

## MÖBIUS COVARIANCE OF ITERATED DIRAC OPERATORS

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(Received 17 July 1991; revised 3 March 1992)

Communicated by P. G. Dodds

### Abstract

Using Fourier transforms, we give a new proof of certain identities for the fundamental solutions of the iterated Dirac operators  $D^\ell = (\sum_{i=1}^n e_i \partial/\partial x_i)^\ell$ ,  $\ell \in \mathbf{Z}_+$  and  $D^\ell = (\partial/\partial x_0 + D)^\ell$ . Based on the close relationship between the fundamental solutions and the conformal weights we then give a simple proof of B. Bojarski's results on the conformal covariance of  $D^\ell$ . We also prove a new conformal covariance result of  $D$ .

1991 *Mathematics subject classification* (Amer. Math. Soc.): 30 G 35.

### 1. Introduction

This paper gives an alternative proof of B. Bojarski's results [1, 2] on Möbius covariance of the iterated Dirac operators  $D^\ell = (\sum_{i=1}^n e_i \partial/\partial x_i)^\ell$ ,  $\ell \in \mathbf{Z}_+$ , by recognizing the close relationship between the conformal weights and the fundamental solutions of the operators. We also give a covariance result on the Dirac operator  $D = \partial/\partial x_0 + D$ , with some observations concerning non-existence of Möbius covariance of its iterations  $D^\ell$ ,  $\ell > 1$  (see also [11]).

We begin by recalling basic knowledge related to Clifford algebras ([1, 2, 4, 3, 6]). The Clifford algebra  $\mathcal{A}_n$  shall be the associative algebra over the real number system  $\mathbb{R}$  generated by  $n$  elements  $e_1, e_2, \dots, e_n$  subject to the relations  $e_i e_j = -e_j e_i$ ,  $i \neq j$ , and  $e_i^2 = -1$ . Each element  $a \in \mathcal{A}_n$  has a unique representation in the form  $a = \sum a_s e_s$ , where  $a_s \in \mathbb{R}$  and the summation is over all ordered subsets  $s = \{0 < i_1 < \dots < i_\ell\} \subset \{1, 2, \dots, n\}$ , and we identify

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$e_s$  with  $e_{i_1} \cdots e_{i_r}$ . For the empty set  $\emptyset$ ,  $e_\emptyset$  is interpreted as the real number 1.  $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2$  can be identified with  $\mathbb{R}$ , the complex field  $\mathbb{C}$  and the quaternions, respectively.

$\mathcal{A}_n$  is a vector space of real dimension  $2^n$ . In the literature there are two ways to identify  $\mathbb{R}^n$  with certain linear subspaces of  $\mathcal{A}_n$ . In this paper we identify  $\mathbb{R}^n$  with  $\text{span}\{e_1, \dots, e_n\}$ . For any element  $x = x_0 + x_1e_1 + \dots + x_n e_n$ , we denote  $x = x_0 + \underline{x}$  with  $\underline{x} = x_1e_1 + \dots + x_n e_n \in \mathbb{R}^n$ . Define two operations on the basic elements:  $(e_{i_1} \cdots e_{i_r})^* = e_{i_1} \cdots e_{i_r}$ ,  $(e_{i_1} \cdots e_{i_r})' = (-1)^r (e_{i_1} \cdots e_{i_r})$  etc., and extend them by linearity to two corresponding operations on  $\mathcal{A}_n$ , still denoted by  $*$  and  $'$ . By combining them we define the third operation  $\bar{\cdot}$  by  $\bar{x} = (x^*)'$ ; it is easy to see that  $\bar{\bar{x}} = x_0 - \underline{x}$  for  $x = x_0 + \underline{x}$ . The natural inner product between  $a$  and  $b$ , denoted by  $\langle a, b \rangle$ , is the number  $\sum_s a_s b_s$  and the norm of  $a$  associated with this inner product is  $|a| = (\sum_s |a_s|^2)^{\frac{1}{2}}$ . We recall that the Clifford group  $\Gamma_n$  is defined as the multiplicative group of all elements in the Clifford algebra which can be written as products of non-zero vectors in  $\mathbb{R}^n$ . For elements  $a, b$  in  $\Gamma_n \cup \{0\}$ ,  $\bar{a}a = |a|^2$  and  $|ab| = |a| \cdot |b|$  (see [1, 2, 4, 3, 6]).

If  $a \in \Gamma_n$ , then it has a representation  $a = \prod_{j=1}^{M(a)} a_j$ , where  $a_j \in \mathbb{R}^n$ . Generally, such a representation is not unique, and neither is the related integer  $M(a)$ . We let  $m(a)$  be the minimum of  $M(a)$  over all such representations. If  $a \in \mathbb{R} \setminus \{0\}$ , then we set  $m(a) = 0$ . So,  $m(\underline{x}) = 1$ , and for  $a \in \Gamma_n$  it follows that  $aa^* = a^*a = (-1)^{m(a)}|a|^2$ .

By the Möbius group we mean the group of orientation preserving transformations acting in the Euclidean space  $\mathbb{R}^n$ , generated by rigid motions, dilations and inversions ([1, 2, 4, 3, 6]).

According to a theorem of Ahlfors ([1, 2, 4, 3]), all Möbius transforms from  $\mathbb{R}^n \cup \{\infty\}$  to  $\mathbb{R}^n \cup \{\infty\}$  are exactly those of form

$$\varphi(\underline{x}) = (a\underline{x} + b)(c\underline{x} + d)^{-1}$$

where  $a, b, c, d \in \Gamma_n \cup \{0\}$  and

$$ad^* - bc^* \in \mathbb{R} \setminus \{0\}, \quad a^*c, cd^*, d^*b, ba^* \in \mathbb{R}^n.$$

See [2], for example. Furthermore, the identification between the  $\varphi$ 's and the Clifford matrices  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  gives a homomorphism under  $2 \times 2$  block matrix multiplication.

Since we are interested in differentiation of transformations of functions, we can assume without loss of generality that the functions under consideration

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are defined in  $\mathbb{R}^n$  or its one-point compactification, with compact supports and  
 having as many orders of differentiability as we need in our argument; the same  
 applies to functions defined in  $\mathbb{R}_1^n = \mathbb{R} \oplus \mathbb{R}^n = \{x = x_0 + \underline{x}; x_0 \in \mathbb{R}, \underline{x} \in \mathbb{R}^n\}$ .

For Clifford number-valued functions defined in  $\mathbb{R}^n$  belonging to the above  
 mentioned nice classes we introduce the following Fourier transform:

$$\hat{f}(\underline{\xi}) = \int_{\mathbb{R}^n} e^{i(\underline{x}, \underline{\xi})} f(\underline{x}) d\underline{x}.$$

The associated symbol of the Dirac operator  $\underline{D}$  is  $i\underline{\xi}$ ; and, accordingly, those of  
 $\underline{D}^\ell$  and  $\underline{D}^{-\ell}$  are  $(i\underline{\xi})^\ell$  and  $(i\underline{\xi})^{-\ell}$ ,  $\ell \in \mathbb{Z}_+$ , respectively.

For functions defined in  $\mathbb{R}_1^n = \{x_0 + \underline{x} : x_0 \in \mathbb{R}, \underline{x} \in \mathbb{R}^n\}$  we use the  
 following definition

$$\hat{f}(\xi) = \int_{\mathbb{R}_1^n} e^{i(x, \xi)} f(x) dx$$

where  $\xi, x \in \mathbb{R}_1^n$ .

With this definition the associated symbols of the iterated Dirac operator  $D^\ell$   
 and  $D^{-\ell}$ ,  $\ell \in \mathbb{Z}_+$ , where  $D = \partial/\partial x_0 + \underline{D}$ , are  $(i\xi)^\ell$  and  $(i\xi)^{-\ell}$ , respectively.

It would be helpful to mention that Hans Jakobson and Michelle Vergne have  
 established an analogue of Theorem 1 for the group  $SU(2, 2)$  ([10]); and Hans  
 Jakobsen has established related results for other Lie groups ([9]).

### 2. Fundamental solutions of $\underline{D}^\ell$ and $D^\ell$

First we deduce the fundamental solutions of  $\underline{D}^\ell$ . We prefer an approach  
 that will not concentrate on integers  $\ell$  at the beginning. It is consistent with the  
 above if we define  $\underline{D}^{-\alpha}$ ,  $\alpha > 0$ , by

$$\underline{D}^{-\alpha} f(\underline{x}) = c_n \int_{\mathbb{R}^n} e^{-i(\underline{x}, \underline{\xi})} (i\underline{\xi})^{-\alpha} \hat{f}(\underline{\xi}) d\underline{\xi},$$

where  $(i\underline{\xi})^{-\alpha}$  is defined by

$$(i\underline{\xi})^{-\alpha} = |\underline{\xi}|^{-\alpha} \chi_+(\underline{\xi}) + (-|\underline{\xi}|)^{-\alpha} \chi_-(\underline{\xi})$$

and

$$\chi_\pm(\underline{\xi}) = \frac{1}{2} \left( 1 \pm \frac{i\underline{\xi}}{|\underline{\xi}|} \right).$$

$$i\underline{\xi} = |\underline{\xi}| \cdot \frac{1}{2} \left( 1 + \frac{i\underline{\xi}}{|\underline{\xi}|} \right) + (-|\underline{\xi}|) \cdot \frac{1}{2} \left( 1 - i \frac{\underline{\xi}}{|\underline{\xi}|} \right)$$

