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A sharp lower bound of Burkholder's functional for K-quasiconformal mappings and its applications

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Abstract In this paper, for K-quasiconformal mappings of a bounded domain into the complex plane, we build a sharp lower bound of Burkholder's functional. As an application, we give two explicit and sharp lower bounds of Burkholder's integrals for two subclasses of K-quasiconformal mappings, respectively. As the second application, we obtain a sharp upper bound of the L^p-integral of $\sqrt{J_f}$ for certain K-quasiconformal mappings.

Keywords Quasiconformal mapping · Beurling-Ahlfors operator · Principal solution · Burkholder's functional · Burkholder's inequality

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1 Introduction

Let Ω and Ω' be two bounded simply connected domains of the complex plane $\mathbb{C}$ and χ_Ω the characteristic function of Ω. Let $\dot{W}^{1,p}(\mathbb{C},\mathbb{C})$, $1 < p < \infty$, be the

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homogenous Sobolev space of complex-valued locally integrable functions in the plane whose distributional first derivatives are in $L^p(\mathbb{C})$. Let $f = u + iv$. We denote the formal partial derivatives of f by

$$\bar{\partial} f = f_z = \frac{1}{2}(f_x + i f_y) = \frac{1}{2}(u_x - v_y + i(u_y + v_x)),$$
$$\partial f = f_{\bar{z}} = \frac{1}{2}(f_x - i f_y) = \frac{1}{2}(u_x + v_y + i(v_x - u_y)),$$

and write

$$Df = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}, \quad |Df| = |f_z| + |f_{\bar{z}}|, \quad J_f = |f_z|^2 - |f_{\bar{z}}|^2.$$

Let $K \geq 1$ and $k = \frac{K-1}{K+1}$. A *K-quasiconformal mapping* is an orientation preserving homeomorphism f of Ω onto Ω' that belongs to the Sobolev space $W^{1,2}_{loc}(\Omega, \Omega')$ and satisfies the distortion inequality

$$|\bar{\partial} f| \leq k |\partial f|, \quad a.e. \text{ on } \Omega,$$

Particularly, if $|\bar{\partial} f|/|\partial f| = k$ $a.e.$ on Ω, then we say that f has a constant-modulus Beltrami coefficient.

A homeomorphism of $\mathbb{C}$ onto itself is called a *principal solution* of the Beltrami equation

$$f_{\bar{z}} = \mu \chi_\Omega f_z, \quad ||\mu||_\infty \leq 1,$$

if f belongs to $W^{1,2}_{loc}(\mathbb{C})$ and satisfies the asymptotical normalization condition at the infinity as

$$f(z) = z + \frac{b_1}{z} + \frac{b_2}{z^2} + \cdots, \quad z \to \infty.$$

The existence of principal solutions can be determined by properties of *Beurling-Ahlfors operator*, which is defined on $L^p(\mathbb{C})$, $1 < p < \infty$, by

$$\mathbf{T} f(z) = -\frac{1}{\pi} \mathrm{pv} \iint\limits_{\mathbb{C}} \frac{f(\zeta)}{(z - \zeta)^2} dm(\zeta),$$

where pv means the Cauchy principal value and m is the Lebesgue measure of $\mathbb{C}$. This operator and its multidimensional and weighted analogues are fundamental tools in several areas including quasiconformal mappings, partial differential equations, calculus of variations and differential geometry (see [1–3,6,11,13,14,16,19] and the references therein for more details).

Set $p^* = \max\{p, \frac{p}{p-1}\}$, $1 < p < \infty$. Define *Burkholder's functional* (p16 in [9]) by

$$\begin{aligned}\mathbf{B}_p(Df) &= (|\partial f| - (p^* - 1)|\bar{\partial} f|)(|\partial f| + |\bar{\partial} f|)^{p-1} \\ &= \frac{2-p^*}{2}|Df|^p + \frac{p^*}{2}|Df|^{p-2}J_f. \end{aligned} \tag{1.1}$$

Bañuelos and Wang conjectured [7]: For every function $f \in \dot{W}^{1,p}(\mathbb{C}, \mathbb{C})$, it is true that *Burkholder's integral* $\iint_{\mathbb{C}} \mathbf{B}_p(Df)dm$ satisfies

$$\iint_{\mathbb{C}} \mathbf{B}_p(Df)dm \leq 0. \tag{1.2}$$

By *Burkholder's inequality* (p. 16–17 in [9])

$$|\partial f|^p - (p^* - 1)^p|\bar{\partial} f|^p \leq p(1 - \frac{1}{p^*})^{p-1}\mathbf{B}_p(Df), \tag{1.3}$$

Bañuelos-Wang conjecture implies the fact that for complex-valued functions $f \in \dot{W}^{1,p}(\mathbb{C}, \mathbb{C})$, it is true that

$$\iint_{\mathbb{C}} |\partial f|^p dm \leq (p^* - 1)^p \iint_{\mathbb{C}} |\bar{\partial} f|^p dm. \tag{1.4}$$

In 1965, Lehto [17] showed that $||\mathbf{T}||_{L^p(\mathbb{C})} \geq p^* - 1$ for $p \in (1, +\infty)$. Consequently, The validity of Bañuelos-Wang conjecture will imply *Iwaniec's conjecture* [15], that is, $||\mathbf{T}||_{L^p(\mathbb{C})} = p^* - 1$.

Due to the rank-one convexity of Burkholder's functional, Bañuelos-Wang conjecture is closely connected with the long standing Morrey conjecture [18] (see Sect. 5 in [5] or [20] for a precise statement of their relations).

There is a fundamental inequality for the Jacobian J_f and the gradient $|Df|$ given in Chapter 19 in [3].

Theorem A *There exists a number $M = M_p \geq 1$ such that if $1 < p < \infty$, then*

$$\int_{\mathbb{C}} |Df|^{p-2}J_f dm \leq \frac{M-1}{M+1}\int_{\mathbb{C}} |Df|^p dm$$

or equivalently,

$$\int_{\mathbb{C}} (|\partial f| - M|\bar{\partial} f|)(|\partial f| + |\bar{\partial} f|)^{p-1}dm \leq 0,$$

for every function $f \in \dot{W}^{1,p}(\mathbb{C}, \mathbb{C})$. Here,

$$M = ||\mathbf{T}||^p_{L^p(\mathbb{C})} > ||\mathbf{T}||_{L^p(\mathbb{C})} \geq p^* - 1$$

if $p \in (1, 2) \cup (2, \infty)$.

Since the value M in Theorem A cannot be $p^* - 1$ if $p \in (1, 2) \cup (2, \infty)$, Theorem A does not give the proof of Bañuelos-Wang conjecture. However, there are some particular subclasses of $\dot{W}^{1,p}(\mathbb{C}, \mathbb{C})$ validating this. For example, Baernstein and Montgomery-Smith [5] proved the following

Theorem B *If $f \in \dot{W}^{1,p}(\mathbb{C}, \mathbb{C})$, $1 < p < \infty$, is harmonic on $\mathbb{C} \cup \infty \backslash \{|z| = 1\}$, then the inequality (1.2) holds.*

Let $\mathbb{D}$ be the unit disk of $\mathbb{C}$, and $\mathbb{D}^c$ denote the exterior of $\overline{\mathbb{D}}$. Set

$$\varphi(z) = \begin{cases} z, & z \in \overline{\mathbb{D}}, \\ 1/\bar{z}, & z \in \mathbb{D}^c. \end{cases} \tag{1.5}$$

Recently, the authors [10] obtained

Theorem C *Suppose g is a locally univalent logharmonic mapping of the unit disk $\mathbb{D}$ in $W^{1,2}_{loc}(\mathbb{D})$. Let $f = g \circ \varphi$. If $f \in \dot{W}^{1,p}(\mathbb{C}, \mathbb{C})$, then Bañuelos-Wang conjecture is valid for f.*

Meanwhile, the estimates of Burkholder's integral confined to a bounded domain also arise interest. For instance, an upper bound of Burkholder's integral for certain classes of K-quasiconformal mappings was recently obtained by Astala et al. [4] as follows

Theorem D *Let $f : \Omega \to \Omega$ be a K-quasiconformal mapping of a bounded open set $\Omega \subset \mathbb{C}$ onto itself, extending continuously up to the boundary, where it coincides with the identity mapping $Id(z) \equiv z$. Then*

$$\iint\limits_{\Omega} \mathbf{B}_p(Df)dm \leq \iint\limits_{\Omega} \mathbf{B}_p(Id)dm = |\Omega|, \quad \text{for } 2 \leq p \leq \frac{1+k}{k}, \quad K = \frac{1+k}{1-k}.$$

Furthermore, the equality occurs for a class of piecewise radial mappings.

Theorem D *Let $f : \mathbb{C} \to \mathbb{C}$ be the principal solution of a Beltrami equation,*

$$f_{\bar{z}} = \mu f_z, \quad |\mu| \leq k\chi_{\mathbb{D}}, \quad 0 \leq k < 1.$$

Then, for all exponents $2 \leq p \leq \frac{1+k}{k}$, we have

$$\iint\limits_{\mathbb{D}} \mathbf{B}_p(Df)dm \leq |\mathbb{D}|.$$

Equality occurs for some fair general piecewise radial mappings.

What about the lower bound of Burkholder's integral for the classes of K-quasiconformal mappings given in Theorem D and Theorem E? The Burkholder inequality implies a lower bound of Burkholder's integral given by a linear combination of the L^p-integrals of ∂f and $\bar{\partial} f$. For a K-quasiconformal mapping with a Beltrami coefficient μ, Burkholder's functional (1.2) can be expressed by

$$\mathbf{B}_p(Df) = (1 - \frac{p^*|\mu|}{1+|\mu|})|Df|^p, \quad p \in (1, +\infty),$$

which implies that the inequality

$$\frac{1+k-pk}{1+k}|Df|^p \leq \mathbf{B}_p(Df) \leq |Df|^p, \quad p \in (2, +\infty) \tag{1.6}$$

holds if $1 + k - pk \geq 0$, $k = \frac{K-1}{K+1}$ [4]. Consequently, the left inequality of (1.6) also gives a lower bound of Burkholder's integral $\iint_\Omega \mathbf{B}_p(Df)dm$ by the L^p-integral of $|Df|$ and the dilatation k.

In this paper, we continue to study the lower estimate of Burkholder's functional for K-quasiconformal mappings and use it to give explicit lower bounds of Burkholder's integral for certain classes of K-quasiconformal mappings.

Burkholder's functional $\mathbf{B}_p(Df)$ is equal to the Jacobian J_f if $p = 2$, and the lower estimate of (1.6) for $\mathbf{B}_p(Df)$ takes equality for all K-quasiconformal mappings with constant-modulus Beltrami coefficients if $p \in [2, \frac{1+k}{k})$. However, the lower estimate of (1.6) for $\mathbf{B}_p(Df)$ is not the Jacobian J_f for all K-quasiconformal mappings when $p = 2$, while, the lower estimate of Burkholder's inequality does not take equality for K-quasiconformal mappings with constant-modulus Beltrami coefficients when $p \in (2, \frac{1+k}{k})$. Hence, for K-quasiconformal mappings, it is natural to ask whether there exists a lower estimate of $\mathbf{B}_p(Df)$ such that it is the Jacobian J_f when $p = 2$ and simultaneously takes equality for quasiconformal mappings with constant-modulus Beltrami coefficients when $p \in (2, \frac{1+k}{k})$.

Firstly, we get such a lower estimate of Burkholder's functional $\mathbf{B}_p(Df)$ for K-quasiconformal mappings as follows

Theorem 1.1 *If f is a K-quasiconformal mapping of a domain $\Omega \subset \mathbb{C}$ into $\mathbb{C}$, then for every $p \in [2, 1 + 1/k)$,*

$$\frac{(1-(p-1)k)(1+k)^{p-1}}{\sqrt{1-k^2}^p}\sqrt{J_f}^p \leq \mathbf{B}_p(Df) \leq \sqrt{J_f}^p. \tag{1.7}$$

The left equality and the right equality of (1.7) hold at the same time if and only if f is a conformal mapping or $p = 2$. Further, for every $p \in [2, 1 + 1/k)$,

$$\frac{(1-(p-1)k)(1+k)^{p-1}}{\sqrt{1-k^2}^p}\iint_\Omega \sqrt{J_f}^p dm \leq \iint_\Omega \mathbf{B}_p(Df)dm \leq \iint_\Omega \sqrt{J_f}^p dm. \tag{1.8}$$

The left equality and the right equality of (1.8) hold simultaneously if and only if f is a conformal mapping or $p = 2$.

Moreover, the left equalities in (1.7) and (1.8) hold for all K-quasiconformal mappings with constant-modulus Beltrami coefficients when p belongs to $[2, 1 + 1/k)$.

We note that both the lower bound and the upper bound of (1.7) are better than the corresponding ones of (1.6) (see Remark 3.1).

Secondly, we apply this result to get explicit lower bounds of Burkholder's integral for the class of K-quasiconformal mappings of Ω onto itself with the identity boundary mapping, and the class of principal solutions with Beltrami coefficients $\mu\chi_{\mathbb{D}}$, $||\mu||_\infty \leq k < 1$ (see Theorem 4.1 and Theorem 1, respectively).

Finally, combining Theorem B with Theorem 4.1, we obtain a sharp upper bound of L^p-integral of $\sqrt{J_f}$ for K-quasiconformal mappings with the identity boundary mapping (see Theorem 5.1). As its corollary, we get the same upper bound of the L^p-integral of $|Df|$ as the one at Corollary 4.1 in [4].

The rest of this paper is organized as follows. In Sect. 2, we give preliminary lemmas used in the following sections. In Sect. 3, a proof of Theorem 1.1 is given. In Sect. 4, we obtain two explicit and sharp lower bounds of Burkholder's integral. The sharp L^p-integral of $\sqrt{J_f}$ for K-quasiconformal mappings with the identity boundary mapping is given in Sect. 5. At last, we utilize Beurling-Ahlfors operator to construct explicit principal solutions with given Beltrami coefficients, and then compare the lower bounds of the Burkholder integrals we obtain (see Remark 6.1).

2 Preliminary lemmas

We first formulate a classical result of Gronwall [12] and Bieberbach [8], which is known as the area formula. One can see a proof at Theorem 2.10.1 in [3].

Lemma A *Suppose that $f \in W^{1,2}_{loc}(\mathbb{C})$ is analytic outside the disk $\mathbb{D}(0, r) = \{z||z| < r\}$ and has the expansion*

$$z + \frac{b_1}{z} + \frac{b_2}{z^2} + \cdots, \quad z \to \infty.$$

Then

$$\iint\limits_{\mathbb{D}(0,r)} J_f dxdy = \pi\left(r^2 - \sum_{n=1}^{\infty} n|b_n|r^{-2n}\right).$$

In particular, if f is orientation-preserving almost everywhere, then

$$\sum_{n=1}^{\infty} n|b_n|^2 r^{-2n} \leq r^2.$$

For a function $f \in L^p(\mathbb{C})$ with $p \geq 2$, *Cauchy's operator* is defined by

$$\mathbf{C}f(z) = -\frac{1}{\pi}\iint\limits_{\mathbb{C}} \left(\frac{f(\zeta)}{\zeta - z} - \frac{\kappa_{\mathbb{C}\setminus\overline{\mathbb{D}}}}{\zeta}\right) dm(\zeta). \tag{2.1}$$

If $p > 2$, then $\mathbf{C}f$ is Hölder continuous with exponent $1 - 2/p$ (see Theorem 4.3.13 of [3]), while, if $p = 2$, then $\mathbf{C}f$ belongs to the space $VMO(\mathbb{C})$ (see Theorem 4.3.9 of [3]).

We need the following two lemmas when we construct explicit principal solutions with given Beltrami coefficients.

Lemma B *If $f \in L^p(\mathbb{C})$, $p \geq 2$, then the relations*

$$\partial \mathbf{C}f = \mathbf{T}f, \quad \bar{\partial}\mathbf{C}f = f. \tag{2.2}$$

hold in the distributional sense.

See p. 52–53 in [1] and p. 112 in [3] for a proof of Lemma B.

Lemma C *Let $\mu = \bar{z}^n z^m$, where n and m are integers, and let φ be given by (1.5). Then the following relations hold. If $n \geq m$, then*

$$\mathbf{C}(\mu\chi_{\mathbb{D}})(z) = z^m \frac{\varphi(\bar{z})^{n+1}}{n+1} \tag{2.3}$$

and

$$\mathbf{T}(\mu\chi_{\mathbb{D}})(z) = \begin{cases} \frac{m}{n+1} z^{m-1}\bar{z}^{n+1}, & m \neq 0, z \in \mathbb{D}, \\ 0, & m = 0, z \in \mathbb{D}, \\ -\frac{n-m+1}{(n+1)z^{n-m+2}}, & \quad z \in \mathbb{D}^c. \end{cases} \tag{2.4}$$

If $n = m - 1$, then

$$\mathbf{C}(\mu\chi_{\mathbb{D}})(z) = -\frac{1 - |z|^{2n+2}}{n+1}\chi_{\mathbb{D}} \tag{2.5}$$

and

$$\mathbf{T}(\mu\chi_{\mathbb{D}})(z) = z^n \bar{z}^{n+1}\chi_{\mathbb{D}}. \tag{2.6}$$

If $n \leq m - 2$, then

$$\mathbf{C}(\mu\chi_{\mathbb{D}})(z) = -\frac{z^{m-(n+1)}}{n+1}(1 - |z|^{2n+2})\chi_{\mathbb{D}} \tag{2.7}$$

and

$$\mathbf{T}(\mu\chi_{\mathbb{D}})(z) = -\frac{m-(n+1)}{n+1} z^{m-(n+2)} + \frac{m}{n+1} z^{m-1}\bar{z}^{n+1}\chi_{\mathbb{D}}. \tag{2.8}$$

One can see Lemma 3.1 in [10] for a proof of Lemma C.

Lemma 2.1 *Let $K = \frac{1+k}{1-k}$, where $0 \le k < 1$. If f is a K-quasiconformal mapping of a domain $\Omega \subset \mathbb{C}$ into $\mathbb{C}$ then the inequality*

$$|f_{\bar{z}}| \le \frac{k}{\sqrt{1-k^2}} J_f^{1/2} \tag{2.9}$$

holds almost everywhere in Ω. The equality holds if and only if f is a quasiconformal mapping with a constant-modulus Beltrami coefficient.

Proof By the definition of a K-quasiconformal mapping, we have the inequality

$$|f_{\bar{z}}|^2 \le k^2(J_f + |f_{\bar{z}}|^2) \tag{2.10}$$

holds almost everywhere in Ω. Hence

$$|f_{\bar{z}}|^2 \le \frac{k^2}{1-k^2} J_f.$$

If the equality of the above inequality holds, then $|f_{\bar{z}}|/|f_z| = k$. The proof of Lemma 2.1 is complete. □

3 Proof of theorem 1.1

Proof If $p \in [2, 1 + 1/k)$, then we have

$$1 + k - pk \ge 0. \tag{3.1}$$

Let f be a K-quasiconformal mapping of Ω into $\mathbb{C}$. Then by the inequality (2.10) and (3.1), we obtain

$$\sqrt{J_f + |f_{\bar{z}}|^2} - (p-1)|f_{\bar{z}}| \ge (1 + k - pk)\sqrt{J_f + |f_{\bar{z}}|^2} \ge 0. \tag{3.2}$$

Rewrite Burkholder's functional

$$\mathbf{B}_p(Df) = (\sqrt{J_f + |f_{\bar{z}}|^2} - (p-1)|f_{\bar{z}}|)(\sqrt{J_f + |f_{\bar{z}}|^2} + |f_{\bar{z}}|)^{p-1}.$$

Let $s = \sqrt{J_f}$ and $|f_{\bar{z}}| = t$. Then $\mathbf{B}_p(Df)$ can be expressed by

$$f_s(t) = (\sqrt{s^2 + t^2} - (p-1)t)(\sqrt{s^2 + t^2} + t)^{p-1}.$$

By the fact that $\sqrt{s^2 + t^2} - (p-1)t > 0$ when $p > 2$ and $t > 0$, we obtain

$$\frac{d \ln(f_s(t))}{dt} = \frac{f_s'(t)}{f_s(t)} = -\frac{p(p-2)t}{(\sqrt{s^2 + t^2} - (p-1)t)\sqrt{s^2 + t^2}} < 0.$$

So we have that $f_s(t)$ decreases in t if $p > 2$ and $t > 0$. By (2.9) of Lemma 2.1, we obtain

$$\mathbf{B}_p(Df) \geq \frac{(1-(p-1)k)(1+k)^{p-1}}{\sqrt{1-k^2}^p}\sqrt{J_f}^p,$$

and then

$$\iint\limits_{\Omega} \mathbf{B}_p(Df)dm \geq \frac{(1-(p-1)k)(1+k)^{p-1}}{\sqrt{1-k^2}^p}\iint\limits_{\Omega} \sqrt{J_f}^p dxdy.$$

On the other hand, we get from the fact $|f_{\bar{z}}| \geq 0$ that

$$\mathbf{B}_p(Df) \leq \sqrt{J_f}^p,$$

and then

$$\iint\limits_{\Omega} \mathbf{B}_p(Df)dm \leq \iint\limits_{\Omega} \sqrt{J_f}^p dxdy.$$

Let $\sigma(p,k) = \frac{(1-(p-1)k)(1+k)^{p-1}}{\sqrt{1-k^2}^p}$. Then it follows

$$\frac{\partial \ln(\sigma(p,k))}{\partial k} = -\frac{(p-2)pk}{(1-k^2)(1-(p-1)k)} < 0, \quad \text{for every} \quad p \in (2, 1+1/k).$$

Hence, when $p \in (2, 1+1/k)$, we have that $\sigma(p,k) < \sigma(p,0) = 1$ for any $k \in (0,1)$, while, $\sigma(2,k) \equiv 1$ for any $k \in (0,1)$. Thus, the left equality and the right equality in (1.7) and (1.8) hold simultaneously if and only if $p = 2$ or $k = 0$.

If f is a K-quasiconformal mapping with a constant-modulus Beltrami coefficient, then

$$\mathbf{B}_p(Df) = (1+k-pk)(1+k)^{p-1}|f_z|^p,$$

and

$$\frac{(1-(p-1)k)(1+k)^{p-1}}{\sqrt{1-k^2}^p}\sqrt{J_f}^p = (1+k-pk)(1+k)^{p-1}|f_z|^p.$$

Hence, the left equalities of (1.7) and (1.8) hold for all K-quasiconformal mappings with a constant-modulus Beltrami coefficient when $p \in [2, 1+1/k)$.

So the proof of Theorem 1.1 is complete. □

Remark 3.1 Let $\mu = f_{\bar{z}}/f_z$ satisfy $||\mu||_\infty \leq k, 0 \leq k < 1$. it is clear that

$$\sqrt{J_f}^p \leq |Df|^p.$$

For every $p \in [2, 1 + 1/k)$, we also have

$$\begin{aligned}&\tfrac{(1-(p-1)k)(1+k)^{p-1}}{\sqrt{1-k^2}^p}\sqrt{J_f}^p - \tfrac{1+k-pk}{1+k}|Df|^p\\ &= \tfrac{(1+k-pk)|Df|^p}{1+k}\left(\left(\tfrac{1-|\mu|}{1+|\mu|}\right)^{p/2}\left(\tfrac{1+k}{1-k}\right)^{p/2} - 1\right) \geq 0.\end{aligned}$$

Hence, both the upper and the lower bounds of (1.7) are better than the corresponding ones of (1.6).

4 Explicit estimates of Burkholder's integral

The upper and lower estimates of Burkholder's integral in Theorem 1.1 depend on the L^p-integral of $\sqrt{J_f}$. Next, combining Theorem D or Theorem E with Theorem 1.1, we will give explicit estimates of the lower and upper bounds of Burkholder's integral for certain classes of K-quasiconformal mappings.

Theorem 4.1 *If f is a K-quasiconformal mapping of Ω onto itself with the identity boundary mapping, then for every $p \in [2, 1 + 1/k)$,*

$$\frac{(1-(p-1)k)(1+k)^{p-1}}{\sqrt{1-k^2}^p}|\Omega| \leq \iint\limits_{\Omega} \mathbf{B}_p(Df)dm \leq |\Omega|. \tag{4.1}$$

Proof By Hölder's inequality, we get

$$\iint\limits_{\Omega} \sqrt{J_f}^p dxdy \geq \left(\frac{\iint_\Omega J_f dxdy}{\Omega}\right)^{p/2} |\Omega|. \tag{4.2}$$

By Theorem 1.1 and the inequality (4.2), it follows from the assumption that f has the identity boundary mapping that

$$\begin{aligned}\iint\limits_{\Omega} \mathbf{B}_p(Df)dm &\geq \frac{(1-(p-1)k)(1+k)^{p-1}}{\sqrt{1-k^2}^p}\left(\frac{\iint_\Omega J_f dxdy}{\Omega}\right)^{p/2}|\Omega|\\ &= \frac{(1-(p-1)k)(1+k)^{p-1}}{\sqrt{1-k^2}^p}|\Omega|.\end{aligned}$$

The right inequality of (4.1) has been proved at Theorem 1.3 in [4]. □

Example 4.1 Let ε be a real and $f(z) = ze^{i\varepsilon \ln(1/|z|)}$ a spiral mapping of the unit disk $\mathbb{D}$ onto itself. Then for every Lebesgue measurable subset $\Omega \subset \mathbb{D}$, the left equality of (4.1) holds.

Proof By a direct calculation, it follows that

$$f_z = (1 - i\varepsilon/2)e^{i\varepsilon \ln(1/|z|)}, \quad f_{\bar{z}} = -i(\varepsilon/2)(z/\bar{z})e^{i\varepsilon \ln(1/|z|)}, \quad |\mu| = k = \frac{|\varepsilon|}{\sqrt{4+\varepsilon^2}}.$$

Hence,

$$\begin{aligned}\iint_\Omega \mathbf{B}_p(Df)dm &= \iint_\Omega \left(\sqrt{1+\frac{\varepsilon^2}{4}} - (p-1)\frac{|\varepsilon|}{2}\right)\left(\sqrt{1+\frac{\varepsilon^2}{4}} + \frac{|\varepsilon|}{2}\right)^{p-1} dm \\ &= (1-(p-1)k)(1+k)^{p-1}\sqrt{1+\tfrac{\varepsilon^2}{4}}^{\,p}|\Omega|.\end{aligned}$$

By the relation $\sqrt{1-k^2} = \frac{2}{\sqrt{4+\varepsilon^2}}$, we obtain

$$\iint_\Omega \mathbf{B}_p(Df)dm = \frac{(1-(p-1)k)(1+k)^{p-1}}{\sqrt{1-k^2}^{\,p}}|\Omega|.$$

The proof of Example 4.1 is complete. □

Remark 4.1 Example 4.1 shows that the lower bound in Theorem 4.1 is sharp. The upper bound in Theorem 4.1 can be reached by a class of expanding piecewise radial mappings (see Sect. 5 in [4] for its proof).

Theorem 4.2 *If $f : \mathbb{C} \to \mathbb{C}$ is a principle solution of the Beltrami equation $f_{\bar{z}} = \mu_f \chi_{\mathbb{D}} f_z$ with $||\mu_f \chi_{\mathbb{D}}||_\infty \le k < 1$ and the normalization*

$$f(z) = z + b_1\frac{1}{z} + b_2\frac{1}{z^2} + \cdots, \quad z \in \mathbb{C}\backslash\bar{\mathbb{D}},$$

then for every $p \in [2, 1+1/k)$, there holds

$$\frac{(1-(p-1)k)(1+k)^{p-1}}{\sqrt{1-k^2}^{\,p}}\left(1 - \sum_{n=1}^{\infty} n|b_n|^2\right)^{p/2}\pi \le \iint_{\mathbb{D}} \mathbf{B}_p(Df)dm \le \pi. \tag{4.3}$$

Proof By Hölder's inequality, we get

$$\iint_{\mathbb{D}} \sqrt{J_f}^{\,p} dxdy \ge \left(\frac{\iint_\Omega J_f dxdy}{\mathbb{D}}\right)^{p/2}|\mathbb{D}|. \tag{4.4}$$

Hence, combing the left inequality of (1.8) with (4.4), we obtain

$$\iint_{\mathbb{D}} \mathbf{B}_p(Df)dm \ge \frac{(1-(p-1)k)(1+k)^{p-1}}{\sqrt{1-k^2}^{\,p}}\left(\frac{\iint_{\mathbb{D}} J_f dxdy}{\mathbb{D}}\right)^{p/2}|\mathbb{D}|.$$

By the assumption that f is a principal solution with the normalization

$$f(z) = z + b_1\frac{1}{z} + b_2\frac{1}{z^2} + \cdots, \quad z \in \mathbb{C}\backslash\bar{\mathbb{D}},$$

we obtain from Lemma A that

$$\begin{aligned}\iint\limits_{\mathbb{D}} \mathbf{B}_p(Df)dm &\geq \frac{(1-(p-1)k)(1+k)^{p-1}}{\sqrt{1-k^2}^p}\left(\frac{\iint_{\mathbb{D}} J_f dxdy}{\pi}\right)^{p/2}\pi \\ &= \frac{(1-(p-1)k)(1+k)^{p-1}}{\sqrt{1-k^2}^p}\left(1-\sum_{n=1}^{\infty} n|b_n|^2\right)^{p/2}\pi.\end{aligned}$$

A proof of the right inequality of (4.3) is given by Theorem 3.5 in [4].

Example 4.2 Let $\mu = k\chi_{\mathbb{D}}$, $0 \leq k < 1$. Then the principal solution f of μ takes the left equality of (4.3) in Theorem 4.2.

Proof Using Lemma B and Lemma C, we can express the principal solution with the Beltrami coefficient $k\chi_{\mathbb{D}}$ as

$$f(z) = \begin{cases} z + k\bar{z}, & |z| \leq 1, \\ z + k\frac{1}{z}, & |z| > 1. \end{cases}$$

Hence, by direct verification, there holds

$$\iint\limits_{\mathbb{D}} \mathbf{B}_p(Df)dm = (1-(p-1)k)(1+k)^{p-1}\pi,$$

and

$$\frac{(1-(p-1)k)(1+k)^{p-1}}{\sqrt{1-k^2}^p}\left(1-\sum_{n=1}^{\infty} n|b_n|^2\right)^{p/2}\pi = (1-(p-1)k)(1+k)^{p-1}\pi.$$

The claim of Example 4.2 follows. □

Remark 4.2 Example 4.2 shows that the lower bound of Burkholder's integral among the class of principal solutions is sharp. Sharp examples for the upper bound of Burkholder's integral among the class of principal solutions are given by the class of expanding piecewise radial mappings (see Sect. 5 in [4] for its proof).

5 Upper bounds of the L^p-integrals of $\sqrt{J_f}$ and $|Df|$

As an application of Theorem 1.1, we next give an upper bound of the L^p-integrals of $\sqrt{J_f}$ and $|Df|$.

Theorem 5.1 *If f is a K-quasiconformal mapping of Ω onto itself with the identity boundary mapping, or a principal solution with a Beltrami equation*

$$f_{\bar{z}} = \mu f_z, \quad |\mu| \leq k\chi_{\mathbb{D}}, \quad 0 < k < 1,$$

then for every $p \in [2, 1 + 1/k)$,

$$\iint\limits_{\Omega} \sqrt{J_f}^p dxdy \leq \frac{\sqrt{1-k^2}^p}{(1-(p-1)k)(1+k)^{p-1}} |\Omega|, \tag{5.1}$$

and the estimate is sharp.

Proof By Theorem 1.1, for every $p \in [2, 1 + 1/k)$, we may write

$$\iint\limits_{\Omega} \sqrt{J_f}^p dxdy \leq \frac{\sqrt{1-k^2}^p}{(1-(p-1)k)(1+k)^{p-1}} \iint\limits_{\Omega} \mathbf{B}_p(Df) dm.$$

By Theorem D, or alternatively, Theorem E, we have

$$\iint\limits_{\Omega} \sqrt{J_f}^p dxdy \leq \frac{\sqrt{1-k^2}^p}{(1-(p-1)k)(1+k)^{p-1}} |\Omega|.$$

Let $f = z|z|^{1/K-1}$. Then f is a K-quasiconformal mapping of the unit disk $\mathbb{D}$ onto itself. Let $z = re^{i\theta}$, then

$$\sqrt{J_f}^p = \left(\frac{1}{K}\right)^{p/2} r^{p(1/K-1)}.$$

Thus,

$$\iint\limits_{\Omega} \sqrt{J_f}^p dxdy = 2\pi \int\limits_0^1 \left(\frac{1}{K}\right)^{p/2} r^{p(1/K-1)+1} dr = \frac{\sqrt{1-k^2}^p}{(1-(p-1)k)(1+k)^{p-1}} \pi.$$

The proof of Theorem 5.1 is complete. □

Theorem 5.1 implies the following corollary (see also Corollary 4.1 in [4]).

Corollary 5.1 *If f is a K-quasiconformal mapping of Ω onto itself with the identity boundary mapping, or a principal solution with a Beltrami equation*

$$f_{\bar{z}} = \mu f_z, \quad |\mu| \leq k\chi_{\mathbb{D}}, \quad 0 < k < 1,$$

then for every $p \in [2, 1 + 1/k)$,

$$\iint\limits_{\Omega} |Df|^p dxdy \leq \frac{1+k}{1-(p-1)k}|\Omega|. \tag{5.2}$$

The estimate is sharp.

Proof By the equality

$$\sqrt{J_f}^p = (|f_z| + |f_{\bar{z}}|)^p \left(\frac{|f_z| - |f_{\bar{z}}|}{|f_z| + |f_{\bar{z}}|}\right)^{\frac{p}{2}},$$

we get from (5.1) that

$$\left(\frac{1-k}{1+k}\right)^{\frac{p}{2}} \iint\limits_{\Omega} |Df|^p dxdy \leq \iint\limits_{\Omega} \sqrt{J_f}^p dxdy \leq \frac{\sqrt{1-k^2}^p}{(1-(p-1)k)(1+k)^{p-1}}|\Omega|.$$

Hence,

$$\iint\limits_{\Omega} |Df|^p dxdy \leq \left(\frac{1+k}{1-k}\right)^{\frac{p}{2}} \iint\limits_{\Omega} \sqrt{J_f}^p dxdy \leq \frac{1+k}{1-(p-1)k}|\Omega|. \tag{5.3}$$

The claim of Corollary 5.1 follows. □

6 Auxiliary examples

In this section, we will use Beurling-Ahlfors operator and Cauchy's operator to give some concrete examples that fullfill the assumptions of Theorem 4.2 and Theorem 4.1, respectively.

Example 6.1 Let $\mu = k\frac{\bar{z}^n}{z^n}\chi_{\mathbb{D}}$ with $0 \leq k < 1$ and $n \in \mathbb{N}^+$, and let $f : \mathbb{C} \to \mathbb{C}$ be the principal solution of the Beltrami equation $f_{\bar{z}} = \mu f_z$. Then we have

$$\iint\limits_{\mathbb{D}} \mathbf{B}_p(Df)dm \geq \frac{(1+k-pk)(1+k)^{p-1}}{\sqrt{1-k^2}^{\frac{np}{n+1}}}\pi$$

for any $p \in [2, 1 + 1/k)$, where the equality takes if $p = 2 + 2/n$.

Proof Set

$$\mathbf{T}_l = \underbrace{\mathbf{T}\mu\chi_{\mathbb{D}}\mathbf{T}\mu\chi_{\mathbb{D}} \ldots \mathbf{T}\mu\chi_{\mathbb{D}}}_{l}, \quad l \in \mathbb{N}^+.$$

Set $\mathbf{T}_0 = Id$. By Lemma C, it follows from induction that

$$\mathbf{C}\mu\chi_{\mathbb{D}}\mathbf{T}_{l-1} = \begin{cases} (-1)^{l-1}\frac{n(2n+1)(3n+2)...((l-1)n+(l-2))}{l!(n+1)^l}\left(k(\frac{\bar{z}}{z})^{n+1}\right)^l z, & |z| \le 1, \\ (-1)^{l-1}\frac{n(2n+1)(3n+2)...((l-1)n+(l-2))}{l!(n+1)^l}\left(k(\frac{1}{z^2})^{n+1}\right)^l z, & |z| > 1. \end{cases}$$

After calculating the sum of the series

$$z + \sum_{l=1}^{\infty} \mathbf{C}\mu\chi_{\mathbb{D}}\mathbf{T}_{l-1},$$

we obtain that the principal solution f of the Beltrami equation $f_{\bar{z}} = k(\frac{\bar{z}}{z})^n \chi_{\mathbb{D}} f_z$ can be expressed explicitly as

$$f(z) = \begin{cases} z(1 + k(\frac{\bar{z}}{z})^{n+1})^{1/(n+1)}, & |z| \le 1, \\ z(1 + k(\frac{1}{z^2})^{n+1})^{1/(n+1)}, & |z| > 1. \end{cases}$$

Moreover, when $|z| < 1$,

$$f_z = \frac{1}{(1 + k(\frac{\bar{z}}{z})^{n+1})^{\frac{n}{n+1}}}, \quad f_{\bar{z}} = k\frac{\bar{z}^n}{z^n(1 + k(\frac{\bar{z}}{z}))^{n+1})^{\frac{n}{n+1}}}.$$

From the above two equalities it follows

$$\iint_{\mathbb{D}} \mathbf{B}_p(Df)dm = (1+k)^{p-1}(1-(p-1)k)\int_0^1 \left[\int_0^{2\pi} \frac{1}{|1 + k(\frac{\bar{z}}{z})^{n+1}|^{\frac{np}{n+1}}} d\theta\right] rdr.$$

Using Hölder's inequality and the identity

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{1}{1 + 2k\cos\theta + k^2} d\theta = \frac{1}{1-k^2},$$

we have

$$\begin{aligned} \iint_{\mathbb{D}} \mathbf{B}_p(Df)dm &\ge (1+k)^{p-1}(1-(p-1)k)\int_0^1 2\pi\left(\frac{1}{\sqrt{1-k^2}}\right)^{\frac{np}{n+1}} rdr \\ &= \frac{(1+k-pk)(1+k)^{p-1}}{\sqrt{1-k^2}^{\frac{np}{n+1}}}\pi, \end{aligned}$$

and the equality holds if $p = 2 + \frac{2}{n}$. □

Example 6.2 Let $f(z) = z + k\frac{\bar{z}^{n+1}}{n+1}$, $0 \le k < 1$, and n is a nonnegative integer. Then f is a K-quasiconformal mapping of the unit disk $\mathbb{D}$ into $\mathbb{C}$ and satisfies

$$\frac{(1-(p-1)k)(1+k)^{p-1}\sqrt{1-\frac{k^2}{n+1}}^{\,p}}{\sqrt{1-k^2}^{\,p}}\pi$$
$$\leq \iint\limits_{\mathbb{D}} \mathbf{B}_p(Df)dm \leq \pi, \quad \text{for } 2 \leq p < \frac{1+k}{k}.$$

Moreover, the equality of the left inequality holds when $n = 0$ and $p = 2$, while the equality of the right inequality holds when $k = 0$.

Proof Assume that $f(z) = z + k\frac{\bar{z}^{n+1}}{n+1}$, $0 \leq k < 1$. Then f can be extended to a principal solution g with Beltrami coefficient $\mu = k\bar{z}^n$ by the formula

$$g(z) = \begin{cases} f(z), & |z| \leq 1, \\ z + \frac{k}{(n+1)z^{n+1}}, & |z| > 1. \end{cases}$$

Hence, from Theorem 4.1 we have

$$\frac{(1-(p-1)k)(1+k)^{p-1}\sqrt{1-\frac{k^2}{n+1}}^{\,p}}{\sqrt{1-k^2}^{\,p}}\pi$$
$$\leq \iint\limits_{\mathbb{D}} \mathbf{B}_p(Df)dm \leq \pi, \quad \text{for } 2 \leq p < \frac{1+k}{k},$$

where the left equality holds when $n = 0$. □

Example 6.3 Let $f(z) = z|z|^{K-1}$. Then f is a K-quasiconformal mapping of the unit disk $\mathbb{D}$ onto itself and satisfies

$$\iint\limits_{\mathbb{D}} \mathbf{B}_p(Df)dm = \frac{(1-(p-1)k)(1+k)^{p-1}}{(1+kp-k)(1-k)^{p-1}}\pi.$$

Proof There follows

$$|f_z| = \frac{K+1}{2}|z|^{K-1}, \quad |f_{\bar{z}}| = \frac{K-1}{2}|z|^{K-1}.$$

Hence,

$$\iint\limits_{\mathbb{D}} \mathbf{B}_p(Df)dm = 2\pi\frac{(1-(p-1)k)(1+k)^{p-1}}{(1-k)^p}\int_0^1 r^{\frac{2kp}{1-k}+1}dr$$
$$= \frac{(1-(p-1)k)(1+k)^{p-1}}{(1-k)^{p-1}(1+kp-k)}\pi.$$

□

Remark 6.1 In order to compare the lower bounds of Burkholder's integral, we set

$$A = (1 + k - pk)(1 + k)^{p-1}\pi, \quad B_n = \frac{(1+k-pk)(1+k)^{p-1}}{\sqrt{1-k^2}^{\frac{np}{n+1}}}\pi,$$
$$B_n' = \frac{(1-(p-1)k)(1+k)^{p-1}\sqrt{1-\frac{k^2}{n+1}}^{p}}{\sqrt{1-k^2}^{p}}\pi, \quad C = \frac{(1-(p-1)k)(1+k)^{p-1}}{\sqrt{1-k^2}^{p}}\pi,$$
$$D = \frac{(1-(p-1)k)(1+k)^{p-1}}{(1+kp-k)(1-k)^{p-1}}\pi.$$

Then for $p \in (2, \frac{1+k}{k})$ and $0 < k < 1$ we obtain

$$A \le B_n \le C, \quad \text{and} \quad \lim_{n\to\infty} B_n = C;$$
$$A \le B_n' \le C, \quad \text{and} \quad \lim_{n\to\infty} B_n' = C;$$
$$B_0 = B_0' = A, \quad \text{and} \quad C < D < \pi.$$

The above relations show that for K-quasiconformal mappings with the identity mapping, their lower bounds of Burkholder's integral may vary from C to π, while, for principal solutions with given Beltrami coefficients , their lower bounds of Burkholder's integral may vary from A to C.

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