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A class of singular integrals on the n-complex unit sphere

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Abstract The operators on the n-complex unit sphere under study have three forms: the singular integrals with holomorphic kernels, the bounded and holomorphic Fourier multipliers, and the Cauchy-Dunford bounded and holomorphic functional calculus of the radial Dirac operator $D = \sum_{k=1}^{n} z_k \frac{\partial}{\partial z_k}$. The equivalence between the three forms and the strong-type (p, p), $1 < p < \infty$, and weak-type $(1,1)$-boundedness of the operators is proved. The results generalise the work of L. K. Hua, A. Korányli and S. Vagi, W. Rudin and S. Gong on the Cauchy-Szegö kernel and the Cauchy singular integral operator.

Keywords: singular integral, Fourier multiplier, the unit sphere in $\mathbb{C}^n$, functional calculus.

The Cauchy-Szegö kernel and integral formula, and the related singular integrals of several complex variables have been widely studied[1—4]. On the unit sphere in $\mathbb{C}^n$, however, there has been only one singular integral, viz. the Cauchy singular integral, while in the other standard underlying spaces such as in $\mathbb{R}^n$ a far reaching singular integral theory has been developed[5]. In this paper we study a class of singular integrals on the unit sphere. The class includes the Cauchy singular integral as a special case, and each of the operators in the class is similar to the Cauchy singular integral.

The class of singular integrals forms an operator algebra, viz. the bounded and holomorphic functional calculus of the radial Dirac operator $D = \sum_{k=1}^{n} z_k \frac{\partial}{\partial z_k}$. It also has the form of bounded and holomorphic Fourier multipliers.

Analogous theories have been established in various contexts, including graphs og Lipschitz functions of one and several real variables, starlike Lipschitz curves in $\mathbb{C}$, and starlike Lipschitz surfaces in the quatesnionic space and in $\mathbb{R}^n$[6—20].

1 Generalisation of Cauchy-Szegö kernel

In the complex plane, set, for $0 \leqslant \omega < \frac{\pi}{2}$,

$$S_\omega = \{z \in \mathbb{C} \mid z \neq 0, \text{ and } |\arg z| < \omega\},$$

$$S_\omega(\pi) = \{z \in \mathbb{C} \mid z \neq 0, \ |\mathrm{Re} z| \leqslant \pi, \text{ and } |\arg(\pm z)| < \omega\},$$

$$W_\omega(\pi) = \{z \in \mathbb{C} \mid z \neq 0, \ |\mathrm{Re} z| \leqslant \pi, \text{ and } \mathrm{Im}(z) > 0\} \cup S_\omega(\pi),$$

$$H_\omega = \{z \in \mathbb{C} \mid z = e^{iw},\ w \in W_\omega(\pi)\}.$$

The sets S_ω, $S_\omega(\pi)$, $W_\omega(\pi)$ and H_ω are, respectively, cone-shaped, bow-tie-shaped, *W*-shaped and heart-shaped regions.

The following function space is relevant:

$$H^\infty(S_\omega) = \{b: S_\omega \to \mathbb{C} \mid b \text{ is holomorphic in } S_\omega, \text{ and } |b(z)| \leqslant C_\mu < \infty \text{ if } z \in S_\mu,\ 0 < \mu < \omega\}.$$

Let

$$\varphi_b(z) = \sum_{k=1}^{\infty} b(k) z^k.$$

The study of this paper is based on the following technical result[17, 18].

The main lemma. *Let* $b \in H^\infty(S_\omega)$. *Then* φ_b *can be holomorphically extended to* H_ω, *and*

$$\left|\left(z\frac{d}{dz}\right)^l \varphi_b(z)\right| \leqslant \frac{C_{\mu'} l!}{\delta^l(\mu,\mu')\,|1-z|^{1+l}},\quad z \in H_\mu,\quad 0<\mu<\mu'<\omega,\quad l = 0, 1, 2, \cdots, \tag{1}$$

where $\delta(\mu,\mu') = \min\left\{\frac{1}{2}, \tan(\mu'-\mu)\right\}$; $C_{\mu'}$ *are the constants in the definition for* $b \in H^\infty(S_\omega)$.

For the reader's convenience we outline the proof below.

Proof (outline).

Step 1. Define

$$\Psi(z) = \frac{1}{2\pi}\int_{\rho\theta} \exp(iz\zeta) b(\zeta) d\zeta,\quad z \in V_\omega,$$

where

$$V_\omega = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\} \cup S_\omega \cup (-S_\omega),$$

and ρ_θ is the ray $r\exp(i\theta)$, $0 < r < \infty$, where θ is chosen so that $\rho_\theta \subset S_\omega$, and $\exp(iz\zeta)$ is exponentially decaying as $\zeta \to \infty$ along ρ_θ. It is easy to verify that Ψ is well defined, independent of choice of ρ_θ, and holomorphic in V_ω, and

$$|\Psi(z)| \leqslant \frac{C_0 \| b|_{S_\beta} \|_\infty}{|z|},\quad z \in V_\alpha,\quad 0 < \alpha < \beta < \omega.$$

Setp 2. Define

$$\psi(z) = 2\pi \sum_{n=-\infty}^{\infty} \Psi(z + 2n\pi),\quad z \in \bigcup_{n=-\infty}^{\infty} (2n\pi + W_\omega),$$

where

$$W_\omega = V_\omega \cap \{z \in \mathbb{C} \mid -\pi \leqslant \mathrm{Re}(z) \leqslant \pi\}.$$

It is easy to show that ψ is holomorphically and 2π-periodcally defined in the described region, and, up to a constant bounded by $c\,\| b|_{S_\beta} \|$, satisfies the estimate

$$|\psi(z)| \leqslant \frac{C_0 \| b|_{S_\beta} \|_\infty}{|z|},\quad z \in W_\alpha,\quad 0 < \alpha < \beta < \omega.$$

Letting $\varphi(z) = \psi\left(\frac{\log z}{i}\right)$, we obtain the desired inequality for $l = 0$.

Step 3. We notice that at local $z \approx 1$ the set H_ω can be approximated by the cone of the opening angle $\pi + 2\omega$ pointing to the positive direction of the x-axis. This is justified by the relation $e^{\eta} - 1 \approx \eta$, where $0 \approx \eta \in \mathbb{C}$. Then for any point $1 \approx z \in \mathrm{H}_{\alpha}$ the disc $B(z, r)$ of radius $r = \delta(\alpha, \beta) | 1 - z |$ centred at z is contained in H_{β}. Using Cauchy's formula, we have

$$\varphi_b^{(l)}(z) = \frac{l!}{2\pi \mathrm{i}} \int_{\partial B(z,\, r)} \frac{\varphi(\eta)}{(\eta - z)^{1+l}} \mathrm{d}\eta .$$

Therefore,

$$| \varphi^{(l)}(z) | \leqslant \frac{l!}{2\pi} \int_0^{2\pi} \frac{C_0 \| b |_{S_\beta} \|_\infty}{| 1 - \eta |} \frac{1}{r^l} \mathrm{d}\theta \leqslant \frac{C_0 \| b |_{S_\beta} \|_\infty l!}{\delta(\alpha, \beta)^l | 1 - z |^{1+l}},$$

where we have used the relation $| 1 - \eta | \geqslant | 1 - z | - | z - \eta | = | 1 - z | - r \geqslant | 1 - z | - \frac{1}{2} | 1 - z | = \frac{1}{2} | 1 - z |$. The proof is complete.

Remark 1. Pointed out by D. Khavinson, this result belongs to the same seminal results of Leau, Le Roy and Lindelöf. He also gave a different approach of proof[21].

From now on we will change notation and use z as a general element of $\mathbb{C}^n$, i.e. $z = (z_1, \cdots, z_n)$, $z_i \in \mathbb{C}$, $i = 1, 2, \cdots, n$, $n \geqslant 2$. Denote $\bar{z} = [\bar{z}_1, \cdots, \bar{z}_n]$. The theory for $n = 1$ on star-shaped Lipschitz curves is studied in ref. [16]. The notation z is considered to be a row vector. Denote by B the open unit ball $\{ z \in \mathbb{C}^n \mid | z | < 1 \}$, where $| z | = \left(\sum_{i=1}^{n} | z_i |^2 \right)^{\frac{1}{2}}$, and ∂B its boundary, i.e. $\partial B = \{ z \in \mathbb{C}^n \mid | z | = 1 \}$. The open ball centred at z with radius r will be denoted by $B(z, r)$. A general element on the unit sphere is usually denoted by ξ or ζ. The constant ω_{2n-1} involved in the Cauchy-Szegö kernel below is the surface area of $\partial B = S^{2n-1}$ and is equal to $\frac{2\pi^n}{\Gamma(n)}$. For $z, w \in \mathbb{C}^n$, we use the notation $zw' = \sum_{k=1}^{n} z_k w_k$. The theory developed in this study is relevant to the radial Dirac operator $D = \sum_{k=1}^{n} z_k \frac{\partial}{\partial z_i}$.

The following is a revision of basis functions in the space of holomorphic function in B and some relevant function spaces on ∂B. We adopt the settings of ref. [1]. Let k be a nonnegative integer. We consider the column vector $z^{[k]}$ with components

$$\sqrt{\frac{k!}{k_1! \cdots k_n!}} z_1^{k_1} \cdots z_n^{k_n}, \qquad k_1 + \cdots + k_n = k .$$

The dimension of $z^{[k]}$ is

$$N_k = \frac{1}{k!} n(n+1) \cdots (n+k-1) = \mathrm{C}_{n+k-1}^{k} .$$

Set

$$\int_B \overline{z^{[k]\prime}} \cdot z^{[k]} \mathrm{d}z = H_1^k ,$$

$$\int_{\partial B} \overline{\xi^{[k]\prime}} \cdot \xi^{[k]} \mathrm{d}\sigma(\xi) = H_2^k ,$$

where $\mathrm{d}z$ is the Lebesgue volume element of $\mathbb{R}^{2n} = \mathbb{C}^n$, and $\mathrm{d}\sigma(\xi)$ the Lebesgue area element of the unit sphere $S^{2n-1} = \partial B$. It is easy to verify that both H_1^k and H_2^k are positive-definite Hermitian matrices of order N_k. There, therefore, exists a matrix Γ such that

$$\overline{\Gamma}' \cdot H_1^k \cdot \Gamma = \Lambda, \quad \overline{\Gamma}' \cdot H_2^k \cdot \Gamma = I, \tag{2}$$

where $\Lambda = [\beta_1^k, \cdots, \beta_n^k]$ is a diagonal matrix and I the identity matrix.

We will set

$$z_{[k]} = z^{[k]} \cdot \Gamma, \quad \xi_{[k]} = \xi^{[k]} \cdot \Gamma,$$

and denote by $\{p_\nu^k(z)\}$ the components of the vectors $z_{[k]}$. From (2), we have

$$\int_B p_\nu^k(z)\, \overline{p_\mu^l(z)}\, \mathrm{d}z = \delta_{\nu\mu} \cdot \delta_{kl} \cdot \beta_\nu^k, \tag{3}$$

$$\int_{\partial B} p_\nu^k(\xi)\, \overline{p_\mu^l(\xi)}\, \mathrm{d}\sigma(\xi) = \delta_{\nu\mu} \cdot \delta_{kl}. \tag{4}$$

The following theorem is well known[1].

Theorem A. *The system of functions*

$$(\beta_\nu^k)^{-\frac{1}{2}} p_\nu^k, \quad k = 0, 1, 2, \cdots, \ \nu = 1, 2, \cdots, N_k$$

is a complete orthonormal system in the space of holomorphic functions in B. The system $\{p_\nu^k(\xi)\}$ is orthonormal, but not complete in the space of continuous functions on ∂B.

The explicit formula of the Cauchy-Szegö kernel

$$H(z, \overline{\xi}) = \frac{1}{\omega_{2n-1}} \frac{1}{(1 - z\overline{\xi}')^n} \tag{5}$$

on ∂B was first deduced in ref. [1] by using the system $\{p_\nu^k\}$ and the relation

$$H(z, \overline{\xi}) = \sum_{k=0}^{\infty} \sum_{\nu=1}^{N_k} p_\nu^k(z)\, \overline{p_\nu^k(\xi)}, \quad z \in B, \ \xi \in \partial B.$$

Our technical result is the following theorem.

Theorem 1. *Let $b \in H^\infty(S_\omega)$ and*

$$H_b(z, \overline{\xi}) = \sum_{k=1}^{\infty} b(k) \sum_{\nu=1}^{N_k} p_\nu^k(z)\, \overline{p_\nu^k(\xi)}, \quad z \in B, \quad \xi \in \partial B. \tag{6}$$

Then

$$H_b(z, \overline{\xi}) = \frac{1}{(n-1)!\,\omega_{2n-1}} (r^{n-1} \varphi_b(r))^{(n-1)} \Big|_{r = z\overline{\xi}'} \tag{7}$$

is holomorphically defined for any $z \in B$ and $\xi \in \partial B$ such that $z\overline{\xi}' \in \mathrm{H}_\omega$, where φ_b is the function defined in the Main Lemma. Moreover,

$$|\mathrm{D}_z^l H_b(z, \overline{\xi})| \leqslant \frac{C_{\mu'}\, l!}{\delta^l(\mu, \mu')\, |1 - z\overline{\xi}'|^{n+l}}, \quad z\overline{\xi}' \in \mathrm{H}_\mu, \quad 0 < \mu < \mu' < \omega,$$
$$l = 0, 1, 2, \cdots, \tag{8}$$

where $\delta(\mu, \mu') = \min\left\{\frac{1}{2}, \tan(\mu' - \mu)\right\}$; $C_{\mu'}$ are the constants in the definition of the function space $H^\infty(\mathrm{S}_\omega)$.

Proof. Setting $z = r\zeta$, $|\zeta| = 1$ in formula (5), we obtain

$$H(r\zeta, \overline{\xi}) = \frac{1}{\omega_{2n-1}} \frac{1}{(1 - r\zeta\overline{\xi}')^n}. \tag{9}$$

Treating $H(r\zeta, \overline{\xi})$ as a function of r, we assert that the entry of r^k in its Taylor expansion is

$$\frac{1}{k!}\left(\frac{\partial}{\partial r}\right)^k \left(\frac{1}{\omega_{2n-1}} \frac{1}{(1 - r\zeta\overline{\xi}')^n}\right)\Big|_{r=0} r^k = \frac{1}{\omega_{2n-1}} \frac{n(n+1)\cdots(n+k-1)}{k!} (r\zeta\overline{\xi}')^k. \tag{10}$$

Letting $r\zeta = z$, we obtain that the projection of $H(z, \overline{\xi})$ onto the space of k-homogeneous functions in the variable z is equal to

$$\sum_{\nu=1}^{N_k} p_\nu^k \overline{p_\nu^k(\xi)} = \frac{1}{\omega_{2n-1}} \frac{n(n+1)\cdots(n+k-1)}{k!}(z\overline{\xi'})^k.$$

A direct computation together with the definition of φ_b then gives the formula for $H_b(z, \overline{\xi})$. The estimates follow from The Main Lemma.

Remark 2. In the previously studies in refs. [6—20] the size of ω is crucial and is related to the Lipschitz constant of the curve or surface under study. In the present case the Lipschitz constant of the unit sphere is zero, and ω can be taken to be any number in the interval $\left(0, \frac{\pi}{2}\right]$. Throughout this paper we will assume that ω is any number in $\left(0, \frac{\pi}{2}\right]$ but fixed throught the discussion, and taking $\mu = (1/2)\omega$ and $\mu' = (3/4)\omega$ will be sufficient to developing our theory.

2 Fourier multiplier and singular integral operators on ∂B

For $z, w \in B \cup \partial B$ denote by $d(z, w)$ the nonisotropic distance between z and ω, defined through

$$d(z, w) = | 1 - z\overline{w'} |^{1/2}.$$

It can be easily shown that d is a metric on $B \cup \partial B$[3]. The ball on ∂B centred at ζ with radius ε using the metric d is denoted by $S(\zeta, \varepsilon)$. The complement set of $S(\zeta, \varepsilon)$ in ∂B is denoted by $S^c(\zeta, \varepsilon)$.

Let $f \in L^p(\partial B)$, $1 \leqslant p < \infty$. Then the Cauchy integral of f,

$$C(f)(z) = \frac{1}{\omega_{2n-1}} \int_{\partial B} \frac{f(\xi)}{(1 - z\overline{\xi'})^n} \mathrm{d}\sigma(\xi),$$

is well defined and holomorphic in B.

It is well known that operator

$$P(f)(\zeta) = \lim_{r \to 1-0} C(f)(r\zeta)$$

is the projection of $L^p(\partial B)$ onto the Hardy space $H^p(\partial B)$ and is bounded from $L^p(\partial B)$ to $H^p(\partial B)$, $1 < p < \infty$[2,3]. Moreover, $P(f)$ has the singular integral expression[3,4]

$$P(f)(\zeta) = \frac{1}{\omega_{2n-1}} \lim_{\varepsilon \to 0} \int_{S^c(\zeta, \varepsilon)} \frac{f(\xi)}{(1 - \zeta\overline{\xi'})^n} \mathrm{d}\sigma(\xi) + \frac{1}{2} f(\zeta) \quad \text{a.e. } \zeta \in \partial B.$$

Set

$$\mathscr{A} = \{ f \mid f \text{ is holomorphic in } B(0, 1 + \delta) \text{ for some } \delta > 0 \}.$$

It is easy to prove that $\mathscr{A}$ is dense in $L^p(\partial B)$, $1 \leqslant p < \infty$. If $f \in \mathscr{A}$, then

$$f(z) = \sum_{k=0}^{\infty} \sum_{\nu=0}^{N_k} c_{k\nu} p_\nu^k(z),$$

where $c_{k\nu}$ are the Fourier coefficients of f:

$$c_{k\nu} = \int_{\partial B} \overline{p_\nu^k(\xi)} f(\xi) \mathrm{d}\sigma(\xi),$$

and, for any positive integer l, the series

$$\sum_{k=0}^{\infty} k^l \sum_{\nu=0}^{N_k} c_{k\nu} p_\nu^k(z)$$

is uniformly and absolutely convergent in any compact ball contained in the ball $B(0,\ 1+\delta)$ in which f is defined.

Denote by $\mathscr{U}$ the unitary group of $\mathbb{C}^n$ consisting of all unitary operators on the Hilbert space $\mathbb{C}^n$ under the complex inner product $\langle z, w\rangle = z\overline{w}'$. These are the linear operators U that preserve inner products:

$$\langle Uz, Uw\rangle = \langle z, w\rangle.$$

Clearly, $\mathscr{U}$ is a compact subset of $O(2n)$. It is easy to verify that $\mathscr{A}$ is invariant under $U\in\mathscr{U}$. If $f\in\mathscr{A}$, then f is determined by its values on ∂B. In below we treat $f|_{\partial B}$ as identical to $f\in\mathscr{A}$. For a given function $b\in S_\omega$ we define an operator M_b: $\mathscr{A}\to\mathscr{A}$ by

$$M_b(f)(\zeta) = \sum_{k=1}^{\infty} b(k)\sum_{\nu=0}^{N_k} c_{k\nu}p_\nu^k(\zeta), \qquad \zeta\in\partial B,$$

where $c_{k\nu}$ are the Fourier coefficients of the test function $f\in\mathscr{A}$.

The result on principle value of the Cauchy integral defined using the surface metric $d(\eta,\ \zeta) = |1-\eta\overline{\zeta}'|^{1/2}$ can be extended to Theorem 2.

Theorem 2. *Operator M_b has a singular integral expression: for $f\in\mathscr{A}$,*

$$M_b(f)(\zeta) = \lim_{\varepsilon\to 0}\left[\int_{S^c(\zeta,\varepsilon)} H_b(\zeta,\ \overline{\xi})f(\xi)\mathrm{d}\sigma(\xi) + f(\zeta)\int_{S(\zeta,\varepsilon)} H_b(\zeta,\overline{\xi})\mathrm{d}\sigma(\xi)\right], \tag{11}$$

where

$$\int_{S(\zeta,\varepsilon)} H_b(\zeta,\overline{\xi})\mathrm{d}\sigma(\xi)$$

is a bounded function of $\zeta\in\partial B$ and ε.

Proof. Let $f\in\mathscr{A}$, $\rho\in(0,\ 1)$. On one hand,

$$M_b(f)(\rho\zeta) = \sum_{k=1}^{\infty} b(k)\sum_{\nu=1}^{N_p} c_{k\nu}p_\nu^k(\rho\zeta),$$

where $c_{k\nu}$ are the Fourier coefficients of f. From the boundedness of sequence $\{b(k)\}_{k=1}^{\infty}$ and the observation made above on the convergence of the Fourier expansion of $f\in\mathscr{A}$ we have

$$\lim_{\rho\to 1-0} M_b(f)(\rho\zeta) = M_b(f)(\zeta). \tag{12}$$

On the other hand, using the formula for the Fourier coefficients and the definition of $H_b(z,\ \overline{\xi})$ given by (5), we have

$$M_b(f)(\rho\zeta) = \int_{\partial B} H_b(\rho\zeta,\ \overline{\xi})f(\xi)\mathrm{d}\sigma(\xi).$$

For any $\varepsilon>0$, we have

$$\begin{aligned} M_b(f)(\rho\zeta) &= \int_{S^c(\zeta,\varepsilon)} H_b(\rho\zeta,\overline{\xi})f(\xi)\mathrm{d}\sigma(\xi) + \int_{S(\zeta,\varepsilon)} H_b(\rho\zeta,\ \overline{\xi})(f(\xi)-f(\zeta))\mathrm{d}\sigma(\xi) \\ &\quad + f(\zeta)\int_{S(\zeta,\varepsilon)} H_b(\rho\zeta,\ \overline{\xi})\mathrm{d}\sigma(\xi) \\ &= I_1(\rho,\ \varepsilon) + I_2(\rho,\ \varepsilon) + f(\zeta)I_3(\rho,\ \varepsilon). \end{aligned}$$

For $\rho\to 1-0$, we have

$$I_1(\rho,\varepsilon)\to\int_{S^c(\zeta,\varepsilon)} H_b(\zeta,\ \overline{\xi})f(\xi)\mathrm{d}\sigma(\xi).$$

Now we consider $I_2(\rho, \varepsilon)$. Since the metric d and the Euclidean metric $|\cdot|$ and the function class $\mathscr{A}$ are all $\mathscr{U}$-invariant, we can assume without loss of generality, that $\zeta = [1, 0, \cdots, 0]$. We will adopt the parameteric system $\xi_1 = re^{i\theta}$, $\xi_2 = v_2, \cdots, \xi_n = v_n$ for the variable $\xi \in \partial B$. We write $v = [v_2, \cdots, v_n]$. The integral region $S(\zeta, \varepsilon)$ is defined by the conditions

$$v\overline{v'} = 1 - r^2, \quad \cos\theta \geqslant \frac{1 + r^2 - \varepsilon^4}{2r}. \tag{13}$$

Now, since $\frac{1 + r^2 - \varepsilon^4}{2r} \leqslant \cos\theta \leqslant 1$, we have $(1 - r)^2 \leqslant \varepsilon^4$. So $1 - r \leqslant \varepsilon^2$, or $1 - \varepsilon^2 \leqslant r$. This implies $v\overline{v'} = 1 - r^2 \leqslant 1 - (1 - \varepsilon^2)^2 = 2\varepsilon^2 - \varepsilon^4$. Denote $a = a(r, \varepsilon) = \arccos\left(\frac{1 + r^2 - \varepsilon^4}{2r}\right)$. Since $(1 - r)^2 \leqslant \varepsilon^4$ and $1 - y = O(\arccos^2(y))$, we obtain $a = O(\varepsilon^2)$.

It is easy to verify that

$$\begin{aligned} |\zeta - \xi|^2 &= |1 - re^{i\theta}|^2 + (|v_2|^2 + \cdots + |v_n|^2) \\ &= (1 + r^2 - 2r\cos(\theta)) + (1 - r^2) \\ &= 2 - 2r\cos(\theta), \end{aligned} \tag{14}$$

$$\begin{aligned} d^4(\zeta, \xi) &= |1 - \zeta\overline{\xi'}|^2 = 1 + r^2 - 2r\cos(\theta) \\ &= (2 - 2r\cos(\theta)) - (1 - r^2) \\ &= |\zeta - \xi|^2 - (1 + r)(1 - r). \end{aligned} \tag{15}$$

Now, (14) implies $1 - r \leqslant d^2(\zeta, \xi)$. This, together with (15), concludes that

$$d^4(\zeta, \xi) + (1 + r)d^2(\zeta, \xi) \geqslant |\zeta - \xi|^2.$$

Since $d^2(\zeta, \xi)$ is less than 2, the last inequality implies

$$|\zeta - \xi| \leqslant 2d(\zeta, \xi). \tag{16}$$

Note that for $f \in \mathscr{A}$ we have

$$|f(\zeta) - f(\xi)| \leqslant C|\zeta - \xi|,$$

therefore,

$$|f(\zeta) - f(\xi)| \leqslant Cd(\zeta, \xi).$$

For any $\rho \in (0, 1)$, owing to (13), we have

$$\begin{aligned} |I_2(\rho, \varepsilon)| &\leqslant \int_{S(\zeta, \varepsilon)} |H_b(\rho\zeta, \overline{\xi})| \, |f(\xi) - f(\zeta)| \, d\sigma(\zeta) \\ &\leqslant C\int_{S(\zeta, \varepsilon)} \frac{1}{d^{2n-1}(\zeta, \xi)} d\sigma(\xi) \\ &\leqslant C\int_{v\overline{v'} \leqslant 2\varepsilon^2 - \varepsilon^4} \int_{-a}^{a} \frac{1}{|1 - re^{i\theta}|^{n-(1/2)}} d\theta dv. \end{aligned}$$

Now we estimate the inside integral. Proceeding as in ref. [4], for $n = 2$, the Hölder inequality gives

$$\begin{aligned} \frac{1}{2a}\int_{-a}^{a} \frac{1}{|1 - re^{i\theta}|^{2-(1/2)}} d\theta &\leqslant \left(\frac{1}{2a}\int_{-a}^{a} \frac{1}{|1 - re^{i\theta}|^2} d\theta\right)^{3/4} \\ &\leqslant \left(\frac{1}{2a}\int_{-\pi}^{\pi} \frac{1}{|1 - re^{i\theta}|^2} d\theta\right)^{3/4} \\ &\leqslant \left(\frac{1}{2a}\right)^{3/4} \frac{1}{(1 - r^2)^{3/4}}. \end{aligned}$$

In this case,

$$\begin{aligned}|I_2(\rho,\varepsilon)| &\leqslant C\int_{v\overline{v'}\leqslant 2\varepsilon^2-\varepsilon^4} a^{1/4}\frac{1}{(1-r^2)^{3/4}}\mathrm{d}v\\ &\leqslant C\varepsilon^{1/2}\int_{v\overline{v'}\leqslant 2\varepsilon^2-\varepsilon^4}\frac{1}{(v\overline{v'})^{3/4}}\mathrm{d}v\\ &\leqslant C\varepsilon^{1/2}\int_0^{\sqrt{2\varepsilon^2-\varepsilon^4}}\frac{t}{t^{3/2}}\mathrm{d}t\\ &\leqslant C\varepsilon\rightarrow 0,\end{aligned}$$

as $\varepsilon\rightarrow 0$.

For $n>2$, we have, since r is close to 1,

$$\begin{aligned}\int_{-a}^{a}\frac{1}{|1-re^{i\theta}|^{n-(1/2)}}\mathrm{d}\theta &\leqslant C\frac{1}{(1-r^2)^{n-2-(1/2)}}\int_{-\pi}^{\pi}\frac{1}{|1-re^{i\theta}|^2}\mathrm{d}\theta\\ &\leqslant C\frac{1}{(1-r^2)^{n-1-(1/2)}},\end{aligned}$$

and hence,

$$|I_2(\rho,\varepsilon)|\leqslant C\int_0^{\sqrt{2\varepsilon^2-\varepsilon^4}} t^{2n-3}\frac{1}{t^{2n-3}}\mathrm{d}t\leqslant C\varepsilon\rightarrow 0,$$

as $\varepsilon\rightarrow 0$.

Now we prove that if $\rho\rightarrow 1-0$, and then $I_3(\rho,\varepsilon)$ has a limit uniformly bounded for ε near zero. Integrating as before, we have

$$\begin{aligned}I_3(\rho,\varepsilon) &= \int_{S(\zeta,\varepsilon)} H_b(\rho\zeta,\overline{\xi})\mathrm{d}\sigma(\xi)\\ &= \int_{v\overline{v'}\leqslant 2\varepsilon^2-\varepsilon^4}\int_{-a}^{a}(t^{n-1}\varphi_b(t))^{(n-1)}\big|_{t=\rho re^{i\theta}}\mathrm{d}\theta\mathrm{d}v\\ &= \frac{1}{\mathrm{i}}\int_{v\overline{v'}\leqslant 2\varepsilon^2-\varepsilon^4}\int_{\rho re^{-ia}}^{\rho re^{ia}}\frac{(t^{n-1}\varphi_b(t))^{(n-1)}}{t}\mathrm{d}t\mathrm{d}v.\end{aligned}$$

Using integration by parts, the inside integral with respect to the variable t becomes

$$\begin{aligned}&\left[\sum_{k=1}^{n-1}(k-1)!\frac{(t^{n-1}\varphi_b(t))^{(n-1-k)}}{t^k}\right]_{\rho re^{-ia}}^{\rho re^{ia}}+(n-1)!\int_{\rho re^{-ia}}^{\rho re^{ia}}\frac{\varphi_b(t)}{t}\mathrm{d}t\\ &=\sum_{k=1}^{n-1}[J_k(t)]_{\rho re^{-ia}}^{\rho re^{ia}}+L(r,a).\end{aligned}$$

We first estimate the integrals with integrand J_k. We have

$$\int_{v\overline{v'}\leqslant 2\varepsilon^2-\varepsilon^4}J_k(\rho re^{\pm ia})\mathrm{d}v\leqslant C\int_{v\overline{v'}\leqslant 2\varepsilon^2-\varepsilon^4}\frac{1}{|1-\rho re^{\pm ia}|^{n-k}}\mathrm{d}v.$$

It can be directly verified that

$$|1-\rho re^{\pm ia}|\geqslant|1-re^{\pm ia}|=\varepsilon^2.$$

So the above integral is dominated by

$$\frac{1}{\varepsilon^{2n-2k}}\int_{v\overline{v'}\leqslant 2\varepsilon^2-\varepsilon^4}\mathrm{d}v\leqslant\frac{1}{\varepsilon^{2n-2k}}\int_0^{\sqrt{2\varepsilon^2-\varepsilon^4}}t^{2n-3}\mathrm{d}t\leqslant C\frac{\varepsilon^{2n-2}}{\varepsilon^{2n-2k}},$$

which is bounded for $k=1$ and tends to zero for $k\geqslant 2$. The existence of the limit as $\rho\rightarrow 1-0$ is guaranteed by the Lebesgue dominated convergence theorem.

Now,

$$(n-1)!\int_{\rho r e^{-ia}}^{\rho r e^{ia}} \frac{\varphi_b(t)}{t}\mathrm{d}t = (n-1)!\mathrm{i}\int_{-a}^{a} \varphi_b(t)\,|_{t=\rho r e^{i\theta}}\mathrm{d}\theta.$$

Using Cauchy's theorem and the estimate of φ_b, we can show that for any $\rho \to 1-0$ this is a bounded function[16]. This implies that

$$\int_{v\overline{v'} \leqslant 2\varepsilon^2 - \varepsilon^4} L(\rho r, a)\mathrm{d}v \to 0,$$

as $\varepsilon \to 0$.

To sum up, we conclude that $\lim\limits_{\rho \to 1-0} I_3(\rho, \varepsilon)$ exists and is bounded for small $\varepsilon > 0$. This proves Theorem 2.

Remark 3. A consequence of (14) is

$$d(\zeta, \xi) \leqslant |\zeta - \xi|^{1/2}.$$

This side of control of the metric d was not used in the proof.

It is easy to see that $M_b = M_b P$. The boundedness result of Korányi and Vagi is extended to Theorem 3.

Theorem 3. *Operator M_b can be extended to a bounded operator from $L^p(\partial B)$ to $L^p(\partial B)$, $1 < p < \infty$, and from $L^1(\partial B)$ to weak-$L^1(\partial B)$.*

Proof. The boundedness of $M_b = M_b P$ from $L^2(\partial B)$ to $H^2(\partial B)$ is a consequence of the orthonormality of system $\{p_\nu^k(\xi)\}$ (Theorem A). We will show that the operator is bounded from $L^1(\partial B)$ to weak-$L^1(\partial B)$, i.e. of weak-type $(1,1)$. The $L^p(\partial B)$-boundedness, $1 < p < 2$, then will follow from the Marcinkiewicz interpolation theorem[5]. The L^p-boundedness for $2 < p < \infty$ is obtained from a standard duality argument using the property of the kernel: $\overline{H_b(\zeta, \overline{\xi})} = H_b(\xi, \overline{\zeta})$ and the bilinear paring

$$(f, g) = \int_{\partial B} f(\zeta)\,\overline{g(\zeta)}\mathrm{d}\sigma(\zeta).$$

The weak-type $(1,1)$ of M_b is based on a Hörmander type inequality. The proof presented below is different from that of the corresponding one for the Cauchy kernel given in ref. [3]. We will be using the non-tangential approaching region

$$D_\alpha(\zeta) = \left\{ z \in \mathbb{C}^n \,\middle|\, |1 - z\overline{\zeta'}| < \frac{\alpha}{2}(1 - |z|^2) \right\}, \quad \zeta \in \partial B, \quad \alpha > 1.$$

Lemma 1. *Suppose that $\xi, \zeta, \eta \in \partial B$, $d(\xi, \zeta) < \delta$, $d(\xi, \eta) > 2\delta$, and $z \in D_\alpha(\eta)$. Then*

$$|H_b(z, \overline{\xi}) - H_b(z, \overline{\zeta})| \leqslant \delta C_\alpha |1 - \xi\overline{\eta'}|^{-n-\frac{1}{2}}.$$

Proof. Owing to the estimate (see Theorem 1)

$$|(r^{n-1}\varphi_b(r))^{(n)}| \leqslant \frac{C_\omega}{|1 - r|^{n+1}},$$

and the mean value theorem, we have for some $t \in (0, 1)$, the real part

$$\begin{aligned} &|\mathrm{Re}(r^{n-1}\varphi_b(r))^{(n-1)}|_{r=z\overline{\xi'}} - \mathrm{Re}(r^{n-1}\varphi_b(r))^{(n-1)}|_{r=z\overline{\zeta'}}| \\ &\leqslant |(r^{n-1}\varphi_b(r))^{(n)}|_{r=z\overline{w'_t}}|\,|(z\overline{\xi'} - z\overline{\zeta'})| \\ &\leqslant \frac{C_\omega |(z\overline{\xi'} - z\overline{\zeta'})|}{|1 - z\overline{w'_t}|^{n+1}}, \end{aligned} \tag{17}$$

where $w_t = t\bar{\xi}' + (1-t)\overline{\zeta}' \in B$.

The imaginary part satisfies an analogous inequality.

Denote by ξ_t the projection point of ω_t onto ∂B. We can easily show that

(i) $|\xi_t - w_t| = 1 - |z_t| = A(t) \rightarrow 0$ as $\delta \rightarrow 0$;

(ii) $\xi_t \in S(\xi, \delta) \cap S(\zeta, \delta)$.

It follows from the notation in (i) that $\xi_t = \frac{1}{1-A(t)} w_t$. Since $D_\alpha(\eta)$ is an open set, for small $\delta > 0$, say $0 < \delta \leqslant \delta_0$, we have $z_t = (1-A(t))z \in D_\alpha(\eta)$. We write

$$|1 - z\overline{w'}_t| = |1 - z_t\overline{\xi'}_t|. \tag{18}$$

On the other hand, from (4) on page 92 of ref. [3], we have

$$\begin{aligned} |z\overline{\xi'} - z\overline{\zeta'}| &= \frac{1}{1-A(t)} |z_t\overline{\xi'} - z_t\overline{\zeta'}| \\ &\leqslant \frac{1}{1-A(t)}(|z_t\overline{\xi'} - z_t\overline{\xi'}_t| + |z_t\overline{\zeta'} - z_t\overline{\xi'}_t|) \\ &\leqslant \frac{6}{1-A(t)}\delta\alpha^{1/2} |1 - z_t\overline{\xi'}_t|^{1/2} \\ &\leqslant \delta C_\alpha |1 - z_t\overline{\xi'}_t|^{1/2}, \end{aligned} \tag{19}$$

and, from (93) on page 92 of ref. [3], we have

$$|1 - z_t\overline{\xi'}_t|^{-1} \leqslant 16\alpha |1 - \xi\overline{\eta'}|^{-1}. \tag{20}$$

The relations (18)—(20) then imply for $\delta \leqslant \delta_0$, that the last part of the inequality chain (17) is dominated by

$$\delta C_\alpha |1 - \xi\overline{\eta'}|^{-n-\frac{1}{2}},$$

as desired.

For $\delta \geqslant \delta_0$, on the right-hand side of the desired inequality, the part

$$\delta |1 - \xi\overline{\eta'}|^{-n-\frac{1}{2}}$$

has a positive lower bound depending on δ_0. It is then easy to choose $C = C_{\alpha,\delta_0}$ for which the iequality holds. The Lemma is thus proved.

The weak-type (1,1) is a special case of the more general Theorem 4.

Theorem 4. *To every $\alpha > 1$ there exists a constant $C_\alpha < \infty$ such that for any $f \in \mathscr{A}$ and $t > 0$, there is*

$$\sigma\{M_\alpha M_b(f) > t\} \leqslant C_\alpha t^{-1} \|f\|_{L^1(\partial B)},$$

where

$$M_\alpha M_b(f)(\zeta) = \sup\{|M_b(f)(z)| : z \in D_\alpha(\zeta)\}$$

is defined to be the non-tangential maximum function of $M_b(f)$ in region $D_\alpha(\zeta)$.

The proof of Theorem 4 is based on Lemma 1 and a covering lemma[3, 5]. The proof in ref. [3] for the corresponding result for the Cauchy operator[3] can be adapted step by step to the present case.

3 Bounded holomorphic functional calculus of the radial dirac operator

We wish to point out that the class of the bounded operator M_b studied in section 2 constitutes an operator algebra that is, in fact, identical to the Cauchy-Dunford bounded holomorphic functional calculus of DP, where D is the radial Dirac operator and P is the projection operator from L^p to H^p.

The operators M_b enjoy the following properties, and thus the class M_b, $b \in H^\infty(S_\omega)$, is called a bounded holomorphic functional calculus.

Let b, b_1, $b_2 \in H^\infty(S_\omega)$, and α_1, $\alpha_2 \in \mathbb{C}$, $1 < p < \infty$, $0 < \mu < \omega$. Then

$$\| M_b \|_{L^p(\partial B) \to L^p(\partial B)} \leqslant C_{p,\mu} \| b \|_{L^\infty(S_\mu)},$$

$$M_{b_1 b_2} = M_{b_1} \circ M_{b_2},$$

$$M_{\alpha_1 b_1 + \alpha_2 b_2} = \alpha_1 M_{b_1} + \alpha_2 M_{b_2}.$$

The first assertion is obtained from Theorems 3. The second and the third are derived by using Taylor series expansions of the test functions.

Denote by

$$R(\lambda, DP) = (\lambda I - DP)^{-1},$$

the resolvent operator of DP at $\lambda \in \mathbb{C}$. For $\lambda \notin [0, \infty)$ we show that $R(\lambda, DP) = M_{\frac{1}{\lambda - (\cdot)}}$. In fact, owing to the relation

$$DP(f)(\zeta) = \sum_{k=1}^{\infty} k \sum_{\nu=1}^{N_k} c_{k\nu} p_\nu^k(\zeta) \quad f \in \mathscr{A},$$

where $c_{k\nu}$ are the Fourier coefficients of f, the Fourier multiplier $\{\lambda - k\}$ is associated with the operator $\lambda I - DP$, and therefore the Fourier multiplier $\{(\lambda - k)^{-1}\}$ is associated with $R(\lambda, DP)$. The property of the functional calculus in relation to the boundedness then asserts that for $1 < p < \infty$,

$$\| R(\lambda, DP) \|_{L^p(\partial B) \to L^p(\partial B)} \leqslant \frac{C_\mu}{|\lambda|}, \quad \lambda \notin S_\mu.$$

Owing to this estimate, for $b \in H^\infty(S_\omega)$ with good decays at both zero and the infinity, the Cauchy-Dunford integral

$$b(DP)f = \frac{1}{2\pi i} \int_\Pi b(\lambda) R(\lambda, DP) d\lambda f$$

is well defined to be a bounded operator, where Π is a path consisting of two rays in S_ω: $\{s\exp(i\theta)$: s is from ∞ to $0\} \cup \{s\exp(-i\theta)$: s is from 0 to $\infty\}$, $0 < \theta < \omega$. The functions b of this sort form a dense subclass of $H^\infty(S_\omega)$ in the sense specified in the Convergence Lemma of McIntosh in ref. [22]. Using the lemma, we can extend the definition given by the Cauchy-Dunford integral and define a functional calculus $b(DP)$ on general functions $b \in H^\infty(S_\omega)$.

Now we show that $b(DP) = M_b$. Assume again that b has good decays at both zero and the infinity, and $f \in \mathscr{A}$. Then the change of order of integration and summation in the following chain of equalities can be justified, and we have

$$\begin{aligned} b(DP)f(\zeta) &= \frac{1}{2\pi i} \int_\Pi b(\lambda) R(\lambda, DP) d\lambda f(\zeta) \\ &= \frac{1}{2\pi i} \int_\Pi b(\lambda) \sum_{k=1}^{\infty} (\lambda - k)^{-1} \sum_{\nu=1}^{N_k} c_{k\nu} p_\nu^k(\zeta) d\lambda \\ &= \sum_{k=1}^{\infty} \left(\frac{1}{2\pi i} \int_\Pi b(\lambda)(\lambda - k)^{-1} d\lambda \right) \sum_{\nu=1}^{N_k} c_{k\nu} p_\nu^k(\zeta) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{\infty} b(k) \sum_{\nu=1}^{N_k} c_{k\nu} p_{\nu}^{k}(\zeta) \\
&= M_b f(\zeta).
\end{aligned}
$$

It follows from the norm estimate of the resolvent $R(\lambda, DP)$ that DP is a type-ω operator[22]. The operator DP is identical to its dual operator on $L^2(\partial B)$ in the dual pair $(L^2(\partial B), L^2(\partial B))$ under the bilinear pairing used in the proof of Theorem 3. That is

$$(DP(f), g) = (f, DP(g)), \quad f, g \in \mathscr{A}.$$

This can be easily derived from Parseval's identity

$$\sum_{k=0}^{\infty} \sum_{\nu=1}^{N_k} c_{k\nu} \overline{c'_{k\nu}} = \int_{\partial B} f(\zeta) \overline{g(\zeta)} \mathrm{d}\sigma(\zeta),$$

deduced from the orthonormality of $\{p_{\nu}^{k}\}$, where $c_{k\nu}$ and $c'_{k\nu}$ are Fourier coefficients of f and g, respectively.

Similar conclusions hold for the Banach space dual pairs $(L^p(\partial B), L^{p'}(\partial B))$, $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, under the same form of bilinear pairings.

Hilbert and Banach space properties of general type-ω operators are well studied, respectively, in refs. [22, 23]. The results of refs. [22, 23] can be verified to be valid for the operator DP without difficulty.

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