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# POINTWISE CONVERGENCE FOR EXPANSIONS IN SPHERICAL MONOGENICS*

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**Abstract** We offer a new approach to deal with the pointwise convergence of Fourier-Laplace series on the unit sphere of even-dimensional Euclidean spaces. By using spherical monogenics defined through the generalized Cauchy-Riemann operator, we obtain the spherical monogenic expansions of square integrable functions on the unit sphere. Based on the generalization of Fueter's theorem inducing monogenic functions from holomorphic functions in the complex plane and the classical Carleson's theorem, a pointwise convergence theorem on the new expansion is proved. The result is a generalization of Carleson's theorem to the higher dimensional Euclidean spaces. The approach is simpler than those by using special functions, which may have the advantage to induce the singular integral approach for pointwise convergence problems on the spheres.

**Key words** spherical monogenics; pointwise convergence of Fourier-Laplace series; generalized Cauchy-Riemann operator; unit sphere; generalization of Fueter's theorem

**2000 MR Subject Classification** 42B05; 30G35

## 1 Introduction

We start with reviewing the basic notations and results. Let $f \in L^1([-\pi,\pi])$, then the Fourier coefficients $c_k$ are all well defined by

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\mathrm{e}^{-\mathrm{i}kt}\mathrm{d}t, \qquad k \in \mathbf{Z}, \tag{1.1}$$

where $\mathbf{Z}$ denotes the set of all integers. By $s_N(f)(x)$, we denote the partial sum

$$s_N(f)(x) = \sum_{|k|\le N} c_k\mathrm{e}^{\mathrm{i}kx}, \qquad x\in[-\pi,\pi],\ N\in\mathbf{N}_0, \tag{1.2}$$

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of the Fourier series of $f$, where $\mathbf{N}_0$ denotes the set of all natural numbers. Then we have,

$$s_N(f)(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)D_N(x-t)\mathrm{d}t, \tag{1.3}$$

where

$$D_N(x) = \begin{cases} \dfrac{\sin(N+\frac{1}{2})x}{2\sin\frac{x}{2}} & \text{for } x \in [-\pi,\pi]\backslash\{0\}, \\ N+\dfrac{1}{2} & \text{for } x = 0 \end{cases}$$

is called the $N$-th Dirichlet kernel.

Since $L^2([-\pi,\pi]) \subset L^1([-\pi,\pi])$, the Fourier coefficients of $L^2$-functions are well defined. There holds the following Carleson's theorem.

**Theorem 1.1** [1] If $f \in L^2([-\pi,\pi])$, then

$$s_N(f)(x) \to f(x), \qquad \text{a.e. } x \in [-\pi,\pi], \text{ as } N \to +\infty.$$

L. Carleson proved this theorem in 1966. The next year, R.A. Hunt [7] further extended this result to $f \in L^p([-\pi,\pi])$, $1 < p < \infty$.

One naturally asks what is the analogous result for the unit sphere $S^n$ in the $(n+1)$-dimensional Euclidean space $\mathbf{R}_1^n$, where

$$\mathbf{R}_1^n = \{x = x_0 + \underline{x} \mid x_0 \in \mathbf{R},\ \underline{x} \in \mathbf{R}^n\}?$$

For any square integrable function $f$ defined on $S^n$, denoted by $f \in L^2(S^n)$, there is an associated Fourier-Laplace series:

$$f \sim \sum_{k=0}^{\infty} f_k, \tag{1.4}$$

where $f_k$ is the homogeneous spherical harmonics of degree $k$. There were comprehensive studies on convergence and summability of Fourier-Laplace series on the unit sphere of higher dimensional Euclidean spaces (see [8], [17], [20]). However, except for the two-dimensional case, pointwise convergence, being the initial motivation of harmonic analysis, may be said to be very little known. The case $n = 1$ seems to be the only well studied case ([1], [21]). Dirichlet ([2]) made the first detailed study for the case $n = 2$, on the so-called Laplace series. Koschmieder ([22]) studied the case $n = 3$. Roetman ([17]) considered the general cases, and, under certain conditions, reduced the convergence problem for $n = 2k+1$ to $n = 1$; and $n = 2k+2$ to $n = 2$. Among others, Meaney ([10]) and Yu ([23]) addressed some related topics, including the $L^p$ cases.

The theory of Fourier-Laplace series is not facilitated with a complex structure like what is in the complex plane. All the known studies on Fourier-Laplace series are heavily dependent on the properties of spherical harmonics, especially Legendre polynomials. In this study, we provide a new approach based on Clifford algebras. Based on the induced complex structure we obtain integral expressions of partial sums in terms of Dirichlet kernels where no knowledge of special functions is involved. Our approach is based on generalizations of Fueter's theorem (or inducing theorem, see below) on inducing monogenic and harmonic functions from those for one complex variable. In particular, the Dirichlet kernels (see Section 3) are induced from those for one complex variable by using inducing theorem. The convergence results obtained here are stronger than those obtained in [17] with simpler proofs, while [17] heavily relies on the special functions machinery. The applicability of the method adopted in this article is restricted to the even dimensional cases. That is because, the Fueter's theorem in those cases reduces to pointwise differentiation, and further reduces to the problems in complex plane. Our proofs are based on this pointwise differentiation approach. On the other hand, for the odd dimensional cases, Fueter's theorem reduces to computation on Fourier multiplier operators, with connections to complex plane; or alternatively to pointwise differentiation but based on the three-dimensional Euclidean space. The odd dimensional cases would require different ideas ([11], [13], [15]).

This article is a further development of [5] and [9]. In [5], the authors obtained the detailed expression of the Dirichlet kernels restricted on the unit sphere. Then, Riemann-Lebesgue theorem, localization principle, and a Dini's type pointwise convergence theorem are proved. In this article, based on the classical Carleson's theorem, we further obtain a weaker condition in terms of certain Sobolev spaces that guarantees the pointwise convergence of the Fourier-Laplace series (1.4) (see Section 4 for details). The result is a generalization of Carleson's theorem to the higher dimensional Euclidean spaces. We include here some propositions and results from [5] without proofs.

## 2 Preliminaries

We will be working with $\mathbf{R}_1^n$, the real-linear span of $\mathbf{e}_0, \mathbf{e}_1, \cdots, \mathbf{e}_n$, where $\mathbf{e}_0$ is identical with 1 and $\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i = -2\delta_{ij}$. $\mathbf{R}_1^n$ is embedded into the Clifford alegbra $\mathbf{R}^{(n)}$ generated by $e_1, \cdots, e_n$. A typical element in $\mathbf{R}_1^n$ is denoted $x = x_0 + \underline{x}$, where $x_0 \in \mathbf{R}$ and $\underline{x} = x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n \in \mathbf{R}^n$. We will study $\mathbf{R}_1^n$-variable and Clifford-valued functions and the concepts of left- and right-monogenic functions are introduced via the generalized Cauchy-Riemann operator $D = \frac{\partial}{\partial x_0}\mathbf{e}_0 + \frac{\partial}{\partial x_1}\mathbf{e}_1 + \cdots + \frac{\partial}{\partial x_n}\mathbf{e}_n$: a function $f$ with continuous first order derivatives is said to be left-monogenic or right-monogenic if $Df = 0$ or $fD = 0$ in its domain, respectively. A function is said to be monogenic, if it is both left- and right-monogenic. The Cauchy kernel is denoted by $E(x) = \frac{1}{\omega_n}\frac{\bar{x}}{|x|^{n+1}}, x \in \mathbf{R}_1^n \setminus \{0\}$, where $\omega_n = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})}$ is the surface area of the $n$-dimensional unit sphere $S^n$ in $\mathbf{R}_1^n$. The Cauchy kernel is both left- and right-monogenic.

Let $f^0$ be a complex-valued function defined in an open set $O$ in the upper-half complex plane. Write $f^0(z) = u(t, s) + \mathrm{i}v(t, s)$, where $u$ and $v$ are real-valued.

Denote, for $x \in \vec{O}$,

$$\vec{f^0}(x) = u(x_0, |\underline{x}|) + \frac{\underline{x}}{|\underline{x}|}v(x_0, |\underline{x}|),$$

where $\vec{O} = \{x \in \mathbf{R}_1^n | (x_0, |\underline{x}|) \in O\}$.

$\vec{f^0}$ is said to be the induced function form $f^0$, and $\vec{O}$ is the induced set form $O$. The concept of intrinsic sets and functions naturally fit into our theory. In the complex plane $\mathbf{C}$, a set is said to be intrinsic if it is open and symmetric with respect to the real axis; and a function $f^0$ an intrinsic function if it is defined in an intrinsic set and satisfies $\overline{f^0(z)} = f^0(\bar{z})$ within its domain (see [11]). In the notation $f^0 = u + \mathrm{i}v$, the above condition is equivalent to require that $u$ is even and $v$ is odd in their second argument. In particular, $v(x_0, 0) = 0$, i.e., $f^0$ is real-valued if it is restricted to the real line in its domain.

Denote by $\tau$ the mapping

$$\tau(f^0) = \kappa^{-1}\Delta^{\frac{n-1}{2}}\vec{f^0},$$

where $\Delta = D\bar{D}$, $\bar{D} = \frac{\partial}{\partial x_0}\mathbf{e}_0 - \frac{\partial}{\partial x_1}\mathbf{e}_1 - \cdots - \frac{\partial}{\partial x_n}\mathbf{e}_n$ and $\kappa_n = (2\mathrm{i})^{n-1}\Gamma^2(\frac{n+1}{2})$ are the normalizing constant that makes $\tau((\cdot)^{-1}) = E$ (see [14]).

The operator $\Delta^{\frac{n-1}{2}}$ is defined via the Fourier multiplier transformation on tempered distributions $M : \mathcal{S}' \to \mathcal{S}'$ induced by the multiplier $m(\xi) = (2\pi\mathrm{i}|\xi|)^{n-1}$ :

$$Mf = \mathcal{R}(m\mathcal{F}f),$$

where

$$\mathcal{F}f(\xi) = \int_{\mathbf{R}_1^n} \mathrm{e}^{2\pi\mathrm{i}\langle x,\xi\rangle} f(x)\mathrm{d}x,$$

and

$$\mathcal{R}h(x) = \int_{\mathbf{R}_1^n} \mathrm{e}^{-2\pi\mathrm{i}\langle \xi,x\rangle} h(\xi)\mathrm{d}\xi.$$

It is noted that both the Fourier transformation $\mathcal{F}$ and its inverse $\mathcal{R}$ are defined on tempered distributions via pairing with rapidly decreasing functions.

If $n$ is an odd integer, then $\Delta^{\frac{n-1}{2}}$ reduces to a point-wise differential operator that was first studied by M. Sce who extended Fueter's result to $\mathbf{R}_1^n$ for odd $n \in \mathbf{Z}^+$ ([18]). The corresponding result for $n$ even was obtained and discussed by [13] and [14]. We have the following ([6], [18], [12–14]).

**Proposition 2.1** Let $f^0(z) = u(s,t) + \mathrm{i}v(s,t)$ be an intrinsic function defined on an intrinsic set $O \subset \mathbf{C}$. Then, the function $\tau(f^0)$ is monogenic in $\vec{O}$.

If we consider $f^0$ to be of the form $z^k, k \in \mathbf{Z}$, then we define the monomial functions by

$$P^{(-k)} = \tau((\cdot)^{-k}), \qquad P^{(k-1)} = I(P^{(-k)}), \quad k \in \mathbf{Z}^+,$$

where $\mathbf{Z}^+$ is the set of positive integers and $I$ is the Kelvin inversion defined by

$$I(f)(x) = E(x)f(x^{-1}).$$

When $n$ is odd, the striking facts ([14]) are

$$\tau((\cdot)^{(k)}) = P^{(k)},\ k = -1, -2, \cdots; \tag{2.1}$$

$$\tau((\cdot)^{(k+n-1)}) = P^{(k)},\ k = 0, 1, 2, \cdots. \tag{2.2}$$

We note that, for any $k$,

$$P^{(k)}(y^{-1}x)E(y)$$

is monogenic both in $x$ and $y$ ([14]). Since $E(y)n(y) = 1$ on the sphere, by Cauchy's theorem, we have

$$\int_{S^n} P^{(k)}(y^{-1}x)\mathrm{d}\sigma(y) = 0, \quad |x| = 1,\ k \neq 0; \tag{2.3}$$

and, since $P^{(0)} = I(P^{(-1)}) = I(E) = 1$, we have

$$\frac{1}{\omega_n}\int_{S^n} P^{(0)}(y^{-1}x)\mathrm{d}\sigma(y) = 1, \quad |x| = 1. \tag{2.4}$$

## 3 Expansions of the Fourier-Laplace Series and Dirichlet Kernels

In the frame of Clifford algebras, we are enable to decompose spherical harmonics into sums of spherical monogenics ([4]). For any $k$-spherical harmonic $f_k$, as appeared in the Fourier-Laplace series of $f \in L^2(S^n)$ in (1.4), one has the unique decomposition

$$f_k = g_k + h_k, \tag{3.1}$$

where $g_k$ is the restriction on the sphere of a left-monogenic function of homogeneity $k$, and $h_k$ the same restriction of a left-monogenic function of homogeneity $-k+1-n$ ([3], [4]). Results in [14] imply that

$$g_k(x) = \frac{1}{\omega_n}\int_{S^n} P^{(k)}(y^{-1}x)f(y)\mathrm{d}\sigma(y),$$

and

$$h_k(x) = \frac{1}{\omega_n}\int_{S^n} P^{(-k)}(y^{-1}x)f(y)\mathrm{d}\sigma(y).$$

Note that the expressions $P^{(k)}(y^{-1}x)$ make sense as the domain of $P^{(k)}$ may be extended to products of vectors ([14]). The partial sum corresponding to (1.4), denoted by $S_N f(x)$, may be expressed as

$$S_N f(x) = \frac{1}{\omega_n}\int_{S^n} D_N^{(n+1)}(y^{-1}x)f(y)\mathrm{d}\sigma(y), \tag{3.2}$$

where

$$D_N^{(n+1)}(x) = \sum_{|k|\le N} P^{(k)}(x) \tag{3.3}$$

is called the $N$-th Dirichlet kernel in $\mathbf{R}_1^n$. When $n$ is an odd number, invoking (2.1) and (2.2),

$$D_N^{(n+1)}(x) = \tau\Big(\sum_{k=-N}^{N+n-1}(\cdot)^{(k)}\Big)(x). \tag{3.4}$$

In this article, we only consider $n$ being odd numbers and therefore $(n-1)/2$ being positive integers. The computation of $D_N^{(n+1)}(x)$ is based on the inducing theorem obtained in [15], as cited in the following proposition.

**Proposition 3.1** Let $h(t,s)$ be harmonic for $t$ and $s$ in a region $O$ in which $s>0, t>0$. Let $n \in \mathbf{Z}^+$ being odd and $x = x_0 + \underline{x} \in \mathbf{R}_1^n$. Define

$$H(x) = \Delta^{(n-1)/2} h(x_0, |\underline{x}|),$$

where the Laplacian $\Delta$ is in the $n+1$ variables, then

$$H(x) = (n-1)!!(\frac{1}{s}\partial_s)^{(n-1)/2} h(t,s)|_{t=x_0, s=|\underline{x}|}, \tag{3.5}$$

and $H$ is harmonic in $x_0, x_1, \cdots, x_n$ in the corresponding region in $\mathbf{R}_1^n$.

Without loss of generality, we assume that functions to be expanded in (1.4) are scalar-valued. Indeed, a Clifford-valued function may be separated into $2^n$ parts of each which is scalar-valued.

Assume that $f \in L^2(S^n)$ and $f$ is associated with an expansion (1.4). From the decomposition (3.1) and the integral formulas for $g_k$ and $h_k$, we have

$$f_k(x) = \frac{1}{\omega_n}\int_{S^n}(P^{(k)} + P^{(-k)})(y^{-1}x)f(y)\mathrm{d}\sigma(y).$$

Since $f$ is scalar-valued, the scalar part of $P^{(k)} + P^{(-k)}$ will produce $f_k$, and the nonscalar parts of $P^{(k)}$ and $P^{(-k)}$ will have to be canceled out. This concludes that only the scalar parts of the Dirichlet kernels are concerned. We denote the scalar part of $D_N^{(n+1)}$ by $\mathcal{D}_N^{(n+1)}$.

Denote

$$f_N^0(z) = \sum_{k=-N}^{N+n-1} z^k = z^{-N} + \cdots + z^{-1} + 1 + z + \cdots + z^{N+n-1},$$

and $f_N^0(z) = U_N(t,s) + \mathrm{i}V_N(t,s)$, where $z = t + \mathrm{i}s$, $t, s \in \mathbf{R}$. Thus, both $U_N$ and $V_N$ are harmonic functions in $t$ and $s$.

Since we only consider $n$ being odd numbers, we write $n = 2l+1, l \in \mathbf{Z}^+$ for the rest of the article. Then $\mathcal{D}_N^{(n+1)} = \mathcal{D}_N^{(2l+2)}$. From [5], we have the detailed expression of $\mathcal{D}_N^{(2l+2)}$.

**Proposition 3.2** For any $x \in S^n$, i.e., $x = x_0 + x_1 + \cdots + x_n$, and $|x| = 1$. Then

$$\begin{aligned}\mathcal{D}_N^{(2l+2)}(x) = {}& \kappa_n^{-1}(2l)!!\sum_{j=1}^{l}\sum_{i=1}^{j} C_l^j(\sin\theta)^{j-2l}Q_i^{(j)}(\sin\theta,\cos\theta)(\frac{\partial}{\partial\theta})^i U_N \\ &+\kappa_n^{-1}(2l)!!\sum_{j=1}^{l}\sum_{i=1}^{j} C_l^j(\sin\theta)^{j-2l}R_i^{(j)}(\sin\theta,\cos\theta)(\frac{\partial}{\partial\theta})^i V_N,\end{aligned} \tag{3.6}$$

where $\theta = \arccos x_0$, $Q_i^{(j)}$ and $R_i^{(j)}$ are homogeneous polynomials of degree $j$ in two variables, $C_l^j = (-1)^{l-j}\frac{(2l-j-1)!(2l-2j-1)!!}{(j-1)!(2l-2j)!}$, $1 \le j \le l-1$; $C_l^l = 1$.

Now, we work out $U_N$ and $V_N$ in polar coordinates. We have

$$\begin{aligned} f_N^0(z) &= \sum_{k=-N}^{N+n-1} z^k = \frac{1}{z^N(1-z)} - \frac{z^{N+n}}{1-z} \\ &= \frac{\cos N\theta - r\cos(N+1)\theta - r^{2N+n}\cos(N+n)\theta + r^{2N+n+1}\cos(N+n-1)\theta}{r^N(r^2 - 2r\cos\theta + 1)} \\ &\quad +\mathrm{i}\frac{-\sin N\theta + r\sin(N+1)\theta - r^{2N+n}\sin(N+n)\theta + r^{2N+n+1}\sin(N+n-1)\theta}{r^N(r^2-2r\cos\theta+1)} \\ &= U_N(r\cos\theta, r\sin\theta) + \mathrm{i}V_N(r\cos\theta, r\sin\theta).\end{aligned}$$

Restricting to the unit sphere, we have

$$\begin{aligned} U_N(\cos\theta,\sin\theta) &= \frac{\cos N\theta - \cos(N+1)\theta - \cos(N+n)\theta + \cos(N+n-1)\theta}{2(1-\cos\theta)} \\ &= \frac{\sin\frac{2N+1}{2}\theta + \sin\frac{2N+2n-1}{2}\theta}{2\sin\frac{\theta}{2}} = \frac{\sin(N+\frac{n}{2})\theta\cos\frac{n-1}{2}\theta}{\sin\frac{\theta}{2}},\end{aligned}$$

and

$$\begin{aligned} V_N(\cos\theta,\sin\theta) &= \frac{-\sin N\theta + \sin(N+1)\theta - \sin(N+n)\theta + \sin(N+n-1)\theta}{2(1-\cos\theta)} \\ &= \frac{\cos\frac{2N+1}{2}\theta - \cos\frac{2N+2n-1}{2}\theta}{2\sin\frac{\theta}{2}} = \frac{\sin(N+\frac{n}{2})\theta\sin\frac{n-1}{2}\theta}{\sin\frac{\theta}{2}}.\end{aligned}$$

## 4 Main Results

From the study in Section 3, we know that the Dirichlet kernels $\mathcal{D}_N^{(n+1)}(x)$ on $S^n$ depend only on the angle $\theta$, where $\theta = \arccos x_0$, $\theta \in [0,\pi]$, which means that $\mathcal{D}_N^{(n+1)}(y^{-1}x)$ on $S^n$ depend only on $\mathrm{Re}(y^{-1}x)$. In addition, when $x, y \in S^n$, we have that $\mathrm{Re}(y^{-1}x) = \mathrm{Re}(\bar{y}x) = \langle y, x\rangle$. Now, let $x$ be a fixed point on $S^n$ and write $y = x\cos\theta + \tilde{y}\sin\theta$, where $\tilde{y}$ is orthogonal to $x$ and $\theta = \arg(y,x)$. In this case, $y^{-1} = \bar{y} = \bar{x}\cos\theta + \bar{\tilde{y}}\sin\theta$ and $\mathrm{Re}(y^{-1}x) = \langle y, x\rangle = \cos\theta$. We take average of a function $f \in L^2(S^n)$ over the $(n-1)$-dimensional sphere whose points $y$ satisfy $\arg(y,x) = \theta$. This average is denoted by

$$\Phi_x(f)(\theta) = \frac{1}{\omega_{n-1}}\int_{S^{n-1}} f(x\cos\theta + \tilde{y}\sin\theta)\mathrm{d}\sigma_{n-1}(\tilde{y}),$$

where $\tilde{y}$ is the spherical variable on $S^{n-1}$, $\mathrm{d}\sigma_{n-1}(\tilde{y})$ is the normalized surface area measure on $S^{n-1}$. We call $\Phi_x(f)(\theta)$ the average of $f$ about $x$ in angle $\theta$.

Note that in considering convergence problems it suffices to assume $f$ to be scalar-valued. Without repeating, we will always assume this for the rest of the article. We briefly write $\Phi_x(f)(\theta) = \Phi_x(\theta)$. In addition, we use $\mathcal{W}_2^{l,1}([0,\pi])$ to denote the Sobolev space

$$\mathcal{W}_2^{l,1}([0,\pi]) = \Big\{g \in L^2([0,\pi]) | (\frac{\partial}{\partial\theta})^k g \in L^1([0,\pi]),\ k = 1, 2, \cdots, l\Big\}.$$

Denote $\Phi_x(0) = \lim\limits_{\theta\to 0}\Phi_x(\theta)$ if the limit exists. Then, the pointwise convergence theorem is as follows.

**Theorem 4.1** Let $x$ be a fixed point on $S^n$. For any $f \in L^2(S^n)$, we assume that $\Phi_x(0)$ exists. In addition, if $\Phi_x(\theta) \in \mathcal{W}_2^{l,1}([0,\pi])$, then $\lim\limits_{N\to\infty} S_N(f)(x) = \Phi_x(0)$. If, in particular, $f$ is continuous at $x$, then $\lim\limits_{N\to\infty} S_N(f)(x) = f(x)$.

**Proof** It suffices to show $\lim\limits_{N\to\infty}(S_N(f)(x) - \Phi_x(0)) = 0$. From the relations (2.3) and (2.4), we have

$$\frac{1}{\omega_n}\int_{S^n} D_N^{(n+1)}(y^{-1}x)\mathrm{d}\sigma(y) = 1.$$

It is reduced to show

$$\lim_{N\to\infty}\frac{1}{\omega_n}\int_{S^n} D_N^{(n+1)}(y^{-1}x)(f(y) - \Phi_x(0))\mathrm{d}\sigma(y) = 0.$$

The last integral is scalar-valued and thus is induced from the scalar part of the $N$-th Dirichilet kernel $\mathcal{D}_N^{(n+1)} = \mathcal{D}_N^{(2l+2)}$, viz.

$$\frac{1}{\omega_n}\int_{S^n} \mathcal{D}_N^{(2l+2)}(y^{-1}x)(f(y) - \Phi_x(0))\mathrm{d}\sigma(y).$$

Substituting the expression of $\mathcal{D}_N^{(2l+2)}(y^{-1}x)$ and writing the integral into an iterated integral, the above becomes

$$\frac{1}{\omega_n}\int_0^\pi (\sin\theta)^{(2l+2-2)}\Big[\kappa_n^{-1}(2l)!!\sum_{j=1}^{l}\sum_{i=1}^{j} C_l^j(\sin\theta)^{j-2l}Q_i^{(j)}(\sin\theta,\cos\theta)(\frac{\partial}{\partial\theta})^i U_N\Big]$$
$$\cdot(\Phi_x(\theta) - \Phi_x(0))\mathrm{d}\theta$$

$$+\frac{1}{\omega_n}\int_0^{\pi}(\sin\theta)^{(2l+2-2)}\Big[\kappa_n^{-1}(2l)!!\sum_{j=1}^{l}\sum_{i=1}^{j}C_l^j(\sin\theta)^{j-2l}R_i^{(j)}(\sin\theta,\cos\theta)(\frac{\partial}{\partial\theta})^iV_N\Big]$$
$$\cdot(\Phi_x(\theta)-\Phi_x(0))\mathrm{d}\theta$$
$$=\frac{1}{\omega_n}\kappa_n^{-1}(2l)!!\sum_{j=1}^{l}\sum_{i=1}^{j}C_l^j\int_0^{\pi}(\frac{\partial}{\partial\theta})^iU_N[(\sin\theta)^jQ_i^{(j)}(\sin\theta,\cos\theta)(\Phi_x(\theta)-\Phi_x(0))]\mathrm{d}\theta$$
$$+\frac{1}{\omega_n}\kappa_n^{-1}(2l)!!\sum_{j=1}^{l}\sum_{i=1}^{j}C_l^j\int_0^{\pi}(\frac{\partial}{\partial\theta})^iV_N[(\sin\theta)^jR_i^{(j)}(\sin\theta,\cos\theta)(\Phi_x(\theta)-\Phi_x(0))]\mathrm{d}\theta.$$

Taking integration by parts repeatedly, we have

$$\lim_{N\to\infty}(S_N(f)(x)-\Phi_x(0))$$
$$=\lim_{N\to\infty}\frac{1}{\omega_n}\kappa_n^{-1}(2l)!!\sum_{j=1}^{l}\sum_{i=1}^{j}(-1)^iC_l^j\int_0^{\pi}U_N(\cos\theta,\sin\theta)(\frac{\partial}{\partial\theta})^i$$
$$\cdot\Big[(\sin\theta)^jQ_i^{(j)}(\sin\theta,\cos\theta)(\Phi_x(\theta)-\Phi_x(0))\Big]\mathrm{d}\theta$$
$$+\lim_{N\to\infty}\frac{1}{\omega_n}\kappa_n^{-1}(2l)!!\sum_{j=1}^{l}\sum_{i=1}^{j}(-1)^iC_l^j\int_0^{\pi}V_N(\cos\theta,\sin\theta)(\frac{\partial}{\partial\theta})^i$$
$$\cdot\Big[(\sin\theta)^jR_i^{(j)}(\sin\theta,\cos\theta)(\Phi_x(\theta)-\Phi_x(0))\Big]\mathrm{d}\theta$$
$$=\lim_{N\to\infty}I_1+\lim_{N\to\infty}I_2.$$

We first consider the second part of the above expression. Since $\Phi_x(\theta)\in\mathcal{W}_2^{l,1}([0,\pi])$, $\Phi_x(\theta)-\Phi_x(0)$ belongs to $\mathcal{W}_2^{l,1}([0,\pi])$. As a consequence, $(\frac{\partial}{\partial\theta})^i[(\sin\theta)^jR_i^{(j)}(\sin\theta,\cos\theta)(\Phi_x(\theta)-\Phi_x(0))]$, $1\le i\le j$, are all integrable. Replacing $V_N$ by $\frac{\sin(N+\frac{n}{2})\theta\sin\frac{n-1}{2}\theta}{\sin\frac{\theta}{2}}$ and applying the classical Riemann–Lebesgue lemma, we have $\lim\limits_{N\to\infty}I_2=0$.

As for $I_1$, since $U_N$ is equal to $\frac{\sin(N+\frac{n}{2})\theta\cos\frac{n-1}{2}\theta}{\sin\frac{\theta}{2}}$, there will be a singular point $\theta=0$, thus the classical Riemann–Lebesgue lemma cannot be directly used. However, for any $j$, $1\le j\le l$, we first consider the second summation $\sum\limits_{i=1}^{j}$. There is a factor $(\sin\theta)^j$ in the integrand and the order of $\frac{\partial}{\partial\theta}$ is from 1 to $j$. When $1\le i\le j-1$, after taking $(\frac{\partial}{\partial\theta})^i$ on $(\sin\theta)^jQ_i^{(j)}(\sin\theta,\cos\theta)(\Phi_x(\theta)-\Phi_x(0))$, there must be a factor $(\sin\theta)^{m_1}$ left, $m_1\ge1$. If $i=j$, after taking $(\frac{\partial}{\partial\theta})^j$ on $(\sin\theta)^jQ_i^{(j)}(\sin\theta,\cos\theta)(\Phi_x(\theta)-\Phi_x(0))$, there also must be a factor $(\sin\theta)^{m_2}$ left, $m_2\ge1$, except the first item that is $j!(\cos\theta)^jQ_i^{(j)}(\sin\theta,\cos\theta)(\Phi_x(\theta)-\Phi_x(0))$.

By using the classical Riemann–Lebesgue lemma for $I_1$ except the first items of the summation $\sum\limits_{i=1}^{j}$ of $I_1$, we have

$$\lim_{N\to\infty}I_1=\lim_{N\to\infty}\frac{1}{\omega_n}\kappa_n^{-1}(2l)!!\sum_{j=1}^{l}(-1)^jC_l^j\int_0^{\pi}U_N(\cos\theta,\sin\theta)$$
$$\cdot[j!(\cos\theta)^jQ_j^{(j)}(\sin\theta,\cos\theta)(\Phi_x(\theta)-\Phi_x(0))]\mathrm{d}\theta$$
$$=\lim_{N\to\infty}\frac{1}{\omega_n}\kappa_n^{-1}(2l)!!\sum_{j=1}^{l}(-1)^jj!C_l^j\int_0^{\pi}\frac{\sin(N+\frac{n}{2})\theta\cos\frac{n-1}{2}\theta}{\sin\frac{\theta}{2}}$$

$$\cdot(\cos\theta)^j Q_j^{(j)}(\sin\theta,\cos\theta)(\Phi_x(\theta)-\Phi_x(0))\mathrm{d}\theta$$
$$=\frac{2\pi}{\omega_n}\kappa_n^{-1}(2l)!!\sum_{j=1}^{l}(-1)^j j!C_l^j\lim_{N\to\infty}\frac{1}{\pi}\int_0^\pi\frac{\sin((N+\frac{n-1}{2})+\frac{1}{2})\theta}{2\sin\frac{\theta}{2}}\Psi_x^{(j)}(\theta)\mathrm{d}\theta,$$

where $\Psi_x^{(j)}(\theta)=\cos\frac{n-1}{2}\theta(\cos\theta)^jQ_j^{(j)}(\sin\theta,\cos\theta)(\Phi_x(\theta)-\Phi_x(0))$, $1\le j\le l$. Since $\Phi_x(\theta)\in L^2([0,\pi])$, then, obviously, $\Psi_x^{(j)}(\theta)\in L^2([0,\pi])$, $1\le j\le l$. By the classical Carleson's theorem, we obtain

$$\lim_{N\to\infty}I_1=\frac{2\pi}{\omega_n}\kappa_n^{-1}(2l)!!\sum_{j=1}^{l}(-1)^j j!C_l^j\Psi_x^{(j)}(0)=0.$$

Therefore, we have $\lim\limits_{N\to\infty}S_N(f)(x)=\Phi_x(0)$.

Furthermore, by the result of R.A. Hunt ([7]), we can obviously extend the main convergence results to $L^p$ cases, $1<p<\infty$.

Now, we illustrate the results of Theorem 4.1 by two examples.

**Example 4.1** It should be observed that for $n+1=2$, i.e., $l=0$ in Theorem 4.1, the average $\Phi_x(\theta)$ becomes the simple evaluation at two endpoints of the interval $(-\theta,\theta)$,

$$\Phi_x(\theta)=\frac{1}{2}[f(\theta_x+\theta)+f(\theta_x-\theta)],$$

where $\theta_x$ is the angle between $x$ and $e_1$, and the Dirichlet kernel $\mathcal{D}_N^{(2)}$ is just equal to $D_N$ in Section 1. From the condition of Theorem 4.1, $\Phi_x\in L^2([0,\pi])$ if and only if $\frac{1}{2}[f(\theta_x+\theta)+f(\theta_x-\theta)]\in L^2([0,\pi])$. In particular, if $x=1$, Theorem 4.1 reduces to the classical Carleson's theorem.

**Example 4.2** Let $l=1$, i.e., $n+1=4$. Theorem 4.1 becomes the following.

Let $q$ be a fixed point on $S^3$. For any $f\in L^2(S^3)$, we assume that $\Phi_q(0)$ exists. In addition, if $\Phi_q(\theta)\in L^2([0,\pi])$ and its derivative $\Psi_q(\theta)$ exists and $\Psi_q(\theta)$ is integrable in $(0,\pi)$, then $\lim\limits_{N\to\infty}S_N(f)(q)=\Phi_q(0)$. Furthermore, $\lim\limits_{N\to\infty}S_N(f)(q)=f(q)$ if $f$ is continuous at $q$.

The convergence results in [9], with the assumption that $\Phi_q(\theta)$ is absolutely continuous in $[0,\pi]$, is a consequence of the above results. In fact, if $\Phi_q(\theta)$ is absolutely continuous in $[0,\pi]$, then $\Phi_q(\theta)\in L^2([0,\pi])$ and its derivative exists almost everywhere and is integrable on $[0,\pi]$.

For $l\ge 1$ the condition in Theorem 4.1 may be further weakened to the following:

$$\begin{cases}\text{(i)}\ \cos\frac{n-1}{2}\theta(\cos\theta)^jQ_j^{(j)}(\sin\theta,\cos\theta)(\Phi_x(\theta)-\Phi_x(0))\in L^2([0,\pi]),\ \ 1\le j\le l;\\ \text{(ii)}\ (\frac{\partial}{\partial\theta})^i[(\sin\theta)^jQ_i^{(j)}(\sin\theta,\cos\theta)(\Phi_x(\theta)-\Phi_x(0))]\in L^1([0,\pi]),\quad 1\le i\le j\le l;\\ \text{(iii)}\ (\frac{\partial}{\partial\theta})^i[(\sin\theta)^jR_i^{(j)}(\sin\theta,\cos\theta)(\Phi_x(\theta)-\Phi_x(0))]\in L^1([0,\pi]),\quad 1\le i\le j\le l.\end{cases}$$

Obviously, $\Phi_x(\theta)\in\mathcal{W}_2^{l,1}([0,\pi])$ implies the above conditions. The main results of [17] then can be easily deduced. In fact, there holds

**Corollary 4.1** If $f$ is differentiable up to order $l$ on $S^n$, $l\ge1$, i.e., $f\in C^l(S^n)$, then

$$\lim_{N\to\infty}S_N(f)(x)=f(x),\qquad\forall x\in S^n.$$

**Proof** By Whitney's extension theorem ([19], also see [17]), we can show that $\Phi_x(\theta)\in C^l((0,\pi))$, and $(\sin\theta)^j(\frac{\partial}{\partial\theta})^i\Phi_x(\theta)$, $1\le i\le j\le l$, converge to zero at 0 and $\pi$. In particular,

$\Phi_x(\theta)$ is continuous on $[0,\pi]$. Thus, we have that $\Phi_x(\theta) \in L^2([0,\pi])$ and $(\frac{\partial}{\partial\theta})^i[(\sin\theta)^j\Phi_x(\theta)]$ $1 \le i \le j \le l$, are integrable in $[0,\pi]$. Then, the conditions (i), (ii), and (iii) are satisfied since $Q_i^{(j)}$ and $R_i^{(j)}$ are polynomials in $\sin\theta$ and $\cos\theta$. In addition, $f$ is continuous on $S^n$, then we have,

$$\lim_{N\to\infty} S_N(f)(x) = f(x), \qquad \forall x \in S^n.$$

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