Backward Shift Invariant Subspaces With Applications to Band Preserving and Phase Retrieval Problems

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Abstract

The band preserving and phase retrieval problems have long been interested and studied. In this paper we, for the first time, give solutions to these problems in terms of backward shift invariant subspaces. The backward shift method among other methods seems to be direct and natural. We show that a function \( g \in L^p(\mathbb{R}), 1 \leq p \leq \infty \), with \( fg \in L^2(\mathbb{R}) \), that makes the band of \( fg \) to be within that of \( f \) if and only if \( g \) or \( g \) divided by an inner function related to \( f \), belongs to some backward shift invariant subspace in relation to \( f \). By the construction of backward shift invariant space the solution \( g \) is further explicitly represented through the span of the rational function system whose zeros are those of the Laplace transform of \( f \).

**Key words**: Backward shift invariant subspace; Band-limited; Laplace transform; Phase retrieval.

1 Introduction

Instantaneous amplitude and phase are basic concepts in functional analysis, signal processing and physics. A classical way of defining these concepts without ambiguity is through analytic signal [1]. Assume that \( f \) is a signal of finite energy. The analytic signal associated with \( f \) is denoted by \( f_+ \), defined as

\[
f_+(x) := f(x) + iHf(x),
\]

where \( Hf \), the Hilbert transform of \( f \), is defined by

\[
H(f)(x) := \frac{1}{\pi} \lim_{\varepsilon \to 0^+} \int_{|t-x|>\varepsilon} \frac{f(t)}{x-t} dt. \tag{1.1}
\]

It can be easily shown that \( Hf \) is well defined by (1.1) if \( f \) is in \( L^p(\mathbb{R}), 1 < p < \infty \). Restricted to such function classes \( H \) is a bounded operator and thus extendable to the whole \( L^p(\mathbb{R}) \). For \( p = 1 \) the definition of Hilbert transformation is based on its weak-boundedness [2]. In the \( p = \infty \) case Hillbert transformation maps bounded functions to BMO functions ([2]). With an analytic signal, there is a unique pair \((\rho, \theta)\) such that

\[
f_+(x) = f(x) + iHf(x) = \rho(x)e^{i\theta(x)},
\]
where \( \rho = \sqrt{f^2 + (Hf)^2} \) and \( \theta = \arctan[(Hf)/f] \) are respectively called the (analytic) amplitude and phase of \( f \). Let \( \mathcal{AH}^2(\mathbb{R}) := \{ f \mid f = g + i Hg, g \in L^2(\mathbb{R}) \} \) be the class of analytic signals in \( L^2(\mathbb{R}) \). It is known that \( f \in \mathcal{AH}^2(\mathbb{R}) \) if and only if \( f \in L^2(\mathbb{R}) \) with \( \text{supp} \hat{f} \subseteq [0, \infty) \), where \( \hat{f} \) is the Fourier transform of \( f \) defined by

\[
\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx
\]

([3, 4]). Indeed, analytic signals are related to analytic (holomorphic) functions in the upper half plane \( \mathbb{C}^+ := \{ z = x + iy \in \mathbb{C} : y > 0 \} \). Let the Laplace transform of \( f \in \mathcal{AH}^2(\mathbb{R}) \) be defined by

\[
F(z) = \frac{1}{2\pi} \int_{0}^{\infty} \hat{f}(\omega)e^{i\omega z}d\omega. \tag{1.2}
\]

In [3, 4], it is shown that \( f \in \mathcal{AH}^2(\mathbb{R}) \) if and only if \( f \) is the non-tangential boundary limit function of \( F(z) \in \mathcal{H}^2(\mathbb{C}^+) \), where \( \mathcal{H}^2(\mathbb{C}^+) \) denotes the Hardy space in the upper half plane.

In applications one often deals with signals of finite energy whose Fourier transform have compact supports, viz, the so called bandlimited signals. If \( f \in L^2(\mathbb{R}) \) is bandlimited with \( \text{supp} f \subset [A, B] \), where \( A = \inf \hat{f} \) and \( B = \sup \hat{f} \), then we say that \( f \) has band \([A, B]\). The band of \( f \) is denoted as \( \text{Band}\{f\} \). We call \( B - A \) the bandwidth of \( f \). For the purpose of this paper, we use \( \mathcal{FH}^2[A, B] = \{ f \in L^2(\mathbb{R}) \mid \text{Band}\{f\} \subset [A, B] \} \) for the set of the bandlimited signals whose bands are contained in \([A, B]\). Two classical problems of long interest in a number of practical areas, including optics, antenna theory and physics, are formulated as follows: The first is to find all functions \( g \) such that \( \text{Band}\{fg\} \subset \text{Band}\{f\} \). The second is referred as phase retrieval problem that is to find all-pass filters \( e^{i\theta(x)} \) such that \( \text{Band}\{fe^{i\theta(x)}\} \subset \text{Band}\{f\} \). According to the descriptions of these two problems, the solution of the second problem is closely related to that of the first problem. About the first problem, we learn that, if \( f \) and \( g \in L^2(\mathbb{R}) \) with, respectively, bands \([A, B]\) and \([C, D]\), then \( fg \) has band \([A + C, B + D]\) by the well-known Titchmarsh’s convolution theorem on compact supported distributions. This shows that if \( A, B \) are finite numbers, then \( g \) cannot be of finite band. In order to get concrete and structural information of \( g \), an efficient and classical way is to make use of knowledge in complex analysis. The Paley-Winer theorem asserts that if \( f \) in \( L^2(\mathbb{R}) \), then \( f \in \mathcal{FH}^2[0, A] \) if and only if \( f \) is the restriction to the real line of an entire functions \( F(z) \) of the exponential type that belongs to \( \mathcal{H}^2(\mathbb{C}^+) \). This allows to use the Hadamard factorization theorem of entire functions. The existing results on band preserving and phase retrieval problems are, therefore, in the form of quotient of two entire functions ([5, 6, 8, 9]). Our results are in terms of backward shift invariant spaces with explicit representations.

There are two contexts for the theory of shift and backward shift operators, viz., the disc case and half complex plane case([10, 12, 13, 11, 14]). For the purpose of this paper, we concentrate in the half-plane case. Denote by \( S \) the forward shift operator on \( \mathcal{H}^p(\mathbb{R}) \), defined by

\[
Sf(t) = e^{iat} f(t), \quad \forall \ a > 0.
\]

Its conjugate operator \( S^* \) is defined in \( \mathcal{H}^p, S^* \frac{1}{p} + \frac{1}{p^*} = 1, \) by

\[
S^* f(t) = e^{-iat} f(t).
\]

The celebrating Beurling-Lax[11] theorem asserts that a subspace \( M \subset \mathcal{H}^2 \) is a forward shift invariant subspace (i.e. \( SM \subset M \)) if and only if \( M = IH^2 \), where \( I \) is an inner function. Owing to this result and the relationship \( \langle f, Sg \rangle = \langle S^* f, g \rangle, f, g \in L^2(\mathbb{R}) \), we can easily deduce that all backward shift invariant subspaces (i.e. \( S^* M \subset M \)) are of the form

\[
\mathcal{H}^2 \cap \tilde{I} \mathcal{H}^2,
\]
where $I$ is an inner function. The above concepts and results are extendable to the $H^p$ spaces, $1 \leq p < \infty$, with the inner product being replaced by the conjugate pairing between $H^p(\mathbb{R})$ and $H^p(\mathbb{R})$,

$$(f, g) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx, \quad f \in H^p(\mathbb{R}), \quad g \in H^p(\mathbb{R}).$$

In this paper, based on the relationship between $\mathcal{H}^2(\mathbb{C}^+)$ and $\mathcal{F}\mathcal{H}^2[A,B]$, we will develop new techniques to study bandlimited signals and the phase retrieval problem. We show that a function $g \in L^p(\mathbb{R}), 1 \leq p \leq \infty$, with $fg \in L^2(\mathbb{R})$, that makes Band $\{fg\} \subseteq \text{Band}\{f\}$ if and only if $g_+ g_+ = g - iHg$, or $g$ divided by the inner function of $f_+$, belong to the backward shift invariant subspace $H^p(\mathbb{R}) \cap IH^p(\mathbb{R})$, where $I$ is some inner function related to $f$. These results will be given in §2. We subsequently treat the phase retrieval problem in §3. In §4 we deal with construction of the related backward shift invariant spaces. Under the imposed constraint $f \in \mathcal{F}\mathcal{H}^2[A,B]$ in the paper, it will be shown that a solution $g$ to the problem is in the closure of the span of the rational system whose zeros are among those of the Laplace transform of $f$, in, respectively, the upper- and lower-half complex plane. This allows us to give a complete characterization of the solutions to the band preserving and phase retrieval problems.

2 Band Preserving in Relation to Backward Shift Invariant Spaces

**Lemma 2.1** Assume that $f \neq 0$ and $f \in \mathcal{F}\mathcal{H}^2[0,A]$. The follow result hold:

(i) If $g \in H^p(\mathbb{R}), 1 \leq p \leq \infty$, and $fg \in L^2(\mathbb{R})$, then $\text{supp} \hat{fg} \subseteq [0, \infty)$;

(ii) If $g \in H^p(\mathbb{R}), 1 \leq p \leq \infty$, and $fg \in L^2(\mathbb{R})$, then $\text{supp} \hat{fg} \subseteq (-\infty, A]$.

**Proof:** (i): If $p = \infty$, then $fg \in H^2(\mathbb{R})$, and consequently, $\text{supp} \hat{fg} \subseteq [0, \infty)$. For $1 \leq p < \infty$, there exists $0 < r < \infty$ such that $\frac{1}{2} + \frac{1}{p} = \frac{1}{r}$, or, equivalently, $\frac{1}{2r} + \frac{1}{p'r} = 1$. It can be easily shown, by Hölder’s inequality and definition of the complex Hardy $\mathcal{H}^r(\mathbb{C}^+)$ space, $f \hat{g} \in H^r(\mathbb{R})$. Since we also have $fg \in L^2(\mathbb{R})$, we have $fg \in H^2(\mathbb{R})$ (Corollary II. 4.3., [3]), and therefore $\text{supp} \hat{fg} \subseteq [0, \infty)$.

(ii): Since $f \in \mathcal{F}\mathcal{H}^2[0,A]$, we have $e^{iAx}f(x) \in \mathcal{F}\mathcal{H}^2[0,A]$. Let $h(x) := e^{iAx}f(x)g(x)$. The result of (i) shows that $\text{supp} \hat{h} \subseteq [0, \infty)$. Since $(\hat{fg})(\omega) = h(A - \omega)$, we have $\text{supp} \hat{fg} \subseteq (-\infty, A]$. The proof is finished.

Set $H^p(\mathbb{R}) := \{f | \overline{f} \in H^p(\mathbb{R})\}$. With Lemma 2.1 and Lemma 2.1, we give a characterization of functions $g$ such Band $\{fg\} \subseteq \text{Band}\{f\}$.

**Theorem 2.2** Let $f, g$ be nonzero functions, $f \in \mathcal{F}\mathcal{H}^2[0,A]$, $\overline{g} \in H^p(\mathbb{R}), 1 \leq p \leq \infty$. Assume that $fg \in L^2(\mathbb{R})$. Then $fg \in \mathcal{F}\mathcal{H}^2[0,A]$ if and only if $\overline{g} \in H^p(\mathbb{R}) \cap I_f^1 H^p(\mathbb{R})$, where $I_f^u := e^{iAfx+bu}B_f^u(x)$ is the inner function of $f$.

Set $f_1(x) := e^{iAx}f(x)$. Then $f_1 \in \mathcal{F}\mathcal{H}^2[0,A]$ if and only if $f \in \mathcal{F}\mathcal{H}^2[0,A]$. Invoking Theorem 2.2, we have

**Corollary 2.3** Let $f, g$ be nonzero functions, $f \in \mathcal{F}\mathcal{H}^2[0,A]$, $g \in H^p(\mathbb{R}), 1 \leq p \leq \infty$. Assume that $fg \in L^2(\mathbb{R})$. Then $fg \in \mathcal{F}\mathcal{H}^2[0,A]$ if and only if $g \in H^p(\mathbb{R}) \cap I_f^1 H^p(\mathbb{R})$, where $I_f^u := e^{iAfx+bu}B_f^u(x)$ is the inner function of $f_1(x) := e^{iAx}f(x)$.

Note that

$$F_i(z) := (\partial^{-1} f_i)(z) = \frac{1}{2\pi} \int_0^A \widehat{f_i}(\omega)e^{i\omega z}d\omega = \frac{1}{2\pi} \int_0^A \overline{\hat{f}(A-\omega)}e^{i\omega z} = e^{iAz} F(\overline{z}).$$
Thus the zeroes of \( F_i(z) \) in the upper-half complex plane are the conjugates of those of \( F(z) \) in the lower-half complex plane. We denote by \( \{ \alpha_k \} \) and \( \{ \beta_k \} \) the sets of the zeros of \( F(z) \) in the upper-half complex plane \( \mathbb{C}^+ \) and in the lower-half complex plane \( \mathbb{C}^- \) (They repeat according to their respective multiples), respectively, where \( F(z) \) is given by (2.2). Then \( B_f^p \) in Theorem 2.2 and \( B_f^l \) in Corollary 2.3 are respectively given by

\[
B_f^p(x) = \prod_{\alpha_k} \frac{|\alpha_k^2 + 1|}{\alpha_k^2 + 1} \frac{x - \alpha_k}{x - \overline{\alpha_k}}, \quad B_f^l(x) = \prod_{\beta_k} \frac{|\beta_k^2 + 1|}{\beta_k^2 + 1} \frac{x - \beta_k}{x - \overline{\beta_k}}.
\] (2.3)

Let \( f \in \mathcal{FH}^2[0, A] \) be a nonzero function. By Theorem 2.2 and Corollary 2.3, we learn that if \( g \in H^p(\mathbb{R}) \) or \( \overline{g} \in H^p(\mathbb{R}) \), \( 1 \leq p \leq \infty \), a function \( g \) making \( fg \in \mathcal{FH}^2[0, A] \) can be completely characterized by a backward shift invariant subspace \( H^p(\mathbb{R}) \setminus \overline{I H^p(\mathbb{R})} \), where \( I \) is an inner function related to \( f \). Next we extend the just obtained results to general functions \( g \in L^p(\mathbb{R}) \), \( 1 \leq p \leq \infty \). Since the operator \( H \) is bounded on \( L^p(\mathbb{R}) \), \( 1 < p < \infty \), for any \( g \in L^p(\mathbb{R}) \), we have the projection Hardy spaces decomposition

\[
g(x) = \frac{1}{2}(g_+(x) + g_-(x)),
\] (2.4)

where \( g_+ := g + iHg \) and \( g_- := g - iHg \) with \( g_+, \overline{g_-} \in H^p(\mathbb{R}) \). They are, respectively, called the analytic signal and the dual analytic signal of \( g \). We first have

**Lemma 2.4** Let \( f \) be nonzero, \( f \in \mathcal{FH}^2[0, A] \). There exists a function \( g \in L^p(\mathbb{R}) \), \( 1 < p < \infty \), such that \( fg \in \mathcal{FH}^2[0, A] \) if and only if both the relations \( fg_+ \in \mathcal{FH}^2[0, A] \) and \( fg_- \in \mathcal{FH}^2[0, A] \) hold.

**Proof:** Suppose that \( fg \in \mathcal{FH}^2[0, A] \), then \( \overline{\hat{f}g} \subseteq [0, A] \). Since \( f \in \mathcal{FH}^2[0, A] \), \( g_+ \in H^p(\mathbb{R}) \) and \( g_- \in \overline{H^p(\mathbb{R})} \), by Lemma 2.1, we have

\[
\text{supp}(\hat{f}g_+) \subseteq [0, \infty), \quad \text{supp}(\hat{f}g_-) \subseteq (-\infty, A].
\]

Thus \( (\hat{f}g)(\omega) = (\hat{f}g)_-(\omega) = 0 \) for \( \omega < 0 \); and \( (\hat{f}g)(\omega) = (\hat{f}g)_+(\omega) \) for \( \omega > A \). These yield that \( fg_- \in \mathcal{FH}^2[0, A] \) and \( fg_+ \in \mathcal{FH}^2[0, A] \).

Conversely, if \( fg_+ \in \mathcal{FH}^2[0, A] \) and \( fg_- \in \mathcal{FH}^2[0, A] \), then \( fg = fg_+ + fg_- \in \mathcal{FH}^2[0, A] \). The proof is complete. \( \square \)

In virtue of Theorem 2.2, Corollary 2.3, and Lemma 2.4, we obtain

**Theorem 2.5** Let \( f, g \) be nonzero, \( f \in \mathcal{FH}^2[0, A] \), \( g \in L^p(\mathbb{R}) \), \( 1 < p < \infty \), and \( fg \in L^2(\mathbb{R}) \). Then \( fg \in \mathcal{FH}^2[0, A] \) if and only if

\[
\overline{g_-} \in H^p(\mathbb{R}) \setminus \bigcup e^{ia_u x} B_f^u(x) H^p(\mathbb{R}),
\]

and

\[
g_+ \in H^p(\mathbb{R}) \setminus \bigcup e^{ia_l x} B_f^l(x) H^p(\mathbb{R}),
\]

where \( a_u \) and \( a_l \) are two nonnegative real constants, \( B_f^u(x) \) and \( B_f^l(x) \) are respectively given in (2.3).

**Remark** Let \( f \in \mathcal{FH}^2[0, A] \). If \( 0 \in \text{supp} \hat{f} \), then \( a_u = 0 \). If \( A \in \text{supp} \hat{f} \), then \( 0 \in \text{supp} \hat{f} \) and \( a_l = 0 \).
The above theorem gives a characterization for the solutions \( g \in L^p(\mathbb{R}), 1 < p < \infty \), to the band preserving problem. It, however, does not cover the cases \( p = 1 \) and \( p = \infty \) due to the failure of the projectional Hardy spaces decomposition. The case \( p = \infty \) is directly related to the phase retrieval problem. Below we will treat the two exceptional cases as follows.

**Theorem 2.6** Let \( f \in \mathcal{F}H^2[0, A] \), \( g \in L^p(\mathbb{R}), 1 \leq p \leq \infty \), be nonzero functions and \( fg \in L^2(\mathbb{R}) \). Then \( fg \in \mathcal{F}H^2[0, A] \) if and only if

\[
g \in T_f^pH^p(\mathbb{R}) \bigcap I_f^pH^p(\mathbb{R}) = T_f^p \left[ H^p(\mathbb{R}) \bigcap I_f^pH^p(\mathbb{R}) \right],
\]

where \( I_f^p := e^{ia_x}B_f^p(x) \) is the inner function of \( f \) and \( I_f^p(x) := e^{ia_x}B_f^p(x) \) is the inner function of \( f_1(x) := e^{iAx}f(x) \).

**Proof:** Let \( h := fg \). Since \( f, h \in \mathcal{F}H^2[0, A] \), we have \( f = O_fI_f^p \) and \( h = O_hI_h^p \), where \( I_f^p \) is the inner function of \( f \) with the form \( e^{ia_x+b}B_f^p(x) \) and \( I_h^p \) is the inner function of \( h \). From the facts that \( g \in L^p(\mathbb{R}), \ln |h| = \ln |fg|, \ln |f| \in L^2(\frac{dx}{1+x^2}) \), we have \( O_g := \frac{O_h}{O_f} \in H^p(\mathbb{R}) \). Thus

\[
g = \frac{h}{f} = \frac{O_gI_h^p}{I_f^p} \in T_f^pH^p(\mathbb{R}).
\]

On the other hand, for \( h_1(x) := e^{iAx}h(x), f_1(x) := e^{iAx}f(x) \), there hold \( h_1, f_1 \in \mathcal{F}H^2[0, A] \). Since \( \ln |h_1| = \ln |h|, \ln |f_1| = \ln |f| \), then \( f_1 = O_fI_f^p \) and \( h = O_hI_h^p \), where \( I_f^p \) is the inner function of \( f_1 \) with the form \( e^{ia_x+b}B_f^p(x) \) and \( I_h^p \) is the inner function of \( h_1 \). Hence

\[
\overline{g} = \frac{\overline{h}}{\overline{f}_1} = \frac{O_hI_h^p}{O_fI_f^p} = \frac{O_hI_h^p}{I_f^p} \in T_f^pH^p(\mathbb{R}).
\]

By combining with \( g \in T_f^pH^p(\mathbb{R}) \), we have \( g \in T_f^pH^p(\mathbb{R}) \bigcap I_f^pH^p(\mathbb{R}) = T_f^p \left[ H^p(\mathbb{R}) \bigcap I_f^pH^p(\mathbb{R}) \right] \).

Conversely, if \( g \in T_f^pH^p(\mathbb{R}) \bigcap I_f^pH^p(\mathbb{R}) \), then there exist \( g_1, g_2 \in H^p(\mathbb{R}) \) such that \( g = T_f^p g_1 \) and \( \overline{g} = T_f^p g_2 \). Let \( f_1(x) := e^{iAx}f(x) \). Since \( f, f_1 \in \mathcal{F}H^2[0, A] \) and \( f g \in L^2(\mathbb{R}) \), as assumed, we have \( fg = O_fI_f^pT_f^p g_1 = g_1O_f \in H^2(\mathbb{R}) \) and \( e^{iAx}f(x)g(x) = O_fI_f^pT_f^p g_2 = O_{f_1}I_f^pT_f^p g_2 \). Hence, \( \text{supp} \overline{g} \subseteq [0, A] \) and \( fg \in \mathcal{F}H^2[0, A] \). The proof is complete.

\[ \square \]

3 Characterization of backward shift invariant subspace and its application to band preserving

From the above analysis we learn that the solutions \( g \) to the two mentioned problems are all in terms of backward shift invariant spaces \( H^p(\mathbb{R}) \bigcap I^pH^p(\mathbb{R}) \), where \( I \) is some inner function. Under the condition \( f \in \mathcal{F}H^2([0, A]) \), the related inner function \( I \) is with the simplified form \( I(x) = e^{iax+b}B(x), a \geq 0, b \in \mathbb{R} \). \( B \) is a Blaschke product. To know more about the solutions \( g \) is to know more about the construction of the backward shift invariant spaces. Many relevant references are in Russian and are for the disc case ([12, 13, 15]). Specifically, for the half-plane case, the literature on construction of \( H^p(\mathbb{R}) \bigcap e^{iax+b}B(x)H^p(\mathbb{R}) \) in terms of the system consisting of shifted Cauchy kernels does not seem to be available. In this paper we provide the proof for such construction on the upper-half plane.
When \( a = 0 \) and \( B(x) \) is given in (??). Let \( B_0(x) = 1 \),

\[
B_n(x) = \prod_{j=1}^{n} \frac{|\alpha_j^2 + 1|}{\alpha_j^2 + 1} \cdot \frac{x - \alpha_j}{x - \alpha_j}, \quad e_n(x) = d_n \frac{\sqrt{2\pi\alpha_n}}{x - \alpha_n} B_{n-1}(x), \quad n \geq 1,
\]

where \( d_n \) are normalizing constants in \( L^2(\mathbb{R}) \). \( \{e_1, \ldots, e_n, \ldots\} \) is obtained through the Gram-Schmidt orthogonalization process on \( \{B_n\} \), called a Takenaka-Malmquist system. We will be working with the induced conjugate paring \( \langle \cdot, \cdot \rangle \) on \( H^p(\mathbb{R}) \) and \( H^{p'}(\mathbb{R}) \), namely,

\[
\langle f(x), g(x) \rangle = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \overline{g(x)} \, dx,
\]

where \( f \in H^p(\mathbb{R}), g \in H^{p'}(\mathbb{R}), 1/p + 1/p' = 1 \). Furthermore, each \( e_n \) in the system is in \( H^p, 1 < p < \infty \), and \( \{e_1, \ldots, e_n, \ldots\} \) is orthogonal with respect to the pairing between \( H^p \) and \( H^{p'} \).

**Theorem 3.1** Let \( \alpha_1, \ldots, \alpha_n, \ldots \) be a sequence of points in the upper-half complex plane that defines a Blaschke product \( B \). Then for \( 1 < p < \infty \),

\[
H^p(\mathbb{R}) \cap BH^p(\mathbb{R}) = (BH^p)' \perp = \text{span}^p \{e_n\}_{n=1}^{\infty},
\]

where the closure is in the \( L^p \) topology and \( (BH^p)' \perp = \{f \in H^p | \langle f, Bh \rangle = 0, \quad \forall \quad g \in H^{p'}\} \).

**Proof:** To prove the first identical relation of (3.6) we first show

\[
H^p(\mathbb{R}) \cap BH^p(\mathbb{R}) \subset (BH^p)' \perp.
\]

This is all clear. In fact, for \( f = B\overline{h}_p \in H^p(\mathbb{R}), h_p \in H^p(\mathbb{R}) \), we have, for any \( g = Bh_p', h_p' \in H^{p'}(\mathbb{R}) \),

\[
\langle f, g \rangle = \langle B\overline{h}_p, Bh_p' \rangle = \langle \overline{h}_p, h_p' \rangle = 0.
\]

Next we show

\[
H^p(\mathbb{R}) \cap BH^p(\mathbb{R}) \supset (BH^p)' \perp.
\]

Let \( f \in (BH^p(\mathbb{R}))' \perp \). Then for any function of the form \( Bh_p', h_p' \in H^{p'}(\mathbb{R}), 1 < p' < \infty \),

\[
0 = \langle f, Bh_p' \rangle = \langle Bf, h_p' \rangle.
\]

From Lemma 4.1, pp241, [3] we know \( \overline{f}B = h \in H^p(\mathbb{R}) \), or \( f = B\overline{h} \). This completes the first identical relation of (3.6).

Now we prove the second identical relation of (3.6). Since \( 1 < p < \infty \), each \( e_n \) is in \( (BH^p)' \perp \). In fact, for any \( f = Bh_p' \),

\[
\langle f, e_n \rangle = \langle Bh_p', e_n \rangle = c \left( \frac{B}{B_{n-1}}h_p' \right) (\alpha_n) = 0,
\]

where \( c \) is a constant and \( B/B_{n-1} \) is a Blaschke product with \( \alpha_n \) as its zero. Since \( (BH^p)' \perp \) is closed in \( H^p \), we have

\[
H^p(\mathbb{R}) \cap (BH^p)' \perp \supset \text{span}^p \{e_n\}_{n=1}^{\infty}.
\]

Next we prove the opposite set inclusion. Let \( f \in H^p(\mathbb{R}) \cap BH^p \). Thus \( f = B\overline{h}_p \in H^p \). We are to show that \( f \) is in the \( L^p \)-closure of \( \{e_n\}_{n=1}^{\infty} \). By Theorem 4.2, Chapter VI, pp242, [3], it suffices to
show that if \( g \in H^p \), \( 1 < p' < \infty \), such that \( \langle g, e_n \rangle = 0 \), then \( \langle g, f \rangle = 0 \). The assumption \( \langle g, e_n \rangle = 0 \) implies that \( g \) has all zeros of \( B \) together with the multiples. Then
\[
\langle g, f \rangle = \langle Tg, \overline{u}_p \rangle = 0.
\]
The proof is complete. \( \square \)

When \( p = 2 \), we have

**Corollary 3.2**

\[
H^2(\mathbb{R}) = \left( H^2(\mathbb{R}) \cap B\mathcal{H}^2(\mathbb{R}) \right) \oplus B^2(\mathbb{R}) = \text{span}^p \{ e_n \}_{n=1}^\infty \oplus B^2(\mathbb{R}). \tag{3.7}
\]

We further have

**Corollary 3.3** For \( 1 < p < \infty \), \( \text{span}^p \{ e_n \}_{n=1}^\infty = H^p(\mathbb{R}) \) if and only if the sequence \( \{ \alpha_n, \ldots, \alpha_n, \ldots \} \) cannot be zeros of a Blaschke product.

**Proof:** If the sequence \( \{ \alpha_n \}_{n=1}^\infty \) consists of the zeros, together with their multiples, of a Blaschke product, say \( B \), then
\[
H^p(\mathbb{R}) \cap B\mathcal{H}^p(\mathbb{R}) = \text{span}^p \{ e_n \}_{n=1}^\infty.
\]
Now the left-hand-side cannot be identical with \( H^p(\mathbb{R}) \) for not all functions in \( H^p(\mathbb{R}) \) are of the form \( B\overline{u}_p \). This shows that the closure of the span is not \( H^p(\mathbb{R}) \).

On the other hand, suppose that the sequence \( \{ \alpha_n \}_{n=1}^\infty \) cannot define a Blaschke product. In the case, if \( f \in H^p(\mathbb{R}), 1 < p < \infty \), is orthogonal with all \( e_n, n = 1, 2, \ldots \), then \( f \) has to be zero function. Otherwise, \( f \) would have zeros of the same multiples at \( \alpha_n \). This shows that the sequence forms the zeros of a Blaschke product of \( f \), contrary to the assumption. \( \square \)

Let \( \{ \alpha_k \} \) and \( \{ \beta_k \} \) respectively denote the zero sequence of \( F(z) = (1/2\pi) \int_0^{2\pi} \hat{f}(\omega) e^{iz\omega} \, dz \) in the upper-half complex plane \( \mathbb{C}^+ \) and in the lower-half complex plane \( \mathbb{C}^- \) (they repeat according to its multiplicities). With Theorems 2.6 and 3.1, we have the following result.

**Theorem 3.4** Let \( f \in \mathcal{F}H^2[0, A] \) and \( g \in L^p(\mathbb{R}), 1 < p < \infty \), be nonzero functions. If the endpoints \( 0, A \in \text{supp} \hat{f} \). Then \( fg \in \mathcal{F}H^2[0, A] \) if and only if
\[
gB_f^p \in \text{span}^p \{ e_n(x) = \frac{\sqrt{2\pi z_n}}{x - z_n} \prod_{k=1}^n \frac{x - \overline{\beta_k}}{x - z_k} | n \in \mathbb{N} \}, \tag{3.8}
\]
where \( \{ z_k \} = \{ \overline{\alpha_k} \} \cup \{ \beta_k \} \) and \( B_f^p(x) \) is given in (2.3).

**Proof:** By Theorem 2.6, we obtain that \( fg \in \mathcal{F}H^2[0, A] \) if and only if \( g \in \mathcal{T}^p_f \left( H^p(\mathbb{R}) \cap \mathcal{I}^p_f I^p_f \mathcal{H}^p(\mathbb{R}) \right) \), where \( I^p_f = e^{ia_nx}B^p_f(x) \) and \( I^p_f = e^{ia_nx}B^p_f \). Since \( 0, A \in \text{supp} \hat{f} \) Thus \( a_l = a_u = 0 \). By Theorem 3.1, the assertion is proved. \( \square \)

Specially, if \( f \) and \( g \) are real functions, we have \( \hat{f}(\omega) = \hat{f}(-\omega), F(z) = F(\bar{z}) \) and \( g = g_+ + \overline{g}_+ \). By Theorems 2.5 and 3.1, we have

**Corollary 3.5** Let \( f \in \mathcal{F}H^2[-A, A] \) and \( g \in L^p(\mathbb{R}), 1 < p < \infty \), be nonzero real functions. If the endpoints \( -A, A \in \text{supp} \hat{f} \). Then \( fg \in \mathcal{F}H^2[-A, A] \) if and only if
\[
g_+ \in \text{span}^p \{ e_n(x) = \frac{\sqrt{2\pi z_n}}{x - \beta_n} \prod_{k=1}^n \frac{x - \overline{\beta_k}}{x - \beta_k} | n \in \mathbb{N} \},
\]
where \( \{ \beta_k \} \) are the zero sequence of \( F(z) = (1/2\pi) \int_{-A}^A \hat{f}(\omega) e^{i\omega z} \, d\omega \) in the lower half plane.
Notice that the space $H^p(\mathbb{R}) \cap B\overline{H^p}(\mathbb{R})$ depends upon the point sets

$$E = \{ \alpha_k : \alpha_k \in \mathbb{C}^+, k \in \mathbb{N} \}.$$ 

Each $\alpha_k$ may repeat a number of times, where the time is identical with its multiple in the Blaschke product. So we could rearrange them and make the repetition explicit by setting

$$E = \{ \alpha_1, \ldots, \alpha_{n_1}, \alpha_2, \ldots, \alpha_{n_2}, \ldots \}.$$ 

Thus we can accordingly form another possible basis to characterize $H^p(\mathbb{R}) \cap B\overline{H^p}(\mathbb{R})$ for $1 < p < \infty$.

**Corollary 3.6** Let $\alpha_k$ be different zeros of $B(z)$ given by (??) of which each has a multiple $n_k$. Then

$$H^p(\mathbb{R}) \cap B\overline{H^p}(\mathbb{R}) = \text{span}^p \{ \frac{1}{(x - \alpha_k)^j} : j = 1, \ldots, n_k; k \in \mathbb{N} \}.$$ 

Indeed, the Takenaka-Malmquist system is the Gram-Schmidt orthogonalization of the system given in Corollary 3.6. Hence, if $g \in L^2(\mathbb{R})$, we can also give an equivalent characterization of Theorem 3.4 in the frequency domain.

**Corollary 3.7** Let $f \in \mathcal{F}H^2[0, A]$, $g \in L^2(\mathbb{R})$ be nonzero functions and the endpoints $0, A \in \text{Supp}\hat{f}$. Suppose $\{z'_k | k \in \mathbb{N}\}$ be different zeros of $F(z) = (1/2\pi)\int_0^A \hat{f}(\omega)e^{i\omega z}d\omega$ in $\mathbb{C} \setminus \mathbb{R}$ and each of which has a multiple $n_k$. Then $fg \in \mathcal{F}H^2[0, A]$ if and only if

$$\hat{g} \in \text{span}^p \{ u[(-1)^d\omega]e^{i\omega z'_k} : j = 0, \ldots, n_k - 1; k \in \mathbb{N} \}, \quad (3.9)$$

where $d = -1$ if $\exists z'_k > 0$ and $d = 0$ if $\exists z'_k < 0$.

The following theorem gives a characterization for $H^2(\mathbb{R}) \cap e^{i\alpha x+B(x)H^2(\mathbb{R})}$.

**Theorem 3.8** Let $a > 0$ and $B(x)$ be given in (??). Then

$$H^2(\mathbb{R}) \cap e^{i\alpha x+B(x)H^2(\mathbb{R})} = \mathcal{F}H^2[0, a] \bigoplus e^{i\alpha x}\text{span}^2 \{ e_n(x) \}_{n=1}^\infty. \quad (3.10)$$

where $\bigoplus$ denotes the direct sum in $H^2(\mathbb{R})$.

**Proof:** Suppose that $f(x) \in H^2(\mathbb{R}) \cap e^{i\alpha x+B(x)H^2(\mathbb{R})}$. Then $f(x) \in H^2(\mathbb{R})$ and $f(x)e^{-i\alpha x} \in B(x)H^2(\mathbb{R})$. Since $f \in H^2(\mathbb{R})$, we have $\text{supp}\hat{f}(\omega) \subseteq [0, \infty)$. Decomposed $\hat{f}(\omega)$ as a sum of $\hat{f}(\omega) = \hat{f}(\omega)\chi_{[0, a]}(\omega) + \hat{f}(\omega)\chi_{(a, \infty)}(\omega) = \hat{f}_1(\omega) + \hat{f}_2(\omega)$, where $\chi_{[a,b]}(x)$ is a characteristic function on $[a, b]$. By the inverse Fourier transform, $f$ can be written as $f(x) = f_1(x) + f_2(x)$, where $f_1 = \frac{\hat{f}_1}{\hat{f}_1}$ and $f_2 = \frac{\hat{f}_2}{\hat{f}_2}$. Obviously, we have $f_2(x)e^{-i\alpha x} \in H^2(\mathbb{R})$ and $f_1(x) \in \mathcal{F}H^2[0, a]$. Note that $f_1(x)e^{-i\alpha x} \in B(x)H^2(\mathbb{R})$ for $f_1 \in \mathcal{F}H^2[0, a]$. Thus $f_2(x)e^{-i\alpha x} = f(x)e^{-i\alpha x} - f_1(x)e^{-i\alpha x} \in B(x)H^2(\mathbb{R})$. Hence $f_2(x)e^{-i\alpha x} \in H^2(\mathbb{R}) \cap B(x)H^2(\mathbb{R})$. By Theorem 3.1, (3.10) holds.

Conversely, suppose that $f(x) \in \mathcal{F}H^2[0, a] \bigoplus e^{i\alpha x}\text{span}^2 \{ e_n(x) \}_{n=1}^\infty$. By Theorem 3.1, we obtain that $f(x) = f_1(x) + f_2(x)$, where $f_1(x) \in \mathcal{F}H^2[0, a]$ and $f_2(x) \in H^2(\mathbb{R}) \cap e^{i\alpha x}B(x)H^2(\mathbb{R})$. From $f_1(x) \in \mathcal{F}H^2[0, a]$, it also follows that $f_1(x) \in H^2(\mathbb{R}) \cap e^{i\alpha x}B(x)H^2(\mathbb{R})$. Hence, $f(x) \in H^2(\mathbb{R}) \cap e^{i\alpha x}B(x)H^2(\mathbb{R})$. The proof is complete.

**Remark:** Since $f \in \mathcal{F}H^2[A,B]$ and $fg \in \mathcal{F}H^2[A,B]$ if and only if $h \in \mathcal{F}H^2[0,B-A]$ and $hg \in \mathcal{F}H^2[0,B-A]$, where $h(x) = e^{-i\alpha x}f(x)$. It is easy to generalize the above discussions to $f \in \mathcal{F}H^2[A,B]$ and $fg \in \mathcal{F}H^2[A,B]$. 

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4 Application to Phase retrieval problem

As application of Theorem 2.2, Corollary 2.3 and Theorem 2.3, next we solve the phase retrieval problem. Namely, under what conditions Band\{f\} and Band\{g\} both are contained in \([0, A]\) for some positive \(A\) and \(|f| = |g|\). Trivial solutions are \(g(t) = cf(t + a)\) and \(g(t) = c\overline{f(-t + a)}\) with \(|c| = 1\) and \(a \in \mathbb{R}\). It has been showed that more complicated solutions could be obtained from any one of them by flipping nonreal zeros of its Laplace transform \([6, 5, 7]\). All these existing results rely on the Paley-Wiener theorem and the Hadamard factorization theorem. Comparatively the backward shift invariant space method is more direct and explicit.

**Corollary 4.1** Let \(f, g\) be nonzero functions, \(f \in \mathcal{F} \mathcal{H}^2[0, A]\), \(g = e^{i\eta(x)} \in \mathcal{H}^\infty(\mathbb{R})\). Then \(fg \in \mathcal{F} \mathcal{H}^2[0, A]\) if and only if

\[
g(x) = c_1e^{ia_1x} \prod_{\alpha_k} \frac{\alpha_k^2 + 1}{\alpha_k^2 + 1} \cdot \frac{x - \alpha_k}{x - \alpha_k'},
\]

where \(|c_1| = 1\), \(-\alpha_a \leq a_1 \leq 0\) and \(\alpha_k'\) is any subsequence of zero sequence \(\alpha_k\) of \(F(z) = 1/2\pi f_0 A \hat{f}(\omega)e^{i\omega x}\) in the upper-half plane.

** Proof:** The “if” part is easy. Now we prove the “only if” part. Theorem 2.2 implies that \(fg \in \mathcal{F} \mathcal{H}^2[0, A]\) if and only if \(g \in \mathcal{H}^\infty(\mathbb{R}) \cap \mathcal{F} e^{ia_1x}B_f(x) \mathcal{H}^\infty(\mathbb{R})\). Since \(g \in \mathcal{H}^\infty(\mathbb{R})\) and \(|g| = 1\), \(g\) is an inner function. We, at the same time, have

\[
e^{ia_1x}B_f(x)g(x) \in \mathcal{H}^\infty(\mathbb{R}).
\]

It follows that \(\overline{g(x)}\) is a divisor of \(e^{ia_1x}B_f(x)\). This completes the proof. 

By choosing \(g = I_y^u = e^{ia_ux + bu}B_f\) in Corollary 4.1, we obtain

**Corollary 4.2** Let a nonzero function \(f \in \mathcal{F} \mathcal{H}^2[0, A]\). Then \(O_f \in \mathcal{F} \mathcal{H}^2[0, A]\)

By the same method we have

**Corollary 4.3** Let \(f, g\) be nonzero functions, \(f \in \mathcal{F} \mathcal{H}^2[0, A]\), \(g = e^{i\eta(x)} \in \mathcal{H}^\infty(\mathbb{R})\). Then \(fg \in \mathcal{F} \mathcal{H}^2[0, A]\) if and only if

\[
g(x) = c_2e^{ia_2x} \prod_{\beta_k} \frac{\beta_k^2 + 1}{\beta_k^2 + 1} \cdot \frac{x - \beta_k}{x - \beta_k'},
\]

where \(|c_2| = 1\), \(0 \leq a_2 \leq a_1\) and \(\beta_k'\) is any subsequence of zero sequence \(\beta_k\) of \(F(z) = 1/2\pi f_0 A \hat{f}(\omega)e^{i\omega x}\) in the lower-half plane.

**Corollary 4.4** Let \(f, g\) be nonzero functions, \(f \in \mathcal{F} \mathcal{H}^2[0, A]\), \(g = e^{i\eta(x)} \in \mathcal{L}^\infty(\mathbb{R})\). Then \(fg \in \mathcal{F} \mathcal{H}^2[0, A]\) if and only if

\[
g(x) = ce^{iax} \prod_{\alpha_k} \frac{\alpha_k^2 + 1}{\alpha_k^2 + 1} \cdot \frac{x - \alpha_k}{x - \alpha_k'} \cdot \prod_{\beta_k} \frac{\beta_k^2 + 1}{\beta_k^2 + 1} \cdot \frac{x - \beta_k}{x - \beta_k'},
\]

where \(|c| = 1\), \(-\alpha_a \leq a \leq \alpha_1\), \(\alpha_k'\) is any subsequence of \(\alpha_k\) and \(\beta_k'\) is any subsequence of \(\beta_k\).
References


