DOI: 10.1007/s11425-006-2053-x

Clifford algebra approach to pointwise convergence of Fourier series on spheres

Dedicated to Professor Sheng GONG on the occasion of his 75th birthday

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Received February 20, 2006; accepted June 30, 2006

Abstract We offer an approach by means of Clifford algebra to convergence of Fourier series on unit spheres of even-dimensional Euclidean spaces. It is based on generalizations of Fueter's Theorem inducing quaternionic regular functions from holomorphic functions in the complex plane. We, especially, do not rely on the heavy use of special functions. Analogous Riemann-Lebesgue theorem, localization principle and a Dini's type pointwise convergence theorem are proved.

Keywords: Clifford algebra, pointwise convergence of Fouirer series, unit sphere, Fueter's Theorem.

MSC(2000): 42B05, 30G35; 42C10

1 Introduction

To explain the idea we start from the very basic facts, denote by $\mathbf{T}$ the unit circle in the complex plane $\mathbf{C}$ and $L^2(\mathbf{T})$ the class of square integrable functions on $\mathbf{T}$. We have the correspondence between a function f in $L^2(\mathbf{T})$ and its Fourier series:

$$f(z) \sim \sum_{k=-\infty}^{\infty} c_k z^k, \tag{1}$$

where

$$c_k = \frac{1}{2\pi i}\int_{\mathbf{T}} \zeta^{-k} f(\zeta)\frac{d\zeta}{\zeta}, \qquad k \in \mathbf{Z},$$

where $\mathbf{Z}$ stands for the set of all integers.

Note that the term $c_k z^k$ is the projection of the function f onto the one-dimensional complex linear space, $P_k(\mathbf{T})$, of restrictions to the unit circle of k-homogeneous holomorphic functions. The projection operator, P_k, is the convolution operator with the kernel $p^{(k)}(z) = z^k$:

$$P_k(f)(z) = c_k z^k = p^{(k)} * f(z) = \frac{1}{2\pi i}\int_{\mathbf{T}} (\zeta^{-1}z)^k f(\zeta)\frac{d\zeta}{\zeta}, \qquad k \in \mathbf{Z}, \tag{2}$$

where the measure $\frac{1}{2\pi i}\frac{d\zeta}{\zeta}$ is the normalized Haar measure of the multiplicative group on the circle.

The N-th symmetric partial sum of the Fourier series is

$$s_N f(z) = \sum_{|k|\leqslant N} c_k z^k. \tag{3}$$

Inserting the integral expression for $c_n z^n$ we have that

$$s_N f(z) = \sum_{|k|\leqslant N} \frac{1}{2\pi i}\int_{\mathbf{T}} p^{(k)}(\zeta^{-1}z) f(\zeta)\frac{d\zeta}{\zeta} = \frac{1}{2\pi i} D_N * f(z),$$

where

$$D_N(z) = \sum_{|k|\leqslant N} p^{(k)}(z) = \sum_{|k|\leqslant N} z^k \tag{4}$$

is the N-th Dirichlet kernel. Under the change of a variable $z \to e^{i\theta}$, on letting $F(\theta) = f(e^{i\theta})$, the relation (1) becomes

$$F(\theta) \sim \sum_{k=-\infty}^{\infty} c_k e^{ik\theta}, \tag{5}$$

and, we have, correspondingly,

$$\begin{aligned} \widetilde{s}_N F(\theta) &= \sum_{|k|\leqslant N} c_k e^{ik\theta} && (6)\\ &= \frac{1}{2\pi}\int_0^{2\pi} F(\phi)\widetilde{D}_N(\theta-\phi)d\phi && (7)\\ &= \frac{1}{2\pi}\widetilde{D}_N * f(\theta), \end{aligned}$$

where for $\theta \not\equiv 0 (\mathrm{mod}\ 2\pi)$

$$\widetilde{D}_N(\theta) = \sum_{|k|\leqslant N} e^{ik\theta} = \frac{\sin(N+\frac{1}{2})\theta}{\sin\frac{1}{2}\theta}, \tag{8}$$

and for θ congruent to 0 (mod 2π), $\widetilde{D}_N(\theta)$ has the value $2N+1$, obtained by the continuous extension of the cases for $\theta \neq 0$ (mod 2π). The function $\widetilde{D}_N$ is the classical N-th Dirichlet kernel in angle on the circle.

It is noted that $D_N(e^{i\theta}) = \widetilde{D}_N(\theta)$, and $\widetilde{D}_N$ is a trigonometric polynomial of degree N which is even in θ and satisfies

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\widetilde{D}_N(\theta)d\theta = \frac{1}{\pi}\int_0^{\pi}\widetilde{D}_N(\theta)d\theta = 1.$$

It can be shown (see, for instance, ref. [1]) that

$$\|\widetilde{D}_N\|_1 = \frac{4}{\pi^2}\log N + O(1) \qquad (N\to\infty).$$

The formula (6) may be rewritten as

$$\widetilde{s}_N F(\theta) = a_0/2 + \sum_{k=1}^{N} a_k\cos k\theta + b_k \sin k\theta, \tag{9}$$

where

$$a_k = \frac{1}{\pi}\int_0^{2\pi} \cos k\phi F(\phi)d\phi, \quad k = 0, 1, \ldots, N,$$

and

$$b_k = \frac{1}{\pi}\int_0^{2\pi} \sin k\phi F(\phi)d\phi, \quad k = 1, \ldots, N.$$

The expansion (9) is the so called real form Fourier series, where a_k, b_k are the corresponding Fourier coefficients. Note that the term $a_k \cos k\theta + b_k \sin k\theta$ is the projection of the function f onto the two-dimensional complex linear space generated by

$$\cos k\theta = \frac{z^k + z^{-k}}{2}, \quad \sin k\theta = \frac{z^k - z^{-k}}{2i},$$

where $z = x + iy, |z| = 1, \theta = \arg z$. The functions $\cos k\theta$ and $\sin k\theta$, consisting of a basis, are "circular" k-homogeneous harmonic functions.

If f is the restriction of a holomorphic function on an annulus containing the unit circle, then the theory of Laurent series of holomorphic functions asserts that the partial sum (1) converges to f at all points on $\mathbf{T}$. Carleson extended this result to $L^2(\mathbf{T})$ in ref. [2] with convergence almost everywhere on $\mathbf{T}$ in a place of convergence everywhere for the Laurent series case. In 1967 Hunt further extended this result to functions in $L^p(\mathbf{T}), 1 < p < \infty$[3].

Being well known, one naturally asks: what is the analogous theory on the n-sphere, S^n, in the $(n+1)$-dimensional Euclidean space $\mathbf{R}_1^n$ (see sec. 2), where $\mathbf{R}_1^n = \{x = x_0 + \underline{x} \mid x_0 \in \mathbf{R}, \underline{x} \in \mathbf{R}^n\}$?

For any square integrable function f on the n-sphere, denoted by $f \in L^2(S^n)$, there is an associated Fourier-Laplace series:

$$f \sim \sum_{k=0}^{\infty} f_k, \tag{10}$$

where f_k is the projection of f onto the multi-dimensional complex linear space $P_k(S^n)$ of k-homogeneous spherical harmonics. The relation (10) is the analogy of (5). There are also analogies in the multi-dimensional cases for (7), (8) and (9), of which all are in terms of special spherical harmonics, and in particular, Legendre polynomials, and Gegenbauer polynomials of order 2 for the space dimension $n \geqslant 3$ (see refs. [4, 5]).

There has been a long history for the study of convergence and summability of Fourier-Laplace series on the spheres (see refs. [4, 5]). However, except for the very lowest dimensions, the pointwise convergence, being the initial motivation of the study, could be said to be very little known. The case $n = 2$ seems to be the only well studied case. Dirichlet[6] gave the first detailed study on the case $n = 3$, on the so called Laplace series. Koschmieder[22] studied the case $n = 4$. In 1976, Roetman considered that the general cases under certain conditions, reduce the convergence problem of $n = 2k + 2$ to $n = 2$ and that of $n = 2k + 3$ to $n = 3$[4]. Among others, Meaney addressed some related topics, including the L^p cases[7].

The existing theory of Fourier-Laplace series is not facilitated with a complex structure like that in the complex plane. All the known studies on Fourier-Laplace series heavily depend on the properties of spherical harmonics, especially Legendre polynomials and other special functions. In the current study we offer a new approach based on Clifford algebra. It is known that Clifford algebra offers a structure on Euclidean space similar to that on the complex plane. Based on the complex structure and related results, we will obtain integral expressions of partial sums in terms of Dirichlet kernels where no knowledge of special functions is involved. Our approach makes use of generalizations of Fueter's Theorem (or inducing theorems, see below) on inducing monogenic and harmonic functions from those of the same type but for one complex variable. In particular, the Dirichlet kernels (see sec. 2) are induced from those for one complex variable by using inducing theorems. We note that in the present paper we stress on the method but do not make deliberate efforts to obtain the results of the best possible under the types of conditions assumed. Compared with the results obtained in ref. [4] that heavily relies on the special function machinery, our approach has at least the equal force, but much simpler.

With the complex structure induced from Clifford algebra we further expect to develop the function theory and the operator theory in higher dimensional spaces analogous to those in the complex plane. The study of this paper is, as a matter of fact, a continuation of those made in refs. [8, 9], in which the Hilbert transformation and an algebra of monogenic singular integral operators on Lipschitz perturbations of spheres are studied.

The preceding study along this line has been done in the quaternionic space case, considered as a counterpart of the four dimensional space[10] . The quaternionic space has certain advantages: They form a non-commutative associative divisible algebra. A higher dimensional Euclidean space is merely a linear algebra, imbedded into a none-commutative and none-divisible algebra. The present paper is to show that the principles of the study in the quaternionic space can be carried out to higher but even dimensional cases. The present study contains richer results: Besides the simplification role of the approach skipping over special functions, we deduce certain relations between the even dimensional Dirichlet kernels and those in dimension 2. Although this study is independent of quaternions, we recommend the author to refer to ref. [10] for a comparison.

The applicability of the method adopted in the present paper is restricted to the even dimensional cases. That is because the Fueter's theorem in those cases reduces the pointwise differentiation and further reduces the problems in the complex plane. Our paper is based on this pointwise differentiation approach. On the other hand, for the odd dimensional cases, Fueter's theorem reduces the calculations on Fourier multiplier operators, with connections to complex plane; or, alternatively, the pointwise differentiation but based on the three dimensional space. Those latter cases would require different ideas[11,12],1).

In sec. 2 we give an account of Fueter's theorem and its generalizations related to

1) Peña D, Qian T, Sommen F. An alternative proof of Fueter's theorem. (preprint)

this work, and the monomial functions in $\mathbf{R}_1^n$ and the convolution integral expressions of Laurent series by the monomial functions. In sec. 3 formulas for the monomial functions and the Dirichlet kernels are deduced. In sec. 4 we prove the Riemann-Lebesgue theorem, localization principle, and a Dini's type convergence theorem in the context.

2 Basic Clifford analysis and inducing theorems

For the reader's convenience we include the basic knowledge of Clifford algebra used in this paper. For a more detailed account, see refs. [9, 13, 14]. Let n be a positive integer, and $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ the basic elements satisfying

$$\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i = -2\delta_{ij}, \ 1 \leqslant i, j \leqslant n.$$

We shall be working with the universal algebra generated by $\mathbf{e}_1, \ldots, \mathbf{e}_n$ over the real number field, called the real-Clifford algebra, denoted by $\mathbf{R}^{(n)}$. Denote by $\mathbf{R}^n$ and $\mathbf{R}_1^n$ the linear subspaces of $\mathbf{R}^{(n)}$ spanned by $\mathbf{e}_1, \ldots, \mathbf{e}_n$ and by $\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$, respectively, where $\mathbf{e}_0$ is the algebraic unit element, i.e. $\mathbf{e}_0 = 1$, viz.

$$\mathbf{R}^n = \{\underline{x} = x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n \mid x_i \in \mathbf{R}, i = 1, 2, \ldots, n\},$$

and

$$\mathbf{R}_1^n = \{\underline{x} = x_0 + \underline{x} \mid x_0 \in \mathbf{R}, \underline{x} \in \mathbf{R}^n\}.$$

Similarly one can define Clifford algebras over the complex number field, $\mathbf{C}^{(n)}$, and correspondingly the linear spaces $\mathbf{C}^n$ and $\mathbf{C}_1^n$.

Elements of $\mathbf{R}^{(n)}$ or $\mathbf{C}^{(n)}$ are denoted by $x, y, \ldots$ and called Clifford numbers. An element in $\mathbf{R}_1^n$ or $\mathbf{C}_1^n$ is called a vector and of the form $x = x_0\mathbf{e}_0 + \underline{x}$, where $x_0 \in \mathbf{R}$ or $\mathbf{C}$, and $\underline{x} \in \mathbf{R}^n$ or $\mathbf{C}^n$. The parts $x_0\mathbf{e}_0$ and $\underline{x}$ are called the real and the imaginary parts of x, respectively. Define two operations on the basic elements: $(\mathbf{e}_{i_1} \cdots \mathbf{e}_{i_l})^* = \mathbf{e}_{i_l} \cdots \mathbf{e}_{i_1}$ and $(\mathbf{e}_{i_1} \cdots \mathbf{e}_{i_l})' = (\mathbf{e}_{i_1})' \cdots (\mathbf{e}_{i_l})'$, where $(\mathbf{e}_0)' = \mathbf{e}_0, (\mathbf{e}_j)' = -\mathbf{e}_j, j = 1, \ldots, n$, and extend them by linearity to $\mathbf{R}^{(n)}$, and hence to $\mathbf{R}_1^n$ and $\mathbf{R}^n$. By combining them we define a third operation $\bar{\ }$ by $\overline{x} = (x^*)'$. If x and y are two Clifford numbers in $\mathbf{R}^{(n)}$, then we have $\overline{xy} = \overline{y}\,\overline{x}$. If $x = x_0 + \underline{x}$, then $\overline{x} = x_0 - \underline{x}$. If x is a vector and $x \neq 0$, then its inverse x^{-1} exists: $x^{-1} = \frac{\overline{x}}{|x|^2}$ and $x^{-1}x = xx^{-1} = 1$. The complex imaginary element i commutes with all the $\mathbf{e}_j, j = 0, 1, \ldots, n$ and $i' = -i$. So we extend the definitions of * and $'$ and therefore $\bar{\ }$ to $\mathbf{C}^{(n)}$. The natural inner product between x and y in $\mathbf{C}^{(n)}$, denoted by $\langle x, y\rangle$, is the complex number $\sum_S x_S\overline{y_S}$, where $x = \sum_S x_S\mathbf{e}_S, y = \sum_S y_S\mathbf{e}_S, S$ runs over all the ordered subsets $(i_1, i_2, \cdots i_l), i_1 < i_2 < \cdots < i_l$, of the set $\{1, 2, \cdots, n\}$ and $\mathbf{e}_S = \mathbf{e}_{i_1}\mathbf{e}_{i_2}\cdots\mathbf{e}_{i_l}$. The norm associated with this inner product is $|x| = \langle x, x\rangle^{\frac{1}{2}} = (\sum_S |x_S|^2)^{\frac{1}{2}}$. The norm and inner products satisfy the relation $\langle x, y\rangle = \frac{1}{4}(|x+y|^2 - |x-y|^2)$. So, if a transform in $\mathbf{C}^{(n)}$ preserves the norm, then it also preserves the inner product. If $x, y, \ldots, u$ are vectors, then $|xy\cdots u| = |x||y|\cdots|u|$. The angle between two vectors x and y, denoted by $\arg(x, y)$, is defined to be $\arccos\frac{\langle x,y\rangle}{|x||y|}$, where the inverse function arccos takes values in $[0, \pi)$. The concept of angle can be extended to any two elements in $\mathbf{R}^{(n)}$ with the same definition, as both the inner product and the norm one are available to the elements in $\mathbf{R}^{(n)}$. By the unit sphere of $\mathbf{R}_1^n$ we mean the set $\{x \in \mathbf{R}_1^n : |x| = 1\}$, denoted by S^n. We use $B_{\mathbf{X}}(x, \delta)$ for the ball in the metric space X centered at x with a radius δ. The substitutions for X are $\mathbf{R}_1^n, \mathbf{R}^n$ and $\mathbf{C}$ in the sequel. If $X = \mathbf{C}$, then x is replaced by z and the balls are called the discs.

We shall be working with $\mathbf{R}_1^n$-variable and $\mathbf{C}^{(n)}$-valued functions. The Cauchy-Riemann operator, or the C-R operator in brief, is defined to be $D = D_0 + \underline{D}$, where $D_0 = \frac{\partial}{\partial x_0}, \underline{D} = \frac{\partial}{\partial x_1}\mathbf{e}_1 + \cdots + \frac{\partial}{\partial x_n}\mathbf{e}_n$. $\underline{D}$ is called the homogeneous C-R operator or the Dirac operator in $\mathbf{R}^n$. The conjugate of the C-R operator is $\overline{D} = D_0 - \underline{D}$. We define the "left" and "right" roles of the operator D on continuously differentiable functions by

$$D_l f = \sum_{i=0}^{n} \sum_{S} \frac{\partial f_S}{\partial x_i} \mathbf{e}_i \mathbf{e}_S$$

and

$$D_r f = \sum_{i=0}^{n} \sum_{S} \frac{\partial f_S}{\partial x_i} \mathbf{e}_S \mathbf{e}_i.$$

If in a domain (open and connected) Ω there holds $D_l f = 0$, then we say that f is left-monogenic; and, if $D_r f = 0$, then right-monogenic, in Ω. A function is said to be monogenic, if it is both left- and right-monogenic.

The integers and positive integers are denoted by $\mathbf{Z}$ and $\mathbf{Z}^+$, respectively. In equalities or inequalities, the capital letter C will be used for constants which may vary from one occurrence to the other. Subscripts, such as ν in C_ν, etc. are used to stress the dependence of the constants. We usually work out the explicit formulas of the constants to specify the dependence.

Assume that Ω is a bounded, open and connected set in $\mathbf{R}_1^n$ with a Lipschitz boundary. Let f, g be respectively the left- and right-regular functions defined in a neighborhood of the closure of Ω. Then there holds the Cauchy Theorem

$$\int_{\partial\Omega} g(x)n(x)f(x)d\sigma(x) = 0,$$

where $d\sigma$ is the surface area measure and $n(x)$ the outward pointing unit normal to $\partial\Omega$ at $x \in \partial\Omega$. Under the set conditions, for $x \in \Omega$, there holds

$$f(x) = \frac{1}{\omega_n} \int_{\partial\Omega} E(y - x)n(y)f(y)d\sigma(y),$$

known as the Cauchy formula, where $E(x) = \frac{\overline{x}}{|x|^{n+1}}$ is the Cauchy kernel, $\omega_n = \frac{2\pi^{(n+1)/2}}{\Gamma[(n+1)/2]}$, the surface area of S_n, the n-dimensional sphere in $\mathbf{R}_1^n$.

Denote by I the Kelvin inversion defined by

$$I(f)(x) = E(x)f(x^{-1}).$$

Define, for $k \in \mathbf{Z}_+$,

$$P^{(-k)}(x) = \frac{(-1)^{k-1}}{(k-1)!} \left(\frac{\partial}{\partial x_0}\right)^{k-1} E(x),$$

and

$$P^{(k-1)}(x) = I(P^{(-k)})(x).$$

The defined functions $P^{(k)}$, $k \in \mathbf{Z}$, are called the monomial functions.

Note that in $\mathbf{R}_1^n$ a vector $x = x_0 + \underline{x}$ whereby $\underline{x}$ can be written in the form $\underline{x} = \mathbf{e}|\underline{x}|$, where $\mathbf{e} = \frac{\underline{x}}{|\underline{x}|}$ and $\mathbf{e}^2 = -1$. This shows that the variable $x_0 + \mathbf{e}|\underline{x}|$ behaves like the complex variable $z = t + is$, under the correspondence $t \to x_0, s \to |\underline{x}|, i \to \mathbf{e}$.

The monomial functions are induced from generalizations of Fueter's theorem. Fueter's theorem provides a method to induce the monogenic (regular) functions of a quaternionic variable from holomorphic functions of a complex variable. Denote by $\mathbf{H}$ the space of quaternions that is the Clifford algebra space $\mathbf{R}^{(2)}$ generated by $\mathbf{e}_1, \mathbf{e}_2$. It can be easily verified that by letting $\mathbf{i} = \mathbf{e}_1, \mathbf{j} = \mathbf{e}_2$ and $\mathbf{k} = \mathbf{e}_1\mathbf{e}_2$, $\mathbf{i}, \mathbf{j}, \mathbf{k}$ satisfy the requirements for the quaternionic basic elements. The quaternionic monogeneity is also called the quaternionic regularity or the brief regularity in literature. Now assume that f^0 is holomorphic, defined in a relatively open set O in the upper half complex plane, $f^0(z) = u(t,s) + iv(t,s), z = t + is$, where u and v are real-valued. Then Fueter's theorem asserts that $\triangle \vec{f^0}$ is a quaternionic regular function in the relatively open set $\vec{O} = \{q = q_0 + \underline{q} \in \mathbf{H} : (q_0, |\underline{q}|) \in O\}$, where $\vec{f^0} = u(q_0, |\underline{q}|) + \mathbf{e}_{\underline{q}} v(q_0, |\underline{q}|)$, and $\mathbf{e}_{\underline{q}} = \frac{\underline{q}}{|\underline{q}|}$. We will call $\vec{f^0}$ the induced function from f^0 and $\vec{O}$ the induced set from O. The following relations hold:

$$\bar{D}\vec{f^0} = \overrightarrow{\left(\frac{df^0}{dz}\right)}, \tag{11}$$

where $\frac{d}{dz} = \frac{1}{2}(\partial_t - i\partial_s)$, and

$$\triangle \vec{f^0}(q) = \frac{2}{|\underline{q}|}\frac{\partial u}{\partial y}(q_0, |\underline{q}|) + 2\mathbf{e}_{\underline{q}}\left(\frac{1}{|\underline{q}|}\frac{\partial v}{\partial y}(q_0, |\underline{q}|) - \frac{1}{|\underline{q}|^2} v(q_0, |\underline{q}|)\right). \tag{12}$$

We refer to refs. [15, 16] for proofs of these equalities and the two-sided regularity of $\triangle \vec{f^0}$. In the sequel we will call the generalizations of Fueter's theorem inducing theorems. Some inducing theorems are contained in refs. [9, 11, 12, 17, 18]. In below we recall some of those results in relation to this work.

The concept of intrinsic functions suits our theory well. An open set in the complex plane $\mathbf{C}$ is said to be intrinsic if it is symmetric with respect to the real-axis; and a function f^0 is said to be intrinsic if the domain of f^0 is an intrinsic set and $\overline{f^0(z)} = f^0(\overline{z})$ in its domain. An open set in $\mathbf{R}_1^n$ is said to be intrinsic if it does not change under the rotations of $\mathbf{R}_1^n$, considered as an $n+1$ dimensional Euclidean space, that keep the $\mathbf{e}_0$-axis unchanged. If O is an open set in the complex plane, then $\vec{O} = \{x \in \mathbf{R}_1^n : (x_0, |\underline{x}|) \in O\}$ is called the induced set from O. It is clear that an induced set is always an intrinsic set in $\mathbf{R}_1^n$. Functions of the form $\sum c_k(z - a_k)^k, k \in \mathbf{Z}, a_k, c_k \in \mathbf{R}$ are intrinsic functions. If $f^0 = u + iv$, where u and v are real-valued, then f^0 is intrinsic if and only if $u(x,-y) = u(x,y), v(x,-y) = -v(x,y)$ in its domain. In particular, $v(x,0) = 0$, i.e. f^0 is real-valued if it is restricted to the real line in its domain. For more information on intrinsic functions in the complex plane and in the quaternionic space, we refer the readers to refs. [19, 20].

Let $f^0(z) = u(t,s)+iv(t,s)$ be an intrinsic function defined on an intrinsic set $U \subset \mathbf{C}$. We may induce a function $\vec{f^0}$ from f^0, defined on the induced set $\vec{U}$, as follows:

$$\overrightarrow{f^0}(x) = u(x_0, |\underline{x}|) + \frac{\underline{x}}{|\underline{x}|} v(x_0, |\underline{x}|), \quad x \in \vec{U}. \tag{13}$$

The function $\overrightarrow{f^0}$ will be called the induced function from f^0.

Denote by τ the mapping

$$\tau(f^0) = \kappa_n^{-1} \Delta^{\frac{n-1}{2}} \vec{f^0}, \tag{14}$$

where $\Delta = D\overline{D}, \overline{D} = D_0 - \underline{D}$ and $\kappa_n = (2i)^{n-1}\Gamma^2(\frac{n+1}{2})$ is the normalizing constant that makes $\tau((\cdot)^{-1}) = E$ (see Proposition 1 in ref. [9]).

The operator $\Delta^{\frac{n-1}{2}}$ is defined via the Fourier multiplier transformation on tempered distributions $M : \mathcal{S}' \to \mathcal{S}'$ induced by the multiplier $m(\xi) = (2\pi i|\xi|)^{n-1}$:

$$Mf = \mathcal{R}(m\mathcal{F}f),$$

where

$$\mathcal{F}f(\xi) = \int_{\mathbf{R}_1^n} e^{2\pi i\langle x,\xi\rangle} f(x)dx$$

and

$$\mathcal{R}h(x) = \int_{\mathbf{R}_1^n} e^{-2\pi i\langle \xi,x\rangle} h(\xi)d\xi.$$

It is noted that both the Fourier transformation $\mathcal{F}$ and its inverse $\mathcal{R}$ are defined on tempered distributions via pairing with rapidly decreasing functions.

If n is an odd integer, then $\Delta^{\frac{n-1}{2}}$ reduces to an ordinary differential operator that was first studied by Sce who extended Fueter's result to $\mathbf{R}_1^n$ for $n \in \mathbf{Z}^+$ being odd[17] . The corresponding result for n being even is obtained and discussed by refs. [9, 11]. We have the following (see refs. [9, 11, 17, 21])

Theorem 1. Let $f^0(z) = u(s,t) + iv(s,t)$ be an intrinsic function defined on an intrinsic set $U \subset \mathbf{C}$. Then the function $\tau(f^0)$ is monogenic in $\vec{U}$.

If we consider f^0 to be of the form $z^k, k \in \mathbf{Z}$, then the monomial functions $\mathbf{R}_1^n$ defined above are $P^{(-k)} = \tau((\cdot)^{-k}),\ P^{(k-1)} = I(P^{(-k)}),\ k \in \mathbf{Z}^+$. We shall write $P_n^{(k)}$ for $P^{(k)}$ in $\mathbf{R}_1^n$ in case we wish to emphasize its dependence on the dimension n. We have

Proposition 1. Let $k \in \mathbf{Z}^+$. Then

(i) $P^{(-1)} = E$;

(ii) $P^{(-k)}(x) = \frac{(-1)^{k-1}}{(k-1)!}(\frac{\partial}{\partial x_0})^{k-1}E(x)$;

(iii) $P^{(-k)}$ and $P^{(k-1)}$ both are monogenic;

(iv) $P^{(-k)}$ is homogeneous of degree $-n+1-k$ and $P^{(k-1)}$ homogeneous of degree $k-1$;

(v) $c_n P_{n-1}^{(-k)}(x_0 + x_1\mathbf{e}_1 + \cdots + x_{n-1}\mathbf{e}_{n-1}) = \int_{-\infty}^{\infty} P_n^{(-k)}(x)dx_n$, where $c_n = \int_{-\infty}^{\infty}(1+t^2)^{-(\frac{n+1}{2})}dt$;

(vi) $P^{(-k)} = I(P^{(k-1)})$;

(vii) if n is odd, then $P^{(k-1)} = \tau((\cdot)^{n+k-2})$.

While the other assertions are easy to prove (see ref. [9]), the proof of (vii) is troublesome and contained in ref. [11].

Remark 1. The definition of the monomial functions together with the properties proved in Proposition 1 provides a generalization of Fueter's result for quaternions. The assertions (ii) and (vi) amount to re-producing Sce's result for $z^k, k \in \mathbf{Z}$. The assertion (viii), in particular, shows that, if n is odd, then $P^{(k-1)}$ may be alternatively defined by using the operator τ, in the pointwise differentiation sense, instead of using the Kelvin inversion.

The following estimates are useful.

Proposition 2. The monomials satisfy the following estimates: for $k \in \mathbf{Z}^+$,

$$|P^{(-k)}(x)| \leqslant C_n k^n |x|^{-(n+k-1)}, \quad |x| > 1,$$

and

$$|P^{(k)}(x)| \leqslant C_n k^n |x|^k, \quad |x| < 1.$$

The operator τ establishes a corresponding relationship between the sequence

$$\{\dots, z^{-3}, z^{-2}, z^{-1}, z^{n-1}, z^n, z^{n+1}, \dots\}$$

and the sequence

$$\{\dots, P^{(-3)}, P^{(-2)}, P^{(-1)}, P^{(0)}, P^{(1)}, P^{(2)}, \dots\}.$$

In $\mathbf{R}_1^n$ we consider the class of functions

$$A(S^n) = \{f(x) : f(x) \quad \text{is left-regular in} \quad 1 - s < |x| < 1 + s \quad \text{for some} \quad s > 0\}.$$

It was proved in ref. [9] that the restrictions to S^n of functions in the class $A(S^n)$ form a dense subclass of $L^2(S^n)$. From ref. [9] we have

Proposition 3. If $f \in A(S^n)$, then

$$f(x) = \sum_{k=-\infty}^{\infty} \frac{1}{\omega_n} \int_{S^n} P^{(k)}(y^{-1}x)E(y)n(y)f(y)d\sigma(y), \; 1 - s < |x| < 1 + s. \tag{15}$$

Note that the expressions $P^{(k)}(y^{-1}x)$ make sense as the domain of $P^{(k)}$ may be extended to products of vectors[9]. Comparing (15) with (2), we see that the functions $P^{(k)}$ and $p^{(k)}$ play the same role in their respective spaces. When n is an odd number, the striking fact is

$$\tau(p^{(k)}) = P^{(k)}, \; k = -1, -2, \dots; \tag{16}$$

and

$$\tau(p^{(k+n-1)}) = P^{(k)}, \; k = 0, 1, 2, \dots. \tag{17}$$

We note that for any k,

$$P^{(k)}(y^{-1}x)E(y)$$

is monogenic in both x and y[9]. Since $E(y)n(y) = 1$ on the sphere, by Cauchy's theorem, we have

$$\int_{S^n} P^{(k)}(y^{-1}x)d\sigma(y) = 0, \quad |x| = 1, \; k \neq 0; \tag{18}$$

and, since $P^{(0)} = I(P^{(-1)}) = I(E) = 1$, we have

$$\frac{1}{\omega_n} \int_{S^n} P^{(0)}(y^{-1}x)d\sigma(y) = 1, \quad |x| = 1. \tag{19}$$

These results will be used in sec. 4.

Thanks to Clifford algebras and the associated complex structure we are enable to decompose spherical harmonics into sums of spherical monogenics. For any k-spherical harmonics f_k, as those appearing in the Fourier-Laplace series of $f \in L^2(S^n)$ in (10), one has the unique decomposition

$$f_k = g_k + h_k, \tag{20}$$

where g_k is the restriction on the sphere of a left-monogenic function of homogeneity k, and h_k the same restriction of a left-monogenic function of homogeneity $-k+1-n$[14]. The results in ref. [9] imply that

$$g_k(x) = \frac{1}{\omega_n}\int_{S^n} P^{(k)}(y^{-1}x)f(y)d\sigma(y),$$

and

$$h_k(x) = \frac{1}{\omega_n}\int_{S^n} P^{(-k)}(y^{-1}x)f(y)d\sigma(y).$$

The partial sum corresponding to (10), denoted by $S_N f(x)$, may be expressed as

$$S_N f(x) = \frac{1}{\omega_n}\int_{S^n} D_N^{(n+1)}(y^{-1}x)f(y)d\sigma(y), \tag{21}$$

where

$$D_N^{(n+1)}(x) = \sum_{|k|\leqslant N} P^{(k)}(x) \tag{22}$$

is called the N-th Dirichlet kernel in $\mathbf{R}_1^n$. When n is an odd number, invoking (16) and (17),

$$D_N^{(n+1)}(x) = \tau\left(\sum_{k=-N}^{N+n-1} p^{(k)}\right)(x), \tag{23}$$

where $p^{(k)}(z) = z^k$. Note that the superscript $n+1$ in the notation $D_N^{(n+1)}$ indicates that the Dirichlet kernels are dependent of the space dimension $n+1$. In the following, we preserve the notation D_N for the case $n=1$, that is $D_N = D_N^{(2)}$.

The equality (12) is sufficient for computing the Dirichlet kernels in the quaternionic space[10]. The analogy of (12) for higher dimensional cases involving the pointwise differential operator $\Delta^{(n-1)/2}$ is rather complicated (also see ref. [9]). In our paper, we only consider n being odd numbers and therefore $(n-1)/2$ being positive integers. The computation for this case is based on the inducing theorem obtained in ref. [12], as cited in the following proposition.

Proposition 4. Let $h(t,s)$ be harmonic for t and s in a region O in which $s>0$, $t>0$. Let $n \in \mathbf{Z}^+$ be odd and $x = x_0 + \underline{x} \in \mathbf{R}_1^n$. Defining

$$H(x) = \Delta^{(n-1)/2} h(x_0, |\underline{x}|),$$

where the Laplacian Δ is in the $n+1$ variables, then

$$H(x) = (n-1)!!\left(\frac{1}{s}\partial_s\right)^{(n-1)/2} h(t,s)|_{t=x_0, s=|\underline{x}|}, \tag{24}$$

and H is harmonic in $x_0, x_1, \ldots, x_n$ in the corresponding region in $\mathbf{R}_1^n$.

Remark 2. It would be interesting to note that not only the Dirichlet kernels, but also the Newton potentials, Poisson kernels and even heat kernels are all deducible from those in the lowest dimensional cases by using the inducing theorems in ref. [12] (see Applications of ref. [12]).

3 Dirichlet kernels and their estimates

Without loss of generality, we assume that the functions to be expanded in (10) are scalar-valued. Indeed, a $\mathbf{C}^{(n)}$-valued function may be separated into 2^n parts of which each is scalar-valued.

Assume that $f \in L^2(S^n)$ and f is associated with an expansion (10). From the decomposition (20) and the integral formulas for g_k and h_k we have

$$f_k(x) = \frac{1}{\omega_n}\int_{S^n}(P^{(k)} + P^{(-k)})(y^{-1}x)f(y)d\sigma(y).$$

Since f is scalar-valued, the scalar part of $P^{(k)} + P^{(-k)}$ will produce f_k, and the non-scalar parts of $P^{(k)}$ and $P^{(-k)}$ will have to be cancelled out. This concludes that only the scalar part of the Dirichlet kernels is concerned. In the following we denote the scalar part of $D_N^{(n+1)}$ by $\mathcal{D}_N^{(n+1)}$.

Denote

$$f_N^0(z) = \sum_{k=-N}^{N+n-1} p^{(k)}(z) = z^{-N} + \cdots + z^{-1} + 1 + z + \cdots + z^{N+n-1},$$

and $f_N^0(z) = U_N(t,s) + iV_N(t,s)$, where $z = t + is$, $t, s \in \mathbf{R}$. Thus, both U_N and V_N are harmonic functions in t and s.

Since we only consider n being odd numbers, we write $n = 2l+1, l \in \mathbf{Z}^+$ for the rest of the paper. Then $\mathcal{D}_N^{(n+1)} = \mathcal{D}_N^{(2l+2)}$. We first have

Lemma 1.

$$\mathcal{D}_N^{(2l+2)}(x) = \kappa_n^{-1}(2l)!!\Big(\frac{1}{s}\partial_s\Big)^l U_N(t,s)|_{t=x_0,s=|\underline{x}|}. \tag{25}$$

Proof.

$$\begin{aligned}\mathcal{D}_N^{(2l+2)}(x) &= \mathrm{Re}\ \tau(f_N^0)(x) = \kappa_n^{-1}\mathrm{Re}\Delta^{(n-1)/2}\overrightarrow{f_N^0}(x)\\ &= \kappa_n^{-1}\Delta^{(n-1)/2}\overrightarrow{U_N}(x) = \kappa_n^{-1}\Delta^{(n-1)/2}U_N(x_0,|\underline{x}|).\end{aligned}$$

By invoking Proposition 4, we have

$$\mathcal{D}_N^{(2l+2)}(x) = \kappa_n^{-1}(2l)!!\Big(\frac{1}{s}\partial_s\Big)^l U_N(t,s)|_{t=x_0,s=|\underline{x}|}.$$

When $l = 1$, i.e. $n = 3$, it is the quaternionic case (see ref. [10]). From ref. [12], we have $(\frac{1}{s}\partial_s)^l = \sum_{j=1}^l C_l^j s^{j-2l}(\partial_s)^j$, where $C_l^j = (-1)^{l-j}\frac{(2l-j-1)!(2l-2j-1)!!}{(j-1)!(2l-2j)!}$, $1 \leqslant j \leqslant l-1$; $C_l^l = 1$. So we can write $\mathcal{D}_N^{(2l+2)}(x)$ in a finite summation:

$$\mathcal{D}_N^{(2l+2)}(x) = \kappa_n^{-1}(2l)!!\sum_{j=1}^{l} C_l^j s^{j-2l}(\partial_s)^j U_N(t,s)|_{t=x_0,s=|\underline{x}|}. \tag{26}$$

For $z = t + is$, $t = r\cos\theta$, $s = r\sin\theta$, we have the following

Lemma 2. There exist homogeneous polynomials $Q_i^{(j)}$ and $R_i^{(j)}$ of degree j about two variables, such that

$$\left(\frac{\partial}{\partial s}\right)^j U_N = \frac{1}{r^j}\sum_{i=1}^{j} Q_i^{(j)}(\sin\theta,\cos\theta)\left(\frac{\partial}{\partial\theta}\right)^i U_N + \frac{1}{r^j}\sum_{i=1}^{j} R_i^{(j)}(\sin\theta,\cos\theta)\left(\frac{\partial}{\partial\theta}\right)^i V_N. \quad (27)$$

Proof. We prove this lemma by the mathematical induction on j.

When $j=1$, we have

$$\frac{\partial U_N}{\partial s} = \frac{\partial U_N}{\partial\theta}\frac{\partial\theta}{\partial s} + \frac{\partial U_N}{\partial r}\frac{\partial r}{\partial s} = \frac{\partial U_N}{\partial\theta}\frac{\cos\theta}{r} + \frac{\partial U_N}{\partial r}\sin\theta.$$

By taking into account Cauchy-Riemann equations in polar coordinates,

$$\frac{\partial U_N}{\partial r} = \frac{1}{r}\frac{\partial V_N}{\partial\theta}, \quad \frac{\partial V_N}{\partial r} = -\frac{1}{r}\frac{\partial U_N}{\partial\theta},$$

we have

$$\frac{\partial U_N}{\partial s} = \frac{1}{r}\cos\theta\frac{\partial U_N}{\partial\theta} + \frac{1}{r}\sin\theta\frac{\partial V_N}{\partial\theta}.$$

So, (27) holds for $j=1$.

Now we assume that

$$\left(\frac{\partial}{\partial s}\right)^j U_N = \frac{1}{r^j}\sum_{i=1}^{j} Q_i^{(j)}(\sin\theta,\cos\theta)\left(\frac{\partial}{\partial\theta}\right)^i U_N + \frac{1}{r^j}\sum_{i=1}^{j} R_i^{(j)}(\sin\theta,\cos\theta)\left(\frac{\partial}{\partial\theta}\right)^i V_N.$$

Then

$$\begin{aligned}
\left(\frac{\partial}{\partial s}\right)^{j+1} U_N &= \frac{\partial}{\partial s}\left(\left(\frac{\partial}{\partial s}\right)^j U_N\right) = \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\left(\left(\frac{\partial}{\partial s}\right)^j U_N\right) + \sin\theta\frac{\partial}{\partial r}\left(\left(\frac{\partial}{\partial s}\right)^j U_N\right)\\
&= \frac{\cos\theta}{r}\left[\frac{1}{r^j}\sum_{i=1}^{j}\frac{\partial}{\partial\theta}Q_i^{(j)}(\sin\theta,\cos\theta)\left(\frac{\partial}{\partial\theta}\right)^i U_N\right.\\
&\quad+\frac{1}{r^j}\sum_{i=1}^{j} Q_i^{(j)}(\sin\theta,\cos\theta)\left(\frac{\partial}{\partial\theta}\right)^{i+1} U_N\\
&\quad+\frac{1}{r^j}\sum_{i=1}^{j}\frac{\partial}{\partial\theta}R_i^{(j)}(\sin\theta,\cos\theta)\left(\frac{\partial}{\partial\theta}\right)^i U_N\\
&\quad\left.+\frac{1}{r^j}\sum_{i=1}^{j} R_i^{(j)}(\sin\theta,\cos\theta)\left(\frac{\partial}{\partial\theta}\right)^{i+1} V_N\right]\\
&\quad+\sin\theta\left[\frac{1}{r^{j+1}}\sum_{i=1}^{j}(-jQ_i^{(j)}(\sin\theta,\cos\theta))\left(\frac{\partial}{\partial\theta}\right)^i U_N\right.\\
&\quad+\frac{1}{r^j}\sum_{i=1}^{j} Q_i^{(j)}(\sin\theta,\cos\theta)\left(\frac{\partial}{\partial\theta}\right)^i\frac{\partial}{\partial r}U_N\\
&\quad+\frac{1}{r^{j+1}}\sum_{i=1}^{j}(-jR_i^{(j)}(\sin\theta,\cos\theta))\left(\frac{\partial}{\partial\theta}\right)^i V_N\\
&\quad\left.+\frac{1}{r^j}\sum_{i=1}^{j} R_i^{(j)}(\sin\theta,\cos\theta)\left(\frac{\partial}{\partial\theta}\right)^i\frac{\partial}{\partial r}V_N\right].
\end{aligned}$$

By Cauchy-Riemann equations in polar coordinates, and by simplifying the above expression, we have

$$\begin{aligned}\left(\frac{\partial}{\partial s}\right)^{j+1} U_N = &\frac{1}{r^{j+1}}\sum_{i=1}^{j}\cos\theta\frac{\partial}{\partial\theta}Q_i^{(j)}(\sin\theta,\cos\theta)\left(\frac{\partial}{\partial\theta}\right)^{i}U_N\\
&+\frac{1}{r^{j+1}}\sum_{i=1}^{j}\cos\theta Q_i^{(j)}(\sin\theta,\cos\theta)\left(\frac{\partial}{\partial\theta}\right)^{i+1}U_N\\
&+\frac{1}{r^{j+1}}\sum_{i=1}^{j}\cos\theta\frac{\partial}{\partial\theta}R_i^{(j)}(\sin\theta,\cos\theta)\left(\frac{\partial}{\partial\theta}\right)^{i}V_N\\
&+\frac{1}{r^{j+1}}\sum_{i=1}^{j}\cos\theta R_i^{(j)}(\sin\theta,\cos\theta)\left(\frac{\partial}{\partial\theta}\right)^{i+1}V_N\\
&+\frac{1}{r^{j+1}}\sum_{i=1}^{j}(-j\sin\theta)Q_i^{(j)}(\sin\theta,\cos\theta)\left(\frac{\partial}{\partial\theta}\right)^{i}U\\
&+\frac{1}{r^{j+1}}\sum_{i=1}^{j}\sin\theta Q_i^{(j)}(\sin\theta,\cos\theta)\left(\frac{\partial}{\partial\theta}\right)^{i+1}V_N\\
&+\frac{1}{r^{j+1}}\sum_{i=1}^{j}(-j\sin\theta)R_i^{(j)}(\sin\theta,\cos\theta)\left(\frac{\partial}{\partial\theta}\right)^{i}V_N\\
&+\frac{1}{r^{j+1}}\sum_{i=1}^{j}(-\sin\theta)R_i^{(j)}(\sin\theta,\cos\theta)\left(\frac{\partial}{\partial\theta}\right)^{i+1}U_N.\end{aligned}$$

Now let

1°. $$\begin{aligned}Q_i^{(j+1)}(\sin\theta,\cos\theta) = &\cos\theta\frac{\partial}{\partial\theta}Q_i^{(j)}(\sin\theta,\cos\theta)+\cos\theta Q_{i-1}^{(j)}(\sin\theta,\cos\theta)\\ &-j\sin\theta Q_i^{(j)}(\sin\theta,\cos\theta)-\sin\theta R_{i-1}^{(j)}(\sin\theta,\cos\theta),\end{aligned}$$

where $1\leqslant i\leqslant j$ and we set $Q_0^{(j)}=R_0^{(j)}=0$(the same in below);

2°. $Q_{j+1}^{(j+1)}(\sin\theta,\cos\theta)=\cos\theta Q_j^{(j)}(\sin\theta,\cos\theta)-\sin\theta R_j^{(j)}(\sin\theta,\cos\theta)$;

3°. $$\begin{aligned}R_i^{(j+1)}(\sin\theta,\cos\theta) = &\cos\theta\frac{\partial}{\partial\theta}R_i^{(j)}(\sin\theta,\cos\theta)+\cos\theta R_{i-1}^{(j)}(\sin\theta,\cos\theta)\\ &+\sin\theta Q_{i-1}^{(j)}(\sin\theta,\cos\theta)-j\sin\theta R_i^{(j)}(\sin\theta,\cos\theta),\quad 1\leqslant i\leqslant j;\end{aligned}$$

4°. $R_{j+1}^{(j+1)}(\sin\theta,\cos\theta)=\cos\theta R_j^{(j)}(\sin\theta,\cos\theta)+\sin\theta Q_j^{(j)}(\sin\theta,\cos\theta)$.

Since $\frac{\partial}{\partial\theta}Q_i^{(j)}$ and $\frac{\partial}{\partial\theta}R_i^{(j)}$, $1\leqslant i\leqslant j$, are still homogeneous polynomials of degree j about variables $\sin\theta$ and $\cos\theta$, obviously, $Q_i^{(j+1)}$ and $R_i^{(j+1)}$, $1\leqslant i\leqslant j+1$, are homogeneous polynomials of degree $j+1$ about variables $\sin\theta$ and $\cos\theta$.

We also have

$$\begin{aligned}\left(\frac{\partial}{\partial s}\right)^{j+1} U_N = &\frac{1}{r^{j+1}}\sum_{i=1}^{j+1}Q_i^{(j+1)}(\sin\theta,\cos\theta)\left(\frac{\partial}{\partial\theta}\right)^{i}U_N\\ &+\frac{1}{r^{j+1}}\sum_{i=1}^{j+1}R_i^{(j+1)}(\sin\theta,\cos\theta)\left(\frac{\partial}{\partial\theta}\right)^{i}V_N.\end{aligned}$$

This completes the proof.

Now, from (26) and (27), $\mathcal{D}_N^{(2l+2)}(x)$ can be written in polar coordinates,

$$\mathcal{D}_N^{(2l+2)}(x) = \kappa_n^{-1}(2l)!!\sum_{j=1}^{l} C_l^j (r\sin\theta)^{j-2l} r^{-j}$$
$$\cdot\Big[\sum_{i=1}^{j} Q_i^{(j)}(\sin\theta,\cos\theta)\Big(\frac{\partial}{\partial\theta}\Big)^i U_N + \sum_{i=1}^{j} R_i^{(j)}(\sin\theta,\cos\theta)\Big(\frac{\partial}{\partial\theta}\Big)^i V_N\Big].$$

Restricting $\mathcal{D}_N^{(2l+2)}(x)$ on the unit sphere($r=1$), we have

$$\mathcal{D}_N^{(2l+2)}(x) = \kappa_n^{-1}(2l)!!\sum_{j=1}^{l}\sum_{i=1}^{j} C_l^j (\sin\theta)^{j-2l} Q_i^{(j)}(\sin\theta,\cos\theta)\Big(\frac{\partial}{\partial\theta}\Big)^i U_N$$
$$+\kappa_n^{-1}(2l)!!\sum_{j=1}^{l}\sum_{i=1}^{j} C_l^j (\sin\theta)^{j-2l} R_i^{(j)}(\sin\theta,\cos\theta)\Big(\frac{\partial}{\partial\theta}\Big)^i V_N.$$

We turn to compute $P^k(x)$. Denote $p^{(k)}(z) = z^k = u_k(t,s)+iv_k(t,s)$, then $\overrightarrow{p^{(k)}}(x) = u_k(x_0,|\underline{x}|) + \frac{\underline{x}}{|\underline{x}|}v_k(x_0,|\underline{x}|)$. Invoking (14), (16) and (17), we obtain

$$P^{(k)}(x) = \kappa_n^{-1}\Delta^l\overrightarrow{p^{(k')}}(x), \tag{28}$$

where $k'=k$ for $k<0$; and $k'=k+n-1$ for $k\geqslant 0$.

From Lemma 1 of ref. [9], we have

Lemma 3.

$$P^{(k)}(x) = \mathrm{Re}\,P^{(k)}(x) + \frac{\underline{x}}{|\underline{x}|}\mathrm{Im}\,P^{(k)}(x), \tag{29}$$

where

$$\mathrm{Re}\,P^{(k)}(x) = \kappa_n^{-1}(2l)!!\Big(\frac{1}{s}\partial_s\Big)^l u_{k'}(t,s)|_{t=x_0,s=|\underline{x}|}, \tag{30}$$

$$\mathrm{Im}\,P^{(k)}(x) = \kappa_n^{-1}(2l)!!s\Big(\frac{1}{s}\partial_s\Big)^l\Big(\frac{v_{k'}(t,s)}{s}\Big)|_{t=x_0,s=|\underline{x}|}, \tag{31}$$

where $k'=k$ for $k<0$; and $k'=k+n-1$ for $k\geqslant 0$.

Clearly, $\mathrm{Re}\,P^{(k)}(x)$ also can be obtained by Proposition 4, like Lemma 1. From the decomposition of $(\frac{1}{s}\partial_s)^l$, we have

$$\mathrm{Re}\,P^{(k)}(x) = \kappa_n^{-1}(2l)!!\sum_{j=1}^{l} C_l^j s^{j-2l}(\partial_s)^j u_{k'}(t,s)|_{t=x_0,s=|\underline{x}|}, \tag{32}$$

$$\mathrm{Im}\,P^{(k)}(x) = \kappa_n^{-1}(2l)!!\sum_{j=1}^{l} C_l^j s^{j-2l+1}(\partial_s)^j\Big(\frac{v_{k'}(t,s)}{s}\Big)|_{t=x_0,s=|\underline{x}|}. \tag{33}$$

In order to express $\mathrm{Re}\,P^{(k)}(x)$ and $\mathrm{Im}\,P^{(k)}(x)$ in polar coordinates, we need another lemma.

Lemma 4. There exist bounded constants $B_i^{(j)}$, $0\leqslant i\leqslant j$, such that

$$\Big(\frac{\partial}{\partial s}\Big)^j\Big(\frac{v_{k'}(t,s)}{s}\Big) = \sum_{i=0}^{j} B_i^{(j)} s^{-(j-i+1)}\Big(\frac{\partial}{\partial s}\Big)^i v_{k'}(t,s). \tag{34}$$

Proof. We prove this lemma by the mathematical induction on j. To simplify the notation, we briefly write $v_{k'}(t,s)=v_k$ in this proof.

When $j=1$, we have

$$\frac{\partial}{\partial s}\Big(\frac{v_k}{s}\Big) = -\frac{1}{s^2}v_k + \frac{1}{s}\frac{\partial}{\partial s}v_k.$$

Let $B_0^{(1)}=-1$, $B_1^{(1)}=1$, so, the lemma is true for $j=1$.

Now we assume that

$$\Big(\frac{\partial}{\partial s}\Big)^j\Big(\frac{v_k}{s}\Big) = \sum_{i=0}^{j} B_i^{(j)} s^{-(j-i+1)}\Big(\frac{\partial}{\partial s}\Big)^i v_k.$$

Then,

$$\begin{aligned}\Big(\frac{\partial}{\partial s}\Big)^{j+1}\Big(\frac{v_k}{s}\Big) &= \frac{\partial}{\partial s}\Big(\Big(\frac{\partial}{\partial s}\Big)^j\Big(\frac{v_k}{s}\Big)\Big)\\ &= \sum_{i=0}^{j}\Big[B_i^{(j)}[-(j-i+1)]s^{-(j-i+2)}\Big(\frac{\partial}{\partial s}\Big)^i v_k + B_i^{(j)} s^{-(j-i+1)}\Big(\frac{\partial}{\partial s}\Big)^{i+1} v_k\Big].\end{aligned}$$

Let $B_i^{(j+1)} = -(j-i+1)B_i^{(j)} + B_{i-1}^{(j)}$, where $0\leqslant i\leqslant j$ and we set $B_{-1}^{(j)}=0$; $B_{j+1}^{(j+1)} = B_j^{(j)}$. Obviously, $B_i^{(j+1)}$, $0\leqslant i\leqslant j+1$, are all bounded since $B_i^{(j)}$, $0\leqslant i\leqslant j$, are all bounded. Summarily, we have

$$\Big(\frac{\partial}{\partial s}\Big)^{j+1}\Big(\frac{v_k}{s}\Big) = \sum_{i=0}^{j+1} B_i^{(j+1)} s^{-(j-i+2)}\Big(\frac{\partial}{\partial s}\Big)^i v_k.$$

This completes the proof.

Now form Lemma 2, Lemma 3 and Lemma 4, restricting $\mathrm{Re}\,P^{(k)}(x)$ and $\mathrm{Im}\,P^{(k)}(x)$ on the unit sphere (r=1), we have

$$\begin{aligned}\mathrm{Re}\,P^{(k)}(x) &= \kappa_n^{-1}(2l)!!\sum_{j=1}^{l}\sum_{i=1}^{j} C_l^j(\sin\theta)^{j-2l}F_i^{(j)}(\sin\theta,\cos\theta)\Big(\frac{\partial}{\partial\theta}\Big)^i u_{k'}\\ &\quad+\kappa_n^{-1}(2l)!!\sum_{j=1}^{l}\sum_{i=1}^{j} C_l^j(\sin\theta)^{j-2l}H_i^{(j)}(\sin\theta,\cos\theta)\Big(\frac{\partial}{\partial\theta}\Big)^i v_{k'},\\ \mathrm{Im}\,P^{(k)}(x) &= \kappa_n^{-1}(2l)!!\sum_{j=1}^{l}\sum_{i=0}^{j}\sum_{m=1}^{i} C_l^j B_i^{(j)}(\sin\theta)^{i-2l}O_m^{(i)}(\sin\theta,\cos\theta)\Big(\frac{\partial}{\partial\theta}\Big)^m v_{k'}\\ &\quad+\kappa_n^{-1}(2l)!!\sum_{j=1}^{l}\sum_{i=0}^{j}\sum_{m=1}^{i} C_l^j B_i^{(j)}(\sin\theta)^{i-2l}W_m^{(i)}(\sin\theta,\cos\theta)\Big(\frac{\partial}{\partial\theta}\Big)^m u_{k'},\end{aligned}$$

where $k'=k$ for $k<0$ and $k'=k+n-1$ for $k\geqslant 0$; $F_i^{(j)}$ and $H_i^{(j)}$, $1\leqslant i\leqslant j$, are homogeneous polynomials of degree j about variables $\sin\theta$ and $\cos\theta$, $O_m^{(i)}$ and $W_m^{(i)}$, $1\leqslant m\leqslant i$, are homogeneous polynomials of degree i about variables $\sin\theta$ and $\cos\theta$.

Now, we work out U_N, V_N, u_k and v_k in polar coordinates. Clearly, it is trivial for u_k and v_k, since $z^k = r^k\cos k\theta + ir^k\sin k\theta$, so,

$$u_k(t,s) = u_k(r\cos\theta, r\sin\theta) = r^k\cos k\theta,\quad v_k(t,s) = v_k(r\cos\theta, r\sin\theta) = r^k\sin k\theta.$$

Restricting to the unit sphere, we have $u_k(\cos\theta,\sin\theta)=\cos k\theta$, $v_k(\cos\theta,\sin\theta)=\sin k\theta$.

On the other hand,

$$
\begin{aligned}
f_N^0(z) &= \sum_{k=-N}^{N+n-1} p^k(z) = \frac{1}{z^N(1-z)} - \frac{z^{N+n}}{1-z} \\
&= \frac{\cos N\theta - r\cos(N+1)\theta - r^{2N+n}\cos(N+n)\theta + r^{2N+n+1}\cos(N+n-1)\theta}{r^N(r^2-2r\cos\theta+1)} \\
&\quad + i\frac{-\sin N\theta + r\sin(N+1)\theta - r^{2N+n}\sin(N+n)\theta + r^{2N+n+1}\sin(N+n-1)\theta}{r^N(r^2-2r\cos\theta+1)} \\
&= U_N(r\cos\theta, r\sin\theta) + iV_N(r\cos\theta, r\sin\theta).
\end{aligned}
$$

Restricting to the unit sphere, we have

$$
\begin{aligned}
U_N(\cos\theta,\sin\theta) &= \frac{\cos N\theta - \cos(N+1)\theta - \cos(N+n)\theta + \cos(N+n-1)\theta}{2(1-\cos\theta)} \\
&= \frac{\sin\frac{2N+1}{2}\theta + \sin\frac{2N+2n-1}{2}\theta}{2\sin\frac{\theta}{2}} = \frac{\sin(N+\frac{n}{2})\theta\cos\frac{n-1}{2}\theta}{\sin\frac{\theta}{2}},
\end{aligned}
$$

and

$$
\begin{aligned}
V_N(\cos\theta,\sin\theta) &= \frac{-\sin N\theta + \sin(N+1)\theta - \sin(N+n)\theta + \sin(N+n-1)\theta}{2(1-\cos\theta)} \\
&= \frac{\cos\frac{2N+1}{2}\theta - \cos\frac{2N+2n-1}{2}\theta}{2\sin\frac{\theta}{2}} = \frac{\sin(N+\frac{n}{2})\theta\sin\frac{n-1}{2}\theta}{\sin\frac{\theta}{2}}.
\end{aligned}
$$

4 Convergence results

The classical Riemann-Lebesgue theorem asserts that for any function in $L^1([0,2\pi])$ its Fourier series coefficients c_k enjoy the property that $|c_k| \to 0$, as $|k| \to \infty$. For any function $f \in L^1(\mathbf{R}^n)$, its Fourier transform $\hat{f}(\xi) \to 0$, as $|\xi| \to \infty$. On the sphere case there does not exist the concept "Fourier coefficients" (not like the one dimensional case where $c_k z^k$ may be separated into two parts c_k and z^k), instead, the projections $P_k(f)$ of a function f onto the multi-dimensional spaces of monogenic functions of k-homogeneity. For each k, the coefficients depend on the base that you choose. Therefore, we should consider the whole projection, and the assertion $P_k(f)(x) \to 0$, as $|k| \to \infty$, is the analogy of the classical Riemann-Lebesgue theorem (see Theorem 2 below).

Averages of spherical functions on $(n-1)$-dimensional spheres are involved. From the study in sec. 3, we know that the terms in $P^{(k)}(x)$ and the Dirichlet kernels $\mathcal{D}_N^{(n+1)}(x)$ on S^n depend only on the angle θ, where $\theta = \arccos x_0$, $\theta \in [0,\pi]$, which means that $P^{(k)}(y^{-1}x)$ and $\mathcal{D}_N^{(n+1)}(y^{-1}x)$ on S^n depend only on $\mathrm{Re}(y^{-1}x)$. In addition, when $x, y \in S^n$, we have that $\mathrm{Re}\,(y^{-1}x) = \mathrm{Re}\,(\bar{y}x) = \langle y, x\rangle$. Now let x be a fixed point on S^n and write $y = x\cos\theta + \tilde{y}\sin\theta$, where $\tilde{y}$ is orthogonal to x and $\theta = \arg(y,x)$. In this case, $y^{-1} = \bar{y} = \bar{x}\cos\theta + \bar{\tilde{y}}\sin\theta$ and $\mathrm{Re}(y^{-1}x) = \langle y, x\rangle = \cos\theta$. We an take average of a function $f \in L^1(S^n)$ over the $(n-1)$-dimensional sphere whose points y satisfy $\arg(y,x) = \theta$. This average is denoted by

$$
\Phi_x(f)(\theta) = \frac{1}{\omega_{n-1}} \int_{S^{n-1}} f(x\cos\theta + \tilde{y}\sin\theta)d\sigma_{n-1}(\tilde{y}),
$$

where $\tilde{y}$ is the spherical variable on S^{n-1}, $d\sigma_{n-1}(\tilde{y})$ is the normalized surface area measure on S^{n-1}. We call $\Phi_x(f)(\theta)$ the average of f about x in angle θ. Let $\mathbf{e}_{\underline{y^{-1}x}} = \frac{\underline{y^{-1}x}}{|\underline{y^{-1}x}|}$, we can similarly define the average $\Phi_x^{\mathbf{e}}(f)(\theta) = \Phi_x(\mathbf{e}_{\underline{(\cdot)^{-1}x}} f(\cdot))(\theta)$.

In accordance with Proposition 3, for any function f integrable on the unit sphere S^n, we can associate it with a Fourier series

$$\frac{1}{\omega_n}\sum_{k=-\infty}^{\infty}\int_{S^n} P^{(k)}(y^{-1}x)E(y)n(y)f(y)d\sigma(y),$$

and the above series is also written as

$$\sum_{k=-\infty}^{\infty} P_k(f)(x),$$

where $P_k(f)(x) = \frac{1}{\omega_n}\int_{S^n} P^{(k)}(y^{-1}x)E(y)n(y)f(y)d\sigma(y)$ is the projection of f onto the space of left-monogenic functions of k-homogeneity.

Note that in considering convergence problems it suffices to assume f to be scalar-valued. Without repeating, we will always assume this for the rest of the paper. In the following we denote $\Phi_x(\theta) = \Phi_x(f)(\theta)$ and $\Phi_x^{\mathbf{e}}(\theta) = \Phi_x^{\mathbf{e}}(f)(\theta)$. In addition, we use $\mathcal{W}^l([0,\pi])$ to denote the Sobolev space

$$\mathcal{W}^l([0,\pi]) = \Big\{g \in L^1([0,\pi]) : \Big(\frac{\partial}{\partial\theta}\Big)^k g \in L^1([0,\pi]),\ k = 1,2,\ldots,l\Big\}.$$

The Riemann-Lebesgue theorem may be formulated as follows.

Theorem 2 (Riemann-Lebesgue Theorem). Assume that $f \in L^1(S^n)$ and let x be a fixed point on S^n. If $\Phi_x(\theta)$, $\Phi_x^{\mathbf{e}}(\theta) \in \mathcal{W}^l([0,\pi])$, then

$$\lim_{|k|\to\infty} P_k(f)(x) = 0.$$

Proof. Since on the unit sphere $E(y)n(y) = 1$, the integration formula of $P_k(f)$ is abbreviated as

$$P_k(f)(x) = \frac{1}{\omega_n}\int_{S^n} P^{(k)}(y^{-1}x)f(y)d\sigma(y),$$

where $y^{-1} = \bar{y} = y_0 - y_1\mathbf{e}_1 - \cdots - y_n\mathbf{e}_n$ and $y^{-1}x = \mathrm{Re}(y^{-1}x) + \mathrm{Im}(y^{-1}x) = < y, x >$ $+\mathrm{Im}(y^{-1}x) = \cos\theta + \mathbf{e}_{\underline{y^{-1}x}}\sin\theta$. Substituting the formula for $P^{(k)}(y^{-1}x)$ into the integral expression and, based on Fubini's theorem, decomposing the integral on the sphere S^n into the iterated integral composed by one in angle θ with respect to the direction of x and the other in the $(n-1)$-dimensional sphere orthogonal with the x direction, we have

$$\begin{aligned}
&P_k\,(f)(x)\\
&= \frac{1}{\omega_n}\int_0^{\pi}(\sin\theta)^{(2l+2-2)}\Big[\kappa_n^{-1}(2l)!!\sum_{j=1}^{l}\sum_{i=1}^{j}C_l^j(\sin\theta)^{j-2l}F_i^{(j)}(\sin\theta,\cos\theta)\Big(\frac{\partial}{\partial\theta}\Big)^i u_{k'}\Big]\\
&\quad\cdot\Phi_x(\theta)d\theta\\
&\quad+\frac{1}{\omega_n}\int_0^{\pi}(\sin\theta)^{(2l+2-2)}\Big[\kappa_n^{-1}(2l)!!\sum_{j=1}^{l}\sum_{i=1}^{j}C_l^j(\sin\theta)^{j-2l}H_i^{(j)}(\sin\theta,\cos\theta)\Big(\frac{\partial}{\partial\theta}\Big)^i v_{k'}\Big]\\
&\quad\cdot\Phi_x(\theta)d\theta
\end{aligned}$$

$$+\frac{1}{\omega_n}\int_0^{\pi}(\sin\theta)^{(2l+2-2)}\Bigg[\kappa_n^{-1}(2l)!!\sum_{j=1}^{l}\sum_{i=0}^{j}\sum_{m=1}^{i}C_l^jB_i^{(j)}(\sin\theta)^{i-2l}$$

$$\cdot O_m^{(i)}(\sin\theta,\cos\theta)\Big(\frac{\partial}{\partial\theta}\Big)^m v_{k'}\Bigg]\Phi_x^{\mathbf{e}}(\theta)d\theta$$

$$+\frac{1}{\omega_n}\int_0^{\pi}(\sin\theta)^{(2l+2-2)}\Bigg[\kappa_n^{-1}(2l)!!\sum_{j=1}^{l}\sum_{i=0}^{j}\sum_{m=1}^{i}C_l^jB_i^{(j)}(\sin\theta)^{i-2l}$$

$$\cdot W_m^{(i)}(\sin\theta,\cos\theta)\Big(\frac{\partial}{\partial\theta}\Big)^m u_{k'}\Bigg]\Phi_x^{\mathbf{e}}(\theta)d\theta$$

$$=\frac{1}{\omega_n}\kappa_n^{-1}(2l)!!\sum_{j=1}^{l}\sum_{i=1}^{j}C_l^j\int_0^{\pi}\Big(\frac{\partial}{\partial\theta}\Big)^i u_{k'}[(\sin\theta)^jF_i^{(j)}(\sin\theta,\cos\theta)\Phi_x(\theta)]d\theta$$

$$+\frac{1}{\omega_n}\kappa_n^{-1}(2l)!!\sum_{j=1}^{l}\sum_{i=1}^{j}C_l^j\int_0^{\pi}\Big(\frac{\partial}{\partial\theta}\Big)^i v_{k'}[(\sin\theta)^jH_i^{(j)}(\sin\theta,\cos\theta)\Phi_x(\theta)]d\theta$$

$$+\frac{1}{\omega_n}\kappa_n^{-1}(2l)!!\sum_{j=1}^{l}\sum_{i=0}^{j}\sum_{m=1}^{i}C_l^jB_i^{(j)}\int_0^{\pi}\Big(\frac{\partial}{\partial\theta}\Big)^m v_{k'}[(\sin\theta)^iO_m^{(i)}(\sin\theta,\cos\theta)\Phi_x^{\mathbf{e}}(\theta)]d\theta$$

$$+\frac{1}{\omega_n}\kappa_n^{-1}(2l)!!\sum_{j=1}^{l}\sum_{i=0}^{j}\sum_{m=1}^{i}C_l^jB_i^{(j)}\int_0^{\pi}\Big(\frac{\partial}{\partial\theta}\Big)^m u_{k'}[(\sin\theta)^iW_m^{(i)}(\sin\theta,\cos\theta)\Phi_x^{\mathbf{e}}(\theta)]d\theta,$$

where $k'=k$ for $k<0$; and $k'=k+n-1$ for $k\geqslant 0$. Since there exist factors $(\sin\theta)^j$, $j\geqslant i$ and $(\sin\theta)^i$, $i\geqslant m$, so, we can take integration by parts repeatedly, then we have

$$P_k(f)(x)=\frac{1}{\omega_n}\kappa_n^{-1}(2l)!!\sum_{j=1}^{l}\sum_{i=1}^{j}(-1)^iC_l^j\int_0^{\pi}u_{k'}(\cos\theta,\sin\theta)$$

$$\cdot\Big(\frac{\partial}{\partial\theta}\Big)^i[(\sin\theta)^jF_i^{(j)}(\sin\theta,\cos\theta)\Phi_x(\theta)]d\theta$$

$$+\frac{1}{\omega_n}\kappa_n^{-1}(2l)!!\sum_{j=1}^{l}\sum_{i=1}^{j}(-1)^iC_l^j\int_0^{\pi}v_{k'}(\cos\theta,\sin\theta)$$

$$\cdot\Big(\frac{\partial}{\partial\theta}\Big)^i[(\sin\theta)^jH_i^{(j)}(\sin\theta,\cos\theta)\Phi_x(\theta)]d\theta$$

$$+\frac{1}{\omega_n}\kappa_n^{-1}(2l)!!\sum_{j=1}^{l}\sum_{i=0}^{j}\sum_{m=1}^{i}(-1)^mC_l^jB_i^{(j)}\int_0^{\pi}v_{k'}(\cos\theta,\sin\theta)$$

$$\cdot\Big(\frac{\partial}{\partial\theta}\Big)^m[(\sin\theta)^iO_m^{(i)}(\sin\theta,\cos\theta)\Phi_x^{e}(\theta)]d\theta$$

$$+\frac{1}{\omega_n}\kappa_n^{-1}(2l)!!\sum_{j=1}^{l}\sum_{i=0}^{j}\sum_{m=1}^{i}(-1)^mC_l^jB_i^{(j)}\int_0^{\pi}u_{k'}(\cos\theta,\sin\theta)$$

$$\cdot\Big(\frac{\partial}{\partial\theta}\Big)^m[(\sin\theta)^iW_m^{(i)}(\sin\theta,\cos\theta)\Phi_x^{e}(\theta)]d\theta.$$

Recall that

$$u_{k'}(\cos\theta,\sin\theta)=\cos k'\theta,\quad v_{k'}(\cos\theta,\sin\theta)=\sin k'\theta,$$

where $k'=k$ for $k<0$; and $k'=k+n-1$ for $k\geqslant 0$. Since $\Phi_x(\theta)$, $\Phi_x^{e}(\theta)\in\mathcal{W}^l([0,\pi])$, then, $(\sin\theta)^j F_i^{(j)}(\sin\theta,\cos\theta)\Phi_x(\theta)$, $(\sin\theta)^j H_i^{(j)}(\sin\theta,\cos\theta)\Phi_x(\theta)$, $(\sin\theta)^i O_m^{(i)}(\sin\theta,\cos\theta)\Phi_x^e(\theta)$, $(\sin\theta)^i W_m^{(i)}(\sin\theta,\cos\theta)\Phi_x^e(\theta)$ also belong to $\mathcal{W}^l([0,\pi])$. So, the assumptions on $\Phi_x(\theta)$ and $\Phi_x^{\mathbf{e}}(\theta)$ allow us to use the classical Riemann-Lebesgue theorem on $[0,2\pi]$, and we obtain

$$\lim_{|k|\to\infty} P_k(f)(x) = 0.$$

Corollary 1. Let $f \in L^2(S^n)$ and x be a fixed point on S^n. If $\Phi_x(\theta) \in \mathcal{W}^l([0,\pi])$, then from (10), for the Fourier-Laplace series of f in spherical harmonics:

$$f(x) \sim \sum_{k=0}^{\infty} f_k(x),$$

we have

$$\lim_{k\to\infty} f_k(x) = 0.$$

Proof. Since $f \in L^2(S^n)$, then we have that $f \in L^1(S^n)$. It is a general result that f_k, as a k-spherical harmonic on the n-dimensional unit sphere, has a unique decomposition $f_k = g_k + h_k$, where g_k and h_k are spherical monogenic functions of, respectively, homogeneity degree k and $-k+1-n$[14]. It turns out that, in the notation of the proof of Theorem 1, $g_k = P_k(f)$, $h_k = P_{-k+1-n}(f)$. In fact, the partial sum $\sum_{k=0}^{N} f_k(x)$ is identical with the partial sum $S_N f(x)$ induced from the N-th Dirichilet kernel $\mathcal{D}_N^{(n+1)}(x)$. Since f_k is scalar-valued, we have

$$f_k = \mathrm{Re}\,(P_k(f)) + \mathrm{Re}\,(P_{-k+1-n}(f)).$$

So, from the proof of Theorem 1, we have

$$\lim_{k\to\infty} f_k(x) = 0.$$

The corollary may be regarded as the Riemann-Lebesgue theorem on the sphere with the traditional setting. We will call the series in the corollary a scalar Fourier series. Due to the fact that all entries of the series, as well as the function itself, are scalar-valued, in the proof we only need to assume the condition on $\Phi_x(\theta)$ is sufficient.

The localization principle in the context is as follows.

Theorem 3. Let $f \in L^1(S^n)$ and x be a fixed point on S^n. If f vanishes in some neighborhood of x, $\Phi_x(\theta) \in \mathcal{W}^l([0,\pi])$, then

$$\lim_{N\to\infty} S_N(f)(x) = 0.$$

Proof. In view of the proof of Corollary 1, the partial sums are scalar-valued and we can restrict ourselves to the scalar-part of the integral under study. Using the formula for $\mathcal{D}_N^{(2l+2)}$, and performing the integration by parts, we have

$$\begin{aligned} &S_N(f)(x) \\ &= \frac{1}{\omega_n}\int_0^{\pi} (\sin\theta)^{(2l+2-2)} \left[\kappa_n^{-1}(2l)!!\sum_{j=1}^{l}\sum_{i=1}^{j} C_l^j (\sin\theta)^{j-2l} Q_i^{(j)}(\sin\theta,\cos\theta)\left(\frac{\partial}{\partial\theta}\right)^i U_N\right] \\ &\quad \cdot\Phi_x(\theta)d\theta \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\omega_n}\int_0^\pi (\sin\theta)^{(2l+2-2)}\left[\kappa_n^{-1}(2l)!!\sum_{j=1}^{l}\sum_{i=1}^{j}C_l^j(\sin\theta)^{j-2l}R_i^{(j)}(\sin\theta,\cos\theta)\left(\frac{\partial}{\partial\theta}\right)^i V_N\right]\\
&\cdot\Phi_x(\theta)d\theta\\
=&\frac{1}{\omega_n}\kappa_n^{-1}(2l)!!\sum_{j=1}^{l}\sum_{i=1}^{j}C_l^j\int_0^\pi\left(\frac{\partial}{\partial\theta}\right)^i U_N\left[(\sin\theta)^j Q_i^{(j)}(\sin\theta,\cos\theta)\Phi_x(\theta)\right]d\theta\\
&+\frac{1}{\omega_n}\kappa_n^{-1}(2l)!!\sum_{j=1}^{l}\sum_{i=1}^{j}C_l^j\int_0^\pi\left(\frac{\partial}{\partial\theta}\right)^i V_N[(\sin\theta)^j R_i^{(j)}(\sin\theta,\cos\theta)\Phi_x(\theta)]d\theta\\
=&\frac{1}{\omega_n}\kappa_n^{-1}(2l)!!\sum_{j=1}^{l}\sum_{i=1}^{j}(-1)^iC_l^j\int_0^\pi U_N(\cos\theta,\sin\theta)\\
&\cdot\left(\frac{\partial}{\partial\theta}\right)^i\left[(\sin\theta)^j Q_i^{(j)}(\sin\theta,\cos\theta)\Phi_x(\theta)\right]d\theta+\frac{1}{\omega_n}\kappa_n^{-1}(2l)!!\sum_{j=1}^{l}\sum_{i=1}^{j}(-1)^iC_l^j\\
&\cdot\int_0^\pi V_N(\cos\theta,\sin\theta)\left(\frac{\partial}{\partial\theta}\right)^i\left[(\sin\theta)^j R_i^{(j)}(\sin\theta,\cos\theta)\Phi_x(\theta)\right]d\theta.
\end{aligned}
$$

Since f vanishes in a neighborhood of x, there exists $\delta > 0$ such that $\Phi_x(\theta) = 0$, $0 \leqslant \theta \leqslant \delta$, $\theta =\arg(y, x)$. So, we have

$$
\begin{aligned}
S_N(f)(x)=&\frac{1}{\omega_n}\kappa_n^{-1}(2l)!!\sum_{j=1}^{l}\sum_{i=1}^{j}(-1)^iC_l^j\int_\delta^\pi U_N(\cos\theta,\sin\theta)\\
&\cdot\left(\frac{\partial}{\partial\theta}\right)^i\left[(\sin\theta)^j Q_i^{(j)}(\sin\theta,\cos\theta)\Phi_x(\theta)\right]d\theta\\
&+\frac{1}{\omega_n}\kappa_n^{-1}(2l)!!\sum_{j=1}^{l}\sum_{i=1}^{j}(-1)^iC_l^j\int_\delta^\pi V_N(\cos\theta,\sin\theta)\\
&\cdot\left(\frac{\partial}{\partial\theta}\right)^i[(\sin\theta)^j R_i^{(j)}(\sin\theta,\cos\theta)\Phi_x(\theta)]d\theta.
\end{aligned}
$$

Recall

$$
\begin{aligned}
U_N(\cos\theta,\sin\theta)&=\frac{\sin(N+\frac{n}{2})\theta\cos\frac{n-1}{2}\theta}{\sin\frac{\theta}{2}},\\
V_N(\cos\theta,\sin\theta)&=\frac{\sin(N+\frac{n}{2})\theta\sin\frac{n-1}{2}\theta}{\sin\frac{\theta}{2}}.
\end{aligned}
$$

From the assumption of this theorem, $\Phi_x(\theta) \in \mathcal{W}^l([0,\pi])$, then $(\frac{\partial}{\partial\theta})^i[(\sin\theta)^j Q_i^{(j)}(\sin\theta, \cos\theta)\Phi_x(\theta)]$ and $(\frac{\partial}{\partial\theta})^i[(\sin\theta)^j R_i^{(j)}(\sin\theta,\cos\theta)\Phi_x(\theta)]$, $1 \leqslant i \leqslant j$, are integrable in the interval (δ,π), on letting N go to infinity and applying the classical Riemann-Lebesgue theorem, we conclude

$$\lim_{N\to\infty} S_N(f)(x) = 0.$$

Denote

$$\Phi_x(0) = \lim_{\theta\to 0}\Phi_x(\theta)$$

if the limit exists.

A Dini's type convergence theorem is as follows.

Theorem 4. Let $f \in L^1(S^n)$ and x be a fixed point on S^n. We assume that, for some $\delta > 0$, $\Phi_x(0)$ exists and $\frac{\Phi_x(\theta)-\Phi_x(0)}{\theta}$ is integrable in $(0,\delta)$. In addition, if $\Phi_x(\theta) \in \mathcal{W}^l([0,\pi])$, then

$$\lim_{N\to\infty} S_N(f)(x) = \Phi_x(0).$$

If, in particular, f is continuous at x, then

$$\lim_{N\to\infty} S_N(f)(x) = f(x).$$

Proof. It suffices to show

$$\lim_{N\to\infty}(S_N(f)(x) - \Phi_x(0)) = 0.$$

From the relations (18) and (19) we have

$$\frac{1}{\omega_n}\int_{S^n} D_N^{(n+1)}(y^{-1}x)d\sigma(y) = 1.$$

We are reduced to show

$$\lim_{N\to\infty}\frac{1}{\omega_n}\int_{S^n} D_N^{(n+1)}(y^{-1}x)(f(y) - \Phi_x(0))d\sigma(y) = 0.$$

The last integral is scalar-valued and thus is the scalar part of the N-th Dirichilet kernel $\mathcal{D}_N^{(n+1)} = \mathcal{D}_N^{(2l+2)}$, viz.

$$\frac{1}{\omega_n}\int_{S^n}\mathcal{D}_N^{(n+1)}(y^{-1}x)(f(y) - \Phi_x(0))d\sigma(y).$$

Substituting the expression of $\mathcal{D}_N^{(2l+2)}(y^{-1}x)$ and writing the integral into an iterated integral, the above becomes

$$\begin{aligned}
&\frac{1}{\omega_n}\int_0^\pi(\sin\theta)^{(2l+2-2)}\Big[\kappa_n^{-1}(2l)!!\sum_{j=1}^{l}\sum_{i=1}^{j}C_l^j(\sin\theta)^{j-2l}Q_i^{(j)}(\sin\theta,\cos\theta)\Big(\frac{\partial}{\partial\theta}\Big)^i U_N\Big]\\
&(\Phi_x(\theta)-\Phi_x(0))d\theta\\
&+\frac{1}{\omega_n}\int_0^\pi(\sin\theta)^{(2l+2-2)}\Big[\kappa_n^{-1}(2l)!!\sum_{j=1}^{l}\sum_{i=1}^{j}C_l^j(\sin\theta)^{j-2l}R_i^{(j)}(\sin\theta,\cos\theta)\Big(\frac{\partial}{\partial\theta}\Big)^i V_N\Big]\\
&\cdot(\Phi_x(\theta)-\Phi_x(0))d\theta\\
=&\frac{1}{\omega_n}\kappa_n^{-1}(2l)!!\sum_{j=1}^{l}\sum_{i=1}^{j}C_l^j\int_0^\pi\Big(\frac{\partial}{\partial\theta}\Big)^i U_N[(\sin\theta)^jQ_i^{(j)}(\sin\theta,\cos\theta)(\Phi_x(\theta)-\Phi_x(0))]d\theta\\
&+\frac{1}{\omega_n}\kappa_n^{-1}(2l)!!\sum_{j=1}^{l}\sum_{i=1}^{j}C_l^j\int_0^\pi\Big(\frac{\partial}{\partial\theta}\Big)^i V_N[(\sin\theta)^jR_i^{(j)}(\sin\theta,\cos\theta)(\Phi_x(\theta)-\Phi_x(0))]d\theta.
\end{aligned}$$

Taking integration by parts repeatedly, we have

$$\begin{aligned}\lim_{N\to\infty}(S_N(f)(x)-\Phi_x(0))= &\lim_{N\to\infty}\frac{1}{\omega_n}\kappa_n^{-1}(2l)!!\sum_{j=1}^{l}\sum_{i=1}^{j}(-1)^i C_l^j\int_0^{\pi}U_N(\cos\theta,\sin\theta)\\ &\cdot\Big(\frac{\partial}{\partial\theta}\Big)^i[(\sin\theta)^j Q_i^{(j)}(\sin\theta,\cos\theta)(\Phi_x(\theta)-\Phi_x(0))]d\theta\\ &+\lim_{N\to\infty}\frac{1}{\omega_n}\kappa_n^{-1}(2l)!!\sum_{j=1}^{l}\sum_{i=1}^{j}(-1)^i C_l^j\int_0^{\pi}V_N(\cos\theta,\sin\theta)\\ &\cdot\Big(\frac{\partial}{\partial\theta}\Big)^i[(\sin\theta)^j R_i^{(j)}(\sin\theta,\cos\theta)(\Phi_x(\theta)-\Phi_x(0))]d\theta\\ =&\lim_{N\to\infty}I_1+\lim_{N\to\infty}I_2.\end{aligned}$$

We first consider the second part of the above expression. Since $\Phi_x(\theta)\in\mathcal{W}^l([0,\pi])$, $\Phi_x(\theta)-\Phi_x(0)$ also belongs to $\mathcal{W}^l([0,\pi])$, as a consequence, $(\frac{\partial}{\partial\theta})^i[(\sin\theta)^j R_i^{(j)}(\sin\theta,\cos\theta)$ $(\Phi_x(\theta)-\Phi_x(0))]$, $1\leqslant i\leqslant j$, is integrable. Replacing V_N by $\frac{\sin(N+\frac{n}{2})\theta\sin\frac{n-1}{2}\theta}{\sin\frac{\theta}{2}}$ and applying the classical Riemann-Lebesgue theorem, we have

$$\lim_{N\to\infty}I_2=0.$$

As for I_1, since U_N is equal to $\frac{\sin(N+\frac{n}{2})\theta\cos\frac{n-1}{2}\theta}{\sin\frac{\theta}{2}}$, there will be a singular point $\theta=0$ when we use the classical Riemann-Lebesgue theorem directly. However, for any j, $1\leqslant j\leqslant l$, we first consider the second summation $\sum_{i=1}^{j}$. There is a factor $(\sin\theta)^j$ in the integrand and the order of $\frac{\partial}{\partial\theta}$ is from 1 to j. When $1\leqslant i\leqslant j-1$, after taking $(\frac{\partial}{\partial\theta})^i$ on $(\sin\theta)^j Q_i^{(j)}(\sin\theta,\cos\theta)(\Phi_x(\theta)-\Phi_x(0))$, there must a factor $(\sin\theta)^{m_1}$ left, $m_1\geqslant 1$. In addition, even if $i=j$, after taking $(\frac{\partial}{\partial\theta})^j$ on $(\sin\theta)^j Q_i^{(j)}(\sin\theta,\cos\theta)(\Phi_x(\theta)-\Phi_x(0))$, there also must be a factor $(\sin\theta)^{m_2}$ left, $m_2\geqslant 1$, except the first item which is $j!(\cos\theta)^j Q_i^{(j)}(\sin\theta,\cos\theta)(\Phi_x(\theta)-\Phi_x(0))$, that is, all the derivatives are taken on $\sin\theta$.

After using the classical Riemann-Lebesgue theorem for I_1 except the first items of the summation $\sum_{i=1}^{j}$ of I_1, we have

$$\begin{aligned}&\lim\nolimits_{N\to\infty}I_1\\ =&\lim_{N\to\infty}\frac{1}{\omega_n}\kappa_n^{-1}(2l)!!\sum_{j=1}^{l}(-1)^j C_l^j\int_0^{\pi}U_N(\cos\theta,\sin\theta)\\ &\cdot[j!(\cos\theta)^j Q_j^{(j)}(\sin\theta,\cos\theta)(\Phi_x(\theta)-\Phi_x(0))]d\theta\\ =&\lim_{N\to\infty}\frac{1}{\omega_n}\kappa_n^{-1}(2l)!!\sum_{j=1}^{l}(-1)^j j!C_l^j\int_0^{\pi}\frac{\sin(N+\frac{n}{2})\theta\cos\frac{n-1}{2}\theta}{\sin\frac{\theta}{2}}\\ &\cdot(\cos\theta)^j Q_j^{(j)}(\sin\theta,\cos\theta)(\Phi_x(\theta)-\Phi_x(0))d\theta\\ =&\lim_{N\to\infty}\frac{1}{\omega_n}\kappa_n^{-1}(2l)!!\sum_{j=1}^{l}(-1)^j j!C_l^j\int_0^{\pi}\sin\Big(N+\frac{n}{2}\Big)\theta\\ &\cdot\cos\frac{n-1}{2}\theta(\cos\theta)^j Q_j^{(j)}(\sin\theta,\cos\theta)\frac{\Phi_x(\theta)-\Phi_x(0)}{\sin\frac{\theta}{2}}d\theta.\end{aligned}$$

Since $\frac{\Phi_x(\theta)-\Phi_x(0)}{\theta}$ is integrable in $(0,\delta)$, for some $\delta>0$, applying the classical Riemann-

Lebesgue theorem, again, we obtain $\lim_{N\to\infty} I_1 = 0$. Therefore, we have

$$\lim_{N\to\infty} S_N(f)(x) = \Phi_x(0).$$

Acknowledgements This work was supported by a research grant from University of macau (Grant No. RG079/04-05S/QT/FST). We wish to convey our special thankfulness to Gong Sheng who visited the University of Macau (UM) a number of times during the last few years. Not only the impressive lectures and pertinent references that he provided but also the participation by him to concrete research topics gave a great impetus to the study of hyper-complex analysis of the analysis group at UM.

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