quad (3.25)$$

To compare the new and the old uncertainty principles we have

Corollary 3.6: Under the assumptions of Theorem 3.2, the following are equivalent.

(i) The equal sign holds in the old uncertainty principle (3.25).

(ii) The equal signs hold simultaneously in the new uncertainty principle (3.15) and in the old uncertainty principle (3.25).

(iii) The function $(t-\langle t\rangle_s)[(D\varphi)(t)-\langle\omega\rangle_s]\rho^2(t)$ is almost everywhere non-negative or almost everywhere non-positive.

As consequence, the lower bound of (3.15) is strictly larger than that of (3.25) if and only if the function values of $(t-\langle t\rangle_s)[(D\varphi)(t)-\langle\omega\rangle_s]\rho^2(t)$ are positive on a Lebesgue measurable set of positive measure, as well as negative on a Lebesgue measurable set of positive measure.

Proof of Corollary 3.6 The equivalence between (i) and (ii) is obvious. The equivalence between (ii) and (iii) is based on the condition on

$$|\int_{-\infty}^{\infty} f(t)dt| = \int_{-\infty}^{\infty}|f(t)|dt$$

for a Lebesgue integrable function f.

C. Examples and Simulations

Theorem 3.2 gives a sharper lower-bound for the uncertainty product in two LCT domains. This fact is illustrated by the following example. It shows that for the signal class (3.18) the equal sign in the uncertainty principle inequality of Theorem 3.2 holds; and, at the same time, for this class of signals the lower bounds in Theorem 3.2 is strictly larger than the lower bound obtained in [22].

Example 3.7: Let

$$s(t)=\begin{cases}(\frac{\alpha}{\pi})^{\frac{1}{4}}e^{-\frac{\alpha}{2}(t-\langle t\rangle_s)^2}e^{i[\frac{1}{2\varepsilon}(t-\langle t\rangle_s)^2+\langle\omega\rangle_s t+\gamma_4]} & \text{if} \quad t\geq\langle t\rangle_s,\\ (\frac{\alpha}{\pi})^{\frac{1}{4}}e^{-\frac{\alpha}{2}(t-\langle t\rangle_s)^2}e^{i[-\frac{1}{2\varepsilon}(t-\langle t\rangle_s)^2+\langle\omega\rangle_s t+\gamma_5]} & \text{if} \quad t<\langle t\rangle_s.\end{cases} \tag{3.26}$$

Then

$$\begin{aligned}\sigma_{t,s}^2 &= \int_{-\infty}^{\infty}(t-\langle t\rangle_s)^2|s(t)|^2dt\\ &= \int_{-\infty}^{\infty}(t-\langle t\rangle_s)^2|(\frac{\alpha}{\pi})^{\frac{1}{4}}e^{-\frac{\alpha}{2}(t-\langle t\rangle_s)^2}|^2dt\\ &= \int_{-\infty}^{\infty}(t-\langle t\rangle_s)^2(\frac{\alpha}{\pi})^{\frac{1}{2}}e^{-\alpha(t-\langle t\rangle_s)^2}dt\\ &= (\frac{\alpha}{\pi})^{\frac{1}{2}}\int_{-\infty}^{\infty}t^2e^{-\alpha t^2}dt\\ &= \frac{1}{2\alpha},\end{aligned}$$

$$\begin{aligned}\sigma_{\omega,s}^2 &= \int_{-\infty}^{\infty}(D\rho)^2(t)dt+\int_{-\infty}^{\infty}[(D\varphi)(t)-\langle\omega\rangle_s]^2|s(t)|^2dt\\ &= \int_{-\infty}^{\infty}(\frac{\alpha}{\pi})^{\frac{1}{2}}\alpha^2(t-\langle t\rangle_s)^2e^{-\alpha(t-\langle t\rangle_s)^2}dt+\int_{\langle t\rangle_s}^{\infty}[\frac{1}{\varepsilon}(t-\langle t\rangle_s)+\langle\omega\rangle_s-\langle\omega\rangle_s]^2|s(t)|^2dt\\ &+ \int_{-\infty}^{\langle t\rangle_s}[-\frac{1}{\varepsilon}(t-\langle t\rangle_s)+\langle\omega\rangle_s-\langle\omega\rangle_s]^2|s(t)|^2dt\\ &= \frac{\alpha}{2}+\frac{1}{\varepsilon^2}\int_{-\infty}^{\infty}(t-\langle t\rangle_s)^2(\frac{\alpha}{\pi})^{\frac{1}{2}}e^{-\alpha(t-\langle t\rangle_s)^2}dt\\ &= \frac{\alpha}{2}+\frac{1}{2\varepsilon^2\alpha}=\frac{\alpha^2\varepsilon^2+1}{2\varepsilon^2\alpha},\end{aligned}$$

$$
\begin{aligned}
& \mathrm{Cov}_s \\
= \ & \int_{-\infty}^{\infty}(t-\langle t\rangle_s)[(D\varphi)(t)-\langle\omega\rangle_s]|s(t)|^2dt \\
= \ & \int_{\langle t\rangle_s}^{\infty}(t-\langle t\rangle_s)[\frac{1}{\varepsilon}(t-\langle t\rangle_s)+\langle\omega\rangle_s-\langle\omega\rangle_s]|s(t)|^2dt \\
& +\int_{-\infty}^{\langle t\rangle_s}(t-\langle t\rangle_s)[-\frac{1}{\varepsilon}(t-\langle t\rangle_s)+\langle\omega\rangle_s-\langle\omega\rangle_s]|s(t)|^2dt \\
= \ & \int_{\langle t\rangle_s}^{\infty}(t-\langle t\rangle_s)[\frac{1}{\varepsilon}(t-\langle t\rangle_s)]|s(t)|^2dt+\int_{-\infty}^{\langle t\rangle_s}(t-\langle t\rangle_s)[-\frac{1}{\varepsilon}(t-\langle t\rangle_s)]|s(t)|^2dt \\
= \ & \frac{1}{\varepsilon}\int_{\langle t\rangle_s}^{\infty}(t-\langle t\rangle_s)^2(\frac{\alpha}{\pi})^{\frac{1}{2}}e^{-\alpha(t-\langle t\rangle_s)^2}dt-\frac{1}{\varepsilon}\int_{-\infty}^{\langle t\rangle_s}(t-\langle t\rangle_s)^2(\frac{\alpha}{\pi})^{\frac{1}{2}}e^{-\alpha(t-\langle t\rangle_s)^2}dt \\
= \ & \frac{1}{\varepsilon}\int_{0}^{\infty}t^2(\frac{\alpha}{\pi})^{\frac{1}{2}}e^{-\alpha t^2}dt-\frac{1}{\varepsilon}\int_{-\infty}^{0}t^2(\frac{\alpha}{\pi})^{\frac{1}{2}}e^{-\alpha t^2}dt \\
= \ & 0,
\end{aligned}
$$

$$
\begin{aligned}
& \mathrm{COV}_s \\
= \ & \int_{-\infty}^{\infty}|(t-\langle t\rangle_s)[(D\varphi)(t)-\langle\omega\rangle_s]||s(t)|^2dt \\
= \ & \int_{\langle t\rangle_s}^{\infty}|(t-\langle t\rangle_s)[\frac{1}{\varepsilon}(t-\langle t\rangle_s)+\langle\omega\rangle_s-\langle\omega\rangle_s]||s(t)|^2dt \\
& +\int_{-\infty}^{\langle t\rangle_s}|(t-\langle t\rangle_s)[-\frac{1}{\varepsilon}(t-\langle t\rangle_s)+\langle\omega\rangle_s-\langle\omega\rangle_s]||s(t)|^2dt \\
= \ & \int_{\langle t\rangle_s}^{\infty}|(t-\langle t\rangle_s)[\frac{1}{\varepsilon}(t-\langle t\rangle_s)]||s(t)|^2dt+\int_{-\infty}^{\langle t\rangle_s}|(t-\langle t\rangle_s)[-\frac{1}{\varepsilon}(t-\langle t\rangle_s)]||s(t)|^2dt \\
= \ & \frac{1}{\varepsilon}\int_{\langle t\rangle_s}^{\infty}(t-\langle t\rangle_s)^2(\frac{\alpha}{\pi})^{\frac{1}{2}}e^{-\alpha(t-\langle t\rangle_s)^2}dt+\frac{1}{\varepsilon}\int_{-\infty}^{\langle t\rangle_s}(t-\langle t\rangle_s)^2(\frac{\alpha}{\pi})^{\frac{1}{2}}e^{-\alpha(t-\langle t\rangle_s)^2}dt \\
= \ & \frac{1}{\varepsilon}\int_{0}^{\infty}t^2(\frac{\alpha}{\pi})^{\frac{1}{2}}e^{-\alpha t^2}dt+\frac{1}{\varepsilon}\int_{-\infty}^{0}t^2(\frac{\alpha}{\pi})^{\frac{1}{2}}e^{-\alpha t^2}dt \\
= \ & \frac{1}{\varepsilon}\int_{-\infty}^{\infty}t^2(\frac{\alpha}{\pi})^{\frac{1}{2}}e^{-\alpha t^2}dt \\
= \ & \frac{1}{2\alpha\varepsilon}.
\end{aligned}
$$

We therefore conclude that

$$\sigma_{t,s}^2\sigma_{\omega,s}^2=\frac{1}{4}+\mathrm{COV}_s^2>\frac{1}{4}+\mathrm{Cov}_s^2=\frac{1}{4}.$$

Furthermore,

$$\sigma^2_{M_1,u,s} = a_1^2\sigma^2_{t,s} + 2a_1b_1\mathrm{Cov}_s + b_1^2\sigma^2_{\omega,s} = a_1^2\frac{1}{2\alpha} + b_1^2\frac{\alpha^2\varepsilon^2+1}{2\varepsilon^2\alpha} = \frac{a_1^2\varepsilon^2 + b_1^2(\alpha^2\varepsilon^2+1)}{2\varepsilon^2\alpha}, \tag{3.27}$$

and

$$\sigma^2_{M_2,u,s} = \frac{a_2^2\varepsilon^2 + b_2^2(\alpha^2\varepsilon^2+1)}{2\varepsilon^2\alpha}. \tag{3.28}$$

It can be calculated that

$$\begin{aligned} \sigma^2_{M_1,u,s}\sigma^2_{M_2,u,s} &= (a_1b_2 - a_2b_1)^2(\frac{1}{4} + \mathrm{COV}_s^2 - \mathrm{Cov}_s^2) \\ &\quad + [a_1a_2\sigma^2_{t,s} + b_1b_2\sigma^2_{\omega,s} + (a_1b_2 + a_2b_1)\mathrm{Cov}_s]^2. \end{aligned} \tag{3.29}$$

The formulas (3.27), (3.28) and (3.29) still hold when $\langle t\rangle_s = 0$ and $\langle\omega\rangle_s = 0$. Under the assumption $\langle t\rangle_s = 0$ and $\langle\omega\rangle_s = 0$ the above signal is reduced to the case that is considered in [22]. Then the lower bound of uncertainty product in [22] is

$$\begin{aligned} & \frac{(a_1b_2 - a_2b_1)^2}{4} + [a_1a_2 \triangle t^2 + b_1b_2 \triangle \omega^2 + b_1b_2K_1 + (a_1b_2 + a_2b_1)K_2]^2 \\ = \; & \frac{(a_1b_2 - a_2b_1)^2}{4} + (a_1a_2\frac{1}{2\alpha} + b_1b_2\frac{\alpha^2\varepsilon^2+1}{2\varepsilon^2\alpha})^2 \\ < \; & (a_1b_2 - a_2b_1)^2(\frac{1}{4} + \frac{1}{2\alpha\varepsilon}) + (a_1a_2\frac{1}{2\alpha} + b_1b_2\frac{\alpha^2\varepsilon^2+1}{2\varepsilon^2\alpha})^2 \\ = \; & (a_1b_2 - a_2b_1)^2(\frac{1}{4} + \mathrm{COV}_s^2 - \mathrm{Cov}_s^2) \\ & + [a_1a_2\sigma^2_{t,s} + b_1b_2\sigma^2_{\omega,s} + (a_1b_2 + a_2b_1)\mathrm{Cov}_s]^2, \end{aligned} \tag{3.30}$$

where

$$\triangle t^2 = \int_{-\infty}^{\infty} t^2|s(t)|^2 dt = \frac{1}{2\alpha},$$

$$\triangle \omega^2 + K_1 = \int_{-\infty}^{\infty} \omega^2|\hat{s}(\omega)|^2 = \frac{\alpha^2\varepsilon^2+1}{2\varepsilon^2\alpha},$$

and

$$K_2 = \int_{-\infty}^{\infty} t(D\varphi)(t)|s(t)|^2 dt = 0.$$

The formula (3.30) indicates that the lower bound of uncertainty principle in Theorem 3.2 is strictly sharper than that in [22].

The following example corresponds to the signal class (3.16) that give rise to some cases in which the equal sign in the old uncertainty principle holds. Mathematically it implies that the equal sign in the new uncertainty principle also holds. The numerical computation also shows so.

Example 3.8: Let

$$s(t) = (\frac{\alpha}{\pi})^{\frac{1}{4}} e^{-\frac{\alpha}{2}(t-\langle t\rangle_s)^2} e^{i[\frac{1}{2\varepsilon}(t-\langle t\rangle_s)^2+\langle\omega\rangle_s t+\gamma]}, \tag{3.31}$$

where $\alpha > 0, \varepsilon, \gamma \in \mathbb{R}$. It is a signal of the form (3.16).

Then

$$\sigma_{t,s}^2 = \frac{1}{2\alpha}, \qquad \sigma_{\omega,s}^2 = \frac{\alpha^2\varepsilon^2+1}{2\varepsilon^2\alpha},$$

and

$$\begin{aligned}
\mathrm{Cov}_s &= \int_{-\infty}^{\infty} (t-\langle t\rangle_s)[(D\varphi)(t)-\langle\omega\rangle_s]|s(t)|^2 dt \\
&= \int_{-\infty}^{\infty} (t-\langle t\rangle_s)[\frac{1}{\varepsilon}(t-\langle t\rangle_s)+\langle\omega\rangle_s-\langle\omega\rangle_s]|s(t)|^2 dt \\
&= \int_{-\infty}^{\infty} \frac{1}{\varepsilon}(t-\langle t\rangle_s)^2|s(t)|^2 dt \\
&= \frac{1}{2\alpha\varepsilon} \\
&= \int_{-\infty}^{\infty} |(t-\langle t\rangle_s)[(D\varphi)(t)-\langle\omega\rangle_s]||s(t)|^2 dt \\
&= \mathrm{COV}_s.
\end{aligned}$$

We therefore conclude that

$$\sigma_{t,s}^2\sigma_{\omega,s}^2 = \frac{1}{4} + \mathrm{COV}_s^2 = \frac{1}{4} + \mathrm{Cov}_s^2 > \frac{1}{4},$$

and

$$\begin{aligned}
\sigma_{M_1,u,s}^2\sigma_{M_2,u,s}^2 &= (a_1b_2-a_2b_1)^2(\frac{1}{4}+\mathrm{COV}_s^2-\mathrm{Cov}_s^2) \\
&\quad +[a_1a_2\sigma_{t,s}^2+b_1b_2\sigma_{\omega,s}^2+(a_1b_2+a_2b_1)\mathrm{Cov}_s]^2 \\
&= \frac{(a_1b_2-a_2b_1)^2}{4} + [a_1a_2\sigma_{t,s}^2+b_1b_2\sigma_{\omega,s}^2+(a_1b_2+a_2b_1)\mathrm{Cov}_s]^2.
\end{aligned} \tag{3.32}$$

The formula (3.32) indicates the equalities in (3.15) and [22] both hold for the signal $s(t)$. We can see that in this example the integrand of $\mathrm{Cov}_{t,\omega}$, viz.

$$(t-\langle t\rangle_s)[(D\varphi)(t)-\langle\omega\rangle_s]|s(t)|^2 = \frac{1}{\varepsilon}(t-\langle t\rangle_s)^2|s(t)|^2$$

keeps the same sign on $\mathbb{R}$, that is the reason why the new and old lower bounds coincide (See Corollary 3.6).

Let $\alpha = \varepsilon = 1, \langle t\rangle_s = \langle\omega\rangle_s = 0, [a_1, b_1, c_1, d_1] = [1, 0, 1, 1]$ and $[a_2, b_2, c_2, d_2] = [1, -1, 0, 1]$ in this example, we can obtain that

$$\sigma^2_{M_1,u,s}\sigma^2_{M_2,u,s} \approx 0.24999999999999999320465138830953;$$

$$(a_1b_2 - a_2b_1)^2(\frac{1}{4} + \mathrm{COV}^2_s - \mathrm{Cov}^2_s) + [a_1a_2\sigma^2_{t,s} + b_1b_2\sigma^2_{\omega,s} + (a_1b_2 + a_2b_1)\mathrm{Cov}_s]^2 \approx 0.25;$$

$$(a_1b_2 - a_2b_1)^2 \cdot \frac{1}{4} + [a_1a_2\sigma^2_{t,s} + b_1b_2\sigma^2_{\omega,s} + (a_1b_2 + a_2b_1)\mathrm{Cov}_s]^2 \approx 0.25;$$

$$\sigma^2_{t,s}\sigma^2_{\omega,s} \approx 0.49999999999999998640930277661906;$$

$$\frac{1}{4} + \mathrm{COV}^2_s \approx 0.49999999999999999320465138830953;$$

$$\frac{1}{4} + \mathrm{Cov}^2_s \approx 0.49999999999999999320465138830953,$$

so we can conclude from the point of numerical simulation that

$$\begin{aligned}\sigma^2_{M_1,u,s}\sigma^2_{M_2,u,s} &\approx (a_1b_2 - a_2b_1)^2(\frac{1}{4} + \mathrm{COV}^2_s - \mathrm{Cov}^2_s)\\ &\quad +[a_1a_2\sigma^2_{t,s} + b_1b_2\sigma^2_{\omega,s} + (a_1b_2 + a_2b_1)\mathrm{Cov}_s]^2\\ &= (a_1b_2 - a_2b_1)^2 \cdot \frac{1}{4}\\ &\quad +[a_1a_2\sigma^2_{t,s} + b_1b_2\sigma^2_{\omega,s} + (a_1b_2 + a_2b_1)\mathrm{Cov}_s]^2, ,\end{aligned}$$

and

$$\sigma^2_{t,s}\sigma^2_{\omega,s} \approx \frac{1}{4} + \mathrm{COV}^2_s \approx \frac{1}{4} + \mathrm{Cov}^2_s > \frac{1}{4}.$$

Example 3.9: Let

$$s(t) = (\frac{\alpha}{\pi})^{\frac{1}{4}} e^{-\frac{\alpha}{2}t^2} e^{i(\frac{1}{2}t^2 - \frac{\alpha}{6}t^4)},$$

where $\alpha > 0$.

Then

$$\langle t\rangle_s = \int_{-\infty}^{\infty} t|s(t)|^2 dt = \int_{-\infty}^{\infty} t(\frac{\alpha}{\pi})^{\frac{1}{2}} e^{-\alpha t^2} dt = 0,$$

$$\langle\omega\rangle_s = \int_{-\infty}^{\infty} (D\varphi)(t)|s(t)|^2 dt = \int_{-\infty}^{\infty} (t - \frac{2\alpha}{3}t^3)(\frac{\alpha}{\pi})^{\frac{1}{2}} e^{-\alpha t^2} dt = 0,$$

$$\sigma^2_{t,s} = \int_{-\infty}^{\infty} t^2|s(t)|^2 dt = \int_{-\infty}^{\infty} t^2(\frac{\alpha}{\pi})^{\frac{1}{2}} e^{-\alpha t^2} dt = \frac{1}{2\alpha},$$

$$
\begin{aligned}
\sigma^2_{\omega,s} &= \int_{-\infty}^{\infty}(D\rho)^2(t)dt + \int_{-\infty}^{\infty}(D\varphi)^2(t)|s(t)|^2dt \\
&= \int_{-\infty}^{\infty}(\frac{\alpha}{\pi})^{\frac{1}{2}}\alpha^2t^2e^{-\alpha t^2}dt + \int_{-\infty}^{\infty}(t-\frac{2\alpha}{3}t^3)^2(\frac{\alpha}{\pi})^{\frac{1}{2}}e^{-\alpha t^2}dt \\
&= \frac{\alpha}{2} + (\frac{\alpha}{\pi})^{\frac{1}{2}}\int_{-\infty}^{\infty}(t^2-\frac{4\alpha}{3}t^4+\frac{4\alpha^2}{9}t^6)e^{-\alpha t^2}dt \\
&= \frac{\alpha}{2} + (\frac{\alpha}{\pi})^{\frac{1}{2}}\int_{-\infty}^{\infty}(t^2-2t^2+\frac{5}{3}t^2)e^{-\alpha t^2}dt \\
&= \frac{\alpha}{2} + \frac{2}{3}(\frac{\alpha}{\pi})^{\frac{1}{2}}\int_{-\infty}^{\infty}t^2e^{-\alpha t^2}dt \\
&= \frac{\alpha}{2} + \frac{1}{3\alpha},
\end{aligned}
$$

where

$$
\int_{-\infty}^{\infty}t^4e^{-\alpha t^2}dt = -\frac{1}{2\alpha}[t^3e^{-\alpha t^2}|_{-\infty}^{\infty} - 3\int_{-\infty}^{\infty}t^2e^{-\alpha t^2}dt] = \frac{3}{2\alpha}\int_{-\infty}^{\infty}t^2e^{-\alpha t^2}dt,
$$

and

$$
\int_{-\infty}^{\infty}t^6e^{-\alpha t^2}dt = -\frac{1}{2\alpha}[t^5e^{-\alpha t^2}|_{-\infty}^{\infty} - 5\int_{-\infty}^{\infty}t^4e^{-\alpha t^2}dt] = \frac{15}{4\alpha^2}\int_{-\infty}^{\infty}t^2e^{-\alpha t^2}dt.
$$

$$
\begin{aligned}
\mathrm{Cov}_s &= \int_{-\infty}^{\infty}t(D\varphi)(t)|s(t)|^2dt \\
&= \int_{-\infty}^{\infty}t(t-\frac{2\alpha}{3}t^3)(\frac{\alpha}{\pi})^{\frac{1}{2}}e^{-\alpha t^2}dt \\
&= (\frac{\alpha}{\pi})^{\frac{1}{2}}\int_{-\infty}^{\infty}t^2e^{-\alpha t^2}dt - \frac{2\alpha}{3}(\frac{\alpha}{\pi})^{\frac{1}{2}}\int_{-\infty}^{\infty}t^4e^{-\alpha t^2}dt \\
&= (\frac{\alpha}{\pi})^{\frac{1}{2}}\int_{-\infty}^{\infty}t^2e^{-\alpha t^2}dt - \frac{2\alpha}{3}(\frac{\alpha}{\pi})^{\frac{1}{2}}\frac{3}{2\alpha}\int_{-\infty}^{\infty}t^2e^{-\alpha t^2}dt \\
&= 0,
\end{aligned}
$$

and

$$\begin{aligned}
\mathrm{COV}_s &= \int_{-\infty}^{\infty} |t(D\varphi)(t)||s(t)|^2 dt \\
&= \int_{-\infty}^{\infty} |t(t-\frac{2\alpha}{3}t^3)|(\frac{\alpha}{\pi})^{\frac{1}{2}} e^{-\alpha t^2} dt \\
&= 2\int_0^{\infty} t^2|1-\frac{2\alpha}{3}t^2|(\frac{\alpha}{\pi})^{\frac{1}{2}} e^{-\alpha t^2} dt \\
&= 2[\int_0^{\sqrt{\frac{3}{2\alpha}}} t^2(1-\frac{2\alpha}{3}t^2)(\frac{\alpha}{\pi})^{\frac{1}{2}} e^{-\alpha t^2} dt + \int_{\sqrt{\frac{3}{2\alpha}}}^{\infty} t^2(\frac{2\alpha}{3}t^2-1)(\frac{\alpha}{\pi})^{\frac{1}{2}} e^{-\alpha t^2} dt] \\
&= 2[(\frac{\alpha}{\pi})^{\frac{1}{2}} \int_0^{\sqrt{\frac{3}{2\alpha}}} t^2 e^{-\alpha t^2} dt - \frac{2\alpha}{3}(\frac{\alpha}{\pi})^{\frac{1}{2}} \int_0^{\sqrt{\frac{3}{2\alpha}}} t^4 e^{-\alpha t^2} dt \\
&\quad + \frac{2\alpha}{3}(\frac{\alpha}{\pi})^{\frac{1}{2}} \int_{\sqrt{\frac{3}{2\alpha}}}^{\infty} t^4 e^{-\alpha t^2} dt - (\frac{\alpha}{\pi})^{\frac{1}{2}} \int_{\sqrt{\frac{3}{2\alpha}}}^{\infty} t^2 e^{-\alpha t^2} dt] \\
&= 2\{(\frac{\alpha}{\pi})^{\frac{1}{2}} \int_0^{\sqrt{\frac{3}{2\alpha}}} t^2 e^{-\alpha t^2} dt + \frac{1}{3}(\frac{\alpha}{\pi})^{\frac{1}{2}} [t^3 e^{-\alpha t^2}|_0^{\sqrt{\frac{3}{2\alpha}}} - 3\int_0^{\sqrt{\frac{3}{2\alpha}}} t^2 e^{-\alpha t^2} dt] \\
&\quad -\frac{1}{3}(\frac{\alpha}{\pi})^{\frac{1}{2}} [t^3 e^{-\alpha t^2}|_{\sqrt{\frac{3}{2\alpha}}}^{\infty} - 3\int_{\sqrt{\frac{3}{2\alpha}}}^{\infty} t^2 e^{-\alpha t^2} dt] - (\frac{\alpha}{\pi})^{\frac{1}{2}} \int_{\sqrt{\frac{3}{2\alpha}}}^{\infty} t^2 e^{-\alpha t^2} dt\} \\
&= 2\cdot 2\cdot \frac{2\alpha}{3}(\frac{\alpha}{\pi})^{\frac{1}{2}} \frac{1}{2\alpha}\frac{3}{2\alpha}\sqrt{\frac{3}{2\alpha}} e^{-\frac{3}{2}} = \frac{\sqrt{6}}{\alpha\sqrt{\pi}} e^{-\frac{3}{2}}.
\end{aligned}$$

Thus we conclude

$$\sigma_{t,s}^2\sigma_{\omega,s}^2 = \frac{1}{4}+\frac{1}{6\alpha^2} > \frac{1}{4}+\frac{6}{\pi e^3\alpha^2} = \frac{1}{4}+\mathrm{COV}_{t,\omega,s}^2 > \frac{1}{4}+\mathrm{Cov}_{t,\omega,s}^2 = \frac{1}{4}.$$

Due to the Lemma 2.6, we have

$$\sigma_{M_1,u,s}^2 = a_1^2\sigma_{t,s}^2 + 2a_1b_1\mathrm{Cov}_s + b_1^2\sigma_{\omega,s}^2 = a_1^2\frac{1}{2\alpha} + b_1^2(\frac{\alpha}{2}+\frac{1}{3\alpha}), \tag{3.33}$$

and

$$\sigma_{M_2,u,s}^2 = a_2^2\sigma_{t,s}^2 + 2a_2b_2\mathrm{Cov}_s + b_2^2\sigma_{\omega,s}^2 = a_2^2\frac{1}{2\alpha} + b_2^2(\frac{\alpha}{2}+\frac{1}{3\alpha}). \tag{3.34}$$

It can be calculated that

$$\begin{aligned}
&\sigma_{M_1,u,s}^2\sigma_{M_2,u,s}^2 - \{(a_1b_2-a_2b_1)^2(\frac{1}{4}+\mathrm{COV}_s^2-\mathrm{Cov}_s^2) + [a_1a_2\sigma_{t,s}^2 + b_1b_2\sigma_{\omega,s}^2 + (a_1b_2+a_2b_1)\mathrm{Cov}_s]^2\} \\
&= (\frac{1}{6\alpha^2}-\frac{6}{\pi e^3\alpha^2})(a_1b_2-a_2b_1)^2 \geq 0,
\end{aligned}$$

then

$$\begin{aligned}
\sigma_{M_1,u,s}^2\sigma_{M_2,u,s}^2 &\geq (a_1b_2-a_2b_1)^2(\frac{1}{4}+\mathrm{COV}_s^2-\mathrm{Cov}_s^2) + [a_1a_2\sigma_{t,s}^2 + b_1b_2\sigma_{\omega,s}^2 + (a_1b_2+a_2b_1)\mathrm{Cov}_s]^2 \\
&> (a_1b_2-a_2b_1)^2\cdot\frac{1}{4} + [a_1a_2\sigma_{t,s}^2 + b_1b_2\sigma_{\omega,s}^2 + (a_1b_2+a_2b_1)\mathrm{Cov}_s]^2.
\end{aligned}$$

Specially, if $\frac{a_1}{b_1} \neq \frac{a_2}{b_2}$, then the product of $\sigma^2_{M_1,u,s}$ and $\sigma^2_{M_2,u,s}$ is strictly larger than the lower bound in the formula (3.15), that is,

$$\sigma^2_{M_1,u,s}\sigma^2_{M_2,u,s} > (a_1b_2 - a_2b_1)^2(\frac{1}{4} + \mathrm{COV}_s^2 - \mathrm{Cov}_s^2) + [a_1a_2\sigma^2_{t,s} + b_1b_2\sigma^2_{\omega,s} + (a_1b_2 + a_2b_1)\mathrm{Cov}_s]^2.$$

Let $\alpha = 1, [a_1, b_1, c_1, d_1] = [1, 0, 1, 1]$ and $[a_2, b_2, c_2, d_2] = [1, -1, 0, 1]$ in this example, we can obtain that

$$\sigma^2_{M_1,u,s}\sigma^2_{M_2,u,s} \approx 0.66666666666666664854573703549208;$$

$$\begin{aligned}&(a_1b_2 - a_2b_1)^2(\frac{1}{4} + \mathrm{COV}_s^2 - \mathrm{Cov}_s^2) + [a_1a_2\sigma^2_{t,s} + b_1b_2\sigma^2_{\omega,s} + (a_1b_2 + a_2b_1)\mathrm{Cov}_s]^2\\ \approx\ & 0.59508629639359624951502191285584;\end{aligned}$$

$$(a_1b_2 - a_2b_1)^2 \cdot \frac{1}{4} + [a_1a_2\sigma^2_{t,s} + b_1b_2\sigma^2_{\omega,s} + (a_1b_2 + a_2b_1)\mathrm{Cov}_s]^2 \approx 0.49999999999999999320465138830953;$$

$$\sigma^2_{t,s}\sigma^2_{\omega,s} \approx 0.41666666666666665534108564718255;$$

$$\frac{1}{4} + \mathrm{COV}_s^2 \approx 0.34508629639359625631037052454887;$$

$$\frac{1}{4} + \mathrm{Cov}_s^2 \approx 0.25,$$

so we can conclude from the point of numerical simulation that

$$\begin{aligned}\sigma^2_{M_1,u,s}\sigma^2_{M_2,u,s} &> (a_1b_2 - a_2b_1)^2(\frac{1}{4} + \mathrm{COV}_s^2 - \mathrm{Cov}_s^2)\\ &\quad +[a_1a_2\sigma^2_{t,s} + b_1b_2\sigma^2_{\omega,s} + (a_1b_2 + a_2b_1)\mathrm{Cov}_s]^2\\ &> (a_1b_2 - a_2b_1)^2 \cdot \frac{1}{4}\\ &\quad +[a_1a_2\sigma^2_{t,s} + b_1b_2\sigma^2_{\omega,s} + (a_1b_2 + a_2b_1)\mathrm{Cov}_s]^2,\end{aligned}$$

and

$$\sigma^2_{t,s}\sigma^2_{\omega,s} > \frac{1}{4} + \mathrm{COV}_s^2 > \frac{1}{4} + \mathrm{Cov}_s^2 = \frac{1}{4}.$$

Example 3.10: Let

$$s(t) = (\frac{1}{\pi})^{\frac{1}{4}} e^{-\frac{\alpha}{2}t^2} e^{i(\frac{1}{2}t^2 - \frac{1}{3}t^6)}.$$

We have

$$\sigma^2_{M_1,u,s}\sigma^2_{M_2,u,s} \approx 59.31249999999999838780354187643 6;$$

$$\begin{aligned}&(a_1b_2 - a_2b_1)^2(\frac{1}{4} + \mathrm{COV}_s^2 - \mathrm{Cov}_s^2) + [a_1a_2\sigma^2_{t,s} + b_1b_2\sigma^2_{\omega,s} + (a_1b_2 + a_2b_1)\mathrm{Cov}_s]^2\\ \approx\ & 15.558041563271434468147663999431;\end{aligned}$$

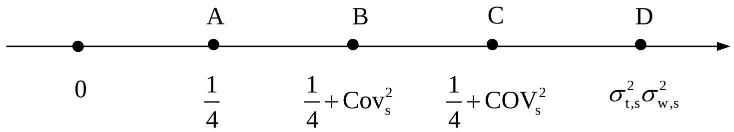

Fig. III.1. (i). Example 3.7 gives a signal s that satisfies $\sigma_{t,s}^2\sigma_{\omega,s}^2 = C > B = A$; (ii). Example 3.8 gives a signal s that satisfies $\sigma_{t,s}^2\sigma_{\omega,s}^2 = C = B > A$; (iii). Example 3.9 gives a signal s that satisfies $\sigma_{t,s}^2\sigma_{\omega,s}^2 > C > B = A$; (iv). Example 3.10 gives a signal s that satisfies $\sigma_{t,s}^2\sigma_{\omega,s}^2 > C > B > A$.

$$(a_1b_2-a_2b_1)^2\cdot\frac{1}{4}+[a_1a_2\sigma_{t,s}^2+b_1b_2\sigma_{\omega,s}^2+(a_1b_2+a_2b_1)\mathrm{Cov}_s]^2 \approx 14.312499999999999617761640592411;$$
$$\sigma_{t,s}^2\sigma_{\omega,s}^2 \approx 55.812499999999998482938422440103;$$
$$\frac{1}{4}+\mathrm{COV}_s^2 \approx 12.058041563271434563282544563098;$$
$$\frac{1}{4}+\mathrm{Cov}_s^2 \approx 10.812499999999999712896521156078,$$

so we can conclude from the point of numerical simulation that

$$\begin{aligned}\sigma_{M_1,u,s}^2\sigma_{M_2,u,s}^2 &> (a_1b_2-a_2b_1)^2(\frac{1}{4}+\mathrm{COV}_s^2-\mathrm{Cov}_s^2)\\ &\quad +[a_1a_2\sigma_{t,s}^2+b_1b_2\sigma_{\omega,s}^2+(a_1b_2+a_2b_1)\mathrm{Cov}_s]^2\\ &> (a_1b_2-a_2b_1)^2\cdot\frac{1}{4}\\ &\quad +[a_1a_2\sigma_{t,s}^2+b_1b_2\sigma_{\omega,s}^2+(a_1b_2+a_2b_1)\mathrm{Cov}_s]^2,\end{aligned}$$

and

$$\sigma_{t,s}^2\sigma_{\omega,s}^2 > \frac{1}{4}+\mathrm{COV}_s^2 > \frac{1}{4}+\mathrm{Cov}_s^2 > \frac{1}{4}.$$

Examples 3.7, 3.8, 3.9 and 3.10 provide signals showing various relations between the old and new lower bounds of the LCT uncertainty product, also give the relations between the four numbers $\frac{1}{4}$, $\frac{1}{4}+\mathrm{Cov}_s^2$, $\frac{1}{4}+\mathrm{COV}_s^2$ and the uncertainty product $\sigma_{t,s}^2\sigma_{\omega,s}^2$ in the classical case.

IV. Concerns About Application

It is commonly understood that a stronger inequality implies the weaker ones, giving more knowledge on the quantity to be estimated. In the classical uncertainty principle case the largest universal lower bound $\frac{1}{4}$ for all signals can be reached only if $\mathrm{COV}_s = 0$. The proposed new theorem gives full characterization of the signals that make the equal relation hold in the uncertainty inequality, including the cases when $\mathrm{COV}_s = 0$. In such a way the new uncertainty principle includes the old ones as particular cases. It thus has the full strength of the old uncertainty principles of various settings. In particular, whenever an application is done by an old uncertainty principle, it can also be done by the new one, and with more information. The philosophy of the LCT uncertainty principles is the same as that for the uncertainty principles in the classical setting. The LCT case is subject to suitably incorporate certain constants induced by the LCT parameters.

An alternative mathematical formulation of the classical Heisenberg uncertainty principle is

$$\frac{1}{4} = \min\{\sigma_{t,s}^2\sigma_{\omega,s}^2 : \|s\|_2^2 = 1, ts(t), \omega\hat{s}(\omega) \in L^2.\},$$

where the minimum value $\frac{1}{4}$ of the uncertain products $\sigma_{t,s}^2\sigma_{\omega,s}^2$ can be reached. For a practical signal s this minimum value $\frac{1}{4}$ should not be reached, and the corresponding uncertainty product is actually larger than $\frac{1}{4}$. Cohen's uncertainty principle provides a better estimate which says that the uncertainty product cannot be smaller than $\frac{1}{4} + \mathrm{Cov}_s^2$. Since $\mathrm{Cov}_s^2 \geq 0$, the latter is a better estimate than $\frac{1}{4}$. Our result shows that, due to the relation $\mathrm{COV}_s^2 \geq \mathrm{Cov}_s^2$, a further better estimate is $\frac{1}{4} + \mathrm{COV}_s^2$. Uncertainty principles are often used to estimate the bandwidths (see [19]). For instance, if $\sigma_{t,s}$ is known, then

$$\sigma_{\omega,s} \geq \frac{\sqrt{\frac{1}{4} + \mathrm{COV}_s^2}}{\sigma_{t,s}} \geq \frac{1}{2\sigma_{t,s}},$$

and, even the middle term of the chain of the inequality usually cannot be reached, except it is a Gaussian type function given in Theorem 3.2.

Alternative to Theorem 3.2, we have

$$\frac{1}{4} + \alpha^2 = \min\{\sigma_{t,s}^2\sigma_{\omega,s}^2 : \|s\|_2^2 = 1, ts(t), \omega\hat{s}(\omega) \in L^2, \mathrm{COV}_s \geq \alpha.\}, \alpha > 0.$$

The last relation shows that if we are dealing with a class of signals whose absolute variations are not less than α, then the uncertainty product cannot be less than $\frac{1}{4} + \alpha^2$.

The situation for the LCT domains is similar with that for the classical case. Below we give an example to illustrate the application aspect of the new result of uncertainty principle in the LCT domains. We will just consider the special case, that is, Corollary 3.4 in the process of wave propagation through an aperture that can be described by an LCT (see [19]). Immediately after crossing the aperture the field has some effective width $\sigma_{t,s}^2$ dictated by the aperture width. After propagation a distance z the transversal distribution of the field can be described by a LCT with parameters $(1, b = \lambda z/2\pi, 0, 1)$. Therefore the relation (3.22) becomes

$$\sigma_{M,u,s}^2 \geq \frac{(\lambda z)^2}{4\pi^2\sigma_{t,s}^2}(\frac{1}{4} + \mathrm{COV}_s^2 - \mathrm{Cov}_s^2) + (\sigma_{t,s} + \frac{\lambda z \mathrm{Cov}_s}{2\pi\sigma_{t,s}})^2 \tag{4.35}$$
$$= \frac{(\lambda z)^2}{4\pi^2\sigma_{t,s}^2}(\frac{1}{4} + \mathrm{COV}_s^2) + \sigma_{t,s}^2 + 2\frac{\lambda z}{2\pi}\mathrm{Cov}_s,$$

implying that for short z the effective spread $2\sigma_{M,u,s}$ is slightly larger than that at the aperture plane $2\sigma_{t,s}$ and for large propagation distances z the spread of the field is proportional to λz and reciprocally proportional to the field spread in the aperture plane.

V. Conclusion

We study uncertainty principles in the LCT domain. The lower-bounds obtained in the uncertainty principles in Theorem 3.2, Corollary 3.4 and Corollary 3.5 are sharper than the existing forms for all the three categories, viz. the Fourier, the fractional Fourier and the LCT transforms, in the literature. It is also shown that the lower bounds are attainable by four classes of complex chirp signals with Gaussian envelop. Examples with simulations are given to illustrate our results. We also concern the applications of the new lower bound of uncertainty principle in LCT domains.

Appendix A
Proof of the Lemma 2.6

Proof of Lemma 2.6

By Lemma 2.5, we may assume that $s_0(t)$ is an absolutely continuous function that is equal to $s(t)$ almost everywhere. Then the LCT of $s_0(t)$, denoted by $S_{0M}(u)$, is equal to $S_M(u)$, the LCT of $s(t)$. Now

$$\begin{aligned}\langle u\rangle_{M,s} &= \int_{-\infty}^{\infty} uS_M(u)\overline{S_M(u)}du \\ &= \int_{-\infty}^{\infty} uS_{0M}(u)\overline{S_{0M}(u)}du \\ &= \int_{-\infty}^{\infty} [ats_0(t) - ibs_0'(t)]\overline{s_0(t)}dt \\ &= \int_{-\infty}^{\infty} [ats(t) - ib(DS)(t)]\overline{s(t)}dt \\ &= \int_{-\infty}^{\infty} at|s(t)|^2 dt - ib\int_{-\infty}^{\infty} (Ds)(t)\overline{s(t)}dt \\ &= a\langle t\rangle_s - ib\int_{-\infty}^{\infty} \frac{(Ds)(t)}{s(t)}|s(t)|^2 dt \\ &= a\langle t\rangle_s + b\int_{-\infty}^{\infty} \mathrm{Im}[\frac{(Ds)(t)}{s(t)}]|s(t)|^2 dt \\ &= a\langle t\rangle_s + b\int_{-\infty}^{\infty} (D\varphi)(t)|s(t)|^2 dt,\end{aligned}$$

where we used the Parseval identity and the property ([14])

$$L_{(a,b,c,d)}(ats_0(t) - ibs_0'(t))(u) = uS_{0M}(u).$$

Then,

$$\begin{aligned}
\sigma^2_{M,u,s} &= \int_{-\infty}^{\infty} uS_M(u)\overline{uS_M(u)}du - \langle u\rangle^2_{M,s} \\
&= \int_{-\infty}^{\infty} uS_{0M}(u)\overline{uS_{0M}(u)}du - \langle u\rangle^2_{M,s} \\
&= \int_{-\infty}^{\infty} [ats_0(t) - ibs_0'(t)]\overline{ats_0(t) - ibs_0'(t)}dt - \langle u\rangle^2_{M,s} \\
&= \int_{-\infty}^{\infty} [ats(t) - ib(Ds)(t)]\overline{ats(t) - ib(Ds)(t)}dt - \langle u\rangle^2_{M,s} \\
&= \int_{-\infty}^{\infty} a^2t^2|s(t)|^2dt + 2ab\int_{-\infty}^{\infty} \mathrm{Im}[t(Ds)(t)\overline{s(t)}]dt + b^2\int_{-\infty}^{\infty} |\frac{(Ds)(t)}{s(t)}|^2|s(t)|^2dt - \langle u\rangle^2_{M,s} \\
&= a^2\langle t^2\rangle_s + 2ab\int_{-\infty}^{\infty} t\mathrm{Im}[\frac{(Ds)(t)}{s(t)}]|s(t)|^2dt \\
&\quad +b^2\int_{-\infty}^{\infty} \{\mathrm{Re}^2[\frac{(Ds)(t)}{s(t)}] + \mathrm{Im}^2[\frac{(Ds)(t)}{s(t)}]\}|s(t)|^2dt - \langle u\rangle^2_{M,s} \\
&= a^2\langle t^2\rangle_s + 2ab\int_{-\infty}^{\infty} t(D\varphi)(t)|s(t)|^2dt + b^2[\int_{-\infty}^{\infty}(D\rho)^2(t)dt + \int_{-\infty}^{\infty}(D\varphi)^2(t)|s(t)|^2dt] \\
&\quad -[a\langle t\rangle_s + b\int_{-\infty}^{\infty}(D\varphi)(t)|s(t)|^2dt]^2 \\
&= a^2[\langle t^2\rangle_s - \langle t\rangle_s^2] + 2ab[\int_{-\infty}^{\infty} t(D\varphi)(t)|s(t)|^2dt - \langle t\rangle_s\int_{-\infty}^{\infty}(D\varphi)(t)|s(t)|^2dt] \\
&\quad +b^2\{\int_{-\infty}^{\infty}(D\rho)^2(t)dt + \int_{-\infty}^{\infty}(D\varphi)^2(t)|s(t)|^2dt - [\int_{-\infty}^{\infty}(D\varphi)(t)|s(t)|^2dt]^2\} \\
&= a^2\sigma^2_{t,s} + 2ab\mathrm{Cov}_s + b^2\{\int_{-\infty}^{\infty}(D\rho)^2(t)dt + \int_{-\infty}^{\infty}[(D\varphi)(t) - \langle\omega\rangle_s]^2|s(t)|^2dt\} \\
&= a^2\sigma^2_{t,s} + 2ab\mathrm{Cov}_s + b^2\sigma^2_{\omega,s}.
\end{aligned}$$

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