Complex Variables and Elliptic Equations
An International Journal

ISSN: 1747-6933 (Print) 1747-6941 (Online) Journal homepage: http://www.tandfonline.com/loi/gcov20

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To cite this article: Yan Yang & Tao Qian (2006) Schwarz lemma in Euclidean spaces, Complex Variables and Elliptic Equations, 51:7, 653-659, DOI: 10.1080/17476930600688623

To link to this article: https://doi.org/10.1080/17476930600688623

Published online: 10 Oct 2011.

Complex Variables and Elliptic Equations
Vol. 51, No. 7, July 2006, 653–659

Taylor & Francis
Taylor & Francis Group

Schwarz lemma in Euclidean spaces

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Communicated by F. Sommen

(*Received 1 February 2006; in final form 10 March 2006*)

In this note a Schwarz lemma for general Euclidean spaces is established. We show that the two-dimensional version of the lemma is equivalent to the Schwarz lemma in the complex plane.

Keywords: Schwarz lemma; Monogenic function; Laurent expansion

AMS Subject Classifications: 30G35; 32A05

1. Introduction

Higher-dimensional version of Schwarz lemma has been sought. Schwarz lemma was studied in the several complex variables context (see [3]). A natural question arises: 'Does there exist a Schwarz lemma in higher dimensional Euclidean spaces?' This note gives an answer to this question. With the Clifford analysis setting we show that a Schwarz lemma exists that is equivalent to the Schwarz lemma in the complex plane.

We first give some basic knowledge in relation to Clifford algebra [1,2]. Let $\mathbf{e}_1, \ldots, \mathbf{e}_m$ be the *basic elements* satisfying $\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i = -2\delta_{ij}$, where $\delta_{ij} = 1$ if $i = j$; and $\delta_{ij} = 0$ otherwise, $i, j = 1, 2, \ldots, m$. Let

$$\mathbf{R}^m = \{\underline{x} = x_1\mathbf{e}_1 + \cdots + x_m\mathbf{e}_m : x_j \in \mathbf{R},\ j = 1, 2, \ldots, m\}$$

be identical with the usual Euclidean space $\mathbf{R}^m$, and

$$\mathbf{R}_1^m = \{x = x_0\mathbf{e}_0 + \underline{x}: x_0 \in \mathbf{R}, \underline{x} \in \mathbf{R}^m\},\ \text{where } \mathbf{e}_0 = 1.$$

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Complex Variables and Elliptic Equations
ISSN 1747-6933 print: ISSN 1747-6941 online
http://www.tandf.co.uk/journals
DOI: 10.1080/17476930600688623

An element in $\mathbf{R}_1^m$ is called a *vector*. The real (or complex) Clifford algebra generated by $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m$, denoted by $\mathbf{R}^{(m)}$ (or $\mathbf{C}^{(m)}$), is the associative algebra generated by $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m$, over the real (or complex) field $\mathbf{R}$ (or $\mathbf{C}$). A general element in $\mathbf{R}^{(m)}$, therefore, is of the form $x = \sum_S x_S \mathbf{e}_S$, where $\mathbf{e}_S = \mathbf{e}_{i_1}\mathbf{e}_{i_2}\cdots\mathbf{e}_{i_l}$, $x_S \in \mathbf{R}$, and S runs over all the ordered subsets of $\{1, 2, \ldots, m\}$, namely

$$S = \{1 \leq i_1 < i_2 < \cdots < i_l \leq m\}, \quad 1 \leq l \leq m.$$

We define the conjugation of $\mathbf{e}_S$ to be $\overline{\mathbf{e}}_S = \overline{\mathbf{e}}_{il}\cdots\overline{\mathbf{e}}_{i1}$, $\overline{\mathbf{e}}_j = -\mathbf{e}_j$. This induces the Clifford conjugate of a vector $x = x_0 + \underline{x}$ to be $\overline{x} = x_0 - \underline{x}$. It is easy to verify that for $0 \neq x \in \mathbf{R}_1^m$ we have

$$x^{-1} = \frac{\overline{x}}{|x|^2}.$$

The *ball with centre x and radius r* in $\mathbf{R}_1^m$ is denoted by $B(x; r)$ and the *closure* of $B(x; r)$ is denoted by $\overline{B}(x; r)$. The natural inner product between x and y in $\mathbf{C}^{(m)}$, denoted by $\langle x, y\rangle$, is the complex number $\sum_S x_S \overline{y_S}$, where $x = \sum_S x_S \mathbf{e}_S$ and $y = \sum_S y_S \mathbf{e}_S$. The norm associated with this inner product is

$$|x| = \langle x,\ x\rangle^{1/2} = \left(\sum_S |x_S|^2\right)^{1/2}.$$

For $x = \sum_S x_S \mathbf{e}_S \in \mathbf{C}^{(m)}$, denoted $[x]_0 = x_0$. It is called the *scalar part* of x. It then follows

$$|x| = \sqrt{[x\overline{x}]_0}.$$

In the following we shall study functions defined in $\mathbf{R}_1^m$ taking values in $\mathbf{C}^{(m)}$. So, they are of the form $f(x) = \sum_S f_S(x)\mathbf{e}_S$, where the f_S are complex-valued functions. We shall use the *generalized Cauchy–Riemann operator* $D = (\partial/\partial x_0)\mathbf{e}_0 + \underline{D}$, where $\underline{D} = (\partial/\partial x_1)\mathbf{e}_1 + \cdots + (\partial/\partial x_m)\mathbf{e}_m$. Define the "left" and "right" roles of the operator D by

$$Df = \sum_{i=0}^{m}\sum_S \frac{\partial f_S}{\partial x_i}\mathbf{e}_i\mathbf{e}_S$$

and

$$fD = \sum_{i=0}^{m}\sum_S \frac{\partial f_S}{\partial x_i}\mathbf{e}_S\mathbf{e}_i.$$

If $Df = 0$ in a domain (open and connected) Ω, then we say that f is *left-monogenic* function in Ω; and, if $fD = 0$ in Ω, we say that f is *right-monogenic* function in Ω. If f is both left- and right-monogenic function, then we say that f is *monogenic*.

In $\mathbf{R}^m$, we use the operator $\underline{D}$ to replace D, which is called the *Dirac operator*.

As a natural generalization of analytic functions to higher-dimensional spaces, left- or right-monogenic functions are the main objects in Clifford analysis. In such framework, there exist a Cauchy theorem and a Cauchy integral formula. Theory of Taylor and Laurent expansions can also be established (see [1,2]).

We call

$$E(x) = \frac{\overline{x}}{|x|^{m+1}}$$

the *Cauchy kernel* in $\mathbf{R}_1^m$. It is easy to verify that $E(x)$ is a monogenic function in $\mathbf{R}_1^m - \{0\}$.

Call $M^+(k, \mathbf{R}_1^m)$ *the space of homogeneous left-monogenic polynomials of degree k in* $\mathbf{R}_1^m$, and $M^-(k, \mathbf{R}_1^m)$ *the space of homogeneous left-monogenic polynomials of degree* $-(k+m)$ *in* $\mathbf{R}_1^m \setminus \{0\}$. Using the Kelvin's inversion formula $If(x) = E(x)f(x^{-1})$, there is a corresponding relation between $M^+(k, \mathbf{R}_1^m)$ and $M^-(k, \mathbf{R}_1^m)$. That is, if $P_k(x) \in M^+(k, \mathbf{R}_1^m)$, then $IP_k(x) = Q_k(x) \in M^-(k, \mathbf{R}_1^m)$; and if $Q_k(x) \in M^-(k, \mathbf{R}_1^m)$, then $IQ_k(x) = P_k(x) \in M^+(k, \mathbf{R}_1^m)$. Both $M^+(k, \mathbf{R}_1^m)$ and $M^-(k, \mathbf{R}_1^m)$ are right-Clifford modules with the same linear dimension the combinatorial number $C_k^{m+k-1} = (m+k-1)!/[(m-1)!k!]$. Note that if $f(x)$ is left-monogenic function, then $If(x)$ is also left-monogenic function (see [1], or from the intertwine results in [4]). In the sequel $\mathbf{N}_0$ denotes the set of non-negative integers.

2. The Schwarz lemma in $\mathbf{R}_1^m$

In this section, we extend Schwarz lemma in $\mathbf{C}$ to higher-dimensional Euclidean spaces. We first obtain a result in $\mathbf{R}_1^m$, then show that when $m=1$ it is equivalent to the Schwarz lemma in the complex plane. We have (see [2])

LEMMA 1 (Laurent expansion) *Let* $\mathbf{n} = (n_1, n_2, \ldots, n_m) \in \mathbf{N}_0^m$, $|\mathbf{n}| = n_1 + n_2 + \cdots + n_m$, *and* $\underline{x}^{\mathbf{n}} = x_1^{n_1} \cdots x_m^{n_m}$. *Assume that* $f(x)$ *is left-monogenic function in the annular domain* $r_1 < |x| < r_2$ $(0 < r_1 < r_2)$. *Then f can be expanded in a unique way into a Laurent series*

$$f(x) = \sum_{|\mathbf{n}|=0}^{\infty} V_{\mathbf{n}}(x) a_{\mathbf{n}} + \sum_{|\mathbf{n}|=0}^{\infty} W_{\mathbf{n}}(x) b_{\mathbf{n}}, \tag{1}$$

where the series converge normally in $B(0; r_2)$ *and in* $\mathbf{R}_1^m \setminus \overline{B}(0; r_1)$, *respectively. Where*

$$V_{\mathbf{n}}(x) = \frac{1}{n_1! \cdots n_m!} \sum_{\pi \in \mathrm{perm}(\mathbf{n})} z_{\pi(n_1)} z_{\pi(n_2)} \cdots z_{\pi(n_m)},$$

perm($\mathbf{n}$) *denotes the set of all distinguishable permutations of the sequence* $(n_1, n_2, \ldots, n_m)$ *and* $z_i = x_i \mathbf{e}_0 - x_0 \mathbf{e}_i$, for $i = 1, 2, \ldots, m$. $W_{\mathbf{n}}(x) = (\partial^{|\mathbf{n}|}/\partial \underline{x}^{\mathbf{n}}) W_0(x)$, $W_0(x) = E(x) = (\bar{x}/|x|^{m+1})$. *The coefficients* $a_{\mathbf{n}}$ *and* $b_{\mathbf{n}}$ *are determined by*

$$a_{\mathbf{n}} = \frac{1}{\omega_m} \int_{\partial B(0,r)} W_{\mathbf{n}}(y) \mathrm{d}\sigma(y) f(y),$$

$$b_{\mathbf{n}} = \frac{1}{\omega_m} \int_{\partial B(0,r)} V_{\mathbf{n}}(y) \mathrm{d}\sigma(y) f(y),$$

where $r \in (r_1, r_2)$ *and* ω_m *is the area of the m-dimensional unit sphere in* $\mathbf{R}_1^m$.

For purely negative powers we precisely have (see 12.1.3, [1]).

LEMMA 2 (Laurent expansion outside a ball) *Let $f(x)$ be left-monogenic function in the domain $|x| > R$ such that*

$$\lim_{|x|\to\infty} f(x) = 0.$$

Then

$$f(x) = \sum_{|\mathbf{n}|=0}^{\infty} W_{\mathbf{n}}(x) b_{\mathbf{n}}.$$

We normally have $|xy| \neq |x||y|$ for x and y in $\mathbf{C}^{(m)}$. However, there holds:

LEMMA 3 *If $\lambda_1 \in \mathbf{R}_1^m$, and $\lambda_2 \in \mathbf{C}^{(m)}$, then $|\lambda_1\lambda_2| = |\lambda_1||\lambda_2|$.*

Proof

$$|\lambda_1\lambda_2| = |\overline{\lambda_2}\ \overline{\lambda_1}| = \sqrt{[\overline{\lambda_2}\ \overline{\lambda_1}\lambda_1\lambda_2]_0} = \sqrt{|\lambda_1|^2[\overline{\lambda_2}\lambda_2]_0} = |\lambda_1|\sqrt{[\overline{\lambda_2}\lambda_2]_0} = |\lambda_1||\lambda_2|. \qquad \blacksquare$$

THEOREM 1 *Suppose that $f(x)$ is left-monogenic function and satisfies $|f(x)| \le 1$ in the domain $|x| > 1$. If, furthermore,*

$$\lim_{|x|\to\infty} f(x) = 0,$$

then there follows

$$|x|^m|f(x)| \le 1 \ (1 < |x| < \infty),$$

and

$$\lim_{|x|\to\infty} |x^m||f(x)| \quad \textit{exists, and} \quad \lim_{|x|\to\infty} |x^m||f(x)| \le 1.$$

If, in particular,

$$\lim_{|x|\to\infty} |x^m||f(x)| = 1,$$

or if there exists x_0, $1 < |x_0| < \infty$, such that $|x_0|^m|f(x_0)| = 1$, then $f(x) = E(x)C_0$ $(|x| > 1)$, where $C_0 \in \mathbf{C}^{(m)}$ is a constant and $|C_0| = 1$.

Proof Since $f(x)$ is left-monogenic function in $|x| > 1$ and satisfies

$$\lim_{|x|\to\infty} f(x) = 0,$$

by Lemma 2, it has a Laurent expansion outside the ball, and

$$\lim_{|x|\to 0} E(x)f(x^{-1}) = b_0.$$

We have that its Kelvin inversion If is left-monogenic function in $|x|<1$. For any $x_0 \in \mathbf{R}_1^m$, $|x_0| > 1$, if $|x_0| > r > 1$, then $|x_0{}^{-1}| < (1/r) < 1$. By the maximum modulus principle ([1]) and Lemma 3, we have

$$\begin{aligned}|E(x_0{}^{-1})||f(x_0)| &= |E(x_0{}^{-1})f(x_0)| \\ &\le \overline{\lim}_{r\to 1} \max_{|x|=1/r} |E(x)f(x^{-1})| \\ &\le \overline{\lim}_{r\to 1} r^n = 1.\end{aligned}$$

Therefore,

$$|b_0| \le 1.$$

Consequently,

$$|x_0|^m |f(x_0)| \le 1 \ (1 < |x_0| < \infty) \quad \text{and} \quad \lim_{|x|\to\infty} |x|^m\, |f(x)| = |b_0| \le 1.$$

In particular, when

$$\lim_{|x|\to\infty} |x^m||f(x)| = 1,$$

or if there exists $x_0, 1 < |x_0| < \infty$, such that $|x_0|^m|f(x_0)| = 1$, then the maximum modulus principle implies

$$E(x)f(x^{-1}) = C_0(|x| < 1) \text{ and } |C_0| = 1.$$

So $f(x) = E(x)C_0$ when $|x| > 1$. ∎

Remark 1 The statement of the lemma and its proof may be adapted word by word to the context $\mathbf{R}^m$.

Let $m = 1$ in the theorem, we obtain.

Corollary 1 *Suppose that $f(z)$ is holomorphic and satisfies $|f(z)| \le 1$ in the domain $|z| > 1$. If*

$$\lim_{|z|\to\infty} f(z) = 0,$$

then $\lim_{|z|\to\infty} |zf(z)| \le 1$ *and* $|f(z)| \le (1/|z|)$ $(1 < |z| < \infty)$. *If, in particular,* $\lim_{|z|\to\infty} |zf(z)| = 1$ *or there exists* $1 < |z_0| < \infty$ *such that* $|f(z_0)| = (1/|z_0|)$, *then*

$$f(z) = e^{i\theta}\frac{1}{z} \ (|z| > 1),$$

where $\theta \in \mathbf{R}$.

The corollary may be proved to be equivalent to:

LEMMA 4 (Schwarz lemma) *Suppose that* $f(z)$ *is holomorphic in* $|z| < 1$ *and* $|f(z)| \leq 1$ *when* $|z| < 1$. *If* $f(0) = 0$, *then* $|f'(0)| \leq 1$ *and* $|f(z)| \leq |z|$ $(|z| < 1)$. *If, in particular, when* $|f'(0)| = 1$ *or there exists* $0 < |z_0| < 1$ *such that* $|f(z_0)| = |z_0|$, *then*

$$f(z) = e^{i\theta} z \ (|z| < 1),$$

where $\theta \in \mathbf{R}$.

The equivalence may be verified through the mapping $z \to 1/z$. Setting $f_1(z) = f(1/z)$, we obtain that $f(z)$ is holomorphic and $|f(z)| \leq 1$ in $|z| < 1$ if and only if $f_1(z)$ is holomorphic and $|f_1(z)| \leq 1$ in $|z| > 1$. Therefore, we have

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots, \tag{2}$$

$$f_1(z) = a_0 + a_1 \frac{1}{z} + a_2 \frac{1}{z^2} + \cdots + a_n \frac{1}{z^n} + \cdots. \tag{3}$$

So $f(0) = 0$ if and only if $\lim_{|z| \to \infty} f_1(z) = 0$. We accordingly have

$$f(z) = a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots, \tag{4}$$

$$f_1(z) = a_1 \frac{1}{z} + a_2 \frac{1}{z^2} + \cdots + a_n \frac{1}{z^n} + \cdots. \tag{5}$$

Obviously, $f(z) \leq |z|$ $(|z| < 1)$ if and only if $f_1(z) \leq 1/|z|$ $(|z| > 1)$. From (4) and (5), we get $|f'(0)| = |a_1| = \lim_{|z| \to \infty} |z f_1(z)|$, and, therefore, $|f'(0)| = 1$ if and only if $\lim_{|z| \to \infty} |z f_1(z)| = 1$. If $0 < |z_0| < 1$, $|f(z_0)| = |z_0|$, then for $z_1 = (1/z_0)$, $1 < |z_1| < \infty$, $|f_1(z_1)| = (1/|z_1|)$. The converse also holds. Finally, $f(z) = e^{i\theta} z$ $(|z| < 1)$ if and only if $f_1(z) = e^{i\theta}(1/z)(|z| > 1)$.

Remark 2 For $m > 1$ the Schwarz lemma inside the unit ball does not hold at least in the original form. For example, the functions

$$f_j(x) = x_j \mathbf{e}_0 - x_0 \mathbf{e}_j, \quad j = 1, 2, \ldots, m$$

are left-monogenic function in $|x| < 1$, and satisfy $|f_j(x)| \leq 1$ when $|x| < 1$. However, for $x = x_0 \mathbf{e}_0 (|x| < 1)$ and non-constant functions f_j there hold $|f_j(x)| = |x|$, $j = 1, \ldots, m$.

Remark 3 In the complex plane, if $f(z)$ is analytic in the annular domain $r_1 < |z| < r_2$, then the Laurent expansion of $f(z)$ is

$$f(z) = \sum_{n=1}^{\infty} a_n z^n + a_0 + \sum_{n=1}^{\infty} b_n z^{-n}.$$

For every $z^k \in P(k, \mathbf{C})$ and $z^{-k} \in Q(k, \mathbf{C})$, the corresponding relation between $P(k, \mathbf{C})$ and $Q(k, \mathbf{C})$ is through the inversion mapping $z \to 1/z$, but rather than Kelvin inversion with the conformal weight $E(z)$, and, for any k, the dimension of $P(k, \mathbf{C})$

or $Q(k, \mathbf{C})$ is 1. So Schwarz lemma for inside and outside of the unit disk are equivalent. While in higher-dimensional spaces, just because $M^-(0, \mathbf{R}_1^m) = \{E(x)b_0\}$ has dimension 1, we are able to have Schwarz lemma for outside of the unit ball. The space $M^+(k, \mathbf{R}_1^m)$ is transformed to $M^-(k, \mathbf{R}_1^m)$ by Kelvin inversion $If(x) = E(x)f(x^{-1})$, and $I(W_0 b_0) = b_0 \in M^+(0, \mathbf{R}_1^m)$. In particular, both spaces $M^{\pm}(k, \mathbf{R}_1^m)$ for $k > 0$ are multi-dimensional. This explains why Schwarz lemma inside the unit ball does not hold for higher-dimensional spaces. It, however, further hints that Schwarz lemma is equivalent to the maximum modulus principle. As a matter of fact, in the proof of Theorem 1 we use the maximum modulus principle as a key step. Now we show that the latter is an immediate consequence of the former, as in the proof of

Corollary 2 (Maximum Modulus Principle) *Assume that f is left-monogenic function in the open and connected set Ω. If there exists a point $a \in \Omega$ such that*

$$|f(x)| \leq |f(a)|, \quad y \in \Omega,$$

then f must be a constant function in Ω.

Proof We may assume $|f(a)| > 0$, for otherwise the assertion is trivial. We show that the set $A = \{x \in \Omega | \, |f(x)| = |f(a)|\}$ is non-empty, and is an open and closed set. Since Ω is open and connected, this will conclude $A = \Omega$. The fact that A being non-empty follows from $a \in A$. If $y \in A$, then there exists an open ball $B(y; r) \subset \Omega$. Construct function $g(x) = (1/|f(a)|)f(y - rx)$. The function g is left-monogenic function and satisfies $|g(x)| \leq 1$ in $|x| < 1$, with $|g(0)| = 1$. The Kelvin inversion of g, that is $Ig(x) = E(x)g(x^{-1})$, is left-monogenically defined in $|x| > 1$ satisfying $|Ig(x)| \leq 1$ in $|x| > 1$. Since

$$\lim_{|x| \to \infty} |x^m||E(x)g(x^{-1})| = \lim_{|x| \to 0} |g(x)| = 1,$$

Theorem 1 may be applied to conclude $g(x^{-1}) = C_0, |C_0| = 1$, for $|x| > 1$, or $g(x) = C_0$ for $|x| < 1$. This shows that $B(y; r) \subset A$. The closeness of A follows from the continuity of f. So, we have $A = \Omega$. In the above argument the usage of Theorem 1, in fact, shows that, not only the norm, but also the function value itself, is equal to a constant. The proof is complete. ■

Acknowledgement

This work was supported by Research Grant of the University of Macau No. RG079/04-05S/QT/FST.

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