Dear Author,

Please check your proof carefully and mark all corrections at the appropriate place in the proof (e.g., by using on-screen annotation in the PDF file) or compile them in a separate list. Note: if you opt to annotate the file with software other than Adobe Reader then please also highlight the appropriate place in the PDF file. To ensure fast publication of your paper please return your corrections within 48 hours.

For correction or revision of any artwork, please consult [http://www.elsevier.com/artworkinstructions](http://www.elsevier.com/artworkinstructions).

Any queries or remarks that have arisen during the processing of your manuscript are listed below and highlighted by flags in the proof. Click on the 'Q' link to go to the location in the proof.

<table>
<thead>
<tr>
<th>Location in article</th>
<th>Query / Remark: click on the Q link to go</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q1</td>
<td>Your article is registered as a regular item and is being processed for inclusion in a regular issue of the journal. If this is NOT correct and your article belongs to a Special Issue/Collection please contact <a href="mailto:p.muthukumar@elsevier.com">p.muthukumar@elsevier.com</a> immediately prior to returning your corrections.</td>
</tr>
<tr>
<td>Q2</td>
<td>Please confirm that given name(s) and surname(s) have been identified correctly.</td>
</tr>
<tr>
<td>Q3</td>
<td>Please check whether the designated corresponding author is correct, and amend if necessary.</td>
</tr>
<tr>
<td>Q4</td>
<td>Please note that “Corollary 4.5” is cited in the text but not provided.</td>
</tr>
</tbody>
</table>

Thank you for your assistance.
1. Introduction

There are two mathematical analysis problems that have long been interested in physical practice and signal processing. The first is: under what conditions on \( f \) and \( g \) does the equality \( H(fg) = fHg \) hold, where \( H \) is the Hilbert transformation defined by

\[
H(g)(x) := \frac{1}{\pi} \lim_{\epsilon \to 0} \int_{|x-t|>\epsilon} \frac{g(t)}{x-t} \, dt.
\]

The second is phrased as a band preserving problem: assume that \( g \in L^2(\mathbb{R}) \) is bandlimited, and has its band \( [A, B] \), viz., \( \text{supp} \hat{g} \subseteq [A, B] \). Then, under what conditions on \( f \) does the relation \( \text{supp} \hat{fg} \subseteq [A, B] \) hold? Here the notation \( \hat{g} \) stands for the Fourier transform of \( g \), defined by

\[
\hat{g}(\omega) = \int_{-\infty}^{\infty} g(x) e^{-i\omega x} \, dx.
\]

and \( \text{supp} \hat{g} \) is the support of \( \hat{g} \). The first problem amounts to generalize the classical results of Bedrosian given in [1]: For \( f, g \in L^2(\mathbb{R}) \), if \( \text{supp} f \subseteq [-A, A] \) and \( \text{supp} \hat{g} \subseteq \mathbb{R} \setminus [-A, A] \) for a positive number \( A \); or, alternatively, if \( \text{supp} f \subseteq [0, \infty) \) and \( \text{supp} \hat{g} \subseteq [0, \infty) \), then \( H(fg) = fHg \). The recent interest on the Bedrosian equation was motivated by the study of mono-components that, by the definition, are analytic signals with non-negative phase derivatives, the latter being defined as instantaneous frequencies [2–4]. Some interesting and useful results for the Bedrosian equation are established in [5–11]. About the second problem, if \( g \in L^2(\mathbb{R}) \) has its band \( [A, B] \), by the Paley–Wiener theorem, it is known that \( fg \in L^2(\mathbb{R}) \) has the same band as that of \( g \) if and only if \( f \) is a quotient of two entire functions of the exponential type. A number of authors devoted to
In the paper, we will solve them by the method of shift invariant subspaces. An interesting thing will be found that the Fourier transform given in the distributional sense coincides with the traditional definition of Fourier transformation.

Let $\hat{L}_p$ be the set of all analytic signals, and $\hat{L}_p$ for $1 \leq p < \infty$ is a constant of the form given in (1.1) as $\hat{L}_p = \hat{L}_p + A_p$, where $A_p$ is a constant of the form $\int_{-\infty}^{\infty} \frac{g(t)}{t+i} dt$. (2.3)

The Fourier transform given in (1.2) is well-defined for functions in $L^p(\mathbb{R})$ for $1 \leq p < 2$. But for $p > 2$, it can be shown that there are functions in $L^p(\mathbb{R})$ such that the Fourier transform given in (1.2) is not a well-defined function [16]. In order to generalize the study of band preserving problem from $L^p(\mathbb{R})$ spaces to $L^p(\mathbb{R})$ spaces, $1 \leq p < \infty$, the tempered distribution should be introduced. The tempered distribution as Fourier transform is defined through the Schwartz class. (2.4)

The Fourier transform given in the distributional sense coincides with the traditional definition of Fourier transformation given in (1.2) for $1 \leq p \leq 2$. The support of a distribution $T$ is defined as the complement of the largest open set on which $T$ vanishes, where $T$ is said to vanish on $U$ if $(T, \phi) = 0$ for any test function $\phi$ whose support is contained in $U$.

Let $L^p_\theta(\mathbb{R}) := \{f \in L^p(\mathbb{R}) | Tf \in L^p(\mathbb{R})\}$ for $1 \leq p \leq \infty$, where $T$ is the Hilbert transform interpreted as $Tf = \begin{cases} \hat{H}_f, & 1 \leq p < \infty; \\ \hat{H}_f, & p = \infty. \end{cases}$ (2.5)

Obviously, when $1 < p < \infty$, the space $L^p_\theta(\mathbb{R})$ coincides with $L^p(\mathbb{R})$. For $p = 1$ and $p = \infty$, the spaces $L^p_\theta(\mathbb{R})$ are respectively proper subspaces of $L^p(\mathbb{R})$. For any function $f(x) \in L^p_\theta(\mathbb{R})$, we have the Hardy space decomposition

$$f(x) = \frac{1}{2} [f_+(x) + f_-(x)] = \frac{1}{2} ([f + i\mathcal{F}] + [f - i\mathcal{F}]).$$ (2.6)

The signals $f_+ := f + i\mathcal{F}$ and $f_- := f - i\mathcal{F}$ are respectively called the analytic signal and the conjugate analytic signal associated with $f$. Let

$${\mathcal{H}}^p(\mathbb{R}) := \{f : f = u + i\mathcal{H}(u) \text{ for certain function } u \in L^p_\theta(\mathbb{R})\}$$

be the set of all analytic signals, and

$${\mathcal{H}}^p(\mathbb{R}) := \{f : f = u - i\mathcal{H}(u) \text{ for certain function } u \in L^p_\theta(\mathbb{R})\}$$

the set of all conjugate analytic signals. It is shown that the space $H^p(\mathbb{R})$ is identical to the Banach subspace of $L^p(\mathbb{R})$ of the functions of non-negative spectrum [17], that is, $H^p(\mathbb{R}) = H^p(\mathbb{R}^+) := \{f \in L^p(\mathbb{R}) | \text{supp} \ f \subseteq [0, \infty)\}$, where $\mathbb{R}^+ := [0, \infty)$ and the support of $f$ is defined in the distributional sense. Alternately, the equivalent operation of (2.6) in the frequency domain is
Moreover, there is a one-to-one correspondence between analytic signals in $H^p(\mathbb{R})$ and functions in the Hardy space $H^p(\mathbb{C}^+)$ [18], where $H^p(\mathbb{C}^+)$ is the class of all holomorphic functions $F(z)$ on the upper half plane $\mathbb{C}^+ := \{z = x + iy, y > 0\}$ satisfying
\[
\sup_{z \in \mathbb{C}^+} |F(z)| < \infty, \quad \text{for } p = \infty.
\]

For functions $f \in H^p(\mathbb{R})$, we have [17]:

**Lemma 2.1.** Let $1 \leq p \leq \infty$. Then $f \in H^p(\mathbb{R})$ if and only if one of the following conditions holds:

1. $T(f) = -if$.
2. $f \in H^p(\mathbb{R}^+)$.
3. $f$ is the non-tangential boundary value of a function $F(z) \in H^p(\mathbb{C}^+)$, where, for $z \in \mathbb{C}^+$, $F(z)$ can be represented by

\[
F(z) = \begin{cases} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{f(t)}{t-z} dt, & 1 \leq p < \infty; \\ \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\ln |F(t)|}{t-z} dt, & p = \infty. \end{cases}
\]

In what follows, let $1 \leq p, q, r, \infty$ such that $r^{-1} = p^{-1} + q^{-1}$. By the Hölder inequality, it is easy to verify that the product of two analytic signals and the product of two conjugate analytic signals is a conjugate analytic signal, as stated below.

**Lemma 2.2.** If $f \in H^p(\mathbb{R})$ and $g \in H^q(\mathbb{R})$ are two nonzero analytic signals, then $fg \in H^r(\mathbb{R})$. If $f \in H^p(\mathbb{R})$ and $g \in H^q(\mathbb{R})$ are nonzero conjugate analytic signals, then $fg \in H^r(\mathbb{R})$.

Naturally, we would encounter the question in what circumstance the product of an analytic signal and a conjugate analytic signal is still analytic or conjugate analytic. The study of this problem is on its own interest, as well as of significant applications. Its solutions are related to characterizations of the solutions of the Bedrosian equation and of the band preserving problem. It is related to the well-known Nevanlinna decomposition theorem for $F \in H^p(\mathbb{C}^+)$ [19]. Any function $F(z) \in H^p(\mathbb{C}^+)$ can be decomposed as $F(z) = O_t(z)I_t(z)$, $z \in \mathbb{C}^+$, where $O_t(z)$, the outer factor of $F$, is of the form

\[
O_t(z) = \exp \left\{ \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{1 + tz \ln |F(t)|}{t-z} dt \right\},
\]

and $I_t(z)$ is the inner factor of $F$, that is, $I_t(z) \in H^p(\mathbb{C}^+)$ and $|I_t(x)| = 1$ for almost all $x \in \mathbb{R}$. Factorizing $I_t(z)$ canonically, we get

\[
I_t(z) = e^{iax-b}B_t(z)S_t(z)
\]

for $z \in \mathbb{C}^+$, where $a$ is a non-negative constant, $b$ is a real constant, $B_t(z)$ is a Blaschke product and $S_t(z)$ is a singular inner function. Let $E$ be the set of all different zeros of $F(z)$ on the upper half plane and $m(\lambda)$ the multiplicity of $\lambda$ at $F(\lambda) = 0$. Then the Blaschke product $B_t$ is of the form

\[
B_t(z) = B_{t,m}(z) = \prod_{\lambda \in E} \left( \frac{|\lambda|^2 + 1}{\lambda^2 + 1} : \frac{z - \lambda}{\lambda} \right)^{m(\lambda)}.
\]

where the pair $(E, m)$ satisfies the Blaschke condition

\[
\sum_{\lambda \in E} m(\lambda)|\lambda|^2 < \infty.
\]

In (2.8) we are with the convention that $(\lambda^2 + 1)/(\lambda^2 + 1)$ takes value 1 when $\lambda = i$. The singular inner function $S_t(z)$ is of the form

\[
S_t(z) := \exp \left\{ \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{1 + tz}{t-z} dt \right\},
\]

where $\mu(t)$ is a real, bounded, increasing function with derivative $\mu'(t) = 0$ almost everywhere on $\mathbb{R}$. A fundamental property of Hardy space functions is that they have non-tangential boundary limits almost everywhere. Hence, for almost all $x \in \mathbb{R}$, the corresponding decomposition of nonzero function $f \in H^p(\mathbb{R})$ is

\[
f(x) = \lim_{y \to 0} F(x + iy) = O_t(x)I_t(x) = e^{iax-b}O_t(x)B_t(x)S_t(x).
\]
Note that, apart from the self-explanatory notation $F(x)$ as non-tangential boundary value of $F(z)$, we also denote the non-tangential boundary value function of $F \in H^p(C^+)$ by $\partial F$, and, for a function $f \in H^p(\mathbb{R})$, denote the corresponding function in $H^p(C^+)$ by $\partial^* f$.

**Theorem 2.3.** Let $f \in H^p(\mathbb{R})$ and $g \in H^p(\mathbb{R})$ be non-zero functions. Then the following statements are equivalent:

1. $fg \in H^p(\mathbb{R})$.
2. $f \in H^p(\mathbb{R}) \cap H^p_f(\mathbb{R})$, where $I_f$ is the inner function of $g$.
3. $\frac{\partial}{\partial \tau} I_f = \frac{\partial}{\partial \tau} g$ and $g \in \varphi_1 H^p(\mathbb{R})$, where $\varphi_1 = B_1 S_1$ and $\varphi_2 = B_2 S_2$ are two co-prime inner functions, that is, $B_1(z)$ and $B_2(z)$ have no common zeros, the singular measures $\mu_1$ and $\mu_2$ of $S_1$ and $S_2$ have no common minorant.

**Proof.** We first assume (1) holds, that is, we have nonzero functions $f \in H^p(\mathbb{R}), g \in H^p(\mathbb{R})$ and $fg \in H^p(\mathbb{R})$. Then $\partial^*(fg)(z) \in H^p(C^+)$ and the outer function of $\partial^*(fg)(z)$ is represented by

$$O_{\partial^*(fg)}(z) = \exp \left\{ \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{1 + 2t}{(z - t)(1 + t^2)} \ln |f(t)g(t)| dt \right\}.$$  

By $\ln |fg| = \ln |g| + \ln |f|$, we have $O_{\partial^*(fg)} = O_{\partial^*g} O_{\partial^*f}$. Hence, $O_{\partial^*g} = O_{\partial^*f}$. By the factorization theorem, we therefore obtain

$$O_{\partial^* f} I_f g = O_{\partial^* f} g I_f g = O_{\partial^* h} I_f g = g h O_{\partial^* h} I_f.$$  

From equation (2.10) and the assumptions $\mu_1 = 0$ and $f \neq 0$, we have

$$f = I_f O_{\partial^* h} g \in H^p(\mathbb{R}).$$

Hence, $f \in H^p(\mathbb{R}) \cap H^p_f(\mathbb{R})$, proving that (1) implies (2). The relation (2.10) also shows

$$\frac{O_{\partial^* f} I_f}{O_{\partial^* f}} = I_f \in H^\infty(\mathbb{R}).$$

That further implies

$$I_f \frac{O_{\partial^* f}}{O_{\partial^* f}} = I_f \in H^\infty(\mathbb{R}).$$

The relation (2.11) shows that $O_{\partial^* f} I_f$ is the quotient of two inner functions $I_f$ and $I_g$. Eliminating the greatest common divisor of the two inner functions $I_f$ and $I_g$, there exist two co-prime inner functions $\varphi_1 = B_1 S_1$ and $\varphi_2 = B_2 S_2$ such that

$$\frac{O_{\partial^* f} I_f}{O_{\partial^* f}} = \frac{\varphi_1}{\varphi_2}.$$  

(2.13)

From the fact that

$$I_f \frac{\varphi_2}{\varphi_1} = I_f \frac{O_{\partial^* f}}{O_{\partial^* f}} \in H^\infty(\mathbb{R}),$$

we obtain that $I_f$ must have a non-trivial divisor $\varphi_1$. Hence, we have $g = O_g I_g \in \varphi_1 H^p(\mathbb{R})$ and $O_{\partial^* f} I_f = \frac{\varphi_1}{\varphi_2}$, where $\varphi_1$ and $\varphi_2$ are two co-prime inner functions. This shows that (1) implies (3).

Now we show that (2) implies (1). If $f \in H^p(\mathbb{R}) \cap H^p_f(\mathbb{R})$, then $f = I_f h$ for some $h \in H^p(\mathbb{R})$. Since $g \in H^p(\mathbb{R})$, we have $g = O_g I_g \in H^p(\mathbb{R})$ and $O_g \in H^p(\mathbb{R})$. Thus $fg = I_f h h O_g I_g = h O_g$. By the Hölder inequality, it follows that $fg \in H^p(\mathbb{R})$.

Now we show that (3) implies (1). If $g \in \varphi_1 H^p(\mathbb{R})$ and $O_{\partial^* f} I_f = \frac{\varphi_1}{\varphi_2}$, then there exists $h \in H^p(\mathbb{R})$ such that $g = \varphi_1 h \in H^p(\mathbb{R})$ and $\varphi_1 = \varphi_2 O_{\partial^* f} I_f$. Since $f \in H^p(\mathbb{R})$, we have $f = I_f = O_{\partial^* f} I_f$. Therefore

$$fg = \varphi_1 h f = h \varphi_2 O_{\partial^* f} I_f = h \varphi_2 O_{\partial^* f}.$$  

(2.14)

By the Hölder inequality, we conclude that $fg \in H^p(\mathbb{R})$. The proof is complete. □

The equivalence relation between (1) and (2) was first proved in [11]. Here we cite the proof again for the self-containing purpose, as well as for the convenience of proving their equivalence with (3). Symmetrically, or as a corollary of Theorem 2.3, we have an analogous result for the conjugate analytic case.
Corollary 2.4. Let \( f \in H^p(R) \) and \( g \in H^q(R) \) be non-zero functions. Then the following statements are equivalent:

1. \( fg \in H^p(R) \),
2. \( g \in H^p(R) \cap \overline{I^p H^q(R)} \), where \( I^p \) is the inner function of \( f \).
3. \( \sum \frac{1}{\varphi_1} I_x = \overline{\frac{1}{\varphi_2}} \) and \( f, g \in \varphi_1 H^p(R) \), where \( \varphi_1 = B_1 S_1 \) and \( \varphi_2 = B_2 S_2 \) are two co-prime inner functions.

For a given nonzero function \( g \in H^p(R) \), we associate it with the spaces

\[
X_g^p := \{ f : f \in H^p(R) \text{ and } fg \in H^p(R) \},
\]

and

\[
Y_g^p := \{ f : f \in H^p(R) \text{ and } fg \in H^p(R) \}.
\]

We note that the induced spaces by \( g, X_g^p \) and \( Y_g^p \), consist of Hardy space functions and hence are subspaces of \( H^p(R) \). By Corollary 2.4, we learn that if \( \sum \frac{1}{\varphi_1} I_x \) is not a quotient of two co-prime inner functions (for example, if \( O_1(z) \) is an outer function with essential singularities in the lower half plane), then \( X_g^p \) is an empty set. If \( \sum \frac{1}{\varphi_1} I_x \) is the quotient of two co-prime inner functions \( \varphi_1 \) and \( \varphi_2 \), then \( X_g^p \) is \( \varphi_1 H^p(R) \). The space \( \varphi_1 H^p(R) \), usually referred as a forward shift invariant subspace, is invariant under the semigroup \( \{ S_a : a > 0 \} \) of forward shift operators

\[
(S_u h)(t) = e^{iut} h(t), \quad h \in H^p(R).
\]

By Theorem 2.3, the space \( Y_g^p \) is \( H^p(R) \cap \overline{I^p H^q(R)} \), where \( I_g \) is the inner function of \( g \). When \( g \in H^p(R) \) is an outer function, that is, \( I_g = e^{i\theta}, b \in \mathbb{R} \) being trivial, the space \( Y_g^p \) is an empty set for \( 1 \leq p < \infty \). When \( g \in H^p(R) \) is not an outer function, that is, \( I_g \) is not a constant, then the space \( Y_g^p \) is not empty, and is an invariant subspace of \( H^p(R) \) under the semigroup \( \{ S_a : a > 0 \} \) of backward shift operators

\[
(S_u^* h)(t) = e^{-iut} h(t), \quad h \in H^p(R).
\]

For \( p^{-1} + p^{-1} = 1 \), the backward shift operator \( S_a^* \) is the adjoint of the forward shift operator \( S_a \) under the conjugate pairing between \( H^p(R) \) and \( \overline{H^p(R)} \),

\[
\langle S_a^* f, g \rangle = \langle f, S_a g \rangle, \quad f \in H^p(R), \quad g \in \overline{H^p(R)}.
\]

The study of forward and backward shift invariant subspaces has a long history, some useful and interesting results can be found in [20–24]. Both forward shift invariant and backward shift invariant subspaces are expressible in terms of inner functions. The most classical characterization of forward shift invariant subspaces is referred to Beurling–Lax Theorem [20]. When the related inner function is just a Blaschke product, the corresponding backward shift invariant subspace can be explicitly characterized (also see [22]).

Lemma 2.5. Let \( 1 \leq p \leq \infty \), and \( B_x \) be the Blaschke product given in (2.8). Then \( H^p(R) \cap \overline{B_x H^p(R)} \) is the closure in \( L^p(R) \) of the sets of all rational functions of which each has its poles as a subset of those of \( B_x \) with multiplicities not exceeding those of the corresponding poles of \( B_x \).

By imposing the restriction that \( \hat{g} \) be compactly supported, that is, \( g \) belongs to bandlimited Hardy space \( FH^p[A, B] \),

\[
FH^p[A, B] := \{ g \in L^q(R) : \text{supp} \hat{g} \subseteq [A, B] \},
\]

we will first give a characterization for the forward shift invariant subspace \( X_g^p \) described in Corollary 2.4 induced by \( g \in FH^p[A, B] \) for \( 0 \leq A < B \).

Corollary 2.6. Let \( g \in FH^p[A, B] \) be a nonzero function. If \( B \in \text{supp} \hat{g} \) and \( 0 \leq A < B \). Then the forward invariant subspace \( X_g^p \) is \( e^{i\theta} H^p(R) \).

Proof. Let \( h(x) := \frac{\hat{g}(x)}{\hat{f}(x)} \). Then \( \hat{h}(\omega) = \frac{\hat{g}(B - \omega)}{\hat{f}(\omega)} \). Since \( g \in FH^p[A, B] \) is a nonzero function with \( B \in \text{supp} \hat{g} \), we have \( h \in FH^p[0, B - A] \subset H^p(R) \) with \( 0 \in \text{supp} \hat{h} \). By the factorization theorem, we have \( O_h = O_g \) and

\[
\frac{O_h(x)}{O_g(x)} = \frac{\hat{g}(x)}{\hat{f}(x)} = \frac{\hat{g}(x)}{\hat{f}(x)} = \frac{e^{i\theta} O_f(x)}{O_h(x) I_h(x)} = \frac{e^{i\theta}}{I_h}.
\]

Since \( 0 \in \text{supp} \hat{h} \), we obtain that \( e^{i\theta} \hat{h} \) and \( I_h \) are two co-prime inner functions. Hence, \( \varphi_1(x) = e^{i\theta} \) and \( X_g^p = e^{i\theta} H^p(R) \).

Let \( g \in FH^p[A, B] \) be a nonzero function, where \( 0 \leq A < B \). By the Paley–Wiener–Schwarz theorem [25], we learn that \( g \in FH^p[A, B] \) if and only if \( g \) is the non-tangential boundary limit of an entire function \( G(z) \) of exponential type (not exceeding \( B \)) that belongs to \( H^p(C^+) \), with the representation

\[ G(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{iz(x-z)} - e^{iz(x-x)}}{z-x} \, dx. \]  

(2.17)

The following lemma is to show that the singular inner factor of \( g \in FH^p[A,B] \) is a constant in such a case.

**Lemma 2.7.** Let \( g(x) \in FH^p[A,B] \) be a nonzero function, where \( 0 \leq A < B \). Then the inner function of \( g \) is \( e^{i(ax+b)}B_g(x) \), where \( 0 \leq a \leq B, b \) is a real constant, \( B_g(x) \) is the boundary Blaschke product formed by the zeros of \( G(z) := (\partial^{-1}g)(z) \) defined by (2.17) in the upper half plane and \( |B_g(x)| = 1 \) on the whole \( \mathbb{R} \).

**Proof.** The following proof is an adaptation of [19, Chapter II, Theorem 6.3]. Since the nonzero function \( g \in FH^p[A,B] \subset H^p(\mathbb{R}) \) for \( 0 \leq A < B \), \( G(z) := (\partial^{-1}g) \in H^p(\mathbb{C}^+) \) is an entire function of exponential type and \( G(z) = e^{i(ax+b)}O_c(z)B_c(z)S_c(z) \) for \( z \in \mathbb{C}^+ \), where \( 0 \leq a \leq B, b \) is a real constant, \( O_c(z) \) is the outer function of \( G(z) \), \( B_c(z) \) is the Blaschke product of \( G(z) \) and \( S_c(z) \) is the singular inner function of \( G(z) \). In what follows, we will show that \( B_c(z) \) is holomorphic across \( \mathbb{R} \) and \( S_c(z) \equiv 1 \).

If the zeros of \( B_c(z) \) had an accumulation point \( z_0 \in \mathbb{R} \), then the zeros of \( G(z) \) would have an accumulation point \( z_0 \in \mathbb{R} \) such that \( G(z_0) = 0 \). This is impossible since \( G(z) \) is non-trivial and holomorphic across \( \mathbb{R} \). Consequently, \( B_c(z) \) is holomorphic across \( \mathbb{R} \) and \( |B_c(x)| = 1 \) on \( \mathbb{R} \).

Below, we will show that \( S_c(z) \equiv 1 \). Let \( \mu \) be the measure determining \( S_c \). If \( \mu \) had a point charge at \( x \), then every derivative \( G^{(n)}(z) \) of \( G(z) \) satisfies \( \lim_{x \to x^+} G^{(n)}(x + iy) = 0 \). This is impossible since \( G(z) \) is analytic across \( \mathbb{R} \) and \( G(z) \equiv \mathbb{Q} \). Thus \( \mu(\{x\}) = 0 \) for all \( x \in \mathbb{R} \). Next suppose that \( \mu(\{x\}) > 0 \) for some compact subset \( K \) of \( \mathbb{R} \). Since \( \mu \) is singular, for \( \mu \)-almost all \( x \), we have \( \lim_{x \to x^+} |S_c(x + iy)| = 0 \), and hence \( \lim_{x \to x^+} |G(x + iy)| = 0 \) (Theorem 6.2, [19]). Therefore, the holomorphic function \( G(z) \) has zero values on infinitely many points of \( K \). The compactness of \( K \) implies that \( G \) has to be a zero function, being contradictory with the assumption that \( G(z) \) is nonzero. Therefore \( \mu(\{x\}) = 0 \). Hence, \( \mu = 0 \) and \( S_c(z) \equiv 1 \). This completes the proof. \( \Box \)

Denote by \( E \) the set of all different zeros of \( G(z) := (\partial^{-1}g)(z) \) in the upper half plane and denote by \( m(\lambda) \) the multiplicity of the zero \( \lambda \) of \( G(z) \). With Lemma 2.5, we have a characterization for the backward shift invariant subspace \( Y^p_g \) when \( g \) is a band-limited signal.

**Corollary 2.8.** Suppose that \( g(x) \) is a nonzero function in \( FH^p[A,B], 0 \leq A < B \). Then the backward shift invariant subspace \( Y^p_g \) induced by \( g \) as given in (2.15) is identical with

\[ H^p(\mathbb{R}) \cap e^{i\alpha x}B_g H^p(\mathbb{R}) = F H^p[0,a] e^{i\alpha x} R_{E,m}, \]

where \( 0 \leq a < B \) and \( B_g \) is the Blaschke product formed by the points in \( E \) and the corresponding multiplicity function \( m(\lambda) \), and \( R_{E,m} \) is the closed subspace

\[ R_{E,m} = \text{span} \left\{ \frac{1}{(x-z)^j}, \lambda \in E, \ j = 1, \ldots, m(\lambda) \right\} \]

of the \( L^p(\mathbb{R}) \) space.

**Proof.** By Lemma 2.7, we get that \( Y^p_g = H^p(\mathbb{R}) \cap e^{i\alpha x}B_g H^p(\mathbb{R}) \), where \( 0 \leq a < B \). \( B_g \) is the Blaschke product formed by the zeros together with their multiplicities of \( G(z) := (\partial^{-1}g)(z) \) in the upper half plane.

Suppose that \( f(x) \in H^p(\mathbb{R}) \cap e^{i\alpha x}B_g H^p(\mathbb{R}) \). Then \( \tilde{f}(x) \in H^p(\mathbb{R}) \) and \( f(x) \in e^{i\alpha x}B_g(x) H^p(\mathbb{R}) \). Since \( \text{supp} \tilde{f} \subseteq [0,\infty) \), we have

\[ \tilde{f}(\alpha) = \tilde{f}(\alpha)X_{0,\alpha} + \tilde{f}(\alpha)X_{[0,\infty]} = \tilde{f}_1(\alpha) + \tilde{f}_2(\alpha), \]

where \( X_{[0,\infty]} \) is the characteristic function of \( [a,b) \). By the inverse Fourier transform in the distributional sense, \( f \) can be written as \( f = f_1 + f_2 \), where \( f_1 \in F H^p[0,a] \) and \( f_2 \in F H^p[a,\infty) \). Hence, we have

\[ f_1(x) e^{-i\alpha x} \in H^p(\mathbb{R}) \text{ and } f_2(x) e^{-i\alpha x} \in H^p(\mathbb{R}). \]

We note that

\[ f_1(x) e^{-i\alpha x} \in \text{span} \left\{ B_g(x) H^p(\mathbb{R}) \right\}, \]

and \( f_1(x) e^{-i\alpha x} \in B_g H^p(\mathbb{R}) \). Therefore, \( f_2(x) e^{-i\alpha x} = f(x) e^{-i\alpha x} - f_1(x) e^{-i\alpha x} \in B_g H^p(\mathbb{R}) \). Being combined with \( f_2(x) e^{-i\alpha x} \in H^p(\mathbb{R}) \), it yields that \( f_2(x) e^{-i\alpha x} \in H^p(\mathbb{R}) \cap B_g H^p(\mathbb{R}). \) By Lemma 2.5, it shows that \( f_2(x) \in e^{i\alpha x} R_{E,m} \).

Conversely, if \( f(x) \in F H^p[0,a] e^{i\alpha x} R_{E,m} \), then \( f(x) = f_1(x) + f_2(x) \), where \( f_1(x) \in F H^p[0,a] \) and \( f_2(x) \in e^{i\alpha x} R_{E,m} \). By Lemma 2.5, we obtain that \( f_2(x) \in e^{i\alpha x} H^p(\mathbb{R}) \cap B_g H^p(\mathbb{R}) \subset H^p(\mathbb{R}) \cap e^{i\alpha x} B_g(x) H^p(\mathbb{R}) \). Since \( f_1(x) \in F H^p[0,a] \), we obtain that \( f_1(x) e^{-i\alpha x} B_g^{-1}(x) \in H^p(\mathbb{R}) \). Hence, \( f_1(x) e^{-i\alpha x} B_g^{-1}(x) \in H^p(\mathbb{R}) \) and \( f(x) \in H^p(\mathbb{R}) \cap e^{i\alpha x} B_g H^p(\mathbb{R}) \). The proof is completed. \( \Box \)

**Remark.** In Corollary 2.8, if the point \( 0 \in \text{supp} \tilde{g} \), then \( a = 0 \).

3. Solutions of the Bedrosian equation

The study about the identity $H(fg) = fHg$ has a long history. In 1963, E. Bedrosian found that: If $f, g \in L^2(\mathbb{R})$ satisfy (i) $\text{supp} \ f \subseteq [0, \infty)$, $\text{supp} \ g \subseteq [0, \infty)$; or, (ii) $\text{supp} \ f \subseteq [-A, A]$, $\text{supp} \ g \subseteq \mathbb{R} \setminus [-A, A]$ for a positive number $A$, then $H(fg) = fHg$. This result is known as the Bedrosian theorem and $H(fg) = fHg$ is called the Bedrosian identity. After that, some simple and useful sufficient conditions for the Bedrosian identity were given in the frequency domain in [5]. In 1986, some necessary and sufficient conditions for the Bedrosian identity $H(fg) = fHg$ were obtained for $f, g \in L^2(\mathbb{R})$ [6]. Recently, as an advent of the EMD (empirical mode decomposition) algorithm and the related transform in terms of the IMFs (intrinsic mode functions) as a product of EMD for non-stationary signal processing [4], the study of the Bedrosian identity received much attention again. Many new and interesting results in relation to the Bedrosian identity are reported in [17,8–11] and the references therein.

Proof. Let $g \in FH^1[A, B]$ be a nonzero function with $0 = A < B$. Then $g = g_+ \in FH^1[A, B]$ is a nonzero function with the point $0 \in \text{supp } g_+$, and $g_+ \equiv 0$. By (5) of Theorem 3.1 and Corollary 2.8, we learn that $T(fg) = fTg$ if and only if

$$f_+ \in \text{span}\left\{ \frac{1}{(x - \lambda)^j} ; \lambda \in E_1, \ j = 1, \ldots, m(\lambda) \right\},$$

where $E_1$ and $E_2$ are respectively the sets of all different zeros of $G_+ := (\partial^- g_+)(z)$ in the upper half plane and $G_- := (\partial^- g_+)(z)$ in the lower half plane, and $m(\lambda)$ denotes the multiplicity at $\lambda$.

Remark. Moreover, if $g$ is a real function in Theorem 3.3, then $g_+ = \overline{g_+}$, $G_2(z) = \overline{G_1(z)}$ and $E_2 = E_1 = \{ \lambda \in E_1 \}$. $\square$

4. An application in construction of bandlimited signals

Another question arising in physical practice is: Given a bandlimited function $g \in FH^1[A, B]$, find all the functions $f \in L^2(\mathbb{R})$ such that $fg \in FH^1[A, B]$. For a nonzero function $g \in FH^1[A, B]$, by the Paley–Wiener theorem, we easily known that $fg \in FH^1[A, B]$ if and only if $f$ has a bandlimit within that of $g$, viz., $fg \in FH^1[A, B]$. $\square$

Lemma 4.1. Let $g \in FH^1[A, B]$ and $f \in L^0_1(\mathbb{R})$ be nonzero functions. Then $fg \in FH^1[A, B]$ if and only if $f_+ g \in FH^1[A, B]$ and $f_\_ g \in FH^1[A, B]$.

Proof. Since $f \in L^0_1(\mathbb{R})$, we have

$$fg = (1/2)g(f_+ + f_-) = (1/2)(gf_+ + gf_-),$$

where $f_+ \in H^0(\mathbb{R})$ and $f_- \in H^0(\mathbb{R})$.

Let $h_1(x) := e^{-i\theta} g(x)$ and $h_2(x) := e^{i\theta} g(x)$. Then $h_1 \in FH^1[0, B - A] \subset H^0(\mathbb{R})$ and $h_2 \in FH^1[0, B - A] \subset H^0(\mathbb{R})$ from the facts

$$h_1(\omega) = \overline{g(\omega + A)}$$

and $h_2(\omega) = \overline{g(B - \omega)}$. Since $f_+$ and $f_- \in H^0(\mathbb{R})$, by Lemmas 2.1 and 2.2, we have $f_+ h_1, f_- h_2 \in H^0(\mathbb{R})$ and

$$\supp f_+ h_1 \subset (0, \infty), \supp f_- h_2 \subset (0, \infty).$$

Hence, $\supp f_+ g \subset [A, \infty)$ and $\supp f_- g \subset (-\infty, B]$ for $f_- g(\omega) = \overline{(h_2 f_+) \omega - A}$ and $f_+ g(\omega) = \overline{g(B - \omega)}$. This gives that $fg \in FH^1[A, B]$ if and only if $f_+ g \in FH^1[A, B]$ and $f_- g \in FH^1[A, B]$. The proof is completed. $\square$

Below we will give some sufficient conditions for functions $f$ that make $fg$ have the same bandlimit as $g$ does.

Theorem 4.2. Suppose that functions $0 = A < g \in FH^1[A, B]$ and $f \in L^0_1(\mathbb{R})$. If the endpoints $A, B \in \text{supp } g$, then $fg \in FH^1[A, B]$ if and only if

$$f \in \text{span}\left\{ \frac{1}{(x - \lambda)^j} ; \lambda \in E, \ j = 1, \ldots, m(\lambda) \right\},$$

where $E$ is the set of all different zeros of $G(z) := (\partial^- g)(z)$ given by (2.17) on $\mathbb{C} \setminus \mathbb{R}$ and $m(\lambda)$ denotes the order of $\lambda$.

Proof. Let $g_1(x) := e^{i\beta} g(x)$ and $g_2(x) := e^{-i\beta} g(x)$. Then we have $g_1 \in FH^1[0, B - A] \subset H^0(\mathbb{R}), g_2 \in FH^1[0, B - A] \subset H^0(\mathbb{R})$, the point $0 \in \text{supp } g_1$, and $0 \in \text{supp } g_2$. Therefore, by Lemma 4.1 and Corollary 2.8, we obtain that $f_+ g_1 \in FH^1[0, B - A]$ and $f_- g_2 \in FH^1[0, B - A]$ if and only if $f_+ \in H^0(\mathbb{R}) \cap I_{g_1}(x) H^0(\mathbb{R})$ and $f_- \in H^0(\mathbb{R}) \cap I_{g_2}(x) H^0(\mathbb{R})$, where $I_{g_1}(x) := e^{i\beta} g_1(x)$ is the inner

function of $g_2(x) := e^{i\theta B_2(x)}$, and $I_{g_2}(x) := e^{i\theta B_2(x)}$ is the inner function of $g_2(x) := e^{-i\theta B_2(x)}$. By (2.17), we have $G_2(z) = e^{i\theta C_2(z)}$ and $G_1(z) = e^{i\theta C_1(z)}$. Thus the zeros of $G_1(z)$ in the upper half plane are the conjugate of the zeros of $G(z)$ in the lower half plane. By Lemma 2.5, we obtain that

$$f \in \text{span}\left\{ \frac{1}{(x-\lambda)}, \quad \lambda \in \mathbb{E}, \quad j = 1, \ldots, m(\lambda) \right\}.$$ 

The proof is finished. □

Theorem 4.2 gives a characterization of the solutions $f \in L^p(\mathbb{R})$, $1 < p < \infty$, in terms of backward shift invariant subspaces.

It, however, does not cover the cases $p = 1$ and $p = \infty$ due to the failure of the projectional Hardy spaces decomposition. Below we will treat the two exceptional cases by using an alternative approach.

Theorem 4.3. Suppose that $g \in FH^0[A, B]$ and $f \in L^p(\mathbb{R})$ are nonzero functions, $1 \leq p \leq \infty$. Then $fg \in FH^0[A, B]$ if and only if

$$f \in \text{span}\left\{ \frac{1}{(x-\lambda)}, \quad \lambda \in \mathbb{E}, \quad j = 1, \ldots, m(\lambda) \right\},$$

where $I_{g_1}(x) := e^{i\theta B_1(x)}$ is the inner function of $g_1(x) := e^{-i\theta B_1(x)}$ and $I_{g_2}(x) := e^{i\theta B_2(x)}$ is the inner function of $g_2(x) := e^{i\theta B_2(x)}$.

Proof. Suppose that $g \in FH^0[A, B]$ and $fg \in FH^0[A, B]$. Let $g_1(x) := e^{-i\theta B_1(x)}$ and $h_1(x) := f(x)g_1(x)$. Then $g_1 \in FH^0[0, B - A]$ and $h_1 \in FH^0[0, B - A]$. Hence, we have $g_1 = O_1 I_1$ and $h_1 = O_1 I_1$ by the decomposition theorem. From the facts that $g \in L^p(\mathbb{R})$ and $\ln |h_1| = \ln |g_1| = \ln |f| + \ln |g|$, we have $O_f = O_g I_1 \in FH^0(\mathbb{R})$. Thus

$$f = \frac{h_1}{g_1} = \frac{O_f I_1}{O_g I_1} \in \text{span}\left\{ \frac{1}{(x-\lambda)}, \quad \lambda \in \mathbb{E}, \quad j = 1, \ldots, m(\lambda) \right\}.$$

For $g_1 \in FH^0[0, B - A]$ and $(A^\theta g_1)(z) = e^{-i\theta G_1(z)}$, by Lemma 2.7, we have $I_{g_1} = e^{i\theta B_1(x)} B_1(x)$. On the other hand, letting $g_2(x) := e^{i\theta B_2(x)}$ and $h_2(x) := f(x)g_2(x)$, there hold $g_2 \in FH^0[0, B - A]$ and $h_2 \in FH^0[0, B - A]$. Since $\ln |h_2| = \ln |h_1|$, it follows that

$$f = \frac{h_2}{g_2} = \frac{O_f I_2}{O_g I_2} \in \text{span}\left\{ \frac{1}{(x-\lambda)}, \quad \lambda \in \mathbb{E}, \quad j = 1, \ldots, m(\lambda) \right\}.$$ 

For $g_2 \in FH^0[0, B - A]$ and $(A^\theta g_2)(z) = e^{i\theta G_2(z)}$, by Lemma 2.7, we have $I_{g_2} = e^{i\theta B_2(x)} B_2(x)$. By combining the above two facts, we have

$$f \in \text{span}\left\{ \frac{1}{(x-\lambda)}, \quad \lambda \in \mathbb{E}, \quad j = 1, \ldots, m(\lambda) \right\}.$$ 

Conversely, if $f \in \text{span}\left\{ \frac{1}{(x-\lambda)}, \quad \lambda \in \mathbb{E}, \quad j = 1, \ldots, m(\lambda) \right\}$, then there exist $f_1, f_2 \in FH^0(\mathbb{R})$ such that $f = f_1 + f_2$. Let $g_1(x) := e^{-i\theta B_1(x)}$ and $g_2(x) := e^{i\theta B_2(x)}$. Since $g_1, g_2 \in FH^0[0, B - A]$, we have $e^{i\theta B_1(x)} = O_1 I_1$ and $e^{i\theta B_2(x)} = O_2 I_2$. By Lemma 2.1 and 2.2, we obtain that $supp \tilde{f} \subseteq [A, B]$. The proof is completed. □

Corollary 4.4. Suppose $0 \# g \in FH^0[A, B]$. If $G(z) := (A^\theta g)(z)$ given by (2.17) has no zero points on $C \setminus \mathbb{R}$, then there exists $f \in L^p(\mathbb{R})$ such that $fg \in FH^0[A, B]$ if and only if

$$f \in e^{i\theta A^\alpha} H^0(\mathbb{R}) \cap e^{i\theta B^\alpha} H^0(\mathbb{R}) = FH^0[A - a, B - b],$$

where $[a, b] := supp \tilde{g}$.

Remark. : From Corollary 4.4, we know that if the end points $A, B \in supp \tilde{g}$ and $G(z) := (A^\theta g)(z)$ given by (2.17) has no zero points on $C \setminus \mathbb{R}$, then there exists no nonzero function $f \in L^p(\mathbb{R})$, $1 \leq p \leq \infty$, such that $fg \in FH^0[A, B]$.

References
