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# Extracting outer function part from Hardy space function

*Dedicated to the memory of Professor CHENG MinDe on the occasion of the centenary of his birth*

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**Abstract** Any analytic signal $f_a(\mathrm{e}^{\mathrm{i}t})$ can be written as a product of its minimum-phase signal part (the outer function part) and its all-phase signal part (the inner function part). Due to the importance of such decomposition, Kumarasan and Rao (1999), implementing the idea of the Szegő limit theorem (see below), proposed an algorithm to obtain approximations of the minimum-phase signal of a polynomial analytic signal

$$f_a(\mathrm{e}^{\mathrm{i}t}) = \mathrm{e}^{\mathrm{i}N_0 t} \sum_{k=0}^{M} a_k \mathrm{e}^{\mathrm{i}kt}, \tag{0.1}$$

where $a_0 \neq 0$, $a_M \neq 0$. Their method involves minimizing the energy

$$E(f_a, h_1, h_2, \dots, h_H) = \frac{1}{2\pi} \int_0^{2\pi} \left| 1 + \sum_{k=1}^{H} h_k \mathrm{e}^{\mathrm{i}kt} \right|^2 |f_a(\mathrm{e}^{\mathrm{i}t})|^2 dt \tag{0.2}$$

with the undetermined complex numbers $h_k$'s by the least mean square error method. In the limiting procedure $H \to \infty$, one obtains approximate solutions of the minimum-phase signal. What is achieved in the present paper is two-fold. On one hand, we rigorously prove that, if $f_a(\mathrm{e}^{\mathrm{i}t})$ is a polynomial analytic signal as given in (0.1), then for any integer $H \geqslant M$, and with $|f_a(\mathrm{e}^{\mathrm{i}t})|^2$ in the integrand part of (0.2) being replaced with $1/|f_a(\mathrm{e}^{\mathrm{i}t})|^2$, the exact solution of the minimum-phase signal of $f_a(\mathrm{e}^{\mathrm{i}t})$ can be extracted out. On the other hand, we show that the Fourier system $\mathrm{e}^{\mathrm{i}kt}$ used in the above process may be replaced with the Takenaka-Malmquist (TM) system, $r_k(\mathrm{e}^{\mathrm{i}t}) := \frac{\sqrt{1-|\alpha_k|^2}\mathrm{e}^{\mathrm{i}t}}{1-\overline{\alpha}_k \mathrm{e}^{\mathrm{i}t}} \prod_{j=1}^{k-1} \frac{\mathrm{e}^{\mathrm{i}t}-\alpha_j}{1-\overline{\alpha}_j \mathrm{e}^{\mathrm{i}t}}, k = 1, 2, \dots,\ r_0(\mathrm{e}^{\mathrm{i}t}) = 1$, i.e., the least mean square error method based on the TM system can also be used to extract out approximate solutions of minimum-phase signals for any functions $f_a$ in the Hardy space. The advantage of the TM system method is that the parameters $\alpha_1, \dots, \alpha_n, \dots$ determining the system can be adaptively selected in order to increase computational efficiency. In particular, adopting the $n$-best rational (Blaschke form) approximation selection for the $n$-tuple $\{\alpha_1, \dots, \alpha_n\}, n \geqslant N$, where $N$ is the degree of the given rational analytic signal, the minimum-phase part of a rational analytic signal can be accurately and efficiently extracted out.

**Keywords** complex Hardy space, analytic signal, Nevanlinna decomposition, inner and outer functions, minimum-phase signal, all-phase signal, Takenaka-Malmquist system

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## 1 Introduction

Let $\boldsymbol{D}$ denote the open unit disc in the complex plane and $\partial\boldsymbol{D}$ its boundary. Let $L^2(\partial\boldsymbol{D})$ denote the Hilbert space of all square-integrable $2\pi$-periodic functions. For $f(\mathrm{e}^{\mathrm{i}t}) \in L^2(\partial\boldsymbol{D})$, we have the Fourier series expansion

$$f(\mathrm{e}^{\mathrm{i}t}) = \sum_{n=-\infty}^{\infty} c_n(f)\mathrm{e}^{\mathrm{i}nt},$$

where $c_n(f) = \frac{1}{2\pi}\int_0^{2\pi} f(\mathrm{e}^{\mathrm{i}t})\mathrm{e}^{-\mathrm{i}nt}dt$ is the $n$-th Fourier coefficient of $f(\mathrm{e}^{\mathrm{i}t})$, and the convergence is in the $L^2$-norm sense. For $f \in L^2(\partial\boldsymbol{D})$, a well-accepted way to define the concepts *instantaneous amplitude*, *phase* and *frequency* of $f$ will now be given. First, we define the *circular Hilbert transform* (or briefly *Hilbert transform*) of $f$ as

$$(\widetilde{H}f)(\mathrm{e}^{\mathrm{i}t}) = \sum_{n=-\infty}^{\infty} (-\mathrm{i})\mathrm{sgn}(n)c_n(f)\mathrm{e}^{-\mathrm{i}nt} = \lim_{\epsilon\to 0+}\frac{1}{2\pi}\int_{\epsilon<|t-x|\leqslant 2\pi} f(\mathrm{e}^{\mathrm{i}x})\cot\left(\frac{t-x}{2}\right)dx, \tag{1.1}$$

where $\mathrm{sgn}(n) := \frac{n}{|n|}$ if $n \neq 0$ and $\mathrm{sgn}(n) = 0$ if $n = 0$. We have the relation

$$f(\mathrm{e}^{\mathrm{i}t}) + \mathrm{i}(\widetilde{H}f)(\mathrm{e}^{\mathrm{i}t}) = -c_0(f) + 2\sum_{n=0}^{\infty} c_n(f)\mathrm{e}^{\mathrm{i}nt}.$$

Further define $f_a$ to be the *associated analytic signal of* $f$:

$$f_a(\mathrm{e}^{\mathrm{i}t}) := \frac{1}{2}(f(\mathrm{e}^{\mathrm{i}t}) + \mathrm{i}(\widetilde{H}f)(\mathrm{e}^{\mathrm{i}t}) + c_0(f)) = \sum_{n=0}^{\infty} c_n(f)\mathrm{e}^{\mathrm{i}nt}. \tag{1.2}$$

The complex-valued function $f_a$, in fact, coincides with $f^+$ in the Hardy space decomposition $f = f^+ + f^-$, where $f^+$ is the non-tangential boundary limit of the in-disc Cauchy integral of $f$, namely,

$$C^+(f)(z) = \frac{1}{2\pi\mathrm{i}}\int_{\partial\boldsymbol{D}} \frac{f(\zeta)}{\zeta - z}d\zeta.$$

Since $f$ is real-valued, one has the simple relationship between $f_a$ and $f$,

$$f(\mathrm{e}^{\mathrm{i}t}) = 2\mathrm{Re}f_a(\mathrm{e}^{\mathrm{i}t}) - c_0(f).$$

In such way analysis of a real-valued signal $f$ reduces to that of the Hardy space function $f_a$, the associated analytic signal of $f$.

There is a unique amplitude-phase representation for the complex-valued $f_a$, i.e., $f_a(\mathrm{e}^{\mathrm{i}t}) = \rho(t)\mathrm{e}^{\mathrm{i}\theta(t)}$, where

$$\rho(t) = |f_a(\mathrm{e}^{\mathrm{i}t})| \geqslant 0;$$

and the phase $\theta(t)$ is accordingly a real-valued function. Note that the phase function $\theta$ is not uniquely defined due to periodicity of the trigonometric exponential function: In general, a canonical and unique phase function does not exist. In some cases the phase derivative $\theta'(t)$, as a function, may be uniquely defined up to a Lebesgue null set (see [25]). The above defined functions $\rho(t)$ and $\theta(t)$ are, respectively, called the (analytic) *amplitude* and *phase* of $f(\mathrm{e}^{\mathrm{i}t})$. The derivative function $\theta'(t)$, if can be defined, is referred as *instantaneous frequencies* of $f(\mathrm{e}^{\mathrm{i}t})$ (see [8, 25]).

By $H^2(\partial\boldsymbol{D})$ we denote the Hilbert space consisting of all functions $f(\mathrm{e}^{\mathrm{i}t}) \in L^2(\partial\boldsymbol{D})$, whose Fourier coefficients $c_n(f) = 0$ for $n < 0$. This space is identical with the space of the non-tangential boundary limits of the functions in the Hardy $H^2(\boldsymbol{D})$ space. Signals in $H^2(\partial\boldsymbol{D})$ are called analytic signals. If an analytic signal $f_a(\mathrm{e}^{\mathrm{i}t}) \in H^2(\partial\boldsymbol{D})$ only has a finite number of non-zero Fourier coefficients, then $f_a(\mathrm{e}^{\mathrm{i}t})$ is a polynomial (analytic) signal. Every polynomial signal can be represented as

$$f_a(\mathrm{e}^{\mathrm{i}t}) = \mathrm{e}^{\mathrm{i}N_0t}\sum_{k=0}^{M} a_k\mathrm{e}^{\mathrm{i}kt},$$

where $N_0$ is a non-negative integer, $a_0 \neq 0$ and $a_M \neq 0$. If such a polynomial analytic signal $f_a(z)$ does not have zero points on $\partial \boldsymbol{D}$, then $f_a(\mathrm{e}^{\mathrm{i}t})$ can be factorized as

$$\begin{aligned} f_a(\mathrm{e}^{\mathrm{i}t}) &= a_0 \mathrm{e}^{\mathrm{i}N_0 t} \prod_{k=1}^{P} (1 - p_k \mathrm{e}^{\mathrm{i}t}) \prod_{k=1}^{Q} (1 - q_k \mathrm{e}^{\mathrm{i}t}) \\ &= \underbrace{A_c \prod_{k=1}^{P} (1 - p_k \mathrm{e}^{\mathrm{i}t}) \prod_{k=1}^{Q} \left(1 - \frac{1}{\overline{q_k}} \mathrm{e}^{\mathrm{i}t}\right)}_{f_{\min}(t)} \times \underbrace{\mathrm{e}^{\mathrm{i}N_0 t} \prod_{k=1}^{Q} \frac{\mathrm{e}^{\mathrm{i}t} - 1/q_k}{1 - (1/\overline{q_k})\mathrm{e}^{\mathrm{i}t}}}_{f_{\mathrm{all}}(t)}, \end{aligned} \tag{1.3}$$

where $P + Q = M$, $A_c = a_0 \prod_{k=1}^{Q} (-q_k)$, $|p_k| < 1$ and $|q_k| > 1$ (see [22]). Putting $p_k = |p_k| \mathrm{e}^{\mathrm{i}\theta_k}$ and $q_k = |q_k| \mathrm{e}^{\mathrm{i}\phi_k}$, it is known in [13] that $f_{\min}(\mathrm{e}^{\mathrm{i}t})$ can be further represented as

$$\begin{aligned} f_{\min}(\mathrm{e}^{\mathrm{i}t}) &= A_c \prod_{k=1}^{P} (1 - p_k \mathrm{e}^{\mathrm{i}t}) \prod_{k=1}^{Q} \left(1 - \frac{1}{\overline{q_k}} \mathrm{e}^{\mathrm{i}t}\right) \\ &= \mathrm{e}^{\mathrm{i}\gamma} \exp\{\ln |A_c| + \mathrm{i}\widetilde{H} \ln |A_c| + \alpha(t) + \mathrm{i}\widetilde{H}\alpha(t)\} \\ &= \mathrm{e}^{\mathrm{i}\gamma} \exp\{\ln |f_a|(\mathrm{e}^{\mathrm{i}t}) + \mathrm{i}\widetilde{H}(\ln |f_a|)(\mathrm{e}^{\mathrm{i}t})\}, \end{aligned} \tag{1.4}$$

where $\gamma$ is a real constant and

$$\alpha(\mathrm{e}^{\mathrm{i}t}) = \sum_{m=1}^{\infty} \left[ \sum_{k=1}^{P} \frac{-|p_k|^m}{m} \cos(mt + m\theta_k) + \sum_{k=1}^{Q} \frac{-1/|q_k|^m}{m} \cos(mt + m\phi_k) \right].$$

While the instantaneous frequency of $f_{\mathrm{all}}(\mathrm{e}^{\mathrm{i}t})$, i.e.,

$$f_{\mathrm{all}}(\mathrm{e}^{\mathrm{i}t}) = \mathrm{e}^{\mathrm{i}\theta_{\mathrm{all}}(t)} = \mathrm{e}^{\mathrm{i}N_0 t} \prod_{k=1}^{Q} \frac{\mathrm{e}^{\mathrm{i}t} - 1/q_k}{1 - (1/\overline{q_k})\mathrm{e}^{\mathrm{i}t}}$$

is

$$\theta'_{\mathrm{all}}(\mathrm{e}^{\mathrm{i}t}) = N_0 + \sum_{k=1}^{Q} \frac{1 - \frac{1}{|q_k|^2}}{1 - 2\frac{1}{|q_k|} \cos(t + \phi_k) + \frac{1}{|q_k|^2}} > 0. \tag{1.5}$$

In signal analysis and system theory (see [22]), a polynomial circular signal $f_a(\mathrm{e}^{\mathrm{i}t})$ can be recognized as the counterpart of the frequency response of a finite impulse response (FIR) filter. The signal $f_{\min}(z)$ and $f_{\mathrm{all}}(z)$ are referred as the minimum-phase (MinP) signal or the outer function part, and the all-phase signal or the inner function part, respectively, of the given analytic signal $f_a(z)$. Moreover, $f_{\min}(z)$ can be uniquely determined by the amplitude $|f_a(\mathrm{e}^{\mathrm{i}t})|$ within a multiplicative constant difference. The characteristic property of minimum-phase signals as analytic functions of one complex variable is that they do not have zeros inside the unit disc. The instantaneous frequency of $f_{\mathrm{all}}(\mathrm{e}^{\mathrm{i}t})$, as a function, is positive everywhere, and, in fact, expressed as a finite sum of the Poisson kernels at the zeros of the corresponding complex inner function (see [12,24]). More generally, if an analytic signal $f_a(\mathrm{e}^{\mathrm{i}t}) \in H^2(\partial \boldsymbol{D})$ has an infinite number of non-zero Fourier coefficients, which usually corresponds to an infinite impulse response filter (IFIR), the analytic signal $f_a(\mathrm{e}^{\mathrm{i}t})$ can still be factorized into two parts of which one is a minimum-phase and the other an all-phase signal. The minimum-phase and the all-phase signals of infinite series possess the same properties as those of analytic polynomials (see [12, 25]).

There have been continuing studies on minimum-phase and all-phase signals of which the recent ones include (see [7, 10, 11, 18, 25]). In particular, a comprehensive study on analytic signals with positive analytic phase derivatives, viz., *mono-components*, and expansions of general analytic signals into mono-components, has been well-pursued in recent publications (see [16, 23–26, 28]). Classical and operator-valued inner and outer functions are crucial subjects in the recent seminal work of Ball and Bolotnikov [4, 5] that brings separate but closely related, and, in fact, equivalent four topics together, viz., operator-valued inner functions, shift-invariant subspaces (the Beurling-Lax theorem), conservative discrete-time input/state/output linear systems, and $C_{\cdot 0}$ Hilbert-space contraction operators. The most recent interest on mono-components expansions in relation to inner and outer functions would be one called *unwinding Blaschke decomposition*, that has a close relationship with the development of the so called *adaptive Fourier decomposition* (AFD) by Alpay et al. [2, 3], Qian and Wang [28] and Qian [26, 27]. Unwinding decomposition was first studied by Nahon [21] in the form of a PhD thesis in Yale University in 2000, which was followed by a formal publication in 2016 (see [9]). Unwinding Blaschke expansion was independently studied by Qian [26] since 2010. It is noted that extracting the minimum-phase part or, equivalently, the all-phase part of an analytic signal is a crucial step of applications of the unwinding decomposition. The unwinding decomposition itself and the related computation issues are described as follows.

Let $F(z) = F_1(z)$ be a function in the complex Hardy $H^2(\boldsymbol{D})$ space and $F_1 = I_1O_1$ its Nevanlinna factorization into a product of an inner function and an outer function. The inner function $I_1$ can be further factorized into $I_1 = B_1S_1$, where $B_1$ is a Blaschke product, collecting all the zeros of $F_1$, finitely or infinitely many, and $S_1$ is a singular inner function induced by a Borel measure singular with respect to the Lebesgue measure on the circle. Note that $O_1$ and $S_1$ do not have zeros inside the unit disc, and all the related functions $B_1, S_1$ and $O_1$ are unique up to unimodular constants (see [12]). Correspondingly, we can also write $F_1 = B_1G_1$, where $G_1 = S_1O_1$. Coifman and Steinerberger [9] reproved this result. If $F_1$ is a finite polynomial or a rational function, or $F_1$ has an analytic continuation across the unit circle, or with other suitable conditions, then $S_1$ becomes trivial and ignorable, being equal to a unimodular constant, and then $F_1 = B_1G_1$ with $G_1 = O_1$. Below we will assume this case at each of the recursive steps. We proceed as

$$\begin{aligned} F(z) = F_1(z) &= B_1(z)G_1(z) \\ &= B_1(z)[G_1(0) + (G_1(z) - G_1(0))] \\ &= a_1B_1(z) + B_1(z)F_2(z) \\ &= a_1B_1(z) + B_1(z)B_2(z)G_2(z) \\ &\cdots \\ &= \sum_{k=1}^{n} a_kB_1(z)\cdots B_k(z) + B_1(z)\cdots B_n(z)F_{n+1}(z), \end{aligned}$$

where for each $k, a_k = G_k(0), F_{k+1}(z) = G_k(z) - G_k(0), F_k = B_kG_k$, and $B_k$ is a (finite or infinite) Blaschke product, and $G_k$ is an outer function. It is shown in [26] (see also [9, Example 4.3]) that

$$\lim_{n\to\infty} \|B_1\cdots B_nF_{n+1}\|_2 = 0,$$

and so, in the $L^2$-sense on the boundary,

$$F(z) = \sum_{k=1}^{\infty} a_kB_1(z)\cdots B_k(z).$$

There are several variations of the described decomposition (see [9, 26]). We note that at each recursive step $B_k$ does not necessarily collect all the zeros of $G_k$. Taking this as granted, Fourier series expansion is seen to be a particular case of the general decomposition. The advantages of such type decompositions include that each entry of the sum has a well-defined positive phase derivative function (see [25]), which is defined to be the instantaneous frequency of the entry; and the fast convergence, being expected to be of the exponential speed. The decomposition is also robust with noise corruption. There are, however, some inconveniences with implementation of the unwinding decomposition due to the difficulty of computing the outer function and the inner function parts. Finding zeros is a difficult task even for polynomials of finite degree, let alone for infinite series with infinitely many zeros. Some attempts along this line are made in [17]. On the other hand, computing the outer function part of an analytic signal involves computation of Hilbert transform. Hilbert transform is a singular integral whose computation itself is a hard problem having attracted ample studies (see [9, 21]). Various ways for additional stabilization have been proposed for the algorithm based on Hilbert transform: the stabilizing effect of adding a small positive constant was investigated by Nahon [21], whereas Letelier and Saito [15] proposed adding a small pure sinusoid. Adaptive Fourier decomposition, $n$-best approximation by Blaschke forms and unwinding Fourier decomposition have effective applications in signal analysis and system identification (see [17, 19, 20]). In the related studies algorithm to extract outer and inner functions has central importance.

In this paper, we study effective algorithms to extract the minimum-phase part from an analytic signal $f_a(\mathrm{e}^{\mathrm{i}t})$. There have been several feasible methods. First, one can find the Fourier coefficients of $f_a(\mathrm{e}^{\mathrm{i}t})$, then find the roots of the obtained polynomial that is to find $p_i$ and $q_i$, and then group them as in (1.3). Alternatively, one can compute the log-envelope of $f_a(\mathrm{e}^{\mathrm{i}t})$, i.e., $\ln|f_a(\mathrm{e}^{\mathrm{i}t})|$, and subsequently compute its Hilbert transform (see [9, 22]), etc. Computation of a Hilbert transform itself, however, is not an easy task in mathematical computation. Recently, based on Szegő's idea, Kumaresan and Rao [13], Rao and Kumaresan [29], and Kumarasan and Wang [14] proposed a new method to extract the minimum-phase part from a polynomial analytic signal

$$f_a(\mathrm{e}^{\mathrm{i}t}) = \mathrm{e}^{\mathrm{i}N_0 t}\sum_{k=0}^{M} a_k \mathrm{e}^{\mathrm{i}kt},$$

which does not require computation of roots of the polynomial, nor Hilbert transform of $\ln|f_a(\mathrm{e}^{\mathrm{i}t})|$. This method, due to its effectiveness, has been adopted in signal analysis and speech analysis. The method, however, has a great room to be discussed and improved.

Their algorithm is based on minimizing the energy $E(f_a, h_1, \ldots, h_H)$. Indeed, for any coefficients $h_1, \ldots, h_H$, we have

$$\begin{aligned} E(f_a, h_1, \ldots, h_H) &= \frac{1}{2\pi}\int_0^{2\pi}\left|1+\sum_{k=1}^{H} h_k \mathrm{e}^{\mathrm{i}kt}\right| |f_{\min}(\mathrm{e}^{\mathrm{i}t})|^2 dt \\ &= \frac{1}{2\pi}\int_0^{2\pi}\left|\left(1+\sum_{k=1}^{H} h_k \mathrm{e}^{\mathrm{i}kt}\right)\left(c_0(f_{\min})+\sum_{k=1}^{\infty} c_k(f_{\min})\mathrm{e}^{\mathrm{i}kt}\right)\right|^2 dt \\ &\geqslant |c_0(f_{\min})|^2. \end{aligned} \tag{1.6}$$

For a fixed positive integer $H$, let

$$P^H_{\min}(\mathrm{e}^{\mathrm{i}t}) = 1 + \sum_{k=1}^{H} h_k^{\min} \mathrm{e}^{\mathrm{i}kt}$$

denote the function at which $E(f_a, h_1, \ldots, h_H)$ attains the minimum energy. From (1.6), we know that the energy $E(f_a, h_1, \ldots, h_H)$ would be able to attain the minimum $|c_0(f_{\min})|^2$ if $P^H_{\min}(\mathrm{e}^{\mathrm{i}t})$ could be chosen as $c_0(f_{\min})/f_{\min}(\mathrm{e}^{\mathrm{i}t})$. For a polynomial $f_{\min}(\mathrm{e}^{\mathrm{i}t})$ and a finite $H$, one, however, cannot expect the relation $P^H_{\min}(\mathrm{e}^{\mathrm{i}t}) = c_0(f_{\min})/f_{\min}(\mathrm{e}^{\mathrm{i}t})$ to hold. This relation can approximately hold when $H$ goes to the infinity. This last assertion can be proved by the well-known Szegő limit theorem (see [30]). The Szegő theorem states that if $|f_a(\mathrm{e}^{\mathrm{i}t})| > 0$ and $f_a(\mathrm{e}^{\mathrm{i}t}) \in L^1(\partial \boldsymbol{D})$, then

$$\lim_{H\to\infty} P^H_{\min}(\mathrm{e}^{\mathrm{i}t}) = \frac{\exp\{\frac{1}{2\pi}\int_0^{2\pi}\ln|f_a(\mathrm{e}^{\mathrm{i}t})|dt\}}{\exp\{\ln|f_a|(\mathrm{e}^{\mathrm{i}t}) + \mathrm{i}\widetilde{H}(\ln|f_a|)(\mathrm{e}^{\mathrm{i}t})\}} = \frac{f_{\min}(0)}{f_{\min}(\mathrm{e}^{\mathrm{i}t})},$$

where the numerator identity is obtained by evaluating the outer space function given by (3.1) at $z = 0$.

We note that both the numerator and the denominator of the right-hand-end are obtained through the representation of the minimum-phase signal in terms of the boundary data $\ln|f_a(\mathrm{e}^{\mathrm{i}t})| = \ln|f_{\min}(\mathrm{e}^{\mathrm{i}t})|$. Hence, for a large $H$, the function $f_{\min}(0)/P^H_{\min}(\mathrm{e}^{\mathrm{i}t})$ can be seen as an approximation to the minimum-phase component of $f_a(\mathrm{e}^{\mathrm{i}t})$.

In the present paper we make a number of improvements to the method proposed by Kumaresan and Rao [13]. We show in Section 2 that if $f_a(\mathrm{e}^{\mathrm{i}t})$ is a polynomial analytic signal, as given by $f_a(\mathrm{e}^{\mathrm{i}t}) = \mathrm{e}^{\mathrm{i}N_0 t}\sum_{k=0}^{M} a_k \mathrm{e}^{\mathrm{i}kt}$ under the mentioned conventions on $N_0, a_0$ and $a_M$, then the minimum-phase signal of

$f_a(\mathrm{e}^{\mathrm{i}t})$ can be exactly extracted out through the least mean square error method solving the extremal problem (0.2) for any fixed $H \geqslant M$, but replacing $f_a(\mathrm{e}^{\mathrm{i}t})$ with $\frac{1}{f_a(\mathrm{e}^{\mathrm{i}t})}$ in (0.2). Furthermore, in Section 3, replacing $\mathrm{e}^{\mathrm{i}kt}$ by the Takenaka-Malmquist (TM) system

$$r_k(\mathrm{e}^{\mathrm{i}t}) := \frac{\sqrt{1-|\alpha_k|^2}\mathrm{e}^{\mathrm{i}t}}{1-\overline{\alpha_k}\mathrm{e}^{\mathrm{i}t}} \prod_{j=1}^{k-1} \frac{\mathrm{e}^{\mathrm{i}t}-\alpha_j}{1-\overline{\alpha_j}\mathrm{e}^{\mathrm{i}t}}$$

for $k \geqslant 1$, we show that the same least mean square error method but based on the TM system can also be used to extract out approximations of the minimum-phase part of any analytic signal. Comparing with the trigonometric basis, the TM system has the advantage of flexible choices of the parameters $\alpha_1, \ldots, \alpha_n, \ldots$ We show that for any rational analytic signal $f_a$ of degree $n$ by using the $n$-best choice of the $n$-tuple $\{\alpha_1, \ldots, \alpha_n\}$ (see [27]), the minimum-phase part of $f_a$ can be exactly and efficiently extracted out. The proofs of the results of this paper implement and further develop the techniques of Szegő [30] in proving analogous results for polynomials.

## 2 The decomposition algorithm based on Trigonometric system

In this section, we assume that $f_a(\mathrm{e}^{\mathrm{i}t})$ is a polynomial analytic signal $f_a(\mathrm{e}^{\mathrm{i}t}) = \mathrm{e}^{\mathrm{i}N_0t}\sum_{k=0}^M a_k\mathrm{e}^{\mathrm{i}kt}$, where $a_0 \neq 0$ and $a_M \neq 0$. We show that if, instead of minimizing the error signal $e(\mathrm{e}^{\mathrm{i}t}) = f_a(\mathrm{e}^{\mathrm{i}t})P_H(\mathrm{e}^{\mathrm{i}t})$ (see [13]) we minimize the energy of an error signal $e(\mathrm{e}^{\mathrm{i}t}) = \frac{P_H(\mathrm{e}^{\mathrm{i}t})}{f_a(\mathrm{e}^{\mathrm{i}t})}$, then the function $f_{\min}(\mathrm{e}^{\mathrm{i}t})$ of $f_a(\mathrm{e}^{\mathrm{i}t})$ can be exactly extracted out for any $H \geqslant M$. The proof uses the knowledge of orthogonal projection into a finite dimensional space in the Hilbert space induced by a weighted Lebesgue measure.

**Theorem 2.1.** *Let $f_a(\mathrm{e}^{\mathrm{i}t}) = \mathrm{e}^{\mathrm{i}N_0t}\sum_{k=0}^M a_k\mathrm{e}^{\mathrm{i}kt}$ be a polynomial circular analytic signal and $|f_a(\mathrm{e}^{\mathrm{i}t})| > 0$, where $a_0 \neq 0$ and $a_M \neq 0$. Then for any $H \geqslant M$ the function*

$$P^H_{\min}(\mathrm{e}^{\mathrm{i}t}) = 1 + \sum_{k=1}^H h^{\min}_k \mathrm{e}^{\mathrm{i}kt}$$

*minimizing the integral*

$$\frac{1}{2\pi}\int_0^{2\pi} \left|\frac{1+\sum_{k=1}^H h_k\mathrm{e}^{\mathrm{i}kt}}{f_a(t)}\right|^2 dt$$

*is identical with*

$$P^H_{\min}(\mathrm{e}^{\mathrm{i}t}) = \frac{f_{\min}(\mathrm{e}^{\mathrm{i}t})}{f_{\min}(0)} = \prod_{k=1}^P (1-p_k\mathrm{e}^{\mathrm{i}t}) \prod_{k=1}^Q \left(1-\frac{1}{\overline{q_k}}\mathrm{e}^{\mathrm{i}t}\right),$$

*where $M = P+Q$, $p_k$ and $q_k$ are, respectively, the zeros of $F(z) = \sum_{k=0}^M a_kz^k$ outside and inside the unit circle.*

*Proof.* Let $\{\phi^\mu_0(\mathrm{e}^{\mathrm{i}t}), \phi^\mu_1(\mathrm{e}^{\mathrm{i}t}), \ldots, \phi^\mu_H(\mathrm{e}^{\mathrm{i}t})\}$ be the orthonormal system obtained through applying the Gram-Schmidt orthogonalization process to $\{1, \mathrm{e}^{\mathrm{i}t}, \mathrm{e}^{\mathrm{i}2t}, \ldots, \mathrm{e}^{\mathrm{i}Ht}\}$ with respect to the measure $d\mu(t) = dt/|f_a(\mathrm{e}^{\mathrm{i}t})|^2$. Then the minimum error energy is

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{P^H_{\min}(\mathrm{e}^{\mathrm{i}t})}{f_a(\mathrm{e}^{\mathrm{i}t})}\right|^2 dt &= \frac{1}{2\pi}\int_0^{2\pi}\frac{|\mathrm{e}^{\mathrm{i}Ht}\overline{P^H_{\min}(\mathrm{e}^{\mathrm{i}t})}|^2}{|f_a(\mathrm{e}^{\mathrm{i}t})|^2}dt \\ &= \min_{\{d_0,d_1,\ldots,d_{H-1}\}}\frac{1}{2\pi}\int_0^{2\pi}\left|\mathrm{e}^{\mathrm{i}Ht}+\sum_{k=0}^{H-1}d_k\mathrm{e}^{\mathrm{i}kt}\right|^2\frac{1}{|f_a(\mathrm{e}^{\mathrm{i}t})|^2}dt \\ &= \frac{1}{2\pi}\int_0^{2\pi}\left|\mathrm{e}^{\mathrm{i}Ht}-\sum_{k=0}^{H-1}\langle \mathrm{e}^{\mathrm{i}Ht}, \phi^\mu_k(\mathrm{e}^{\mathrm{i}t})\rangle_\mu \phi^\mu_k(\mathrm{e}^{\mathrm{i}t})\right|^2\frac{1}{|f_a(\mathrm{e}^{\mathrm{i}t})|^2}dt.\end{aligned}$$

This implies that

$$\mathrm{e}^{\mathrm{i}Ht}\overline{P^H_{\min}(\mathrm{e}^{\mathrm{i}t})} = \mathrm{e}^{\mathrm{i}Ht} - \sum_{k=0}^{H-1}\langle \mathrm{e}^{\mathrm{i}Ht}, \phi^\mu_k(\mathrm{e}^{\mathrm{i}t})\rangle_\mu \phi^\mu_k(\mathrm{e}^{\mathrm{i}t}).$$

As a consequence, $\mathrm{e}^{\mathrm{i}Ht}\overline{P^H_{\min}(\mathrm{e}^{\mathrm{i}t})}$ is orthogonal with $\phi^\mu_k(t)$ with respect to the measure $d\mu(t)=dt/|f_a(\mathrm{e}^{\mathrm{i}t})|^2$ for $k=0,1,\ldots,H-1$. In the following, we show the identification

$$\mathrm{e}^{\mathrm{i}Ht}\overline{P^H_{\min}(\mathrm{e}^{\mathrm{i}t})}=\mathrm{e}^{\mathrm{i}Ht}\overline{\left[\frac{f_{\min}(\mathrm{e}^{\mathrm{i}t})}{f_{\min}(0)}\right]}$$

for $H\geqslant M$, where $f_{\min}(\mathrm{e}^{\mathrm{i}t})$ is the polynomial minimum-phase component of order $M$ given in (1.4). We first show that $\mathrm{e}^{\mathrm{i}Ht}\overline{[\frac{f_{\min}(\mathrm{e}^{\mathrm{i}t})}{f_{\min}(0)}]}$ is orthogonal to $\phi^\mu_k(\mathrm{e}^{\mathrm{i}t})$ with respect to the measure $d\mu(t)=dt/|f_a(\mathrm{e}^{\mathrm{i}t})|^2$. Since $\phi^\mu_k(\mathrm{e}^{\mathrm{i}t})$ is a trigonometric polynomial of degree $k$, it suffices to prove that $\mathrm{e}^{\mathrm{i}Ht}\overline{[\frac{f_{\min}(\mathrm{e}^{\mathrm{i}t})}{f_{\min}(0)}]}$ is orthogonal with an arbitrary polynomial signal $\rho_k(\mathrm{e}^{\mathrm{i}t})$ of order $k$ with respect to the measure

$$d\mu(t)=dt/|f_a(\mathrm{e}^{\mathrm{i}t})|^2=1/|f_{\min}(\mathrm{e}^{\mathrm{i}t})|^2.$$

Let $\rho_k(\mathrm{e}^{\mathrm{i}t})=\sum_{m=0}^k b_k\mathrm{e}^{\mathrm{i}mt}$ and $k\leqslant H-1$. By the Cauchy theorem, we have

$$\frac{1}{2\pi\overline{f_{\min}(0)}}\int_0^{2\pi}\mathrm{e}^{\mathrm{i}Ht}\overline{f_{\min}(\mathrm{e}^{\mathrm{i}t})\rho_k(\mathrm{e}^{\mathrm{i}t})}\frac{1}{f_{\min}(\mathrm{e}^{\mathrm{i}t})\overline{f_{\min}(\mathrm{e}^{\mathrm{i}t})}}dt=\frac{1}{2\pi\overline{f_{\min}(0)}}\oint_{|z|=1}\frac{z^H\overline{\rho_k(1/\overline{z})}}{f_{\min}(z)}\frac{dz}{\mathrm{i}z}=0,$$

where $\rho_k(z)=\sum_{m=0}^k b_kz^k$ and

$$f_{\min}(z)=\prod_{k=1}^P(1-p_kz)\prod_{k=1}^Q\left(1-\frac{1}{\overline{q_k}}z\right).$$

We note that $P^H_{\min}(\mathrm{e}^{\mathrm{i}t})$ is assumed to have the form $P^H_{\min}(z)=1+\sum_{k=1}^H h^{\min}_kz^k$. Thus, $P^H_{\min}(0)=1=\frac{f_{\min}(0)}{f_{\min}(0)}$. This proves the relation

$$\mathrm{e}^{\mathrm{i}Ht}\overline{P^H_{\min}(\mathrm{e}^{\mathrm{i}t})}=\mathrm{e}^{\mathrm{i}Ht}\overline{\left[\frac{f_{\min}(t)}{f_{\min}(0)}\right]}.$$

Therefore, by (1.4), we have

$$P^H_{\min}(\mathrm{e}^{\mathrm{i}t})=\frac{f_{\min}(\mathrm{e}^{\mathrm{i}t})}{f_{\min}(0)}=\prod_{k=1}^P(1-p_k\mathrm{e}^{\mathrm{i}t})\prod_{k=1}^Q\left(1-\frac{1}{\overline{q_k}}\mathrm{e}^{\mathrm{i}t}\right),$$

where $p_k$ and $q_k$, respectively denote the zeros of $F(z)=\sum_{k=0}^M a_kz^k$ inside and outside the unit circle. □

From the above theorem, we learn that given a polynomial signal $f_a(\mathrm{e}^{\mathrm{i}t})=\mathrm{e}^{\mathrm{i}N_0t}\sum_{k=0}^M a_k\mathrm{e}^{\mathrm{i}kt}$, its minimum-phase signal $f^{-1}_{\min}(0)f_{\min}(\mathrm{e}^{\mathrm{i}t})$ can be exactly extracted out through minimizing the energy of the error signal $e(t)=\frac{h_H(\mathrm{e}^{\mathrm{i}t})}{f_a(\mathrm{e}^{\mathrm{i}t})}$ for $H\geqslant M$. In order to show the efficiency of the new algorithm corresponding to Theorem 2.1, we apply the algorithm to an example used in [13]. Before that we first review the algorithm design. By the properties of the least mean square error (see [30]), the coefficients $\{h^{\min}_1,\ldots,h^{\min}_H\}$ at which the integral (0.2) attains the minimum can be obtained by solving the following linear equation:

$$\begin{pmatrix}\langle l_1,l_1\rangle_\mu & \langle l_1,l_2\rangle_\mu & \cdots & \langle l_1,l_H\rangle_\mu\\ \langle l_2,l_1\rangle_\mu & \langle l_2,l_2\rangle_\mu & \cdots & \langle l_2,l_H\rangle_\mu\\ \vdots & \vdots & \ddots & \vdots\\ \langle l_H,l_1\rangle_\mu & \langle l_H,l_2\rangle_\mu & \cdots & \langle l_H,l_H\rangle_\mu\end{pmatrix}\begin{pmatrix}h^{\min}_1\\ h^{\min}_2\\ \vdots\\ h^{\min}_H\end{pmatrix}=-\begin{pmatrix}\langle l_1,l_0\rangle_\mu\\ \langle l_2,l_0\rangle_\mu\\ \vdots\\ \langle l_H,1_0\rangle_\mu\end{pmatrix}, \tag{2.1}$$

where $l_k(t)=\mathrm{e}^{\mathrm{i}kt}$ for $k=0,1,\ldots,H$, $d\mu(t)=dt/|f_a(\mathrm{e}^{\mathrm{i}t})|^2$ and the inner product $\langle l_i,l_j\rangle_\mu$ is defined by

$$\langle l_i,l_j\rangle_\mu=\frac{1}{2\pi}\int_0^{2\pi}l_i(t)\overline{l_j(t)}\mu(t)dt,\quad i=1,\ldots,H,\quad j=0,1,\ldots,H.$$

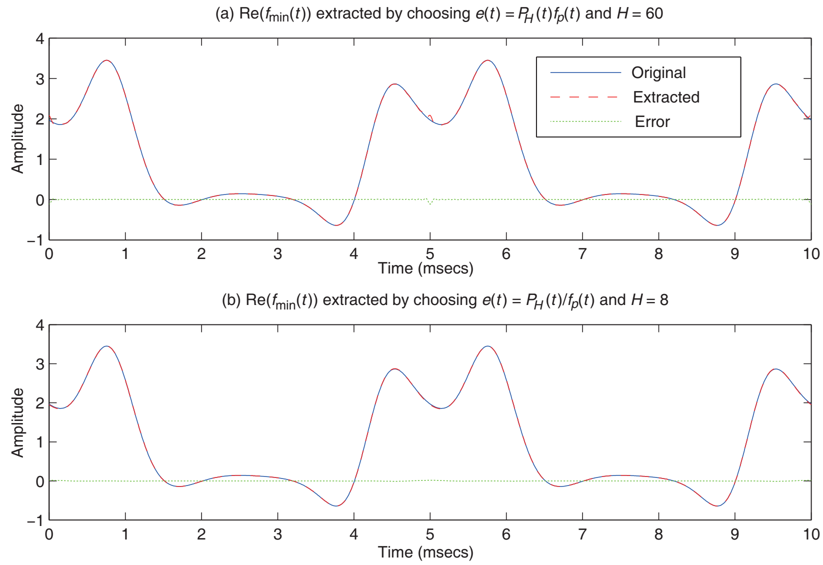


**Figure 1** The real part of the minimum phase signal of $f_p(t)$

Denoting $\boldsymbol{L}, \boldsymbol{h}_H^{\min}$ and $\boldsymbol{e}$ the matrices/vectors from left to right in (2.1), the solution vector $\boldsymbol{h}_H^{\min}$ is given by

$$\boldsymbol{h}_H^{\min} = \boldsymbol{L}^{-1}\boldsymbol{e},$$

where $\boldsymbol{L}^{-1}$ denotes the inverse matrix of $\boldsymbol{L}$. In order to implement the algorithm by the computer, we need discrete sample values of the signal. Let $\mu[n]$ and $l_k[n]$ respectively denote the samples of the given functions $\mu(t)$ and $l_k(t)$ at $t_k = k\Delta t$, where $\Delta t = \frac{2\pi}{N}$ and $n = 0, 1, \ldots, N$. Through replacing $\langle l_i, l_j\rangle_\mu$ by $\langle l_i, l_j\rangle_\mu \approx \frac{1}{N}\sum_{k=0}^{N-1} l_i[k]\overline{l_j[k]}\mu[n]$, a simulation result of the vector $\widetilde{\boldsymbol{h}}_H^{\min}$ can be given by

$$\widetilde{\boldsymbol{h}}_H^{\min} \approx -(\boldsymbol{H}^{\mathrm{T}}\boldsymbol{H})^{-1}\boldsymbol{H}^{\mathrm{T}}\boldsymbol{U} \tag{2.2}$$

where $\boldsymbol{U} = [\mu^{1/2}(0), \mu^{1/2}(1), \ldots, \mu^{1/2}(N)]^{\mathrm{T}}$ and

$$\boldsymbol{H} = \begin{pmatrix} \mu^{1/2}(0), & 0 & \cdots\cdots & & 0 \\ 0 & \mu^{1/2}(1) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \mu^{1/2}(N) \end{pmatrix} \times \begin{pmatrix} l_1(0) & l_2(0) & \cdots & l_H(0) \\ l_1(1) & l_2(1) & \cdots & l_H(1) \\ \vdots & \vdots & \ddots & \vdots \\ l_1(N) & l_2(N) & \cdots & l_H(N) \end{pmatrix}, \tag{2.3}$$

where the capital letter $T$ on the right-shoulder of a matrix stands for the conjugate transpose of the matrix.

Kumaran and Rao [13] used the following example to test their algorithm. We will now use the same example for comparison between their algorithm and ours. We note that their examples are on practical functions of, in general, $T$-periodicity, not $2\pi$-periodicity, the latter being convenient for implementation of function theory of one complex variable as we have been doing in this paper. These two settings are easily converted to each other through a linear change of variable. In the general $T$-periodicity case, in the following example and the next one as well, we adopt the easy function notation $f_a(\mathrm{e}^{\mathrm{i}\Omega t}), \Omega T = 2\pi$, instead of $f_a(\mathrm{e}^{\mathrm{i}t})$, and so on.

Let $f_p(\mathrm{e}^{\mathrm{i}\Omega t})$ be the polynomial consisting of nine harmonic waves as

$$\begin{aligned} f_p(\mathrm{e}^{\mathrm{i}\Omega t}) = 1 &+ 3.37\mathrm{e}^{-\mathrm{i}0.3}\mathrm{e}^{\mathrm{i}\Omega t} + 3.42\mathrm{e}^{-\mathrm{i}1.3}\mathrm{e}^{\mathrm{i}2\Omega t} + 9.45\mathrm{e}^{-\mathrm{i}3.1}\mathrm{e}^{\mathrm{i}3\Omega t} + 15.76\mathrm{e}^{\mathrm{i}2.8}\mathrm{e}^{\mathrm{i}4\Omega t} + 5.4\mathrm{e}^{\mathrm{i}2.7}\mathrm{e}^{\mathrm{i}5\Omega t} \\ &+ 5.4\mathrm{e}^{-1.3}\mathrm{e}^{\mathrm{i}6\Omega t} + 3.72\mathrm{e}^{-\mathrm{i}0.9}\mathrm{e}^{\mathrm{i}7\Omega t} + 1.5\mathrm{e}^{-\mathrm{i}0.6}\mathrm{e}^{\mathrm{i}8\Omega t}, \end{aligned}$$

where $\Omega = 2\pi \times 200$ Hz. By a simple computation, $f_p(\mathrm{e}^{\mathrm{i}\Omega t})$ corresponds to a mixed phase signal consisting of four zeros inside and four zeros outside the unit circle. They use $H = 60$ with the error signal $e(\mathrm{e}^{\mathrm{i}t}) = P_H(\mathrm{e}^{\mathrm{i}\Omega t}) f_p(\mathrm{e}^{\mathrm{i}\Omega t})$ to approximate the minimum-phase component of $f_p(\mathrm{e}^{\mathrm{i}\Omega t})$. The real part of the minimum-phase component of $f_p(\mathrm{e}^{\mathrm{i}\Omega t})$ extracted by their algorithm is shown in Figure 1(a). If we employ $e(t) = \frac{P_H(\mathrm{e}^{\mathrm{i}\Omega t})}{f_p(\mathrm{e}^{\mathrm{i}\Omega t})}$, using the algorithm corresponding to Theorem 2.1 under $H = 8$, the minimum-phase component of $f_p(\mathrm{e}^{\mathrm{i}\Omega t})$ can be exactly extracted out. The simulation result is shown in Figure 1(b).

## 3 The decomposition algorithm based on the Takenaka-Malmquist system

It is a fact that the minimum-phase signal of an analytic signal $f_a(\mathrm{e}^{\mathrm{i}t})$ is uniquely determined by the log-amplitude of $f_a(\mathrm{e}^{\mathrm{i}t})$, i.e.,

$$f_{\min}(\mathrm{e}^{\mathrm{i}t}) = \exp\{\ln|f_a|(\mathrm{e}^{\mathrm{i}t}) + \mathrm{i}\widetilde{H}\ln|f_a|(\mathrm{e}^{\mathrm{i}t})\}.$$

If the values of $|f_a(\mathrm{e}^{\mathrm{i}t})|$ are not almost everywhere enclosed in a compact interval of $(0,\infty)$, then $\ln|f_a(\mathrm{e}^{\mathrm{i}t})|$ will be essentially unbounded. The unboundedness causes unstability of the decomposition algorithm. In order to avoid such situation, we focus on the subspace $A_0(\partial\boldsymbol{D})$ of $H^2(\partial\boldsymbol{D})$, where $A_0(\partial\boldsymbol{D})$ consists of all functions $f_a(\mathrm{e}^{\mathrm{i}t})$ in $H^2(\partial\boldsymbol{D})$ that are, moreover, continuous on $\partial\boldsymbol{D}$ and $|f_a(\mathrm{e}^{\mathrm{i}t})| > 0$. Obviously, the polynomial circular analytic signals discussed in Section 2 belong to $A_0(\partial\boldsymbol{D})$. For general $f_a(\mathrm{e}^{\mathrm{i}t}) \in A_0(\partial\boldsymbol{D})$, it can be shown that its all-phase part is the non-tangential boundary limit of a finite Blaschke product, i.e., $f_a(\mathrm{e}^{\mathrm{i}t})$ can be decomposed as

$$f_a(\mathrm{e}^{\mathrm{i}t}) = \underbrace{\mathrm{e}^{\mathrm{i}\gamma}\exp\{\ln|f_a|(\mathrm{e}^{\mathrm{i}t}) + \mathrm{i}\widetilde{H}(\ln|f_a|)(\mathrm{e}^{\mathrm{i}t})\}}_{f_{\min}(\mathrm{e}^{\mathrm{i}t})} \underbrace{\prod_{k=1}^{N}\frac{\mathrm{e}^{\mathrm{i}t}-\alpha_k}{1-\overline{\alpha_k}\mathrm{e}^{\mathrm{i}t}}}_{f_{\mathrm{all}}(\mathrm{e}^{\mathrm{i}t})},$$

where $\gamma$ is a real constant, $\alpha_1,\ldots,\alpha_N$ are points in the unit disc $\mathbb{D}$ (see [31]). The minimum-phase signal $f_{\min}(\mathrm{e}^{\mathrm{i}t})$, on the other hand, is the non-tangential boundary limit function of the outer function $f_{\min}(z)$, $z = r\mathrm{e}^{\mathrm{i}t}, 0 \leqslant r < 1$, with

$$f_{\min}(z) = \mathrm{e}^{\mathrm{i}\gamma}\exp\left\{\frac{1}{2\pi}\int_0^{2\pi}\frac{\mathrm{e}^{\mathrm{i}\theta}+z}{\mathrm{e}^{\mathrm{i}\theta}-z}\ln|f_a(\mathrm{e}^{\mathrm{i}\theta})|d\theta\right\}. \tag{3.1}$$

In this section, we will show that, replacing the trigonometric series $\mathrm{e}^{\mathrm{i}kt}$ with the rational orthogonal basis $r_k(\mathrm{e}^{\mathrm{i}t})$ in (0.2), an approximation to the minimum-phase signal of $f_a(\mathrm{e}^{\mathrm{i}t}) \in A_0(\partial\boldsymbol{D})$ can also be obtained by minimizing the integral

$$\frac{1}{2\pi}\int_0^{2\pi}\left|1+\sum_{k=1}^{H}h_k r_k(\mathrm{e}^{\mathrm{i}t})\right|^2\frac{1}{|f_a(\mathrm{e}^{\mathrm{i}t})|^2}dt, \tag{3.2}$$

where $\{h_k\}_{k=1}^{H}$ is a set of $H$ complex numbers, $r_k(\mathrm{e}^{\mathrm{i}t})$ is given by

$$r_0(\mathrm{e}^{\mathrm{i}t}) = 1, \quad r_k(\mathrm{e}^{\mathrm{i}t}) = \frac{\sqrt{1-|\alpha_k|^2}\mathrm{e}^{\mathrm{i}t}}{1-\overline{\alpha_k}\mathrm{e}^{\mathrm{i}t}}\prod_{j=1}^{k-1}\frac{\mathrm{e}^{\mathrm{i}t}-\alpha_j}{1-\overline{\alpha_j}\mathrm{e}^{\mathrm{i}t}}, \quad k \geqslant 1. \tag{3.3}$$

When all $\alpha_k$ are chosen to be zero, the rational orthogonal system $\{r_k(\mathrm{e}^{\mathrm{i}t})\}_{k=0}^{\infty}$ reduces to the trigonometric series $\{\mathrm{e}^{\mathrm{i}kt}\}_{k=0}^{\infty}$. When all $\alpha_k = b$, $0 < |b| < 1$, the rational orthogonal system $\{r_k(\mathrm{e}^{\mathrm{i}t})\}_{k=0}^{\infty}$ reduces to the Lagurre basis

$$\left\{1, \frac{\sqrt{1-|b|^2}\mathrm{e}^{\mathrm{i}t}}{1-\bar{b}\mathrm{e}^{\mathrm{i}t}}\left(\frac{\mathrm{e}^{\mathrm{i}t}-b}{1-\bar{b}\mathrm{e}^{\mathrm{i}t}}\right)^{k-1}\right\}_{k=1}^{\infty}.$$

The rational orthogonal series $\{r_k(\mathrm{e}^{\mathrm{i}t})\}_{k=0}^{\infty}$, sometimes being referred as Takenaka-Malmquist (TM) system, has been widely studied and applied to system identification, rational approximation, positive-frequency representation of signals, and so on (see [1,6,28,32]). If the points $\alpha_k \in \mathbb{D}$ satisfy the condition

$$\sum_{k=1}^{\infty}(1-|\alpha_k|)=\infty, \tag{3.4}$$

then $\{r_k(\mathrm{e}^{\mathrm{i}t})\}_{k=0}^{\infty}$ is an orthonormal basis of $H^2(\partial \boldsymbol{D})$. We will call each $r_k(\mathrm{e}^{\mathrm{i}t})$ a *modified Blaschke product* and $\{r_k(\mathrm{e}^{\mathrm{i}t})\}_{k=0}^{H}$ an $H$-TM system, in brief.

Assume that $R_{\min}^H(\alpha_1,\ldots,\alpha_H,\mathrm{e}^{\mathrm{i}t})$ is a function at which the integral (3.2) attains the minimum energy. Then we have

$$\begin{aligned}\mathcal{E}_{\min}^H(f_a,\alpha_1,\ldots,\alpha_H)&=\frac{1}{2\pi}\int_0^T|R_{\min}^H(\alpha_1,\ldots,\alpha_H,\mathrm{e}^{\mathrm{i}t})|^2\frac{1}{|f_a(\mathrm{e}^{\mathrm{i}t})|^2}dt\\&=\min_{h_1,\ldots,h_H}\frac{1}{2\pi}\int_0^{2\pi}\left|1+\sum_{k=1}^{H}h_kr_k(\mathrm{e}^{\mathrm{i}t})\right|^2\frac{1}{|f_a(\mathrm{e}^{\mathrm{i}t})|^2}dt\\&=\min_{R_H(\mathrm{e}^{\mathrm{i}t})\in\mathcal{NR}_H(\partial\boldsymbol{D})}\frac{1}{2\pi}\int_0^{2\pi}|R_H(\mathrm{e}^{\mathrm{i}t})|^2\frac{1}{|f_a(\mathrm{e}^{\mathrm{i}t})|^2}dt,\end{aligned}$$

where $\mathcal{NR}_H(\partial\boldsymbol{D})=\{r(\mathrm{e}^{\mathrm{i}t})\in\mathcal{R}_H(\partial\boldsymbol{D})\,|\,r(0)=1\}$ is the space of zero-term normalized rational functions in $\mathcal{R}_H(\partial\boldsymbol{D})$, the latter stands for the rational function space spanned by $\{r_k(\mathrm{e}^{\mathrm{i}t})\}_{k=0}^{H}$. A minimizer function $R_{\min}^H(\alpha_1,\ldots,\alpha_H,\mathrm{e}^{\mathrm{i}t})$ belongs to $\mathcal{NR}_H(\partial\boldsymbol{D})$.

Let $\{\phi_0^r(\mathrm{e}^{\mathrm{i}t}),\phi_1^r(\mathrm{e}^{\mathrm{i}t}),\ldots,\phi_H^r(\mathrm{e}^{\mathrm{i}t})\}$ be an orthonormal system obtained by the Gram-Schmidt orthogonalization of $\{r_0(\mathrm{e}^{\mathrm{i}t}),r_1(\mathrm{e}^{\mathrm{i}t}),\ldots,r_H(\mathrm{e}^{\mathrm{i}t})\}$ with respect to the positive measure $d\mu(t)=dt/|f_a(\mathrm{e}^{\mathrm{i}t})|^2$. Denote the reproducing kernel of $\mathcal{R}_H(\partial\boldsymbol{D})$ by $S_H(w,z)$, i.e.,

$$S_H(w,z)=\sum_{k=0}^{H}\overline{\phi_k^r(w)}\phi_k^r(z).$$

In the following theorem, we prove that $R_{\min}^H(\alpha_1,\ldots,\alpha_H,\mathrm{e}^{\mathrm{i}t})$ is a minimum-phase signal, and, furthermore, we establish the relation between $R_{\min}^H(\alpha_1,\ldots,\alpha_H,\mathrm{e}^{\mathrm{i}t})$ and $S_H(0,z)$.

**Theorem 3.1.** *Assume that $f_a\in A_0(\partial\boldsymbol{D})$, $H$ is an arbitrary but fixed positive integer, and $R_{\min}^H(\alpha_1,\ldots,\alpha_H,\mathrm{e}^{\mathrm{i}t})$ is a function in $\mathcal{NR}_H(\partial\boldsymbol{D})$ at which the integral* (3.2) *attains the minimum. Then $R_{\min}^H(\alpha_1,\ldots,\alpha_H,\mathrm{e}^{\mathrm{i}t})$ is a minimum-phase signal and*

$$R_{\min}^H(\alpha_1,\ldots,\alpha_H,\mathrm{e}^{\mathrm{i}t})=\frac{S_H(0,\mathrm{e}^{\mathrm{i}t})}{S_H(0,0)}=c\exp\{\ln|R_{\min}^H(\alpha_1,\ldots,\alpha_H,\mathrm{e}^{\mathrm{i}t})|+\mathrm{i}\widetilde{H}(\ln|R_{\min}^H(\alpha_1,\ldots,\alpha_H,\cdot)|)(\mathrm{e}^{\mathrm{i}t})\}.$$

*Furthermore,*

$$\mathcal{E}_{\min}^H(f_a,\alpha_1,\ldots,\alpha_H)=\frac{1}{2\pi}\int_0^{2\pi}|R_{\min}^H(\alpha_1,\ldots,\alpha_H,\mathrm{e}^{\mathrm{i}t})|^2\frac{1}{|f_a(\mathrm{e}^{\mathrm{i}t})|^2}dt=\frac{1}{S_H(0,0)},$$

*where $c$ is a complex constant of modular* 1.

*Proof.* Since $R_{\min}^H(\alpha_1,\ldots,\alpha_H,\mathrm{e}^{\mathrm{i}t})$ is a function in $\mathcal{NR}_H(\partial\boldsymbol{D})$, it has $H$ nonzero roots and can be factorized as

$$R_{\min}^H(\alpha_1,\ldots,\alpha_H,\mathrm{e}^{\mathrm{i}t})=\frac{\prod_{j=1}^{H}(1-\beta_i^{-1}\mathrm{e}^{\mathrm{i}t})}{\prod_{j=1}^{H}(1-\overline{\alpha_j}\mathrm{e}^{\mathrm{i}t})}.$$

Now we show that for all $k=1,\ldots,H$, $|\beta_k|\geqslant 1$. If there held, for some $k$, $|\beta_k|<1$, then we have

$$|R_{\min}^H(\alpha_1,\ldots,\alpha_H,\mathrm{e}^{\mathrm{i}t})|=\left|\frac{\prod_{j=1}^{H}(1-\beta_j^{-1}\mathrm{e}^{\mathrm{i}t})}{\prod_{j=1}^{H}(1-\overline{\alpha_j}\mathrm{e}^{\mathrm{i}t})}\right|$$

$$
\begin{aligned}
&= \left| \frac{(1-\beta_k^{-1}\mathrm{e}^{\mathrm{i}t})\prod_{\substack{j=1\\ j\neq k}}^{H}(1-\beta_j^{-1}\mathrm{e}^{\mathrm{i}t})}{\prod_{j=1}^{H}(1-\overline{\alpha_j}\mathrm{e}^{\mathrm{i}t})} \right| \\
&= |\beta_k^{-1}R'_H(\mathrm{e}^{\mathrm{i}t})|, \qquad (3.5)
\end{aligned}
$$

where

$$
R'_H(\mathrm{e}^{\mathrm{i}t}) = \frac{(1-\overline{\beta_k}\mathrm{e}^{\mathrm{i}t})\prod_{\substack{j=1\\ j\neq k}}^{H}(1-\beta_j^{-1}\mathrm{e}^{\mathrm{i}t})}{\prod_{j=1}^{H}(1-\overline{\alpha_j}\mathrm{e}^{\mathrm{i}t})}.
$$

The last equality (3.5) is due to the unimodular property of the Möbius transform at $\beta_k$, that amounts to the relation

$$
|\beta_k^{-1}(1-\overline{\beta_k}\mathrm{e}^{\mathrm{i}t})| = |1-\beta_k^{-1}\mathrm{e}^{\mathrm{i}t}|.
$$

The assumption $|\beta_k| < 1$ then would imply

$$
\frac{1}{2\pi}\int_0^{2\pi}\frac{|R_{\min}^H(\alpha_1,\ldots,\alpha_H,\mathrm{e}^{\mathrm{i}t})|^2}{|f_a(\mathrm{e}^{\mathrm{i}t})|^2}dt = \frac{|\beta_k|^{-2}}{2\pi}\int_0^{2\pi}\frac{|R'_H(\mathrm{e}^{\mathrm{i}t})|^2}{|f_a(\mathrm{e}^{\mathrm{i}t})|^2}dt > \frac{1}{2\pi}\int_0^{2\pi}\frac{|R'_H(\mathrm{e}^{\mathrm{i}t})|^2}{|f_a(\mathrm{e}^{\mathrm{i}t})|^2}dt.
$$

This contradicts with the assumption that $R_{\min}^H(\alpha_1,\ldots,\alpha_H,z)$ gives rise to the minimum integral value. Hence, $R_{\min}^H(\alpha_1,\ldots,\alpha_H,z)$ contains no zeros in the unit disk and so $R_{\min}^H(\alpha_1,\ldots,\alpha_H,\mathrm{e}^{\mathrm{i}t})$ is a minimum-phase signal. It, therefore, can be expressed as

$$
R_{\min}^H(\alpha_1,\ldots,\alpha_H,\mathrm{e}^{\mathrm{i}t}) = c\exp\{\ln|R_{\min}^H(\alpha_1,\ldots,\alpha_H,\mathrm{e}^{\mathrm{i}t})| + \mathrm{i}\widetilde{H}(\ln|R_{\min}^H(\alpha_1,\ldots,\alpha_H,\cdot)|)(\mathrm{e}^{\mathrm{i}t})\},
$$

where $c$ is a unimodular constant.

Since $\frac{S_H(0,\mathrm{e}^{\mathrm{i}t})}{S_H(0,0)} \in \mathcal{NR}_H(\partial\boldsymbol{D})$, we have

$$
\begin{aligned}
&\min_{R_H(\mathrm{e}^{\mathrm{i}t})\in\mathcal{NR}_H(\partial\boldsymbol{D})}\frac{1}{2\pi}\int_0^{2\pi}|R_H(\mathrm{e}^{\mathrm{i}t})|^2\frac{1}{|f_a(\mathrm{e}^{\mathrm{i}t})|^2}dt \\
&\quad\leqslant \frac{1}{2\pi}\int_0^{2\pi}\frac{|S_H(0,\mathrm{e}^{\mathrm{i}t})|^2}{|S_H(0,0)|^2}\frac{1}{|f_a(\mathrm{e}^{\mathrm{i}t})|^2}dt \\
&\quad\leqslant \frac{1}{|S_H(0,0)|^2}\frac{1}{2\pi}\int_0^{2\pi}\left[\sum_{k=0}^{H}\overline{\phi_k^r(0)}\phi_k^r(\mathrm{e}^{\mathrm{i}t})\right]\left[\sum_{k=0}^{H}\phi_k^r(0)\overline{\phi_k^r(\mathrm{e}^{\mathrm{i}t})}\right]\frac{1}{|f_a(\mathrm{e}^{\mathrm{i}t})|^2}dt \\
&\quad= \frac{1}{\sum_{k=0}^{H}|\phi_k^r(0)|^2}.
\end{aligned}
$$

On the other hand, any $R_H(\mathrm{e}^{\mathrm{i}t}) \in \mathcal{NR}_H(\partial\boldsymbol{D})$ can be expressed as

$$
R_H(z) = \sum_{k=0}^{H}\langle R_H(\mathrm{e}^{\mathrm{i}t}),\phi_k^r(\mathrm{e}^{\mathrm{i}t})\rangle_{\frac{1}{|f_a(\mathrm{e}^{\mathrm{i}t})|^2}}\phi_k^r(z),
$$

where

$$
\langle R_H(\mathrm{e}^{\mathrm{i}t}),\phi_k^r(\mathrm{e}^{\mathrm{i}t})\rangle_{\frac{1}{|f_a(\mathrm{e}^{\mathrm{i}t})|^2}} = \frac{1}{2\pi}\int_0^{2\pi}R_H(\mathrm{e}^{\mathrm{i}t})\overline{\phi_k^r(\mathrm{e}^{\mathrm{i}t})}\frac{1}{|f_a(\mathrm{e}^{\mathrm{i}t})|^2}dt
$$

is the weighted inner product. Hence we have

$$
\frac{1}{2\pi}\int_0^{2\pi}|R_H(\mathrm{e}^{\mathrm{i}t})|^2\frac{1}{|f_a(\mathrm{e}^{\mathrm{i}t})|^2}dt = \sum_{k=0}^{H}|\langle R_H(\mathrm{e}^{\mathrm{i}t}),\phi_k^r(\mathrm{e}^{\mathrm{i}t})\rangle_{\frac{1}{|f_a(\mathrm{e}^{\mathrm{i}t})|^2}}|^2.
$$

Since $R_H(0) = 1$, we have

$$
1 = \left|\sum_{k=0}^{H}\langle R_H(\mathrm{e}^{\mathrm{i}t}),\phi_k^r(\mathrm{e}^{\mathrm{i}t})\rangle_{\frac{1}{|f_a(\mathrm{e}^{\mathrm{i}t})|^2}}\phi_k^r(0)\right| \leqslant \left[\sum_{k=0}^{H}|\langle R_H(\mathrm{e}^{\mathrm{i}t}),\phi_k^r(\mathrm{e}^{\mathrm{i}t})\rangle_{\frac{1}{|f_a(\mathrm{e}^{\mathrm{i}t})|^2}}|^2\right]^{1/2}\left[\sum_{k=0}^{H}|\phi_k^r(0)|^2\right]^{1/2}.
$$

Hence,

$$\frac{1}{2\pi}\int_0^{2\pi}|R_H(\mathrm{e}^{\mathrm{i}t})|^2\frac{1}{|f_a(\mathrm{e}^{\mathrm{i}t})|^2}dt=\sum_{k=0}^{H}|\langle R_H(\mathrm{e}^{\mathrm{i}t}),\phi_k^r(\mathrm{e}^{\mathrm{i}t})\rangle_{\frac{1}{|f_a(\mathrm{e}^{\mathrm{i}t})|^2}}|^2\geqslant\frac{1}{\sum_{k=0}^{H}|\phi_k^r(0)|^2}.$$

We obtain

$$\min_{R_H(\mathrm{e}^{\mathrm{i}t})\in\mathcal{NR}_H(2\pi)}\frac{1}{2\pi}\int_0^{2\pi}\frac{|R_H(\mathrm{e}^{\mathrm{i}t})|^2}{|f_a(\mathrm{e}^{\mathrm{i}t})|^2}dt=\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{S_H(0,\mathrm{e}^{\mathrm{i}t})}{S_H(0,0)}\right|^2\frac{1}{|f_a(\mathrm{e}^{\mathrm{i}t})|^2}dt=\frac{1}{\sum_{k=0}^{H}|\phi_k^r(0)|^2},$$

as desired. The proof is completed. □

If the points $\alpha_k$ defining the TM system satisfy the hyperbolic non-separable condition $\sum_{k=1}^{\infty}(1-|\alpha_k|)=\infty$, then the classical Szegő theorem can be generalized to the TM system, i.e., as $H$ goes to infinity, $R_{\min}^H(\alpha_1,\dots,\alpha_H,\mathrm{e}^{\mathrm{i}t})$ goes to $f_{\min}^{-1}(0)f_{\min}(\mathrm{e}^{\mathrm{i}t})$ in the $L^2$-norm sense, where $f_{\min}(0)=\mathrm{e}^{\mathrm{i}\gamma}\exp\{\frac{1}{2\pi}\int_0^{2\pi}\ln|f_a(\mathrm{e}^{\mathrm{i}t})|dt\}$.

**Theorem 3.2.** *Let a nonzero function $f_a(\mathrm{e}^{\mathrm{i}t})\in A_0(\partial\boldsymbol{D})$. If the points $\alpha_k$'s satisfy $\sum_{k=1}^{\infty}(1-|\alpha_k|)=\infty$, then*

$$\lim_{H\to\infty}\mathcal{E}_{\min}^H(f_a,\alpha_1,\dots,\alpha_H)=\frac{1}{|f_{\min}(0)|^2}=\exp\left\{-\frac{1}{\pi}\int_0^{2\pi}\ln|f_a(\mathrm{e}^{\mathrm{i}t})|dt\right\}$$

*and*

$$\lim_{H\to\infty}\left\|R_H^{\min}(\alpha_1,\dots,\alpha_H,\mathrm{e}^{\mathrm{i}t})-\frac{1}{f_{\min}(0)}\exp\{\ln|f_a(\mathrm{e}^{\mathrm{i}t})|+\mathrm{i}\widetilde{H}\ln|f_a(\mathrm{e}^{\mathrm{i}t})|\}\right\|^2=0.$$

*Proof.* The monotonous property $\mathcal{E}_{\min}^{H+1}(f_a,\alpha_1,\dots,\alpha_{H+1})\leqslant\mathcal{E}_{\min}^H(f_a,\alpha_1,\dots,\alpha_H)$ implies the existence of the limit $\lim_{H\to\infty}\mathcal{E}_{\min}^H(f_a,\alpha_1,\dots,\alpha_H)=\mathcal{E}_{\min}(f_a,\alpha)$, and $\mathcal{E}_{\min}(f_a,\alpha)\geqslant0$, where

$$\alpha=\{\alpha_1,\alpha_2,\dots,\alpha_H,\dots\}.$$

By Jessen's inequality, we have

$$\begin{aligned}\mathcal{E}_{\min}^H(f_a,\alpha_1,\dots,\alpha_H)&=\frac{1}{2\pi}\int_0^{2\pi}\frac{|R_{\min}^H(\alpha_1,\dots,\alpha_H,\mathrm{e}^{\mathrm{i}t})|^2}{|f_a(\mathrm{e}^{\mathrm{i}t})|^2}dt\\&\geqslant\exp\left\{\frac{1}{2\pi}\int_0^{2\pi}\ln|f_a(\mathrm{e}^{\mathrm{i}t})|^{-2}dt\right\}\exp\left\{\frac{1}{2\pi}\int_0^{2\pi}\ln|R_{\min}^H(\alpha_1,\dots,\alpha_H,\mathrm{e}^{\mathrm{i}t})|^2dt\right\}\\&=\frac{1}{|f_{\min}(0)|^2}|R_{\min}^H(\alpha_1,\dots,\alpha_H,0)|^2=\frac{1}{|f_{\min}(0)|^2}.\end{aligned}\tag{3.6}$$

Therefore, $\mathcal{E}_{\min}^H(f_a,\alpha_1,\dots,\alpha_H)\geqslant|f_{\min}(0)|^{-2}$ and $\mathcal{E}_{\min}(f_a,\alpha)\geqslant|f_{\min}(0)|^{-2}$.

On the other hand, since $f_a(\mathrm{e}^{\mathrm{i}t})\in A_0(\partial\boldsymbol{D})$, there exists $\delta>0$ such that $|f_a(\mathrm{e}^{\mathrm{i}t})|\geqslant\delta$, and, if the points $\alpha_k$ satisfy

$$\sum_{k=1}^{\infty}(1-|\alpha_k|)=\infty,$$

there exists a rational function $\phi(\mathrm{e}^{\mathrm{i}t})\in\mathcal{R}_H(\partial\boldsymbol{D})$ such that $|\phi(\mathrm{e}^{\mathrm{i}t})|\geqslant\frac{\delta}{2}$ and $|f_a(\mathrm{e}^{\mathrm{i}t})-\phi(\mathrm{e}^{\mathrm{i}t})|<\epsilon$. Let $\phi_{\min}(\mathrm{e}^{\mathrm{i}t})$ be the minimum-phase of $\phi(\mathrm{e}^{\mathrm{i}t})$. From the above inequalities involving $f_a$ we have

$$\left|1-\frac{|\phi(\mathrm{e}^{\mathrm{i}t}|)}{|f_a(\mathrm{e}^{\mathrm{i}t})|}\right|<\delta^{-1}\epsilon.$$

Using the elementary inequality $\ln y<y-1$ for $y\approx1$, we have

$$\begin{aligned}|\phi_{\min}(0)|=|f_{\min}(0)|\frac{|\phi_{\min}(0)|}{|f_{\min}(0)|}&=|f_{\min}(0)|\exp\left\{\frac{1}{2\pi}\int_0^{2\pi}\ln\left|\frac{\phi(\mathrm{e}^{\mathrm{i}t})}{f_a(\mathrm{e}^{\mathrm{i}t})}\right|dt\right\}\\&\leqslant|f_{\min}(0)|\mathrm{e}^{\delta^{-1}\epsilon}\end{aligned}$$

and

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{|\phi(\mathrm{e}^{\mathrm{i}t})|^2}{|f_a(\mathrm{e}^{\mathrm{i}t})|^2}dt \leqslant \frac{1}{2\pi}\int_0^{2\pi}|1+\delta^{-1}\epsilon|^2dt < (1+\delta^{-1}\epsilon)^2.$$

This shows that

$$\begin{aligned}\frac{1}{|f_{\min}^2(0)|} &\leqslant \mathcal{E}_{\min}(f_a,\alpha) \leqslant \mathcal{E}_{\min}^H(f_a,\alpha_1,\ldots,\alpha_H)\\ &\leqslant \frac{1}{|\phi_{\min}(0)|^2}\frac{1}{2\pi}\int_0^{2\pi}\frac{|\phi(\mathrm{e}^{\mathrm{i}t})|^2}{|f_a(t)|^2}dt \leqslant \frac{1}{|f_{\min}^2(0)|}e^{2\delta^{-1}\epsilon}(1+\epsilon\delta^{-1})^2.\end{aligned}$$

Since $\epsilon$ is an arbitrary small positive number,

$$\mathcal{E}_{\min}(f_a,\alpha) = \lim_{H\to\infty}\mathcal{E}_{\min}^H(f_a,\alpha_1,\ldots,\alpha_H) = \frac{1}{|f_{\min}(0)|^2} = \exp\left\{\frac{-2}{2\pi}\int_0^{2\pi}\ln|f_a(\mathrm{e}^{\mathrm{i}t})|dt\right\}. \tag{3.7}$$

Now

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{R_H^{\min}(\alpha_1,\ldots,\alpha_H,\mathrm{e}^{\mathrm{i}t})}{f_{\min}(\mathrm{e}^{\mathrm{i}t})} - f_{\min}^{-1}(0)\right|^2dt &= \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{R_H^{\min}(\alpha_1,\ldots,\alpha_H,\mathrm{e}^{\mathrm{i}t})}{f_{\min}(\mathrm{e}^{\mathrm{i}t})}\right|^2dt + \frac{1}{|f_{\min}(0)|^2}\\ &\quad - 2\Re\left\{\frac{1}{\overline{f_{\min}(0)}}\frac{1}{2\pi}\int_0^{2\pi}\frac{R_H^{\min}(\alpha_1,\ldots,\alpha_H,\mathrm{e}^{\mathrm{i}t})}{f_{\min}(\mathrm{e}^{\mathrm{i}t})}dt\right\}\\ &= \mathcal{E}_{\min}^H(f_a,\alpha_1,\ldots,\alpha_H) - \frac{1}{|f_{\min}(0)|^2}.\end{aligned} \tag{3.8}$$

By (3.7), we obtain

$$\lim_{H\to\infty}\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{R_{\min}^H(\alpha_1,\ldots,\alpha_H,\mathrm{e}^{\mathrm{i}t})}{f_{\min}(\mathrm{e}^{\mathrm{i}t})} - f_{\min}^{-1}(0)\right|^2dt = 0.$$

The proof is completed. □

For a fixed $H$, the minimum energy $\mathcal{E}_{\min}^H(f_a,\alpha_1,\ldots,\alpha_n)$ based on the $H$-TM system depends on the choices of $\alpha_1,\ldots,\alpha_H$. The following theorem shows that if $f_a$ is a rational function, then based on an $n$-best choice of $\{\alpha_1,\ldots,\alpha_n\}$, the exact solution of the minimum-phase part of $f_a$ can be extracted out.

**Theorem 3.3.** *Let $f_a(\mathrm{e}^{\mathrm{i}t})$ be a rational circular analytic signal with the form*

$$f_a(\mathrm{e}^{\mathrm{i}t}) = a_0\frac{\prod_{k=1}^P(1-p_k\mathrm{e}^{\mathrm{i}t})}{\prod_{k=1}^U(1-u_k\mathrm{e}^{\mathrm{i}t})}\prod_{k=1}^Q(1-q_k\mathrm{e}^{\mathrm{i}t}),$$

*where $a_0,p_k,u_k$ and $q_k$ are complex numbers, $|p_k|<1$, $|u_k|<1$ and $|q_k|>1$. Then the energy*

$$\min_{\alpha_1,\ldots,\alpha_n\in\mathbb{D}}\mathcal{E}_{\min}^n(f_a,\alpha_1,\ldots,\alpha_n) = \min_{\alpha_1,\ldots,\alpha_n\in\mathbb{D}}\frac{1}{2\pi}\int_0^{2\pi}\frac{|R_n^{\min}(\alpha_1,\ldots,\alpha_n,\mathrm{e}^{\mathrm{i}t})|^2}{|f_a(\mathrm{e}^{\mathrm{i}t})|^2}dt = \frac{1}{|f_{\min}^2(0)|}$$

*and*

$$R_{\min}^n(\alpha_1^{\min},\alpha_2^{\min},\ldots,\alpha_n^{\min},\mathrm{e}^{\mathrm{i}t}) = f_{\min}^{-1}(0)f_{\min}(\mathrm{e}^{\mathrm{i}t}) = \frac{\prod_{k=1}^P(1-p_k\mathrm{e}^{\mathrm{i}t})\prod_{k=1}^Q(1-1/\overline{q_k}\mathrm{e}^{\mathrm{i}t})}{\prod_{k=1}^U(1-u_k\mathrm{e}^{\mathrm{i}t})}, \tag{3.9}$$

*where $n\geqslant\max\{P+Q,U\}$, $\alpha_1^{\min},\ldots,\alpha_n^{\min}$ are the parameters at which $\mathcal{E}_{\min}^n(f_a,\alpha_1,\ldots,\alpha_n)$ attains minimum energy.*

*Proof.* For any given sequence $\{\alpha_1,\ldots,\alpha_n\}$ in the unit disc $\boldsymbol{D}$, by (3.6), we have that

$$\min_{\alpha_1,\ldots,\alpha_n}\mathcal{E}_{\min}^n(f_a,\alpha_1,\ldots,\alpha_n) \geqslant \frac{1}{|f_{\min}(0)|^2}.$$

Let

$$h(t) = \frac{f_{\min}(\mathrm{e}^{\mathrm{i}t})}{f_{\min}(0)}.$$

The condition $n \geqslant \max\{P+Q, U\}$ implies

$$\begin{aligned}\min_{\alpha_1,\ldots,\alpha_n} \mathcal{E}^n_{\min}(f_a,\alpha_1,\ldots,\alpha_n) &\leqslant \mathcal{E}^n_{\min}(f_a,u_1,\ldots,u_U)\\ &\leqslant \frac{1}{2\pi}\int_0^{2\pi} \frac{|\prod_{k=1}^P(1-p_k\mathrm{e}^{\mathrm{i}t})\prod_{k=1}^Q(1-1/\overline{q_k}\mathrm{e}^{\mathrm{i}t})|^2}{|\prod_{k=1}^U(1-u_k\mathrm{e}^{\mathrm{i}t})|^2}\frac{dt}{|f_a(\mathrm{e}^{\mathrm{i}t})|^2}\\ &\leqslant \frac{1}{2\pi}\int_0^{2\pi}\frac{|f_{\min}(\mathrm{e}^{\mathrm{i}t})|^2}{|f_{\min}(0)|^2|f_a(\mathrm{e}^{\mathrm{i}t})|^2}dt = \frac{1}{|f_{\min}(0)|^2}.\end{aligned}$$

Combining the above two results, we have

$$\min_{\alpha_1,\ldots,\alpha_n}\frac{1}{2\pi}\int_0^{2\pi}\frac{|R^n_{\min}(\alpha_1,\ldots,\alpha_n,\mathrm{e}^{\mathrm{i}t})|^2}{|f_a(\mathrm{e}^{\mathrm{i}t})|^2}dt = \frac{1}{|f_{\min}(0)|^2}.$$

Assume that $\alpha_1^{\min},\ldots,\alpha_n^{\min}$ are the parameters at which

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{|R^n_{\min}(\alpha_1^{\min},\ldots,\alpha_n^{\min},\mathrm{e}^{\mathrm{i}t})|^2}{|f_a(\mathrm{e}^{\mathrm{i}t})|^2}dt = \frac{1}{|f_{\min}(0)|^2}.$$

By (3.8), we have that

$$\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{R^n_{\min}(\alpha_1^{\min},\ldots,\alpha_n^{\min},\mathrm{e}^{\mathrm{i}t})}{f_{\min}(\mathrm{e}^{\mathrm{i}t})} - f^{-1}_{\min}(0)\right|^2 dt = 0.$$

Hence,

$$R^n_{\min}(\alpha_1^{\min},\ldots,\alpha_n^{\min},\mathrm{e}^{\mathrm{i}t}) = f^{-1}_{\min}(0)f_{\min}(\mathrm{e}^{\mathrm{i}t}) = \frac{\prod_{k=1}^P(1-p_k\mathrm{e}^{\mathrm{i}t})\prod_{k=1}^Q(1-1/\overline{q_k}\mathrm{e}^{\mathrm{i}t})}{\prod_{k=1}^U(1-u_k\mathrm{e}^{\mathrm{i}t})}.$$

This completes the proof. □

Compared with the algorithm of Kumaresan and Rao [13] based on Fourier series, the algorithm based on TM system has the advantage of adaptive choices of $\{\alpha_1,\alpha_2,\ldots,\alpha_n\}$. We use the example

$$f_r(\mathrm{e}^{\mathrm{i}\Omega t}) = \frac{(1-0.3\mathrm{e}^{\mathrm{i}\Omega t})(1-2\mathrm{e}^{\mathrm{i}\Omega t})}{(1-0.2\mathrm{e}^{\mathrm{i}\Omega t})(1-0.9\mathrm{e}^{\mathrm{i}\Omega t})(1-0.99\mathrm{e}^{\mathrm{i}\Omega t})}$$

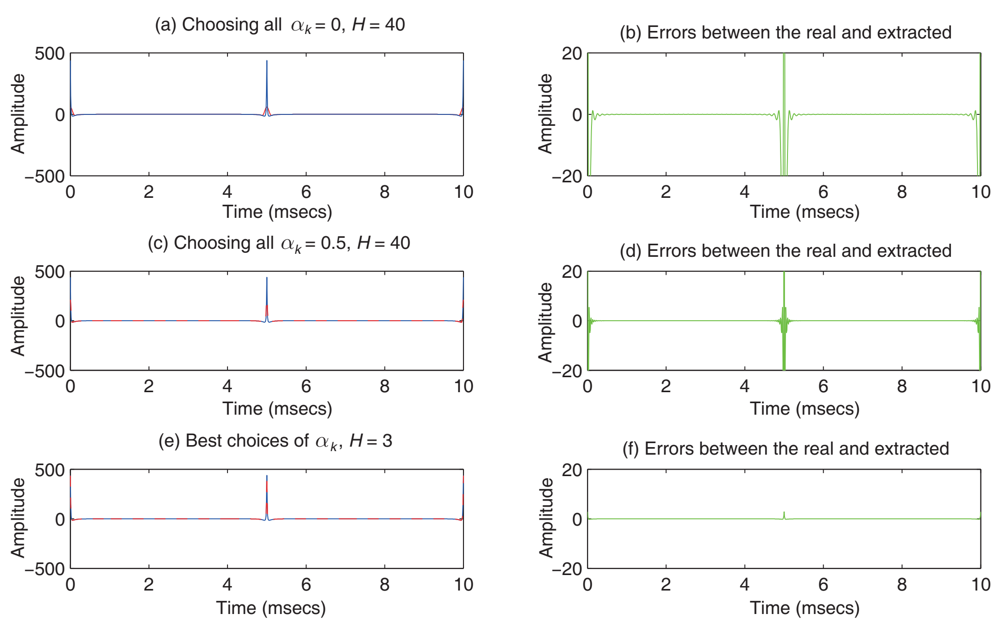


**Figure 2** The column on the left is the real part of the minimum phase signal of $f_r(t)$; the column on the left is the error between the real one and the extracted by different choices of $\alpha_k$

to analyze the advantage of flexible choices of the parameters $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$, where $\Omega = 2\pi \times 0.2$. We first observe that since the poles of $f_r(\mathrm{e}^{\mathrm{i}\Omega t})$ are very close to 1, the function values $\ln|f_r(\mathrm{e}^{\mathrm{i}\Omega t})|$ are very large at some points. It is shown in [21] that the algorithm based on Hilbert transform will be unstable for this case. The minimizing energy method under this study can suppress the unstability in numerical calculation by choosing suitable points $\alpha_k$. On fixed parameters $\{\alpha_1, \dots, \alpha_n\}$ the TM-system minimum energy method

$$\min_{h_k \in \mathbb{C}} \frac{1}{T} \int_0^T \left|1 + \sum_{k=1}^n h_k r_k(\mathrm{e}^{\mathrm{i}\Omega t})\right|^2 \frac{1}{|f_r(\mathrm{e}^{\mathrm{i}\Omega t})|^2} dt,$$

to extracted minimum-phase signal of $f_r(t)$ is sometimes still not so good to fit the spikes. This can be seen in Figures 2(a)–2(d) by respectively choosing all $\alpha_k = 0$ or all $\alpha_k = 0.5$, $H = 40$. When all $\alpha_k = 0$, the algorithm corresponds to what is given in Theorem 2.1 that already improves the method by Kumarasan and Rao [13]. But if the parameters $\alpha_k$ are chosen based on the best rules, i.e.,

$$\min_{\alpha_k \in \mathbb{D}} \min_{h_k \in \mathbb{C}} \left\{ \int_0^T \left|1 + \sum_{k=1}^n h_k r_k(\mathrm{e}^{\mathrm{i}\Omega t})\right|^2 \frac{1}{|f_r(\mathrm{e}^{\mathrm{i}\Omega t})|^2} dt \right\},$$

where $r_k(\mathrm{e}^{\mathrm{i}\Omega t})$ is given as in (3.3), then the minimum-phase component of $f_r(\mathrm{e}^{\mathrm{i}\Omega t})$ can be extracted much more efficiently. The effectiveness of the algorithm corresponding to Theorem 3.3 is explicitly shown in Figures 2(e) and 2(f) by choosing

$$H = 3 \quad \text{and} \quad \{|\alpha_k| < 0.995\}_{k=1}^3.$$

The best rules of choosing $\alpha_k$ may be consulted with the $n$-best rational approximation methods as given, for example, in [6, 27].

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