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Complex Analysis and Operator Theory

Characterizations of Mono-Components: the Blaschke and Starlike Types

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Abstract Since the last decade, motivated by attempts of positive frequency decomposition of signals, complex periodic functions $s(e^{it}) = \rho(t)e^{i\theta(t)}$ satisfying the conditions

$$H(\rho(t)\cos\theta(t)) = \rho(t)\sin\theta(t), \quad \rho(t) \geq 0, \ \theta'(t) \geq 0, \ a.e.,$$

have been sought, where H is the circular Hilbert transform and the phase derivative $\theta'(t)$ is suitably defined and interpreted as instantaneous frequency of the signal $\rho(t)\cos\theta(t)$. Functions satisfying the above conditions are called mono-components. Mono-components have been found to form a large pool and used to decompose and analyze signals. This note in a great extent concludes the study of seeking for mono-components through characterizing two classes of mono-components of which one is phrased as the Blaschke type and the other the starlike type. The Blaschke type

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mono-components are of the form $\rho(t)\cos\theta(t)$, where $\rho(t)$ is a real-valued (generalized) amplitude functions and $e^{i\theta(t)}$ is the boundary limit of a finite or infinite Blaschke product. For the starlike type mono-components, we assume the condition $\int_0^{2\pi}\theta'(t)dt = n\pi$, where n is a positive integer. It shows that such class of mono-components is identical with the class consisting of products between p-starlike and boundary $(n-2p)$-starlike functions. The results of this paper explore connections between harmonic analysis, complex analysis, and signal analysis.

Keywords Circular analytic signal · Mono-component · Blaschke product · Bedrosian identity · Multivalent starlike functions · Boundary starlike functions

1 Introduction

Fourier series expansion (for periodic functions) and inverse Fourier transform (for non-periodic transform), in fact, are signal decompositions into basic signals of non-negative instantaneous frequencies. In those decompositions the basic signals are trigonometric functions that are of constant frequencies. With combined efforts of researchers, based on the Hardy space theory and analysis of functions of one complex variable, a comprehensive mono-component function theory and the related signal decompositions, especially lately the adaptive ones, were established (see [1–7], and related references). We now recall the mono-component concept in the unit disc context. Now require that the function $x(e^{it}) = \rho(t)\cos\theta(t)$ satisfies $H(\rho(t)\cos\theta(t)) = \rho(t)\sin\theta(t)$, where H is the circular Hilbert transformation. The last requirement amounts to the circular analytic signals has a form of $s = x + iHx = \rho e^{i\theta}$. Let $H^p(\mathbb{T})$ be the set of all circular analytic signals in $L^p(\mathbb{T})$. In the case $\rho(t)\cos\theta(t)$ is the real part of the boundary limit of a good analytic function, say a Hardy $H^p(\mathbf{D})$ space function in the unit disc $\mathbf{D}$, $1 \leq p \leq \infty$ [8]. We further write $s(z) = \rho_r(t)e^{i\theta_r(t)}$, $z = re^{it}$, $r < 1$, $t \in [0, 2\pi)$. If, moreover, the following limit exists and satisfies

$$\lim_{r\to 1^-}\theta_r'(t) \geq 0, \quad \text{a.e.,} \tag{1.1}$$

then $x(e^{it})$ is said to be a *mono-component*, and the limit is defined to be $\theta'(t)$, as the *analytical instantaneous frequency*, or simply, *instantaneous frequency* (IF) [9,10]. If $x(e^{it})$ is a mono-component, then we say that the associated analytic signal $s(e^{it})$ is also a mono-component, and vice versa. It is noted that the necessity of taking limit from inside the unit disc is due to the fact that the classical derivative $\theta'(t)$ as a measurable function may not exist when we extract $\theta(t)$ from the boundary limit of a Hardy space function.

The first type mono-components with nonlinear phases are Möbius transforms. They are of the form

$$s(e^{it}) = \frac{e^{it} - a}{1 - \overline{a}e^{it}}, \quad |a| < 1.$$

It is a known fact that its phase derivative is identical with the Poisson kernel. Möbius transforms turn to be the very basic constructive blocks of most other types of mono-component. In particular, it immediately implies that finite Blaschke products are mono-components. They are also constructive blocks of infinite Blaschke products. It took a while to find a proof for the fact that infinite Blaschke products are also mono-components. In [10], Qian proves that for all inner functions positivity of their phase derivatives are reducible to the Wolff–Julia–Carathéodory Theorem. We note that only in the limiting sense as given in (1.1) this property can be proved. Owing to this result unit module mono-components have all been found, as summarized in the following [8,10]:

Theorem 1.1 *Let $s(e^{it})$ be of the form $s(e^{it}) = e^{i\theta(t)}$, where $\theta(t)$ is a real-valued measurable function. Then $s(e^{it})$ is a mono-component if and only if $H(\cos\theta) = \sin\theta$, or, equivalently, $H(e^{i\theta}) = -ie^{i\theta}$, and if and only if $s(e^{it})$ is the non-tangential boundary limit of an inner function in* $\mathbf{D}$.

Signals of unit module are called *phase signals* [11]. The above theorem characterizes all phase signals that are mono-components. What is left now is to characterize non-unimodular mono-component signals $s(e^{it})$, viz. those with non-constant module $\rho(t) = |s(e^{it})|$.

We restrict ourselves to the Hardy spaces $H^p(\mathbf{D})$, $1 \leq p \leq \infty$. We divide non-unimodule mono-components into the Bedrosian and non-Bedrosian categories. We call $s(e^{it}) = \rho(t)e^{i\theta(t)}$ as a *Bedrosian type mono-component* if both the original signal $\rho(t)e^{i\theta(t)}$ and its phase signal part $e^{i\theta(t)}$ are mono-components, and ρ is real-valued, and $\theta'_\rho(t) = 0$, a.e. where $\rho = |\rho|e^{i\theta_\rho}$. We call $s(e^{it}) = \rho(t)e^{i\theta(t)}$ as a *non-Bedrosian type mono-component* if $\rho(t)e^{i\theta(t)}$ is a mono-component, but $e^{i\theta(t)}$ is not.

Note that in the Bedrosian category we release the usual restriction $\rho = |s| \geq 0$, a.e., and, instead, replace it with a real-valued function $\rho = |\rho|e^{i\theta_\rho}$ such that $\theta'_\rho = 0$, a.e. In the sequel we call such a function ρ as a *generalized amplitude*.

To justify the terminology "Bedrosian type" we recall the Bedrosian Theorem on the real line (There is a counterpart result in the unit circle context): if 1°. For some $\sigma > 0$, supp $\hat{f} \subset [-\sigma, \sigma]$ and supp $\hat{g} \subset \mathbb{R}\backslash(-\sigma, \sigma)$; or 2°. $f, g \in H^2(\mathbb{C}^+)$, then

$$H(fg) = fHg. \tag{1.2}$$

Note that 1° and 2° are only sufficient conditions for the Bedrosian identity (1.2) (see [12,13]).

Now, if $e^{i\theta(t)}$ for a real-valued $\theta(t)$ is a mono-component, or, equivalently, $H(\cos\theta) = \sin\theta$ or $e^{i\theta(t)}$ is the boundary limit of an inner function, then

$$H(\rho\cos\theta) = \rho\sin\theta$$

holds if and only if

$$H(\rho\cos\theta) = \rho H(\cos\theta) \tag{1.3}$$

holds. The last relation amounts to saying that for $f = \rho$ and $g = \cos\theta$ the Bedrosian identity (1.2) holds. On the other hand, if $H(\cos\theta) \neq \sin\theta$, then for ρ and $\cos\theta$ the Bedrosian identity (1.3) does not hold. The last mentioned case corresponds to the non-Bedrosian type mono-components.

In this paper, to study the Bedrosian type mono-components, we concentrate to the case that $e^{i\theta(t)}$ is the non-tangential boundary limit of an infinite Blaschke product (The finite Blaschke case has previously been proved in [14]), and ρ is real-valued and changes its sign only at a finitely many points. We will call such a Bedrosian type mono-component as a Blaschke type mono-component.

The non-Bedrosian type mono-components are essentially multi-starlike and multi-boundary starlike functions, or products of those two types of functions. The non-Bedrosian type corresponds to the case where a Hardy space signal $s(e^{it})$ has a non-trivial outer function factor, i.e. θ'_{outer} is not identical with the zero function but $(\theta_{\text{outer}}(t) + \theta_{\text{inner}}(t))' \geq 0$, a.e. Under certain conditions to avoid non-trivial singular inner functions, we present a class of non-Bedrosian type mono-components. Due to their relation with starlike functions, the non-Bedrosian type mono-components are also called starlike type mono-components.

For signals defined on the whole real line there is a counterpart theory. In this paper we will concentrate on the unit disc context corresponding to periodic signals.

2 Mono-Components of the Blaschke Type

Efforts of finding mono-components of the Bedrosian type, in fact, form a new phase of study of the Bedrosian identity [5,7,13,14].

The essential structure of such type mono-components is presented in the following example:

$$s(e^{it}) = \left(\frac{1}{1-\overline{a}_1 e^{it}} + \frac{1}{1-a_1\overline{e^{it}}}\right)\frac{e^{it}-a_1}{1-\overline{a}_1 e^{it}}\,\frac{e^{it}-a_2}{1-\overline{a}_2 e^{it}}, \quad a_1, a_2 \in \mathbf{D}.$$

In verifying that $s(e^{it})$ is a Bedrosian type mono-component, the key point is that

$$\frac{1}{1-\overline{a}_1 z}\,\frac{z-a_1}{1-\overline{a}_1 z}\,\frac{z-a_2}{1-\overline{a}_2 z}$$

is an analytic function in the disc; and for $|z| = 1$, the product

$$\frac{1}{1-a_1\overline{z}}\,\frac{z-a_1}{1-\overline{a}_1 z}\,\frac{z-a_2}{1-\overline{a}_2 z}$$

has an analytic continuation to the whole interior part of the disc. As a result, $s(z)$ is analytic, and, in fact, an H^∞-function in the disc. Since $\frac{1}{1-\overline{a}_1 z} + \frac{1}{1-a_1\overline{z}}$ is real-valued and has finitely many sign-change pints on $|z| = 1$, it is a generalized amplitude on the circle $\mathbb{T} := \{z|\ |z| = 1\}$.

In the literature the following result for Blaschke products of finitely many zeros is known [5,14].

Theorem 2.1 *Let $e^{i\theta(t)}$ be the boundary limit of a Blaschke product with finitely many zeros $a_1, \ldots, a_n$ in the unit disc $\mathbf{D}$, where multiples of zeros are counted. Let, in particular, the multiple of the zero $z = 0$ is m: $1 \le m \le n$. Hence*

$$e^{i\theta(t)} = e^{imt} \prod_{\substack{k=m+1 \\ a_k \neq 0}}^{n} \frac{|a_k|}{a_k} \frac{a_k - e^{it}}{1 - \overline{a}_k e^{it}} = B_n(e^{it}). \tag{2.1}$$

Then $\rho(t)$, as a real-valued generalized amplitude function in $L^p[-\pi, \pi]$, $1 \le p \le \infty$, gives rise to a mono-component $\rho(t)e^{i\theta(t)}$ if and only if

$$\rho(t) = c_1 + \sum_{k=2}^{n} \left[c_k e_k(e^{it}) + \overline{c_k e_k(e^{it})} \right], \tag{2.2}$$

where $\{e_k(e^{it})\}_{k=1}^{n}$ denotes the rational orthonormal (Takenaka-Mulmquist) system

$$\left\{ 1, e^{it}, \ldots, e^{i(m-1)t}, \frac{\sqrt{1 - |a_{m+1}|^2} e^{imt}}{1 - \overline{a_{m+1}} e^{it}}, \ldots, \right.$$
$$\left. \frac{\sqrt{1 - |a_{m+r}|^2} e^{imt}}{1 - \overline{a_{m+r}} e^{it}} \prod_{j=m+1}^{m+r-1} \frac{e^{it} - a_j}{1 - \overline{a_j} e^{it}}, \ldots, \; 1 \le r \le n - m \right\},$$

generated by $a_1, \ldots, a_n$, and $c_1, \ldots, c_n$ are arbitrary complex numbers.

It is known that the linear space spanned by the rational orthonormal sequence $\{e_k(e^{it}), k = 1, \ldots, n\}$ is identical with the space $H^p(\mathbb{T}) \bigcap B_n(e^{it})\overline{H^p(\mathbb{T})}$, referred as backward shift invariant subspace in $H^p(\mathbb{T})$, $1 \le p \le \infty$ [15,16].

Next, we will extend the above result to infinite Blaschke products. Recall that an infinite Blaschke product is defined

$$B(z) = z^m \prod_{\substack{k=m+1 \\ a_k \neq 0}}^{\infty} \frac{|a_k|}{a_k} \frac{a_k - z}{1 - \overline{a_k} z}, \tag{2.3}$$

where the points $a_k \in \mathbf{D}$ satisfy the condition $\sum_{k=1}^{\infty} (1 - |a_k|) < \infty$. It is known the non-tangential boundary limit

$$B(e^{it}) = \lim_{r \to 1^-} B(z), \quad z = re^{it}, \tag{2.4}$$

exists and $|B(e^{it})| = 1$ for almost all $t \in [0, 2\pi]$ [17]. To extend Theorem 2.1 to the infinite Blaschke products case we first quote two lemmas from [18]. We provide our proofs below that are shorter than what are found in [18], as well as for the self-containing purpose.

We need the truncated Blaschke products

$$B_n(z) = z^m \prod_{\substack{k=m+1 \\ a_k \neq 0}}^{n} \frac{|a_k|}{a_k} \frac{a_k - z}{1 - \overline{a}_k z}. \tag{2.5}$$

Lemma 2.2 *Let $B_n(e^{it})$ and $B(e^{it})$ be given by (2.5) and (2.4). Then*

$$\lim_{n \to \infty} \int_0^{2\pi} |B_n(e^{it}) - B(e^{it})|^p dt = 0$$

for $1 < p < \infty$.

Proof For $p = 2$, as $n \to \infty$, we have

$$\begin{aligned}
\int_0^{2\pi} |B_n(e^{it}) - B(e^{it})|^2 dt &= \int_0^{2\pi} B_n(e^{it}) \overline{B_n(e^{it})} dt \\
&\quad - 2\mathrm{Re} \left[\int_0^{2\pi} B(e^{it}) \overline{B_n(e^{it})} dt \right] + \int_0^{2\pi} B(e^{it}) \overline{B(e^{it})} dt \\
&= 4\pi - 4\pi \frac{B(0)}{B_n(0)} \\
&= 4\pi \left(1 - \prod_{k=n+1}^{\infty} |a_k| \right) \to 0,
\end{aligned}$$

where the last equality employs the fact that $\sum_{k=1}^{\infty}(1 - |a_k|) < \infty$ implies that $\prod_{k=1}^{\infty} |a_k|$ converges.

Next, we prove that for any $p > 1$, there holds

$$\lim_{n \to \infty} \int_0^{2\pi} \left| B_n(e^{it}) - B(e^{it}) \right|^p dt = 0.$$

First assume $p > 2$. Then, as $n \to \infty$, due to $|B_n(z)| \leq 1$, $|B(z)| \leq 1$ for $|z| \leq 1$, we have

$$\int_0^{2\pi} \left| B_n(e^{it}) - B(e^{it}) \right|^p dt \leq 2^{p-2} \int_0^{2\pi} \left| B_n(e^{it}) - B(e^{it}) \right|^2 dt \to 0.$$

For $1 < p < 2$, by Hölder's inequality, we have, as $n \to \infty$,

$$\int_0^{2\pi} \left|B_n(e^{it}) - B(e^{it})\right|^p dt \le \left(\int_0^{2\pi} |B_n(e^{it}) - B(e^{it})|^2 dt\right)^{\frac{p}{2}} \left(\int_0^{2\pi} 1dt\right)^{\frac{2-p}{2}}$$
$$= (2\pi)^{1-\frac{p}{2}} \left(\int_0^{2\pi} \left|B_n(e^{it}) - B(e^{it})\right|^2 dt\right)^{p/2} \to 0.$$

The proof is completed. □

Lemma 2.3 *Let $B_n(e^{it})$ and $B(e^{it})$ be defined as in* Lemma 2.2. *Then for any $h(e^{it}) \in L^p(\mathbb{T}),\ 1 < p < \infty$, we have*

$$\lim_{n\to\infty} \int_0^{2\pi} \left|B_n(e^{it}) - B(e^{it})\right|^p \left|h(e^{it})\right|^p dt = 0.$$

Proof For any $\delta > 0$, let $E_n(\delta) = E\left\{t \mid |B_n(e^{it}) - B(e^{it})| > \delta, t \in [0, 2\pi]\right\}$ and $E_n^*(\delta) = [0, 2\pi]\backslash E_n(\delta)$. Since

$$\delta^p mes(E_n(\delta)) \le \int_{E_n(\delta)} |B_n(e^{it}) - B(e^{it})|^p dt \le \int_0^{2\pi} \left|B_n(e^{it}) - B(e^{it})\right|^p dt,$$

by Lemma 2.2, we have $\lim_{n\to\infty} mes(E_n(\delta)) = 0$, where $mes(E)$ denotes the measure of E.

For any $h(e^{it}) \in L^p(\mathbb{T})$, the integral can be decomposed as

$$\int_0^{2\pi} \left|B_n(e^{it}) - B(e^{it})\right|^p \left|h(e^{it})\right|^p dt \le 2^p \int_{E_n(\delta)} |h(e^{it})|^p dt + \delta^p \int_{E_n^*(\delta)} |h(e^{it})|^p dt. \tag{2.6}$$

By choosing a small δ we can first have

$$\delta^p \int_{E_n^*(\delta)} |h(e^{it})|^p dt \le \delta^p \int_0^{2\pi} |h(e^{it})|^p dt < \frac{\epsilon}{2}.$$

Then, by absolute continuity of integral and $\lim_{n\to\infty} mes(E_n(\delta)) = 0$, for a large $N = N(\epsilon, \delta)$, if $n > N$, we have

$$2^p \int_{E_n(\delta)} |h(e^{it})|^p dt < \frac{\epsilon}{2},$$

and thus

$$\int_0^{2\pi} |B_n(e^{it}) - B(e^{it})|^p |h(e^{it})|^p dt < \epsilon.$$

This completes the proof. □

Theorem 2.4 *Let B be an infinite Blaschke product with zeros $a_1, \ldots, a_n, \ldots$, counting the multiples, where $a_1 = 0$, with the multiple $m \geq 1$. Let $\{e_k(e^{it})\}_{k=1}^{\infty}$ be the rational orthogonal system*

$$\left\{1, e^{it}, \ldots, e^{i(m-1)t}, \frac{\sqrt{1-|a_{m+1}|^2}e^{imt}}{1-\overline{a_{m+1}}e^{it}}, \ldots, \right.$$
$$\left. \frac{\sqrt{1-|a_{m+r}|^2}e^{imt}}{1-\overline{a_{m+r}}e^{it}} \prod_{j=m+1}^{m+r-1} \frac{e^{it}-a_j}{1-\overline{a_j}e^{it}}, \ldots, \ 1 \leq r < \infty \right\}$$

generated by the sequence $\{a_i\}_{i=1}^{\infty}$. Then we have

(1) *$\rho(t)$ is a real-valued function such that $\rho(t)B(e^{it}) \in H^p(\mathbb{T})$, $1 \leq p \leq \infty$, if and only if ρ is the real part of some function in the backward shift invariant space induced by the Blaschke product, viz., $\rho \in \mathrm{Re}\{H^p(\mathbb{T}) \bigcap B(e^{it})\overline{H^p(\mathbb{T})}\}$;*
(2) *let $1 < p < \infty$, $\rho \in \mathrm{Re}\{H^p(\mathbb{T}) \bigcap B(e^{it})\overline{H^p(\mathbb{T})}\}$ if and only if*

$$\rho(t) = c_1 + \sum_{k=2}^{\infty}\left[c_k e_k(e^{it}) + \overline{c_k e_k(e^{it})}\right], \tag{2.7}$$

where $c_k = \langle \rho(t), e_k(e^{it})\rangle = \frac{1}{2\pi}\int_0^{2\pi} \rho(t)\overline{e_k(e^{it})}dt$ for $k = 1, 2, \ldots$, and the convergence is in the $L^p(\mathbb{T})$ norm sense.

Proof (1) Let $\rho_+ = \frac{1}{2}(\rho + iH\rho)$, where H is the Hilbert transformation on the circle. Then $\rho_+ \in H^p(\mathbb{T})$ and thus $\rho_+ B \in H^p(\mathbb{T})$. Based on the last relation, $\rho B \in H^p(\mathbb{T})$ if and only if $(H\rho)B \in H^p(\mathbb{T})$, and hence, if and only if $\overline{\rho_+}B \in H^p(\mathbb{T})$. The last relation is equivalent to $\rho_+ \in B\overline{H^p(\mathbb{T})}$, or $\rho_+ \in H^p(\mathbb{T}) \cap B\overline{H^p(\mathbb{T})}$.

(2) If ρ has the representation (2.7), then surely $\rho \in \mathrm{Re}\{H^p(\mathbb{T}) \bigcap B(e^{it})\overline{H^p(\mathbb{T})}\}$ (see [19]). What is needed is to show the "only if" part. We assume that $\rho \in \mathrm{Re}\{H^p(\mathbb{T}) \bigcap B(e^{it})\overline{H^p(\mathbb{T})}\}$, and we show that ρ has the expression (2.7).

Since $\rho_+ \in H^p(\mathbb{T})$, we have, by invoking the Plemelj formula and relation $H\rho_+ = -i\rho_+$ (see, for instance [17]),

$$\rho_+(t) = \lim_{r\to 1^-} \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{1-e^{-ix}z}\rho_+(x)dx, \quad z = re^{it} \in \mathbf{D}. \tag{2.8}$$

On the other hand, since $\rho_+ \in B(e^{it})\overline{H^p(\mathbb{T})}$, we have that $B^{-1}\rho_+$ is the boundary limit of some function in the Hardy space H^p outside the unit disc. The Cauchy formula of the outside Hardy space function on the boundary gives,

$$\begin{aligned}
&\lim_{r\to 1^-} \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{-ix}zB^{-1}(e^{ix})B(z)}{1-e^{-ix}z}\rho_+(x)dx \\
&= zB(z)\lim_{r\to 1^-}\frac{1}{2\pi i}\int_{\partial\mathbf{D}} \frac{B^{-1}(\zeta)\rho_+(\zeta)}{\zeta(\zeta - z)}d\zeta, \quad z = re^{it} \in \mathbf{D}, \\
&= 0.
\end{aligned} \tag{2.9}$$

Adding up (2.8) and (2.9), we obtain

$$\rho_+(t) = \lim_{r\to 1^-} \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - e^{-ix} z B^{-1}(e^{ix}) B(z)}{1 - e^{-ix} z} \rho_+(x) dx, \quad z = re^{it} \in \mathbf{D}.$$

Let

$$\rho_+^{(n)}(t) = \lim_{r\to 1^-} \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - e^{-ix} z B_n^{-1}(e^{ix}) B_n(z)}{1 - e^{-ix} z} \rho_+(x) dx, \quad z = re^{it} \in \mathbf{D}.$$

Then

$$\begin{aligned}
&\left\| \rho_+(t) - \rho_+^{(n)}(t) \right\|^p \\
&= \left\| \lim_{r\to 1^-} \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{-ix} z B_n^{-1}(e^{ix}) B_n(z) - e^{-ix} z B^{-1}(e^{ix}) B(z)}{1 - e^{-ix} z} \rho_+(x) dx \right\|^p \\
&\le \left\| \lim_{r\to 1^-} \frac{1}{2\pi} \int_0^{2\pi} \frac{\left[B_n^{-1}(e^{ix}) - B^{-1}(e^{ix}) \right] e^{-ix} z B_n(z)}{1 - z e^{-ix}} \rho_+(x) dx \right\|^p \\
&\quad + \left\| \lim_{r\to 1^-} \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{-ix} z B^{-1}(e^{ix}) \left[B_n(z) - B(z) \right]}{1 - e^{-ix} z} \rho_+(x) dx \right\|^p.
\end{aligned}$$

Due to (2.9) the last integral and thus the corresponding L^p-norm is zero. By the Plemelj formula and L^p-boundedness of the Hilbert transformation for $1 < p < \infty$, we have

$$\begin{aligned}
\left\| \rho_+(t) - \rho_+^{(n)}(t) \right\|^p &\le A_p \left\| \left[B_n^{-1}(e^{it}) - B^{-1}(e^{it}) \right] \rho_+(t) \right\|^p \\
&= A_p \left\| \left[B_n(e^{it}) - B(e^{it}) \right] \overline{\rho_+(t)} \right\|^p,
\end{aligned}$$

where A_p is a finite constant. By Lemma 2.3, we have

$$\lim_{n\to\infty} \left\| \rho_+(t) - \rho_+^{(n)}(t) \right\|^p = 0. \tag{2.10}$$

Since $\{e_k(e^{it}), k \in \mathbb{Z}^+\}$ is an orthonormal sequence, by the Christoffel–Darboux formula, we have

$$\sum_{k=1}^{n} \overline{e_k(e^{ix})} e_k(e^{it}) = \frac{1 - e^{-i(x-t)} B_n^{-1}(e^{ix}) B_n(e^{it})}{1 - e^{-i(x-t)}}.$$

Let $c_k = \langle \rho(x), e_k(e^{ix}) \rangle = \frac{1}{2\pi} \int_0^{2\pi} \rho(x) \overline{e_k(e^{ix})} dx$. Then

$$\begin{aligned}
\rho_+^n(t) &= \frac{1}{2} \sum_{k=1}^{n} \left[\langle \rho(x), e_k(e^{ix}) \rangle e_k(e^{it}) + i \langle H\rho(x), e_k(e^{ix}) \rangle e_k(e^{it}) \right] \\
&= \frac{1}{2} \sum_{k=1}^{n} \left[\langle \rho(x), e_k(e^{ix}) \rangle e_k(e^{it}) - i \langle \rho(x), \widetilde{H} e_k(e^{ix}) \rangle e_k(e^{it}) \right] \\
&= \frac{1}{2} \langle \rho(x), e_1(e^{ix}) \rangle + \sum_{k=2}^{n} \langle \rho(x), e_k(e^{ix}) \rangle e_k(e^{it}) \\
&= \frac{1}{2} c_1 + \sum_{k=2}^{n} c_k e_k(e^{it}).
\end{aligned}$$

With (2.10), we obtain that

$$\lim_{n \to \infty} \left\| \rho(t) - c_1 - \sum_{k=2}^{n} \left[c_k e_k(e^{it}) + \overline{c_k e_k(e^{it})} \right] \right\|^p = 0.$$

This completes the proof. □

Corollary 2.5 *Let* $\rho \in \mathrm{Re}\{H^p(\mathbb{T}) \bigcap B(e^{it}) \overline{H^p(\mathbb{T})}\}$ *be continuous on* $\mathbb{T} = \partial \mathbf{D}$ *with its zeros forming a null set. Then* $\rho(t) B(e^{it})$ *is a mono-component.*

Proof If ρ is continuous and a.e. non-zero, then $\{t \in (0, 2\pi) \colon \mathrm{Re}(\rho) > 0\} \cup \{t \in (0, 2\pi) \colon \mathrm{Re}(\rho) < 0\}$ has measure 2π, being the union of some constructive open intervals. On each of the intervals the phase is a constant. Hence the phase derivative of ρ is a.e. zero. In the case ρ is a generalized amplitude, and therefore, $\rho(t) B(e^{it})$ is a mono-component.

The above Corollary enables us to easily construct mono-components of the Bedrosian-type. For instance, if the series (2.7) contains finitely many non-zero terms, then it gives rise to mono-component ρB.

Remark The proof of (2) of Theorem 2.4 provides an alternative proof of the recently established result that any rational orthonormal system $\{e_k\}_{k=1}^{\infty}$ is a Schauder basis in the L^p closure of the span of the system, viz., $\overline{\mathrm{span}}^p \{e_k\}_{k=1}^{\infty}$, $1 < p < \infty$ [20]. The condition $a_1 = \cdots = a_m = 0$ with $m \geq 1$ is to guarantee what is obtained is a mono-component. Technically, the proof of (2) itself does not rely on this assumption.

3 Mono-Components of the Starlike Type

Let Ω be a domain and let a belong to the closure of Ω. We say that Ω is *starlike with respect to* a if for each $z \in \Omega$, every point $tz + (1-t)a \in \Omega$, with $0 < t \leq 1$, belongs to Ω.

Definition 3.1 A univalent function $s(z)$ is said to be a starlike function if $s(z)$ is holomorphic in $\mathbf{D}$, $s(0) = 0$ and $s(\mathbf{D})$ is starlike with respect to 0.

Let $\mathcal{S}^*$ denote the set of starlike functions. It is known that $s(z) \in \mathcal{S}^*$ if and only if $s(z)$ is univalent and holomorphic in $\mathbf{D}$, $s(0) = 0$ and

$$\frac{\partial}{\partial t}\left[\arg s(z)\right] = \operatorname{Re}\left\{z\frac{s'(z)}{s(z)}\right\} > 0, \tag{3.1}$$

for all $z = re^{it} \in \mathbf{D}$, where $\operatorname{Re}(z)$ denotes the real part of z.

The definitions of n-starlike function and weakly n-starlike function can be found in [21]:

Definition 3.2 For a given positive integer n, let $\mathcal{S}(n)$ be the class of $n-$valent starlike functions $s(z)$ that is holomorphic in $\mathbf{D}$ and satisfies

(i) there exists some positive r such that for all $r < |z| < 1$, $\operatorname{Re}\left\{z\frac{s'(z)}{s(z)}\right\} > 0$;
(ii) $\int_0^{2\pi} \operatorname{Re}\left\{z\frac{s'(z)}{s(z)}\right\} dt = 2\pi n$, $z = qe^{it}$ for each q, $r < q < 1$.

Definition 3.3 A function $s(z)$ is said to be weakly n-valent starlike function if and only if $s(z)$ is holomorphic in the unit disc $\mathbf{D}$, has exactly n zeros at points $a_1, \ldots, a_n$ in $\mathbf{D}$ (multiples are counted), and

$$s(z) = [h(z)]^n \prod_{k=1}^{n} \frac{(z - a_j)(1 - \overline{a_j} z)}{z},$$

where $h(z) \in \mathcal{S}^*$.

Let $\mathcal{S}_w(n)$ denote the class of all weakly $n-$valent starlike functions. It is observed in [21] that $\mathcal{S}(n)$ is a proper sub-class of $\mathcal{S}_w(n)$. It is shown in [21] that

$$f(z) = \frac{z^n}{(1-z)^{2n}} \prod_{k=1}^{n} \frac{(z - a_k)(1 - \overline{a_k} z)}{z} \in \mathcal{S}_w(n) \backslash \mathcal{S}(n),$$

where a_k are points in $\mathbf{D}$ for $k = 1, \ldots, n$.

The relation between 1-starlike functions and mono-components is apparent as indicated in [9] (also see Lemma 3.1). In [4], it is stated that all n-starlike functions are mono-components. Further studies along this direction in connection with n-starlike functions, or weakly n-starlike functions, are presented in [5,7], as well as the related papers. The work [7] establishes the identical relation between the class of weakly n-starlike functions and the class of the so called H-2n atoms (see below) which form a subclass of mono-components.

In this paper, we will study a wider type mono-components involving *boundary starlike functions*. Univalent starlike functions with respect to the boundary point was first introduced and investigated by Roberson [22] and were further studied by Lyzzaik and Lecko [23–25].

Definition 3.4 A univalent function $s(z)$ is said to be a starlike function with respect to the boundary point 0 if $s(z)$ is holomorphic in $\mathbf{D}$, $\lim_{r\to 1^-} s(r) = 0$, $s(\mathbf{D})$ is starlike with respect to the origin, and $\operatorname{Re}\{e^{i\alpha} s(z)\} > 0$ for some real α and all $z \in \mathbf{D}$.

Denote by $\mathcal{G}^*$ the class of all starlike functions $s(z)$ with respect to the boundary point. It was proved by Lyzzaik in [23] that $s(z) \in \mathcal{G}^*$ if and only if $s(z)$ is univalent and holomorphic in $\mathbf{D}$, $\lim_{r\to 1^-} s(r) = 0$ and

$$\mathrm{Re}\left\{2z\frac{s'(z)}{s(z)} + \frac{z+1}{z-1}\right\} > 0, \quad z \in \mathbf{D}.$$

Let $h(z) = \frac{-zs^2(z)}{(1-z)^2}$. By simple computation, we have

$$\frac{\partial}{\partial t}[\arg h(z)] = \mathrm{Re}\left\{z\frac{h'(z)}{h(z)}\right\} = \mathrm{Re}\left\{2z\frac{s'(z)}{s(z)} + \frac{z+1}{z-1}\right\}.$$

This gives the corresponding relationship that $s(z) \in \mathcal{G}^*$ if and only if $h(z) = \frac{-zs^2(z)}{(z-1)^2} \in \mathcal{S}^*$.

Below we introduce the definition of Hilbert-n atoms that form the wider class that we mentioned earlier.

Definition 3.5 Let $s(e^{it}) = \rho(t)e^{i\theta(t)} \in L^p(\mathbb{T})$ be a nonzero complex function, where $p > 1$. Then $s(e^{it}) = \rho(t)e^{i\theta(t)}$ is called an H-n atom if it satisfies the conditions

(i) $H[\rho(t)\cos\theta(t)] = \rho(t)\sin\theta(t)$ modulo constants;
(ii) $\rho(t) \geq 0, \theta'(t) \geq 0$ for almost all $t \in [0, 2\pi]$;
(iii) $\int_0^{2\pi} \theta'(t)dt = n\pi.$

It is obvious that all H-n atoms are mono-components. The following result is proved in [9].

Lemma 3.1 *A complex function $s(e^{it}) = \rho(t)e^{i\theta(t)}$ satisfying $\rho(t) \neq 0, 0 \leq t < 2\pi$, where $\rho(t)$, $\theta(t)$ are absolutely continuous functions and $\int_0^{2\pi} s(e^{it})dt = 0$, is an H-2 atom if and only if $s(e^{it})$ is the boundary limit of a starlike function $s(z)$ whose boundary is a bounded rectifiable closed Jordan curve.*

In order to conveniently prove our structural result we assume that $s(z)$ is holomorphic on the closed unit disc $cl\{\mathbf{D}\} := \{\mathbf{z} | \, |\mathbf{z}| \leq \mathbf{1}\}$. Let $\mathbb{A}$ denote the class of functions that are holomorphic in the closed unit disc.

Lemma 3.2 *There holds $\mathbb{A} \cap \mathcal{S}(n) = \mathbb{A} \cap \mathcal{S}_w(n)$.*

Proof It is shown in [21] that $\mathcal{S}(n) \subseteq \mathcal{S}_w(n)$. Conversely, if $s(z) \in \mathcal{S}_w(n)$, then $s(z)$ can be represented as

$$s(z) = h^n(z)\prod_{k=1}^{n}\frac{(z-a_k)(1-\overline{a_k}z)}{z},$$

where $h(z) \in \mathcal{S}^*$. Obviously, $s(z)$ contains only n zeros $a_1, \ldots, a_n$ in $\mathbf{D}$. Since $s(z)$ is holomorphic on $cl\{\mathbf{D}\}$, we have that $h(z) \in \mathcal{S}^*$ is holomorphic on $cl\{\mathbf{D}\}$. For both

$s(z)$ and $h(z)$, their respective limiting phase derivatives given as in (1.1) coincide with their boundary phase derivatives along the circle. Note that the factor

$$\prod_{k=1}^{n} \frac{(z-a_k)(1-\overline{a_k}z)}{z}$$

on the circle has constant phase. We hence have

$$\text{Re}\left\{e^{it}\frac{s'(e^{it})}{s(e^{it})}\right\} = n\ \text{Re}\left\{e^{it}\frac{h'(e^{it})}{h(e^{it})}\right\} \geq 0.$$

Hence there exists some $0 < r < 1$, $\max\{|a_1|, |a_2|, \ldots, |a_n|\} < r < 1$ such that for all $r < |z| \leq 1$, we have $\text{Re}\left\{z\frac{s'(z)}{s(z)}\right\} > 0$, due to the minimum value principle of harmonic functions. By the argument principle, we have $\int_0^{2\pi} \text{Re}\left\{z\frac{s'(z)}{s(z)}\right\} dt = 2\pi n$, $z = qe^{it}$ for each q, $r < q < 1$. This shows that $s(z) \in \mathcal{S}(n)$. The proof is completed. □

Under the assumption that $s(z)$ is holomorphic on $cl\{\mathbf{D}\}$, we now study relations between H-n atoms with certain products of multi-starlike functions and multi-boundary starlike functions. In order to interpret H-n atoms more easily, we first give some characterizations for H-2 atoms. Below is a slightly modified version of Lemma 3.1 with a proof. It is also for the self-containing purpose.

Lemma 3.3 *Let a nonzero function $s(z)$ be holomorphic on the closed disc $cl\{\mathbf{D}\}$ and $|s(e^{it})| > 0$. Then its boundary function $s(e^{it})$ is an H-2 atom if and only if one of the following conditions holds:*

(1) *$s(z_0) = 0$ for some $z_0 \in \mathbf{D}$, and $s(\mathbf{D})$ is a starlike domain bounded by a rectifiable Jordan curve;*
(2) *$s(z) = (z - z_0)(\frac{1}{z} - \overline{z_0})h(z) \in \mathcal{S}_w(1)$, where $h(z) \in \mathcal{S}^*$ is holomorphic on $cl\{\mathbf{D}\}$;*
(3) *$s(z) \in \mathcal{S}(1)$.*

Proof If $s(e^{it}) = \rho(t)e^{i\theta(t)}$ is an H-2-atom, we have $\theta'(t) \geq 0$ and $\int_0^{2\pi} \theta'(t)dt = 2\pi$. Since $s(z)$ is holomorphic in $cl\{\mathbf{D}\}$ and $|s(e^{it})| > 0$, by the argument principle and the nondecreasing property of $\theta(t)$, we know that $s(z)$ has only one zero in $\mathbf{D}$ and $s(\mathbf{D})$ is a starlike domain bounded by a rectifiable Jordan curve. Without loss of generality, we assume that $s(z_0) = 0$ for some $z_0 \in \mathbf{D}$. This proves that H-2-atom implies (1).

Assuming (1), we show that (2) holds. Let $h(z) = z\frac{s(z)}{(z-z_0)(1-\overline{z_0}z)}$. Since $s(z)$ is holomorphic on $cl\{\mathbf{D}\}$ and contains only one zero at $z = z_0$ in $cl\{\mathbf{D}\}$. Then $h(z)$ is holomorphic in $cl\{\mathbf{D}\}$, containing only one zero at $z = 0$ in $cl\{\mathbf{D}\}$. By

$$\frac{\partial}{\partial t}[\arg(h(e^{it}))] = \frac{\partial}{\partial t}[\arg(s(e^{it}))] \geq 0$$

and the argument principle, we know that $h(\mathbf{D})$ is a starlike domain bounded by a rectifiable Jordan curve $h(\mathbb{T})$. To prove that $h(z)$ is univalent in $\mathbf{D}$, we consider

$$\chi(h(z), c) = \frac{1}{2\pi i}\oint_{|z|=1} \frac{h'(z)dz}{h(z)-c} = \frac{1}{2\pi i}\oint_J \frac{d\omega}{\omega - c}.$$

Since $h(z)$ is holomorphic in $cl\{\mathbf{D}\}$, by the argument principle, $\chi(h(z), c)$ is identical with the number of times that $h(z)$ takes the value c in the open disk. By the residual theorem, we have $\chi(h(z), c) = 0$ if $c \neq h(\mathbf{D})$ and $\chi(h(z), c) = \pm 1$ if $c \in h(\mathbf{D})$. This follows that $h(z)$ is a univalent holomorphic function in $cl\{\mathbf{D}\}$, and $h(\mathbf{D})$ is a starlike domain bounded by a rectifiable Jordan curve. Hence, we have $h(z) \in \mathcal{S}^*$ is holomorphic on $cl\{\mathbf{D}\}$. This proves that (1) implies (2).

Finally, we prove that (2) implies that $s(e^{it}) := \rho(t)e^{i\theta(t)}$ is an H-2 atom. If $s(z) = \frac{(z-z_0)(1-\overline{z_0}z)}{z}h(z)$, where $h(z) \in \mathcal{S}^*$ is holomorphic on $cl\{\mathbf{D}\}$, then $s(z)$ is holomorphic and has only one zero at $z = z_0$ in $cl\{\mathbf{D}\}$. Since $s(z) \in H^2(\mathbf{D})$, we have $H[\rho(t)\cos\theta(t)] = \rho(t)\sin\theta(t)$. By the property of starlike functions $h(z)$ given in (3.1), for all $t \in [0, 2\pi]$, we have

$$\frac{d}{dt}[\arg(s(e^{it}))] = \frac{d}{dt}[\arg(h(e^{it}))] = \mathrm{Re}\left\{e^{it}\frac{h'(e^{it})}{h(e^{it})}\right\} \geq 0.$$

Furthermore, by the argument principle, we obtain $\int_0^{2\pi}\theta'(t)dt = 2\pi$. This shows that $s(e^{it})$ is an H-2 atom.

(3) is a direct result of Lemma 3.2. The proof is completed. □

The next theorem, as a main result of this paper, generalizes the above results by involving boundary starlike functions. In fact, the two types of starlike functions, viz., those with zeros inside the unit disc and on the unit circle, are unified together with the concept H-n atoms.

Theorem 3.4 *Let $s(z)$ be holomorphic on the closed unit disc $cl\{\mathbf{D}\}$ and have a number of p zeros inside the unit disc $\mathbf{D}$. Then its boundary value function $s(e^{it}) = \rho(t)e^{i\theta(t)}$ is an H-n-atom ($n \geq 1$) if and only if*

$$\begin{aligned} s^2(z) &= \left[\prod_{i=1}^{p} h_i(z)\right]^2 \prod_{j=1}^{n-2p} g_j^2(z) \\ &= \left[\prod_{k=1}^{p}(z-a_k)\left(\frac{1}{z}-\overline{a_k}\right)\right]^2\left[\prod_{k=1}^{n-2p}(z-b_k)\left(\frac{1}{z}-\overline{b_k}\right)\right][h(z)]^n, \end{aligned} \tag{3.2}$$

where $\{a_k\}_{k=1}^{p}$ are zeros of $s(z)$ in $\mathbf{D}$, $\{b_k\}_{k=1}^{n-2p}$ are zeros of $s(z)$ on the unit circle $\mathbb{T}$ (some maybe multiple), $h(z) \in \mathcal{S}^*$, $h_i(z) \in \mathcal{S}_w(1)$ and $g_j(b_j z) \in \mathcal{G}^*$, all being holomorphic on $cl\{\mathbf{D}\}$ for $i = 1, \ldots, p$ and $j = 1, \ldots, n-2p$.

Proof Assume that $s(z)$ has an expression given by (3.2). Since on the unit circle the phases of

$$(z-a_k)\left(\frac{1}{z}-\overline{a_k}\right) \quad \text{and} \quad (z-b_k)\left(\frac{1}{z}-\overline{b_k}\right)$$

are all zero, we have

$$2 \arg s(e^{it}) = n \arg h(e^{it}). \tag{3.3}$$

If $h(z)$ is a starlike function in $\mathcal{S}^*$ and holomorphic in $cl\{\mathbf{D}\}$, by Lemma 3.3, $h(e^{it})$ is an H-2 atom, that is $\int_0^{2\pi} \frac{d}{dt}[\arg h(e^{it})]dt = 2\pi$ and $\frac{d}{dt}[\arg h(e^{it})] \geq 0$. From (3.3), it follows that $s(e^{it})$ is an H-n atom.

We now show the converse result. Since $s(z)$ is holomorphic in $cl\{\mathbf{D}\}$, $s(z)$ only has a finite number of isolated zeros in the closed unit disc $cl\{\mathbf{D}\}$. Assuming that $s(e^{it})$ is an H-n atom, by the argument principle, $s(z)$ has exactly p zeros in $\mathbf{D}$ and $n-2p$ zeros on $\mathbb{T}$. We denote by $\{a_1, a_2, \ldots, a_p\}$ the zeros of $s(z)$ in $\mathbf{D}$ and $\{b_1, b_2, \ldots, b_{n-2p}\}$ the zeros of $s(z)$ on $\mathbb{T}$. Let

$$g(z) = \frac{s^2(z)}{\prod_{k=1}^{p} \left[(z-a_j)(1-\overline{a_j}z)\right]^2 \prod_{k=1}^{n-2p} (z-b_j)(1-\overline{b_j}z)}.$$

Note that $g(z)$ is holomorphic and nonzero in $cl\{\mathbf{D}\}$, so is $[g(z)]^{1/n}$. Set $h(z) = z[g(z)]^{1/n}$. Then

$$s^2(z) = [h(z)]^n \prod_{k=1}^{p} \left[\frac{(z-a_j)(1-\overline{a_j}z)}{z}\right]^2 \prod_{k=1}^{n-2p} \frac{(z-b_k)(1-\overline{b_k}z)}{z},$$

where $h(z)$ is holomorphic and contains only one zero at $z = 0$ in $cl\{\mathbf{D}\}$. Moreover, by (3.3) and Lemma 3.3, we have $h(z) \in \mathcal{S}^*$ is holomorphic in $cl\{\mathbf{D}\}$.

Let $h_i(z) = (z-a_i)(\frac{1}{z}-\overline{a_i})h(z)$ for $i = 1, \ldots, p$ and $g_j^2(z) = (z-b_j)(\frac{1}{z}-\overline{b_j})h(z)$ for $j = 1, \ldots, n-2p$. Then $s(z)$ can be further written as

$$s^2(z) = \left[\prod_{i=1}^{p} h_i(z)\right]^2 \prod_{j=1}^{n-2p} g_j^2(z).$$

By the definition of $\mathcal{S}_w(1)$ and $\mathcal{G}^*$, we know that $h_i(z) \in \mathcal{S}_w(1)$, $i = 1, \ldots, p$, and $g_j(b_j z) \in \mathcal{G}^*$, $j = 1, \ldots, n-2p$, are holomorphic on $cl\{\mathbf{D}\}$. The proof is completed. □

The cases that $s(z)$ has no zero in $\mathbf{D}$ or no zero on $\mathbb{T}$ correspond to the following two corollaries.

Corollary 3.5 *Let $s(z)$ be holomorphic on the closed unit disc $cl\{\mathbf{D}\}$ and $s(z) \neq 0$ for $z \in \mathbb{T}$. Then the boundary value function $s(e^{it}) = \rho(t)e^{i\theta(t)}$ is an H-2n atom if and only if $s(z) \in \mathcal{S}(n)$ or*

$$s(z) = \prod_{k=1}^{n} h_k(z) = h^n(z) \prod_{k=1}^{n} (z-a_k)\left(\frac{1}{z} - \overline{a_k}\right) \in \mathcal{S}_w(n),$$

where $\{a_1, a_2, \ldots, a_n\}$ *are the zeros of* $s(z)$ *in the unit disc* $\mathbf{D}$*,* $h_k(z) \in \mathcal{S}_w(1)$ *and* $h(z) \in \mathcal{S}^*$ *are holomorphic on* $cl\{\mathbf{D}\}$ *for* $k = 1, \ldots, n$*.*

Corollary 3.5 is obtained in, and as a main result of [7].

Corollary 3.6 *Let* $s(z)$ *be holomorphic in the closed unit disc* $cl\{\mathbf{D}\}$ *and* $s(z) \neq 0$ *for* $z \in \mathbf{D}$*. Then the boundary value function* $s(e^{it}) = \rho(t)e^{i\theta(t)}$ *is an* H*-m atom if and only if*

$$s^2(z) = \prod_{k=1}^{m} g_k^2(z) = h^m(z) \prod_{k=1}^{m} (z - b_k) \left(\frac{1}{z} - \overline{b_k} \right),$$

where $\{b_1, \ldots, b_m\}$ *are the zeros of* $s(z)$ *on the unit circle* $\mathbb{T}$*,* $g_k(b_k z) \in \mathcal{G}^*$ *and* $h(z) \in \mathcal{S}^*$ *are holomorphic on* $cl\{\mathbf{D}\}$ *for all* $k = 1, \ldots, m$*.*

Remark If $s(z)$ is holomorphic in the closed disc $cl\{\mathbf{D}\}$ and $s(z) \neq 0$ for $z \in \mathbf{D}$. Then $s(e^{it})$ is an H-1 atom if and only if $s(e^{it_k} z) \in \mathcal{G}^*$ for some $t_k \in [0, 2\pi)$. Corollary 3.6 reveals the identical relation between the class of mono-components as H-1 atoms and the corresponding class of starlike functions with respect to the boundary points.

Finally we provide some conditions that guarantee a.e. existence of the phase derivative $\theta'(t)$ defined by (1.1).

Theorem 3.7 *Let nonzero functions* $s(z) \in H^1(\mathbf{D})$ *and* $s'(z) \in H^1(\mathbf{D})$*. Then* $\theta'(t)$ *is a well defined measurable function by* (1.1) *and* $\theta'(t) = \mathrm{Re}\left[\frac{e^{it}s'(e^{it})}{s(e^{it})}\right]$ *exists for a.e.* $t \in [0, 2\pi]$*.*

Proof If $s(z) \in H^1(\mathbf{D})$ and $s'(z) \in H^1(\mathbf{D})$, then the non-tangential boundary limits $s(e^{it})$ and $s'(e^{it})$ exist almost everywhere. Since $s(z), s'(z) \in H^1(\mathbf{D})$ are nonzero functions, it follows that $s(e^{it})$ and $s'(e^{it})$ are a.e. non-zero on the unit circle $\mathbb{T} = \{z||z| = 1\}$. Hence, the non-tangential limit

$$\theta'(t) = \lim_{r \to 1^-} \mathrm{Re}\left[\frac{re^{it}s'(re^{it})}{s(re^{it})}\right] = \mathrm{Re}\left[\frac{e^{it}s'(e^{it})}{s(e^{it})}\right]$$

exists for almost all $t \in [0, 2\pi]$. □

In [17] it is shown that $s(z) \in H^1(\mathbf{D})$ and $s'(z) \in H^1(\mathbf{D})$ if and only if $s(z)$ is holomorphic in $|z| < 1$ and absolutely continuous on $\mathbb{T}$. These conditions do not imply that $s(z)$ has finitely many zeros on $cl\{\mathbf{D}\}$ as assumed in our main Theorem 3.4 concerning H-n atoms. Generally, $s(z)$ may have infinitely many zeros on $cl\{\mathbf{D}\}$. It is an open question on how to extend Theorem 3.4 to the infinitely many zeros case.

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