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An alternative proof of Fueter's theorem§

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In this article we establish an alternative proof of the generalized Fueter method presented in a former paper [Qian, T. and Sommen, F., 2003, Deriving harmonic functions in higher dimensional spaces. *Zeitschrift für Analysis und ihre Anwendungen*, **22**(2), 275–288] leading to the construction of special harmonic and monogenic functions in higher dimensions. At the same time, we also obtain a generalization of this result.

Keywords: Dirac operators; Monogenic functions; Vekua systems

AMS Subject Classifications: 30G35; 32A25; 42B20

1. Introduction

It is a remarkable fact that the Cauchy–Riemann system in the plane generates monogenic functions. This was first observed by Fueter in [1] in the setting of quaternionic analysis.

Assume f to be holomorphic in an open set of the upper-half complex plane and substitute $f(z) = u(x, y) + iv(x, y)$ ($z = x + iy$) where as usual $u = \operatorname{Re} f$, $v = \operatorname{Im} f$. Then, Fueter's theorem asserts that in the corresponding region the following relation holds:

$$D\Delta\left(u(q_0, |q|) + \frac{q}{|q|}v(q_0, |q|)\right) = 0$$

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§Dedicated to Professor Guochun Wen on the occasion of his 75th birthday.

with

$$\begin{aligned} \underline{q} &= q_1 i + q_2 j + q_3 k \\ D &= \partial_{q_0} + i\partial_{q_1} + j\partial_{q_2} + k\partial_{q_3} \\ \Delta &= \partial_{q_0}^2 + \partial_{q_1}^2 + \partial_{q_2}^2 + \partial_{q_3}^2 \end{aligned}$$

and i, j, k are the basic elements of the Hamilton quaternionic space.

Let $e_j, j = 1, 2, \dots, m$ be the generating basic elements of the 2^m -dimensional real linear associative but non-commutative Clifford algebra $\mathbb{R}_{0,m}$, with the multiplication rules

$$e_i e_j + e_j e_i = -2\delta_{ij}, \quad i, j = 1, 2, \dots, m.$$

Any element $a \in \mathbb{R}_{0,m}$ may be written as

$$a = \sum_A a_A e_A, \quad a_A \in \mathbb{R},$$

where $e_A = e_{i_1} e_{i_2} \cdots e_{i_k}$, $A = \{i_1, i_2, \dots, i_k\} \subset \{1, 2, \dots, m\}$, $i_1 < \dots < i_k$ and for $A = \emptyset$, $e_\emptyset = 1$ is the identity element of $\mathbb{R}_{0,m}$.

For $k = 0, 1, \dots, m$ fixed, we call

$$\mathbb{R}_{0,m}^{(k)} = \left\{ a \in \mathbb{R}_{0,m} : \sum_{|A|=k} a_A e_A \right\},$$

the subspace of k -vectors and thus we have that

$$\mathbb{R}_{0,m} = \sum_{k=0}^m \oplus \mathbb{R}_{0,m}^{(k)}.$$

For $a \in \mathbb{R}_{0,m}$, thus we may write

$$a = \sum_{k=0}^m [a]_k,$$

where $[a]_k$ is the projection of a on $\mathbb{R}_{0,m}^{(k)}$.

The subspace $\sum_{k \text{ even}}^m \oplus \mathbb{R}_{0,m}^{(k)}$, called ‘even subalgebra’ is denoted by $\mathbb{R}_{0,m}^+$.

The Euclidean space \mathbb{R}^m is embedded in the Clifford algebra $\mathbb{R}_{0,m}$ by identifying (x_1, x_2, \dots, x_m) with the vector variable \underline{x} given by

$$\underline{x} = \sum_{j=1}^m x_j e_j.$$

The first-order linear differential operator

$$\partial_{x_0} + \partial_{\underline{x}} = \partial_{x_0} + \sum_{j=1}^m e_j \partial_{x_j},$$

called the Cauchy–Riemann operator, splits the Laplace operator in \mathbb{R}^{m+1} :

$$\Delta = \partial_{x_0}^2 + \sum_{j=1}^m \partial_{x_j}^2 = (\partial_{x_0} + \partial_{\underline{x}})(\partial_{x_0} - \partial_{\underline{x}}).$$

A continuously differentiable $\mathbb{R}_{0,m}$ -valued function g defined in some open set of \mathbb{R}^{m+1} solution of the equation $(\partial_{x_0} + \partial_{\underline{x}})g = 0$ is called a left monogenic function (see e.g. [2,3]).

The operator $\partial_{\underline{x}}$ is called the Dirac operator in \mathbb{R}^m . For a differentiable k -vector-valued function $F_k = \sum_{|A|=k} e_A F_{k,A}$ and a differentiable $\mathbb{R}_{0,m}$ -valued function g , we have that (see e.g. [4])

$$\partial_{\underline{x}}(F_k g) = (\partial_{\underline{x}} F_k)g + 2 \sum_{j=1}^m [e_j F_k]_{k-1} \partial_{x_j} g + (-1)^k F_k (\partial_{\underline{x}} g).$$

This Leibniz rule, in the particular case of a differentiable scalar-valued function ϕ reads as:

$$\partial_{\underline{x}}(\phi g) = (\partial_{\underline{x}} \phi)g + \phi(\partial_{\underline{x}} g), \tag{1}$$

and for a vector-valued function $\underline{f} = \sum_{j=1}^m e_j f_j$:

$$\partial_{\underline{x}}(\underline{f} g) = (\partial_{\underline{x}} \underline{f})g - 2 \sum_{j=1}^m f_j \partial_{x_j} g - \underline{f}(\partial_{\underline{x}} g). \tag{2}$$

In [5], Sce extended the Fueter's theorem to $\mathbb{R}_{0,m}$ for m odd, i.e. under the same assumptions on f , the function f

$$\Delta^{(m-1)/2}(u(x_0, r) + \underline{\omega} v(x_0, r)) \quad (r = |\underline{x}|, \underline{r}\underline{\omega} = \underline{x}),$$

is left monogenic.

In [6], Sommen generalized the Sce's result: If m is an odd positive integer, then

$$\Delta^{k+(m-1)/2}[(u(x_0, r) + \underline{\omega} v(x_0, r))P_k(\underline{x})]$$

is also left monogenic function, where $P_k(\underline{x})$ is a homogeneous left-monogenic polynomial of degree k in \mathbb{R}^m .

The Fueter's theorem has also been considered for m even (see [7,8]) and also for non-integer powers (see [7]).

The Fueter's theorem provides us with the so-called axial monogenic functions of degree k (see [9,10]), i.e. monogenic functions of the form

$$(A(x_0, r) + \underline{\omega} B(x_0, r))P_k(\underline{x}),$$

A and B being \mathbb{R} -valued and satisfying the Vekua-type system

$$\begin{aligned} \partial_{x_0} A - \partial_r B &= \frac{2k + m - 1}{r} B, \\ \partial_{x_0} B + \partial_r A &= 0. \end{aligned}$$

We split up \mathbb{R}^m as $\mathbb{R}^m = \sum_{s=1}^d \oplus \mathbb{R}^{p_s}$, $\sum_{s=1}^d p_s = m$. Therefore, the vector variable \underline{x} may be written as

$$\underline{x} = \sum_{s=1}^d \underline{x}^{(s)}, \quad \underline{x}^{(s)} = \sum_{j=1}^{p_s} e_j^{(s)} x_j^{(s)},$$

and the Dirac operator $\partial_{\underline{x}}$ as

$$\partial_{\underline{x}} = \sum_{s=1}^d \partial_{\underline{x}^{(s)}}, \quad \partial_{\underline{x}^{(s)}} = \sum_{j=1}^{p_s} e_j^{(s)} \partial_{x_j^{(s)}}.$$

Let

$$r_s = |\underline{x}^{(s)}|, \quad \omega_s = \frac{\underline{x}^{(s)}}{r_s}, \quad s = 1, 2, \dots, d.$$

Next, we consider the following harmonic multivector field

$$\underline{g}(x_0, r_1, \dots, r_d) = (g_0(x_0, r_1, \dots, r_d), g_1(x_0, r_1, \dots, r_d), \dots, g_d(x_0, r_1, \dots, r_d))$$

i.e. \underline{g} satisfies the Riesz system

$$\begin{aligned} \partial_{x_0} g_0 - \sum_{s=1}^d \partial_{r_s} g_s &= 0, \\ \partial_{x_0} g_s + \partial_{r_s} g_0 &= 0, \\ \partial_{r_s} g_j - \partial_{r_j} g_s &= 0, \end{aligned}$$

$s, j = 1, 2, \dots, d, s \neq j$.

Looking for a version of Fueter’s theorem in the poly-axial case, Qian and Sommen proved in [11] the following result: If p_s ($s = 1, 2, \dots, d$) are odd, then the function

$$\Delta^{(m-d)/2} \left(g_0(x_0, r_1, \dots, r_d) + \sum_{j=1}^d \omega_j g_j(x_0, r_1, \dots, r_d) \right) \tag{3}$$

is left monogenic with respect to $\partial_{x_0} + \partial_{\underline{x}}$, where $\Delta = \partial_{x_0}^2 + \sum_{s=1}^d \Delta_{\underline{x}^{(s)}}$, $\Delta_{\underline{x}^{(s)}} = \sum_{j=1}^{p_s} \partial_{x_j^{(s)}}^2$.
 The present article, extends the previous result as given in the following.

THEOREM 1 Let g_s , p_s ($s = 1, 2, \dots, d$) as above. Then

$$\Delta^{k+(m-d)/2} \left[\left(g_0(x_0, r_1, \dots, r_d) + \sum_{j=1}^d \omega_j g_j(x_0, r_1, \dots, r_d) \right) \mathbf{P}_k(\underline{x}) \right],$$

is also left monogenic with respect to $\partial_{x_0} + \partial_{\underline{x}}$, where $\mathbf{P}_k(\underline{x}) = \prod_{j=1}^d P_{k_j}(\underline{x}^{(j)})$, $k = \sum_{j=1}^d k_j$ and $P_{k_j}(\underline{x}^{(j)})$ is a homogeneous left-monogenic polynomial of degree k_j in \mathbb{R}^{p_j} with values in \mathbb{R}_{0,p_j}^+ .

In [11], the authors gave an elegant proof of the Fueter's theorem based on the fact that the function

$$g_0(x_0, r_1, \dots, r_d) + \sum_{j=1}^d \omega_j g_j(x_0, r_1, \dots, r_d),$$

may be written locally as $(\partial_{x_0} - \partial_{\underline{x}})h(x_0, r_1, \dots, r_d)$ for some scalar harmonic function in the $d+1$ variables x_0, r_1, \dots, r_d . Therefore, function (3) is left monogenic if

$$\Delta^{(m-d+2)/2} h(x_0, r_1, \dots, r_d) = 0,$$

see [11], Theorem 3.

The present article is not just an extension of the mentioned result in [11], but also proves it in a different way.

The sketch of the proof of Theorem 1 is as follows: First to prove that

$$\Delta^{k+(m-d)/2} \left[\left(g_0(x_0, r_1, \dots, r_d) + \sum_{j=1}^d \omega_j g_j(x_0, r_1, \dots, r_d) \right) \mathbf{P}_k(\underline{x}) \right],$$

has the form

$$\left(A_0(x_0, r_1, \dots, r_d) + \sum_{j=1}^d \omega_j A_j(x_0, r_1, \dots, r_d) \right) \mathbf{P}_k(\underline{x}),$$

and then verify that A_j ($j = 0, 1, \dots, d$) satisfy the corresponding Vekua system for poly-axial monogenic functions of degree k .

2. Proof of Theorem 1

The proof of Theorem 1 is divided into several steps:

LEMMA 1 Let $A_j(x_0, r_1, \dots, r_d)$ ($j = 0, 1, \dots, d$) \mathbb{R} -valued, then the function

$$\left(A_0(x_0, r_1, \dots, r_d) + \sum_{j=1}^d \omega_j A_j(x_0, r_1, \dots, r_d) \right) \mathbf{P}_k(\underline{x}),$$

is left monogenic if the A_j ($j = 0, 1, \dots, d$) are solutions of the system

$$\begin{aligned} \partial_{x_0} A_0 - \sum_{j=1}^d \partial_{r_j} A_j &= \sum_{j=1}^d \frac{2k_j + p_j - 1}{r_j} A_j, \\ \partial_{x_0} A_l + \partial_{r_l} A_0 &= 0, \\ \partial_{r_l} A_j - \partial_{r_j} A_l &= 0, \end{aligned}$$

$l, j = 1, 2, \dots, d, l \neq j$.

Proof Applying (1) and (2) yields

$$\begin{aligned} \partial_{\underline{x}}(A_0 \mathbf{P}_k(\underline{x})) &= \left(\sum_{l=1}^d \underline{\omega}_l \partial_{r_l} A_0 \right) \mathbf{P}_k(\underline{x}) \\ \partial_{\underline{x}}(A_j \underline{\omega}_j \mathbf{P}_k(\underline{x})) &= ((\partial_{\underline{x}} A_j) \underline{\omega}_j + A_j (\partial_{\underline{x}} \underline{\omega}_j)) \mathbf{P}_k(\underline{x}) - 2 \frac{A_j}{r_j} \sum_{l=1}^{p_j} x_l^{(j)} \partial_{x_l^{(j)}} \mathbf{P}_k(\underline{x}) \\ &= \left(\sum_{l=1}^d \partial_{r_l} A_j \underline{\omega}_l \underline{\omega}_j - \frac{p_j - 1}{r_j} A_j \right) \mathbf{P}_k(\underline{x}) - 2 \frac{k_j}{r_j} A_j \mathbf{P}_k(\underline{x}). \end{aligned}$$

Therefore,

$$\begin{aligned} (\partial_{x_0} + \partial_{\underline{x}}) \left(\left(A_0 + \sum_{j=1}^d \underline{\omega}_j A_j \right) \mathbf{P}_k(\underline{x}) \right) &= \left(\partial_{x_0} A_0 - \sum_{j=1}^d \left(\partial_{r_j} A_j + \frac{2k_j + p_j - 1}{r_j} A_j \right) \right) \mathbf{P}_k(\underline{x}) \\ &\quad + \left(\sum_{j=1}^d (\partial_{x_0} A_j + \partial_{r_j} A_0) \underline{\omega}_j \right) \mathbf{P}_k(\underline{x}) \\ &\quad + \left(\sum_{l=1}^d \sum_{j=l+1}^d (\partial_{r_l} A_j - \partial_{r_j} A_l) \underline{\omega}_l \underline{\omega}_j \right) \mathbf{P}_k(\underline{x}), \end{aligned}$$

which gives the desired result. ■

LEMMA 2 Let $h(x_0, r_1, \dots, r_d)$ be a scalar function. Then

- (i) $\partial_{r_s}^2 [D_{r_s}(\mu)\{h\}] = D_{r_s}(\mu)\{\partial_{r_s}^2 h\} - 2\mu D_{r_s}(\mu + 1)\{h\}$,
- (ii) $\partial_{r_s} [D_{r_s}(\mu - 1)\{h/r_s\}] = D^{r_s}(\mu)\{h\}$,
- (iii) $D^{r_s}(\mu)\{\partial_{r_s} h\} = \partial_{r_s} [D_{r_s}(\mu)\{h\}]$,
- (iv) $D_{r_s}(\mu)\{\partial_{r_s} h\} - \partial_{r_s} D^{r_s}(\mu)\{h\} = (2\mu/r_s) D^{r_s}(\mu)\{h\}$.

where $D_{r_s}(\mu)\{h\} = ((1/r_s) \partial_{r_s})^\mu \{h\}$ and

$$\begin{aligned} D^{r_s}(0)\{h\} &= h, \\ D^{r_s}(1)\{h\} &= \partial_{r_s} \left(\frac{h}{r_s} \right), \\ D^{r_s}(\mu)\{h\} &= \partial_{r_s} \left(\frac{D^{r_s}(\mu - 1)\{h\}}{r_s} \right), \quad \mu \geq 2, \end{aligned}$$

$s = 1, \dots, d$.

Proof To prove (i), we use mathematical induction. When $\mu = 1$, we have

$$\begin{aligned} \partial_{r_s}^2 [D_{r_s}(1)\{h\}] &= \frac{\partial_{r_s}^3 h}{r_s} - 2 \frac{\partial_{r_s}^2 h}{r_s^2} + 2 \frac{\partial_{r_s} h}{r_s^3} \\ &= D_{r_s}(1)\{\partial_{r_s}^2 h\} - 2D_{r_s}(2)\{h\}, \end{aligned}$$

as desired.

Now, we proceed to show that when the case (i) holds for a positive integer μ , then (i) also holds for $\mu + 1$. Indeed,

$$\begin{aligned} \partial_{r_s}^2 [D_{r_s}(\mu + 1)\{h\}] &= D_{r_s}(1)\{\partial_{r_s}^2 [D_{r_s}(\mu)\{h\}]\} - 2D_{r_s}(2)\{D_{r_s}(\mu)\{h\}\} \\ &= D_{r_s}(1)\left\{D_{r_s}(\mu)\{\partial_{r_s}^2 h\} - 2\mu D_{r_s}(\mu + 1)\{h\}\right\} - 2D_{r_s}(\mu + 2)\{h\} \\ &= D_{r_s}(\mu + 1)\{\partial_{r_s}^2 h\} - 2(\mu + 1)D_{r_s}(\mu + 2)\{h\}, \end{aligned}$$

where, we have used the mathematical induction hypothesis on μ .

(ii) comes easily from the definition of $D^{r_s}(\mu)\{h\}$. Next, using (ii), we get

$$D^{r_s}(\mu)\{\partial_{r_s} h\} = \partial_{r_s} \left[D_{r_s}(\mu - 1) \left\{ \frac{\partial_{r_s} h}{r_s} \right\} \right] = \partial_{r_s} [D_{r_s}(\mu)\{h\}].$$

Finally to obtain (iv) we use (i) and (ii), respectively. In fact,

$$\begin{aligned} D_{r_s}(\mu)\{\partial_{r_s} h\} - \partial_{r_s} D^{r_s}(\mu)\{h\} &= D_{r_s}(\mu)\{\partial_{r_s} h\} - \partial_{r_s}^2 \left[D_{r_s}(\mu - 1) \left\{ \frac{h}{r_s} \right\} \right] \\ &= D_{r_s}(\mu)\{\partial_{r_s} h\} - D_{r_s}(\mu - 1) \left\{ \partial_{r_s}^2 \left\{ \frac{h}{r_s} \right\} \right\} + 2(\mu - 1)D_{r_s}(\mu) \left\{ \frac{h}{r_s} \right\} \\ &= 2\mu D_{r_s}(\mu) \left\{ \frac{h}{r_s} \right\} = \frac{2\mu}{r_s} D^{r_s}(\mu)\{h\}, \end{aligned}$$

and this completes the proof. ■

LEMMA 3 *Let $h(x_0, r_1, \dots, r_d)$ be a scalar function harmonic in the $d+1$ variables x_0, r_1, \dots, r_d . Then*

$$(i) \quad \partial_{x_0}^2 \prod_{s=1}^d D_{r_s}(\mu_s)\{h\} + \sum_{j=1}^d \partial_{r_j}^2 \prod_{s=1}^d D_{r_s}(\mu_s)\{h\} = -2 \sum_{j=1}^d \mu_j \prod_{s=1, s \neq j}^d D_{r_s}(\mu_s)D_{r_j}(\mu_j + 1)\{h\},$$

$$\begin{aligned} (ii) \quad &\partial_{x_0}^2 \prod_{s=1, s \neq c}^d D_{r_s}(\mu_s)D^{r_c}(\mu_c)\{h\} + \sum_{j=1}^d \partial_{r_j}^2 \prod_{s=1, s \neq c}^d D_{r_s}(\mu_s)D^{r_c}(\mu_c)\{h\} \\ &= -2 \sum_{j=1, j \neq c}^d \mu_j \prod_{s=1, s \neq c, j}^d D_{r_s}(\mu_s)D^{r_c}(\mu_c)D_{r_j}(\mu_j + 1)\{h\} \\ &\quad - 2\mu_c \prod_{s=1, s \neq c}^d D_{r_s}(\mu_s)D^{r_c}(\mu_c + 1)\{h\}. \end{aligned}$$

Proof Using Lemma 2 and the assumption on h , we can prove that

$$\begin{aligned} & \partial_{x_0}^2 \prod_{s=1}^d D_{r_s}(\mu_s)\{h\} + \sum_{j=1}^d \partial_{r_j}^2 \prod_{s=1}^d D_{r_s}(\mu_s)\{h\} = \prod_{s=1}^d D_{r_s}(\mu_s)\{\partial_{x_0}^2 h\} \\ & + \sum_{j=1}^d \prod_{s=1, s \neq j}^d D_{r_s}(\mu_s) \left\{ D_{r_j}(\mu_j)\{\partial_{r_j}^2 h\} - 2\mu_j D_{r_j}(\mu_j + 1)\{h\} \right\} \\ & = \prod_{s=1}^d D_{r_s}(\mu_s)\{\partial_{x_0}^2 h\} + \sum_{j=1}^d \prod_{s=1}^d D_{r_s}(\mu_s)\{\partial_{r_j}^2 h\} - 2 \sum_{j=1}^d \mu_j \prod_{s=1, s \neq j}^d D_{r_s}(\mu_s) D_{r_j}(\mu_j + 1)\{h\} \\ & = -2 \sum_{j=1}^d \mu_j \prod_{s=1, s \neq j}^d D_{r_s}(\mu_s) D_{r_j}(\mu_j + 1)\{h\}. \end{aligned}$$

Similarly (ii) can be proved in the same way. In fact,

$$\begin{aligned} & \partial_{x_0}^2 \prod_{s=1, s \neq c}^d D_{r_s}(\mu_s) D^{r_c}(\mu_c)\{h\} + \sum_{j=1}^d \partial_{r_j}^2 \prod_{s=1, s \neq c}^d D_{r_s}(\mu_s) D^{r_c}(\mu_c)\{h\} \\ & = \prod_{s=1, s \neq c}^d D_{r_s}(\mu_s) D^{r_c}(\mu_c)\{\partial_{x_0}^2 h\} + \sum_{j=1, j \neq c}^d \prod_{s=1, s \neq c}^d D_{r_s}(\mu_s) D^{r_c}(\mu_c)\{\partial_{r_j}^2 h\} \\ & - 2 \sum_{j=1, j \neq c}^d \mu_j \prod_{s=1, s \neq c, j}^d D_{r_s}(\mu_s) D^{r_c}(\mu_c) D_{r_j}(\mu_j + 1)\{h\} \\ & + \prod_{s=1, s \neq c}^d D_{r_s}(\mu_s) \left\{ \partial_{r_c}^3 \left[D_{r_c}(\mu_c - 1) \left\{ \frac{h}{r_c} \right\} \right] \right\}. \end{aligned}$$

As,

$$\begin{aligned} \partial_{r_c}^3 \left[D_{r_c}(\mu_c - 1) \left\{ \frac{h}{r_c} \right\} \right] & = \partial_{r_c} \left[D_{r_c}(\mu_c - 1) \left\{ \partial_{r_c}^2 \left\{ \frac{h}{r_c} \right\} \right\} \right] - 2(\mu_c - 1) \partial_{r_c} D_{r_c}(\mu_c) \left\{ \frac{h}{r_c} \right\} \\ & = D^{r_c}(\mu_c)\{\partial_{r_c}^2 h\} - 2\mu_c D^{r_c}(\mu_c + 1)\{h\}, \end{aligned}$$

we get that

$$\begin{aligned} & \partial_{x_0}^2 \prod_{s=1, s \neq c}^d D_{r_s}(\mu_s) D^{r_c}(\mu_c)\{h\} + \sum_{j=1}^d \partial_{r_j}^2 \prod_{s=1, s \neq c}^d D_{r_s}(\mu_s) D^{r_c}(\mu_c)\{h\} \\ & = \prod_{s=1, s \neq c}^d D_{r_s}(\mu_s) D^{r_c}(\mu_c)\{\partial_{x_0}^2 h\} + \sum_{j=1}^d \prod_{s=1, s \neq c}^d D_{r_s}(\mu_s) D^{r_c}(\mu_c)\{\partial_{r_j}^2 h\} \\ & - 2 \sum_{j=1, j \neq c}^d \mu_j \prod_{s=1, s \neq c, j}^d D_{r_s}(\mu_s) D^{r_c}(\mu_c) D_{r_j}(\mu_j + 1)\{h\} \end{aligned}$$

$$\begin{aligned}
 & -2\mu_c \prod_{s=1, s \neq c}^d D_{r_s}(\mu_s) D^{r_c}(\mu_c + 1)\{h\} \\
 = & -2 \sum_{j=1, j \neq c}^d \mu_j \prod_{s=1, s \neq c, j}^d D_{r_s}(\mu_s) D^{r_c}(\mu_c) D_{r_j}(\mu_j + 1)\{h\} \\
 & -2\mu_c \prod_{s=1, s \neq c}^d D_{r_s}(\mu_s) D^{r_c}(\mu_c + 1)\{h\},
 \end{aligned}$$

and we have our result. ■

LEMMA 4 *Let $h(x_0, r_1, \dots, r_d)$ be a scalar function harmonic in the $d+1$ variables x_0, r_1, \dots, r_d . Then*

$$\begin{aligned}
 \Delta^\mu [h \mathbf{P}_k(\underline{x})] &= \left(\sum \frac{\mu!}{\mu_1! \cdots \mu_d!} \prod_{s=1}^d d_{p_s, k_s}(\mu_s) \prod_{s=1}^d D_{r_s}(\mu_s)\{h\} \right) \mathbf{P}_k(\underline{x}) \\
 \Delta^\mu [\omega_j h \mathbf{P}_k(\underline{x})] &= \omega_j \left(\sum \frac{\mu!}{\mu_1! \cdots \mu_d!} \prod_{s=1}^d d_{p_s, k_s}(\mu_s) \prod_{s=1, s \neq j}^d D_{r_s}(\mu_s) D^{r_j}(\mu_j)\{h\} \right) \mathbf{P}_k(\underline{x}),
 \end{aligned}$$

where the summation runs over all possible $\mu_1, \dots, \mu_d \in \mathbb{N}_0$ such that

$$\sum_{s=1}^d \mu_s = \mu,$$

and

$$\begin{aligned}
 d_{p_s, k_s}(\mu_s) &= (2k_s + p_s - 1)(2k_s + p_s - 3) \cdots (2k_s + p_s - (2\mu_s - 1)) \\
 d_{p_s, k_s}(0) &= 1.
 \end{aligned}$$

Proof The proof follows by induction using Lemma 3, and the following identities

$$\begin{aligned}
 \Delta [h \mathbf{P}_k(\underline{x})] &= \left(\partial_{x_0}^2 h + \sum_{s=1}^d \partial_{r_s}^2 h + \sum_{s=1}^d \frac{2k_s + p_s - 1}{r_s} \partial_{r_s} h \right) \mathbf{P}_k(\underline{x}) \\
 \Delta [\omega_j h \mathbf{P}_k(\underline{x})] &= \omega_j \left(\partial_{x_0}^2 h + \sum_{s=1}^d \partial_{r_s}^2 h + \sum_{s=1, s \neq j}^d \frac{2k_s + p_s - 1}{r_s} \partial_{r_s} h + (2k_j + p_j - 1) \partial_{r_j} \left\{ \frac{h}{r_j} \right\} \right) \mathbf{P}_k(\underline{x}),
 \end{aligned}$$

which are valid for any scalar function h . ■

Remark If, in addition to the assumption in Lemma 4, we assume that p_1, p_2, \dots, p_d are odd, then

$$\Delta^{k+(m-d+2)/2} [h \mathbf{P}_k(\underline{x})] = \Delta^{k+(m-d+2)/2} [\omega_j h \mathbf{P}_k(\underline{x})] = 0.$$

Indeed, in that case, all the terms in the expansions of Lemma 4 are zero. In fact, if there is a non-zero term in those expansions, then we have $2k_s + p_s - (2\mu_s - 1) \geq 2$

for $s = 1, 2, \dots, d$. Since $\sum_{s=1}^d p_s = m$, $\sum_{s=1}^d k_s = k$ and $\sum_{s=1}^d \mu_s = \mu = k + ((m - d + 2)/2)$, adding up the previous inequality together produces the false relation $2 \leq 0$.

Proof of Theorem 1 From Lemma 4 we have

$$\Delta^{k+((m-d)/2)} \left[\left(g_0 + \sum_{j=1}^d \omega_j g_j \right) \mathbf{P}_k(\underline{x}) \right] = \left(A_0 + \sum_{j=1}^d \omega_j A_j \right) \mathbf{P}_k(\underline{x}), \quad (4)$$

with

$$A_0 = (2k + m - d)!! \prod_{s=1}^d D_{r_s} \left(k_s + \frac{p_s - 1}{2} \right) \{g_0\}$$

$$A_j = (2k + m - d)!! \prod_{s=1, s \neq j}^d D_{r_s} \left(k_s + \frac{p_s - 1}{2} \right) D^{r_j} \left(k_j + \frac{p_j - 1}{2} \right) \{g_j\},$$

$j = 1, 2, \dots, d$.

Now, taking into account Lemma 2 and the fact that \underline{g} satisfy the Riesz system, it is easy to check that (4) satisfy the Vekua system for poly-axial monogenic functions of degree k . ■

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