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Hilbert Transforms on the Sphere with the Clifford Algebra Setting

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Abstract Through a double-layer potential argument the inner and outer Poisson kernels, the Cauchy-type conjugate inner and outer Poisson kernels, and the kernels of the Cauchy-type inner and outer Hilbert transformations on the sphere are deduced. We also obtain Abel sum expansions of the kernels and prove the L^p-boundedness of the inner and outer Hilbert transformations for $1 < p < \infty$.

Keywords Poisson kernel · Conjugate Poisson kernel · Schwarz kernel · Hilbert transformation · Cauchy singular integral · Double-layer potential · Clifford analysis

Mathematics Subject Classification (2000) 62D05 · 30D10 · 42B35

1 Introduction

When studying boundary values of analytic functions of one complex variable, singular Cauchy integrals are unavoidably involved. The same is true when studying boundary values of monogenic functions for several real variables in the Clifford algebra setting. Closely related studies were carried out under the notion of conjugate harmonic systems in, for instance, [21, 22]. Closely related studies, among a large

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quantity of literature, include [2, 10, 15–18, 20, 23]. They should also include, in particularly, the systematic study of Alan McIntosh and his collaborators, especially A. Axelsson, on functional calculus of first and second order partial differential operators, Hodge decomposition, Kato's conjecture and related topics.

When the boundary curve or the boundary surface of the domain under study is flat, such as the real line in the complex plane or the hyper-plane $\mathbf{R}^{m-1}$ in $\mathbf{R}^m$, then the singular Cauchy integral reduces to the Hilbert transform (see the example in Sect. 3). In the literature the usage of the terminology and understanding of Hilbert transformation, however, are not uniform. Our definition is consistent with [4] in the complex plane, and consistent with [2] in higher dimensional spaces. A series of papers of Brackx et al., including [5, 6, 9] and [8], and one by Constales [11], study conjugate harmonic functions, and in particular conjugate Poisson kernels and the kernels of Hilbert transformations on the sphere. Their methods are based on different representations of the Cauchy-Riemann and the Laplace operators, including in the Cartesian and the polar coordinates ones. By integrating different forms of the Cauchy-Riemann equations they obtain their harmonic conjugates. As is well known, not like the one complex variable case, harmonic conjugates in higher dimensions are not unique modulo constants, but unique modulo a linear space of infinite dimensions. The comparison and classification of the harmonic conjugates obtained by using different methods have not been properly addressed. It can be shown that the harmonic conjugates in the half space, and in and out the unit sphere obtained by using the Brackx et al. and Constales' methods coincide with our Cauchy-type harmonic conjugates defined for general Lipschitz domains. In [3] we further show that the just mentioned harmonic conjugates also coincide with those of the Hodge type in these two mentioned contexts, but they do not coincide the other contexts. In [3] we further extend the results of the present paper to harmonic k-forms in Lipschitz domains. For harmonic conjugates of the Hodge type in general we refer the reader to Axelsson's paper [2] and thesis [1]. The present study proposes a systematic approach based merely on the Cauchy integral, via Plemelj-Sokhotzki Theorem in the context. What are obtained, called harmonic conjugates of the Cauchy-type, with existence and uniqueness in a large range of domains, are in particular related to Clifford Hardy spaces. Indeed, the CMcM Theorem implies that the resulted monogenic functions belong to the corresponding Hardy spaces [10, 18]. We further deduce the Abel sum expansions of the kernels and prove the L^p-boundedness of the inner and outer Hilbert transformations.

Brackx et al. and Constales' methods are based on solving the Cauchy-Riemann equations with different types of decompositions. It is standard, as it follows the same idea as in the one complex variable case, though with more complicated forms. For the unit sphere case the harmonic conjugates obtained from their Cauchy-Riemann methods and those from our Cauchy integral method coincide. In the forthcoming paper [3] we reveal some connections between harmonic conjugates obtained from different methods.

In Sect. 2 we give a short introduction to some basic notation and terminology of Clifford algebra used in the paper. In Sect. 3 we deduce the inner and outer Poisson kernels and their Cauchy-type conjugates, the latter having a direct relation to the kernels of the (Cauchy-type) Hilbert transformations on the sphere. In Sect. 4 we deduce the Abel sum representations of the kernels. In Sect. 5 we prove the L^p-boundedness, $1 < p < \infty$, of the inner and outer Hilbert transformations by using two methods, of which one is based on the bounded holomorphic functional calculus of the spherical Dirac operator [20], and the other based on existence and boundedness of an inverse operator in relation to double-layer potential [15, 23].

2 Preliminary

We will be working with real-Clifford algebras. Let $\mathbf{e}_1, \ldots, \mathbf{e}_m$ be *basic elements* satisfying $\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i = -2\delta_{ij}$, where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise, $i, j = 1, 2, \ldots, m$. Let

$$\mathbf{R}^m = \{\underline{x} : \underline{x} = x_1\mathbf{e}_1 + \cdots + x_m\mathbf{e}_m : x_j \in \mathbf{R},\ \ j = 1, 2, \ldots, m\}$$

be identical with the usual m-dimensional Euclidean space.

An element in $\mathbf{R}^m$ is called a *vector*. The real-Clifford algebra generated by $\mathbf{e}_1, \ldots, \mathbf{e}_m$, denoted by $\mathbf{R}^{(m)}$, is the non-commutative associative algebra generated by $\mathbf{e}_1, \ldots, \mathbf{e}_m$, over the real field $\mathbf{R}$. A general element in $\mathbf{R}^{(m)}$, therefore, is of the form $x = \sum_T x_T\mathbf{e}_T$, where $x_T \in \mathbf{R}$, and $\mathbf{e}_T = \mathbf{e}_{i_1}\mathbf{e}_{i_2}\cdots\mathbf{e}_{i_l}$, being called *induced products*, where T runs over all the ordered subsets of $\{1, \ldots, m\}$, namely

$$T = \{1 \le i_1 < \cdots < i_l \le m\}, \quad 1 \le l \le m.$$

When $T = \emptyset$, we set $\mathbf{e}_\emptyset = \mathbf{e}_0 = 1$. We denote $|T| = l$ where l is the number of the indices involved. A general Clifford number x may be decomposed into

$$x = \sum_{l=0}^{2^m} x^{(l)}, \quad x^{(l)} = \sum_{|T|=l} x_T\mathbf{e}_T.$$

A Clifford number of the form $x^{(l)}$ is called a Clifford number of the l-form. A Clifford number of 2-form is also called a *bi-vector*. Set

$$\mathbf{R}_1^m = \{x = x_0 + \underline{x} : x_0 \in \mathbf{R},\ \underline{x} \in \mathbf{R}^m\}.$$

Elements in $\mathbf{R}_1^m$ are called *para-vectors*. The Clifford conjugation of the para-vectors $x = x_0 + \underline{x}$ in $\mathbf{R}_1^m$ is denoted by $\overline{x}$, defined to be $\overline{x} = x_0 - \underline{x}$. Thus the Clifford conjugate of a vector $\underline{x}$ is $\overline{\underline{x}} = -\underline{x}$. The conjugation of $\mathbf{e}_T = \mathbf{e}_{i_1}\cdots\mathbf{e}_{i_l}$ is defined to be $\overline{\mathbf{e}}_T = \overline{\mathbf{e}}_{i_l}\cdots\overline{\mathbf{e}}_{i_1}$, where $\overline{\mathbf{e}}_j = -\mathbf{e}_j$, and extended to the Clifford algebra $\mathbf{R}^{(m)}$ by linearity. A sum of a 0-form and a 2-form is called a *para-bivectors*. Accordingly, for para-bivectors we formally have the same relation $\overline{c + x} = c - x$, where c is a scalar and x is a bivector. We adopt the convention that any Clifford number $x \in \mathbf{R}^{(m)}$ has the decomposition $x = \mathrm{Sc}[x] + \mathrm{NSc}[x]$, where $\mathrm{Sc}[x] = x_0 \in \mathbf{R}$, the scalar part of x, and $\mathrm{NSc}[x]$ the non-scalar part. If $\underline{x}, \underline{y}$ are vectors, then

$$\underline{x}\underline{y} = -\langle\underline{x}, \underline{y}\rangle + \underline{x} \wedge \underline{y}, \tag{1}$$

where

$$\mathrm{Sc}[\underline{x}\,\underline{y}] = -\langle \underline{x}, \underline{y}\rangle, \qquad \mathrm{NSc}[\underline{x}\,\underline{y}] = \underline{x} \wedge \underline{y} = \sum_{i<j}(x_i y_j - x_j y_i)\mathbf{e_i}\mathbf{e_j}.$$

It is easy to verify that $0 \neq x \in \mathbf{R}_1^m$ implies

$$x^{-1} = \frac{\overline{x}}{|x|^2}.$$

The open ball with center 0 and radius 1 in $\mathbf{R}^m$ is denoted by B^m, and the unit sphere in $\mathbf{R}^m$ is denoted by S^{m-1}, whose surface area, denoted by σ_{m-1}, is of the value $2\pi^{\frac{m}{2}}/\Gamma(\frac{m}{2})$.

The natural inner product between $x = \sum_T x_T \mathbf{e}_T$ and $y = \sum_T y_T \mathbf{e}_T$ in $\mathbf{R}^{(m)}$, denoted by $\langle x, y\rangle$, is the real number $\sum_T x_T y_T$. The norm associated with this inner product is

$$|x| = \langle x, x\rangle^{\frac{1}{2}} = \left(\sum_T |x_T|^2\right)^{\frac{1}{2}}.$$

In below, except in the example in Sect. 3, we will study functions defined in subsets of $\mathbf{R}^m$ taking values in $\mathbf{R}^{(m)}$. Thus they are of the form $f(\underline{x}) = \sum_T f_T(\underline{x})\mathbf{e}_T$, where f_T's are real-valued functions. We will use the *homogeneous Dirac operator*, $\underline{D}$, where

$$\underline{D} = \frac{\partial}{\partial x_1}\mathbf{e}_1 + \cdots + \frac{\partial}{\partial x_m}\mathbf{e}_m.$$

We define the "left" and "right" roles of the operators $\underline{D}$, respectively, by

$$\underline{D}f = \sum_{i=1}^m \sum_T \frac{\partial f_T}{\partial x_i}\mathbf{e}_i\mathbf{e}_T$$

and

$$f\underline{D} = \sum_{i=1}^m \sum_T \frac{\partial f_T}{\partial x_i}\mathbf{e}_T\mathbf{e}_i.$$

If f has all continuous first order partial derivatives and $\underline{D}f = 0$ in a (connected and open) domain Ω, then we say that f is *left-monogenic* in Ω; and, if $f\underline{D} = 0$ in Ω, we say that f is *right-monogenic* in Ω. The function theories for left-monogenic functions and for right-monogenic functions are parallel. In the sequel we will briefly write "left-monogenic" as "monogenic".

We call

$$E(\underline{x}) = \frac{\overline{\underline{x}}}{|\underline{x}|^m}$$

the *Cauchy kernel* in $\mathbf{R}^m$. It is easy to verify that $E(\underline{x})$ is a monogenic function in $\mathbf{R}^m \setminus \{0\}$.

There is a slightly different function theory for the *inhomogeneous Dirac operator* $D = \frac{\partial}{\partial x_0} + \underline{D}$, or Cauchy-Riemann operator, in $\mathbf{R}_1^m$. Complex analysis of one complex variable in relation to the Cauchy-Riemann equations is of this setting. In many cases, but not always, the two settings can be converted to each other.

As holomorphic in one complex variable, monogenic functions are of central interest in Clifford analysis. In the framework of the latter the Cauchy theorem, Cauchy integral formula, Taylor and Laurent expansions of monogenic functions, etc. are all valid. For details we refer the reader to [7, 14] and [12].

3 Poisson and Conjugate Poisson Kernels on the Unit Sphere

Hilbert transformation is naturally related to the Hardy space theory [3, 14]. It is, in particular, related to the over-determinedness of the Hardy spaces in view of the Burkholder-Gundy-Silverstein Theorem. The latter amounts to determine the unique harmonic conjugate so that the induced monogenic function belongs to the corresponding Hardy space. For concerned aspects in higher dimensional spaces we refer to [14] (especially the conjecture proposed to the end of its Chap. 2). The conjugate harmonic functions of the Cauchy-type proposed in this work satisfy, at the same time, the existence and uniqueness, and the Hardy-space-related requirements. We concentrate to the basic but essential case that the boundary data is real-valued (below also called scalar-valued), and, as consequence, the corresponding Cauchy integral is para-bivector-valued. The forthcoming paper [3] generalizes the scalar-valued boundary data case to the k-form data cases.

Let us now try to follow the pattern given in [4] to determine a Hilbert transformation on the sphere. We assume that our boundary data u is scalar-valued and belongs to $\in L^p(S^{m-1})$, $1 < p < \infty$. Then the Dirichlet problem in the ball B^m with the given boundary data u (as non-tangential limit) has a unique solution, U. Suppose that (i) Under some normalization conditions we can uniquely determine a function V with zero scalar part such that $U + V$ is left-monogenic; and, (ii) V has a non-tangential boundary limit v, then the mapping $u \to v$ is defined to be the *inner Hilbert transformation* on the sphere under the normalization conditions. The same formulation but with respect to the exterior of the unit ball gives rise to the *outer Hilbert transformation* on the sphere. Both the existence and uniqueness of U inside the unit ball have no problem. The uniqueness of U in the unbounded domain, viz. outside the ball in our case, requires certain normalization conditions. What are usually used is either U belongs to the h^p space, or U simply satisfies the condition $U(x) \to 0$ as $x \to \infty$. The uniqueness of V in both the bounded and unbounded cases require certain normalization conditions. In the one complex variable and the bounded domain case the usual one is to require V taking a special value at a spacial point. In higher dimensional spaces, like ours now, there, however, exist more than one harmonic conjugates V that form a linear space of infinite dimensions, and existence of a non-tangential boundary value of a harmonic conjugate is again another issue. Practically the above mentioned process encounters difficulty. Our strategy is to seek for the solution by revealing the natural connection between the Cauchy integral, Poisson kernel, Dirichlet problem, and recent development of operator theory in relation to double-layer potential.

To illustrate our process we include the upper-half space case as an example [21, 22]. Set

$$\mathbf{R}^m_{1,+} = \{x = x_0 + \underline{x} : x_0 > 0,\ \underline{x} \in \mathbf{R}^m\}.$$

The unique harmonic extension of a given boundary data $u \in L^p(\mathbf{R}^m)$, $1 < p < \infty$, that tends to zero at infinity, is given by the Poisson integral

$$U(y + \underline{x}) = P_y * u(\underline{x}), \quad y \in (0, \infty),\ \underline{x} \in \mathbf{R}^m,$$

where P_y is the Poisson kernel for $\mathbf{R}^m_{1,+}$,

$$P_y(\underline{x}) = \frac{2}{\sigma_m} \frac{y}{(y^2 + |\underline{x}|^2)^{\frac{m+1}{2}}}.$$

Its unique harmonic conjugate, V, such that $U + V$ belongs to the Hardy space $H^p(\mathbf{R}^m_{1,+})$, is given by

$$V(y + \underline{x}) = P_y * Hu(\underline{x}),$$

where Hu is the Hilbert transform on $\mathbf{R}^m$ given by

$$Hu = \sum_{k=1}^{m} \mathbf{e_k} R_k u,$$

where $R_k u$ is the k-th Riesz transform of u:

$$R_k u(\underline{x}) = \frac{2}{\sigma_m} p.v. \int_{\mathbf{R}^m} \frac{x_k - y_k}{|\underline{x} - \underline{y}|^{m+1}} u(\underline{y}) d\underline{y}.$$

Note that if we denote $V_k = -P_y * \mathbf{e_k R_k u}$, then $U, V_1, \ldots, V_m$ form a conjugate harmonic system in the sense of Stein and Weiss [21, 22]. According to the Plemelj-Sokhotzki Theorem in the context, the Hilbert transform Hu in the case is the non-scalar part of the boundary value of the Cauchy integral in the upper-half space:

$$\lim_{y \to 0+} \frac{1}{\sigma_m} \int_{\mathbf{R}^m} E(\underline{t} - (y + \underline{x})) n(\underline{t}) u(\underline{t}) d\underline{t} = \frac{1}{2} u(\underline{x}) + \frac{1}{2} Hu(\underline{x}), \quad \text{a.e.}, \tag{2}$$

where $n(\underline{t})$ is the outward unit normal to the integral surface, and in our case $n(\underline{t}) = -\mathbf{e_0} = -\mathbf{1}$, and E is the Cauchy kernel in $\mathbf{R}^m_1$, stands for

$$E(y + \underline{x}) = \frac{\overline{y + \underline{x}}}{|y + \underline{x}|^{m+1}}.$$

In the context we can further determine the associated harmonic conjugate of the Poisson kernel, Q_y, with the expression

$$Q_y(\underline{x}) = \frac{2}{\sigma_m} \frac{\underline{x}}{(y^2 + |\underline{x}|^2)^{\frac{m+1}{2}}}$$

and the properties

$$Q_y = HP_y, \quad P_y(\underline{x}) + Q_y(\underline{x}) = \frac{2}{\sigma_m} E(y + \underline{x}).$$

The above defined functions P_y, Q_y, and thus the functions U and V, as well as the associated Hilbert transformation H, are all based on the Cauchy integral.

We restrict ourselves to the following two types of Lipschitz domains (see [3]) $\Omega^{\pm} \subset \mathbf{R}^m$: (i) A graph domain, Ω^+, that is above a Lipscitz graph, and Ω^- is the one below; and, (ii) Ω^+ is a Lipschitz perturbation of the unit sphere and Ω^- is its exterior. Denote by $\Sigma := \partial\Omega^+ = \partial\Omega^-$. When working with such a general Lipschitz domain Ω ($\Omega = \Omega^+$ or $\Omega = \Omega^-$) with the boundary surface Σ the formula (2) becomes

$$\lim_{\Omega \ni \underline{x}' \to \underline{x}} \frac{1}{\sigma_m} \int_\Sigma E(\underline{t} - \underline{x}')n(\underline{t})u(\underline{t})d\sigma(\underline{t}) = \frac{1}{2}u(\underline{x}) + \frac{1}{2}\mathcal{C}u(\underline{x}), \quad \text{a.e.,} \tag{3}$$

where $d\sigma(\underline{t})$ is the surface area measure and $n(\underline{t})$ is the outward-pointing unit normal of Σ with respect to Ω, at the point $\underline{t} \in \partial\Omega$, and the limit takes the non-tangential sense. Note that the curly $\mathcal{C}$ stands for the singular Cauchy integral on the surface (compare its definition in (7) for the sphere case). The result (3) is based on the corresponding result of [10] (or more generally that of [16] or [17]), known as the Plemelj-Sokhotzki Theorem in the context. What is different from (2) is that $\mathcal{C}u$ now is combined with a non-zero scalar part, due to existence of non-zero curvature on the surface, and, therefore, $\mathcal{C}u$ does not coincide with the Hilbert transform of u. The latter, however, may be worked out through dividing the right-hand-side of (3) into its scalar and non-scalar parts, as being given in (8) for the sphere case, and solving (9) and (10). This is the basic idea of the approach.

From now on we will be working with the unit sphere in the homogeneous space $\mathbf{R}^m$, $m \geq 2$. We take $\Omega^+ = B^m$ and $\Omega^- = \overline{R^m}^c$, and $\Sigma = S^{m-1}$. Denote, for a scalar-valued function f in $L^p(S^{m-1})$, $1 \leq p \leq \infty$, the two Cauchy integrals by

$$M^{\pm} f(\underline{x}) = \frac{1}{\sigma_{m-1}} \int_{S^{m-1}} E(\underline{y} - \underline{x})n^{\pm}(\underline{y})f(\underline{y})d\sigma(\underline{y}), \quad \underline{x} \in \Omega^{\pm}, \tag{4}$$

where $n^{\pm}(\underline{y}) = \pm\underline{y}$. By using (1), the above is further written as

$$\begin{aligned} M^{\pm} f(\underline{x}) &= \frac{1}{\sigma_{m-1}} \int_{S^{m-1}} \langle E(\underline{x} - \underline{y}), \pm\underline{y}\rangle f(\underline{y})d\sigma(\underline{y}) \\ &\quad + \frac{1}{\sigma_{m-1}} \int_{S^{m-1}} (E(\underline{y} - \underline{x}) \wedge \pm\underline{y}) f(\underline{y})d\sigma(\underline{y}), \quad \underline{x} \in \Omega^{\pm}. \end{aligned} \tag{5}$$

The right-hand-side of formula (5) is a decomposition of the Cauchy integral into its scalar and the 2-form parts.

The general result (3) implies that the non-tangential boundary values of $M^{\pm} f(\underline{x})$ exist, denoted by the curly $\mathcal{M}^{\pm}$, being equal to

$$\mathcal{M}^{\pm} f(\underline{x}) = \frac{1}{2}[f(\underline{x}) \pm \mathcal{C} f(\underline{x})], \quad \text{a.e. } \underline{x} \in S^{m-1}, \tag{6}$$

respectively, where the operator $\mathcal{C}$ is the *principal value Cauchy singular integral operator* on the sphere given by

$$\begin{aligned}\mathcal{C}f(\underline{x}) &= \frac{2}{\sigma_{m-1}} \lim_{\epsilon\to 0+} \int_{|\underline{y}-\underline{x}|>\epsilon,\, \underline{y}\in S^{m-1}} E(\underline{y}-\underline{x})\underline{y}f(\underline{y})d\sigma(\underline{y}) \\ &= \frac{2}{\sigma_{m-1}} p.v. \int_{S^{m-1}} \langle E(\underline{x}-\underline{y}), \underline{y}\rangle f(\underline{y})d\sigma(\underline{y}) \\ &\quad + \frac{2}{\sigma_{m-1}} p.v. \int_{S^{m-1}} (E(\underline{y}-\underline{x}) \wedge \underline{y}) f(\underline{y})d\sigma(\underline{y}), \quad \text{a.e. } \underline{x}\in S^{m-1}. \end{aligned} \tag{7}$$

The last expression is a divided form into its scalar and 2-form parts.

Note that since the boundary data f is scalar-valued, the Cauchy integrals $M^{\pm}f$, as well as the boundary values $\mathcal{M}^{\pm}f$, are all para-bivector-valued. We will be using the mappings $\mathrm{Sc} : f \to \mathrm{Sc}[f]$ and $\mathrm{NSc} : f \to \mathrm{NSc}[f]$. On both the para-vector- and para-bivector-valued functions f there hold the relations

$$\mathrm{Sc} f = \frac{1}{2}[f + \overline{f}], \qquad \mathrm{NSc} f = \frac{1}{2}[f - \overline{f}].$$

Using this notation, for scalar-valued data function f, we can rewrite the relation (6) into

$$\mathcal{M}^{\pm}f = \frac{1}{2}(f \pm \mathrm{Sc}[\mathcal{C}f]) \pm \frac{1}{2}\mathrm{NSc}[\mathcal{C}f] = u^{\pm} + v^{\pm}, \tag{8}$$

where $u^{\pm}$ and $v^{\pm}$ are non-tangential boundary values of $U^{\pm}$ and its Cauchy-type harmonic conjugation $V^{\pm}$, $\pm$ refers to "inner" or "outer" part of the sphere, respectively. We will call $\mathrm{Sc}[M^{\pm}f]$ and $\mathrm{NSc}[M^{\pm}f]$ the harmonic representations (harmonic extensions to) inside the unit ball of $\frac{1}{2}(f \pm \mathrm{Sc}[\mathcal{C}f])$ and $\pm\frac{1}{2}\mathrm{NSc}[\mathcal{C}f]$, respectively. We have the operator equations

$$u^{\pm} = \frac{1}{2}(I \pm \mathrm{Sc}[\mathcal{C}])f, \qquad v^{\pm} = \frac{1}{2}\mathrm{NSc}[\mathcal{C}]f, \tag{9}$$

and, therefore, at least formally,

$$v^{\pm} = H^{\pm}u^{\pm} = \mathrm{NSc}[\mathcal{C}](I \pm \mathrm{Sc}[\mathcal{C}])^{-1}u^{\pm}, \tag{10}$$

being the Hilbert transforms of $u^{\pm}$, respectively.

Remark 1 The study of the Cauchy-type harmonic conjugates and Hilbert transforms on Lipschitz domains will be based on the fact that the related operators $\mathrm{Sc}[\mathcal{C}]$, $\mathrm{NSc}[\mathcal{C}]$ and $(I \pm \mathrm{Sc}[\mathcal{C}])^{-1}$ are all bounded from L^p to L^p, $2-\epsilon < p < 2+\delta$, where $0 < \epsilon < 1, 0 < \delta$, depend on the Lipschitz constant, and $1 < p < \infty$ on the unit sphere [15, 23]. We call the range of p in which the operators $\mathrm{Sc}[\mathcal{C}]$, $\mathrm{NSc}[\mathcal{C}]$ and $(I \pm \mathrm{Sc}[\mathcal{C}])^{-1}$ are all well defined and bounded *the admissible range of p for the Lipschitz domain.*

Definition 1 Let Ω be a Lipschitz domain of the described types in $\mathbf{R}^m$. If U is a scalar-valued harmonic function in Ω and V is a second harmonic function in Ω whose scalar part is zero such that

$$U + V = M^{\pm} f,$$

where f is scalar-valued and

$$M^{\pm} f(\underline{x}) = \frac{1}{\sigma_{m-1}} \int_{\partial\Omega} E(\underline{y} - \underline{x}) n^{\pm}(\underline{y}) f(\underline{y}) d\sigma(\underline{y}), \quad \underline{x} \in \Omega^{\pm}, \tag{11}$$

then V is said to be a Cauchy-type harmonic conjugate of U. The function f is called a *dipole* (function) of U and V.

Theorem 1 *Let U be a scalar-valued harmonic function in a described Lipschitz domain with non-tangential boundary value $u \in L^p$, where p is in the range of the admissible range of the Lipschitz domain, then there uniquely exists a Cauchy-type harmonic conjugate V.*

Proof The first equation in (9) with the existence of the inverse operator $(I \pm \mathrm{Sc}[\mathcal{C}])^{-1}$ gives a solution for f in terms of u. Using f to define a Cauchy integral Mf in Ω. Then the scalar part of Mf is U and the non-scalar part is V. Thus we obtain a Cauchy-type harmonic conjugate V, proving the existence. If there are two Cauchy-type harmonic conjugates of U, say V_1 and V_2, with correspondingly two dipole functions f_1 and f_2, then the first equation in (9) with the existence of the inverse operator asserts $f_1 = f_2$, and thus $V_1 = V_2$. The proof is complete. □

Theorem 2 *On the unit sphere the inner Poisson kernel and its Cauchy-type harmonic conjugate are, respectively,*

$$P^{+}(\underline{x}, \underline{\omega}) = \frac{1}{\sigma_{m-1}} \frac{1 - |\underline{x}|^2}{|\underline{x} - \underline{\omega}|^m}, \tag{12}$$

and

$$Q^{+}(\underline{x}, \underline{\omega}) = \frac{1}{\sigma_{m-1}} \left(\frac{2}{|\underline{x} - \underline{\omega}|^m} - \frac{m-2}{r^{m-1}} \int_0^r \frac{\rho^{m-2}}{|\rho\underline{\xi} - \underline{\omega}|^m} d\rho \right) \underline{x} \wedge \underline{\omega}, \tag{13}$$

where $\underline{x} = r\underline{\xi}$, $\underline{\omega}, \underline{\xi} \in S^{m-1}$, $0 \le r < 1$.

Proof We first prove (12). Study the harmonic extension to the unit ball of the boundary data $\frac{1}{2}(f + \mathrm{Sc}[\mathcal{C}^{+} f])$. On one hand, it is $\mathrm{Sc}[C^{+} f]$. On the other hand, it is the sum of that of $\frac{1}{2} f$ and that of $\mathrm{Sc}[\mathcal{C}^{+} f]$. The harmonic extension of $\frac{1}{2} f$ is given by a half of the Poisson integral. Now we deduce the harmonic extension of $\mathrm{Sc}[\mathcal{C}^{+} f]$. Through a simple computation of the double layer potential $\langle E(\underline{x} - \underline{y}), \underline{y}\rangle$, as given in (7), the part $\mathrm{Sc}[\mathcal{C}^{+} f]$ is given by the integral

$$\mathrm{Sc}[\mathcal{C}^{+} f](\underline{x}) = \frac{1}{\sigma_{m-1}} \int_{S^{m-1}} \frac{f(\underline{\omega})}{|\underline{x} - \underline{\omega}|^{m-2}} d\sigma(\underline{\omega}), \quad \underline{x} \in S^{m-1}.$$

Denote

$$S(\underline{x},\underline{\omega}) = \frac{1}{\sigma_{m-1}} \frac{1}{|\underline{x}-\underline{\omega}|^{m-2}}, \quad \underline{x} \in B^m,\ \underline{\omega} \in S^{m-1}.$$

The harmonic extension of $\mathrm{Sc}[\mathcal{C}^+ f]$ into the ball, therefore, is the integral

$$\mathrm{Sc}[\mathcal{C}^+ f](\underline{x}) = \int_{S^{m-1}} S(\underline{x},\underline{\omega}) f(\underline{\omega}) d\sigma(\underline{\omega}), \quad \underline{x} \in B^m.$$

Below, for simplicity, we sometimes write P^+ and its Cauchy-type harmonic conjugate Q^+ as P and Q, respectively. By comparing with (5), we obtain

$$\frac{1}{\sigma_{m-1}} \frac{\langle \underline{\omega}-\underline{x}, \underline{\omega}\rangle}{|\underline{\omega}-\underline{x}|^m} = \frac{1}{2} P(\underline{x},\underline{\omega}) + \frac{1}{2} S(\underline{x},\underline{\omega}). \tag{14}$$

Straightforward computation then gives (12).

Now we prove (13). By taking the Cauchy-type conjugate to each function in equality (14), we obtain

$$\frac{1}{\sigma_{m-1}} \frac{(\underline{x}-\underline{\omega}) \wedge \underline{\omega}}{|\underline{x}-\underline{\omega}|^m} = \frac{1}{2} Q(\underline{x},\underline{\omega}) + \frac{1}{2} \tilde{S}(\underline{x},\underline{\omega}),$$

where $Q(\underline{x},\underline{\omega})$ and $\tilde{S}(\underline{x},\underline{\omega})$ are respectively the Cauchy-type harmonic conjugates of $P(\underline{x},\underline{\omega})$ and $S(\underline{x},\omega)$.

To obtain $Q(\underline{x},\underline{\omega})$ it suffices to first obtain $\tilde{S}(\underline{x},\underline{\omega})$, that is given by the following Lemma 1 and Lemma 2. Below we employ the functions $P^{(k)}(\underline{\omega}^{-1}\underline{\xi})$ and $P^{(-k)}(\underline{\omega}^{-1}\underline{\xi})$ that are defined by (28) and (32) in the beginning of Sect. 4, while the product form $\underline{\omega}^{-1}\underline{\xi}$ of the variables of the functions are justified in [20, p. 392]. □

Lemma 1 *For* $r = |\underline{x}| < 1$,

$$\begin{aligned} &\sum_{k=0}^{\infty} r^k \frac{m-2}{m+k-2} P^{(k)}(\underline{\omega}^{-1}\underline{\xi}) \\ &\quad = \frac{m-2}{r^{m-2}} \int_0^r \rho^{m-3} E(\underline{\omega}-\rho\underline{\xi})\underline{\omega} d\rho \\ &\quad = \frac{1}{|\underline{x}-\underline{\omega}|^{m-2}} + \frac{m-2}{r^{m-1}} \left(\int_0^r \frac{\rho^{m-2}}{|\rho\underline{\xi}-\underline{\omega}|^m} d\rho \right) \underline{x} \wedge \underline{\omega}; \end{aligned} \tag{15}$$

and, for $r = |\underline{x}| > 1$,

$$\begin{aligned} &\sum_{k=1}^{\infty} \frac{m-2}{k} \frac{1}{r^{m-2+k}} P^{(-k)}(\underline{\omega}^{-1}\underline{\xi}) \\ &\quad = \frac{m-2}{r^{m-2}} \int_r^{\infty} \rho^{m-3} E(\underline{\omega}-\rho\underline{\xi})\overline{\underline{\omega}} d\rho \end{aligned}$$

$$= \frac{1}{|\underline{x}-\underline{\omega}|^{m-2}} - \frac{1}{r^{m-2}} - \frac{m-2}{r^{m-1}} \left(\int_r^\infty \frac{\rho^{m-2}}{|\rho\underline{\xi}-\underline{\omega}|^m} d\rho \right) \underline{x} \wedge \underline{\omega}. \tag{16}$$

Proof For $r < 1$, writing

$$\frac{r^k}{m+k-2} = \frac{1}{r^{m-2}} \int_0^r \rho^{m+k-3} d\rho$$

and using the identity [20]

$$\sum_{k=0}^\infty r^k P^{(k)}(\underline{\omega}^{-1}\underline{\xi}) = E(1-\underline{\omega}^{-1}\underline{x}) = E(\underline{\omega}-\underline{x})\underline{\omega}, \quad r = |\underline{x}| < 1, \tag{17}$$

we have

$$\begin{aligned}
\text{LHS of (15)} &= \frac{m-2}{r^{m-2}} \int_0^r \rho^{m-3} \frac{\langle \underline{\omega}-\rho\underline{\xi}, \underline{\omega}\rangle}{|\underline{\omega}-\rho\underline{\xi}|^m} d\rho \\
&\quad + \frac{m-2}{r^{m-1}} \left(\int_0^r \frac{\rho^{m-2}}{|\rho\underline{\xi}-\underline{\omega}|^m} d\rho \right) \underline{x} \wedge \underline{\omega} \\
&= \frac{m-2}{r^{m-2}} \int_0^r \left\langle \frac{\underline{\omega}/\rho-\underline{\xi}}{|\underline{\omega}/\rho-\underline{\xi}|^m}, \underline{\omega} \right\rangle d\left(\frac{-1}{\rho}\right) \\
&\quad + \frac{m-2}{r^{m-1}} \left(\int_0^r \frac{\rho^{m-2}}{|\rho\underline{\xi}-\underline{\omega}|^m} d\rho \right) \underline{x} \wedge \underline{\omega} \\
&= \frac{1}{r^{m-2}} \int_0^r d\left(\frac{1}{|\underline{\omega}/\rho-\underline{\xi}|^{m-2}} \right) + \frac{m-2}{r^{m-1}} \left(\int_0^r \frac{\rho^{m-2}}{|\rho\underline{\xi}-\underline{\omega}|^m} d\rho \right) \underline{x} \wedge \underline{\omega} \\
&= \frac{1}{r^{m-2}} \frac{1}{|\underline{\omega}/r-\underline{\xi}|^{m-2}} + \frac{m-2}{r^{m-1}} \left(\int_0^r \frac{\rho^{m-2}}{|\rho\underline{\xi}-\underline{\omega}|^m} d\rho \right) \underline{x} \wedge \underline{\omega}.
\end{aligned}$$

Similarly, for $r > 1$, using

$$\frac{1}{r^{m+k-2}k} = \frac{1}{r^{m-2}} \int_r^\infty \frac{1}{\rho^{k+1}} d\rho,$$

and the identity [20]

$$\sum_{k=1}^\infty \frac{1}{r^{m-2+k}} P^{(-k)}(\underline{\omega}^{-1}\underline{\xi}) = -E(1-\underline{\omega}^{-1}\underline{x}) = E(\underline{\omega}-\underline{x})\overline{\underline{\omega}}, \quad |x| > 1, \tag{18}$$

we have

$$\text{LHS of (16)} = -\frac{m-2}{r^{m-2}}\int_r^\infty \rho^{m-3}\frac{\langle \underline{\omega}-\rho\underline{\xi}, \underline{\omega}\rangle}{|\underline{\omega}-\rho\underline{\xi}|^m}d\rho$$

$$-\frac{m-2}{r^{m-1}}\left(\int_r^\infty \frac{\rho^{m-2}}{|\rho\underline{\xi}-\underline{\omega}|^m}d\rho\right)\underline{x}\wedge\underline{\omega}$$

$$= -\frac{m-2}{r^{m-2}}\int_r^\infty \left\langle \frac{\underline{\omega}/\rho-\underline{\xi}}{|\underline{\omega}/\rho-\underline{\xi}|^m}, \underline{\omega}\right\rangle d\left(\frac{-1}{\rho}\right)$$

$$-\frac{m-2}{r^{m-1}}\left(\int_r^\infty \frac{\rho^{m-2}}{|\rho\underline{\xi}-\underline{\omega}|^m}d\rho\right)\underline{x}\wedge\underline{\omega}$$

$$= \frac{-1}{r^{m-2}}\int_r^\infty d\left(\frac{1}{|\underline{\omega}/\rho-\underline{\xi}|^{m-2}}\right) - \frac{m-2}{r^{m-1}}\left(\int_r^\infty \frac{\rho^{m-2}}{|\rho\underline{\xi}-\underline{\omega}|^m}d\rho\right)\underline{x}\wedge\underline{\omega}$$

$$= \frac{1}{r^{m-2}}\frac{1}{|\underline{\omega}/r-\underline{\xi}|^{m-2}} - \frac{1}{r^{m-2}} - \frac{m-2}{r^{m-1}}\left(\int_r^\infty \frac{\rho^{m-2}}{|\rho\underline{\xi}-\underline{\omega}|^m}d\rho\right)\underline{x}\wedge\underline{\omega}.$$

The proof is complete. □

Lemma 2 (i) *The two-form part*

$$\frac{m-2}{r^{m-1}}\left(\int_0^r \frac{\rho^{m-2}}{|\rho\underline{\xi}-\underline{\omega}|^m}d\rho\right)\underline{x}\wedge\underline{\omega}$$

of the last entry of the equality chain (15) *is the Cauchy-type harmonic conjugate of the Newton potential*

$$\frac{1}{|\underline{x}-\underline{\omega}|^{m-2}}$$

for $r<1$ *at* $\underline{\omega}$;

(ii) *The two-form part*

$$-\frac{m-2}{r^{m-1}}\left(\int_r^\infty \frac{\rho^{m-2}}{|\rho\underline{\xi}-\underline{\omega}|^m}d\rho\right)\underline{x}\wedge\underline{\omega}$$

of the last entry of the equality chain (16) *is the Cauchy-type harmonic conjugate of the harmonic function*

$$\frac{1}{|\underline{x}-\underline{\omega}|^{m-2}} - \frac{1}{r^{m-2}}$$

for $r>1$ *at* $\underline{\omega}$.

Proof (i) We first show that for a fixed $\underline{\omega}$ the left-monogenic function $E(\underline{\omega}-\rho\underline{\xi})\underline{\omega}$ for the variable $\underline{x}=\rho\underline{\xi}$ has the Poisson integral expression

$$E(\underline{\omega}-\rho\underline{\xi})\underline{\omega} = \frac{1}{\sigma_{m-1}}\int_{S^{m-1}} E(\underline{\omega}-r\underline{\zeta})\underline{\omega}P^+\left(\frac{\rho}{r}\underline{\xi}, \underline{\zeta}\right)d\sigma(\underline{\zeta}). \tag{19}$$

To this end we borrow the Abel sum expansion of the Poisson kernel (34). (Whose proof in Theorem 4 does not involve explicit formulas of the related Cauchy-type harmonic conjugates. Also see Remark 2.) Substituting in the right-hand-side of (19) the Abel sum expansion of $P^+(\frac{\rho}{r}\underline{\xi},\underline{\zeta})$ and the series expansion of $E(\underline{\omega}-r\underline{\zeta})\underline{\omega}$ in (17) and using the orthogonality properties of the $P^{(k)}$ functions we soon arrive the left-hand-side of (19). Multiplying ρ^{m-3} and integrating in ρ the both sides of (19), we have

$$\int_0^r \rho^{m-3}E(\underline{\omega}-\rho\underline{\xi})\underline{\omega}d\rho=\frac{1}{\sigma_{m-1}}\int_{S^{m-1}}E(\underline{\omega}-r\underline{\zeta})\underline{\omega}\int_0^r \rho^{m-3}P\left(\frac{\rho}{r}\underline{\xi},\underline{\zeta}\right)d\rho d\sigma(\underline{\zeta}).$$

The right-hand-side of the last equality is a Cauchy integral of a scalar-valued function. Invoking Definition 1, we conclude that

$$\frac{m-2}{r^{m-1}}\left(\int_0^r \frac{\rho^{m-2}}{|\rho\underline{\xi}-\underline{\omega}|^m}d\rho\right)\underline{x}\wedge\underline{\omega}$$

is the Cauchy-type harmonic conjugate of the Newton potential

$$\frac{1}{|\underline{x}-\underline{\omega}|^{m-2}}$$

for $r<1$ at $\underline{\omega}$.

The proof of (ii) is similar. The proof of the lemma is complete. □

According to Lemma 2, we have

$$\tilde{S}(\underline{x},\underline{\omega})=\frac{m-2}{\sigma_{m-1}r^{m-1}}\int_0^r \frac{\rho^{m-2}}{|\rho\underline{\xi}-\underline{\omega}|^m}d\rho\ \underline{x}\wedge\underline{\omega},\quad r<1, \tag{20}$$

and thus

$$\begin{aligned}Q(\underline{x},\underline{\omega})&=\frac{2}{\sigma_{m-1}}\frac{(\underline{x}-\underline{\omega})\wedge\underline{\omega}}{|\underline{x}-\underline{\omega}|^m}-\frac{m-2}{r^{m-1}\sigma_{m-1}}\int_0^r \frac{\rho^{m-2}}{|\rho\underline{\xi}-\underline{\omega}|^m}d\rho\underline{x}\wedge\underline{\omega}\\&=\frac{1}{\sigma_{m-1}}\left(\frac{2}{|\underline{x}-\underline{\omega}|^m}-\frac{m-2}{r^{m-1}}\int_0^r \frac{\rho^{m-2}}{|\rho\underline{\xi}-\underline{\omega}|^m}d\rho\right)\underline{x}\wedge\underline{\omega},\end{aligned}$$

as desired.

Theorem 3 *On the unit sphere the outer Poisson kernel and its Cauchy-type harmonic conjugates are*, *respectively,*

$$P^-(\underline{x},\underline{\omega})=\frac{1}{\sigma_{m-1}}\frac{|\underline{x}|^2-1}{|\underline{x}-\underline{\omega}|^m}, \tag{21}$$

and

$$Q^-(\underline{x},\underline{\omega})=\frac{1}{\sigma_{m-1}}\left(-\frac{2}{|\underline{x}-\underline{\omega}|^m}+\frac{m-2}{r^{m-1}}\int_0^r \frac{\rho^{m-2}}{|\rho\underline{\xi}-\underline{\omega}|^m}d\rho\right)\underline{x}\wedge\underline{\omega},\quad r>1. \tag{22}$$

Proof For $r > 1$, $n(\underline{\omega}) = \overline{\underline{\omega}} = -\underline{\omega}$, we have

$$\begin{aligned}M^- f(\underline{x}) &= \frac{1}{\sigma_{m-1}} \int_{S^{m-1}} E(\underline{y} - \underline{x}) n(\underline{\omega}) f(\underline{y}) d\sigma(\underline{y}) \\ &= \frac{1}{\sigma_{m-1}} \int_{S^{m-1}} \frac{\langle \underline{\omega} - \underline{x}, n(\underline{\omega}) \rangle}{|\underline{\omega} - \underline{x}|^m} f(\underline{\omega}) d\sigma(\underline{\omega}) \\ &\quad + \frac{1}{\sigma_{m-1}} \int_{S^{m-1}} \frac{(\underline{x} - \omega) \wedge n(\underline{\omega})}{|\underline{\omega} - \underline{x}|^m} f(\underline{\omega}) d\sigma(\underline{\omega})\end{aligned}$$

and

$$\begin{aligned}\mathcal{M}^-(r\underline{\xi}) &= \left[\frac{1}{2} f(\underline{\xi}) - \frac{1}{2\sigma_{m-1}} \int_{S^{m-1}} \frac{f(\underline{\omega})}{|\underline{\xi} - \underline{\omega}|^{m-2}} d\sigma(\underline{\omega}) \right] \\ &\quad - \left[\frac{1}{\sigma_{m-1}} p.v. \int_{S^{m-1}} \frac{\underline{\xi} \wedge \underline{\omega}}{|\underline{\xi} - \underline{\omega}|^m} f(\underline{\omega}) d\sigma(\underline{\omega}) \right].\end{aligned}$$

Therefore,

$$\frac{1}{\sigma_{m-1}} \frac{\langle \underline{\omega} - \underline{x}, -\underline{\omega} \rangle}{|\underline{\omega} - \underline{x}|^m} = \frac{1}{2} P^-(\underline{x}, \underline{\omega}) - \frac{1}{2} S^-(\underline{x}, \underline{\omega}), \quad |x| > 1, \tag{23}$$

where $S^-(\underline{x}, \underline{\omega})$ is the harmonic representation of $\frac{1}{\sigma_{m-1}} \frac{1}{|\cdot - \underline{\omega}|^{m-2}}$ outside the closed ball. We therefore have

$$\begin{aligned}P^-(\underline{x}, \underline{\omega}) &= \frac{-2}{\sigma_{m-1}} \frac{\langle \underline{\omega} - \underline{x}, \underline{\omega} \rangle}{|\underline{\omega} - \underline{x}|^m} + \frac{1}{\sigma_{m-1} |\underline{x} - \underline{\omega}|^{m-2}} \\ &= \frac{1}{\sigma_{m-1}} \frac{|\underline{x}|^2 - 1}{|\underline{x} - \underline{\omega}|^m}, \quad |x| > 1.\end{aligned}$$

Similarly to the case for the inner conjugate Poisson kernel, we have

$$\frac{1}{\sigma_{m-1}} \frac{(\underline{x} - \underline{\omega}) \wedge (-\underline{\omega})}{|\underline{\omega} - \underline{x}|^m} = \frac{1}{2} Q^-(\underline{x}, \underline{\omega}) - \frac{1}{2} \tilde{S^-}(\underline{x}, \underline{\omega}). \tag{24}$$

To obtain $Q^-(\underline{x}, \underline{\omega})$ we are to find $\tilde{S^-}(\underline{x}, \underline{\omega})$ that is the Cauchy-type harmonic conjugate of $\frac{1}{\sigma_{m-1} |\cdot - \underline{\omega}|^{m-2}}$ outside the ball. The relation (16), however, cannot be directly used due to presence of the extra term $1/r^{m-2}$. On the other hand, $\tilde{S}(\underline{x}, \underline{\omega})$, given by (20), has a unique harmonic extension to outside of the ball $|\underline{x}| \geq 1$, except the point $\underline{x} = \underline{\omega}$. The extension can be obtained by the same integral formula (20) except for those with $r = |\underline{x}| > 1$ and $\underline{x}/|\underline{x}| = \underline{\omega}$. We therefore obtain

$$\begin{aligned}Q^-(\underline{x}, \underline{\omega}) &= \frac{1}{\sigma_{m-1}} \left(\frac{-2}{|\underline{x} - \underline{\omega}|^m} \right. \\ &\quad \left. + \frac{m-2}{r^{m-1}} \int_0^r \frac{\rho^{m-2}}{|\rho \underline{\xi} - \underline{\omega}|^m} d\rho \right) \underline{x} \wedge \underline{\omega}, \quad \text{a.e. } |\underline{x}| > 1.\end{aligned} \tag{25}$$

□

4 Abel Sum Formulas of the Kernels

For $f \in L^2(S^{m-1})$, $\underline{x} = r\underline{\xi}, 0 \le r < 1, \underline{y} = \underline{\omega} \in S^{m-1}$, we have

$$M^+ f(\underline{x}) = \sum_{k=0}^{\infty} \frac{|\underline{x}|^k}{\sigma_{m-1}} \int_{S^{m-1}} C^+_{m,k}(\underline{\xi}, \underline{\omega}) f(\underline{\omega}) d\sigma(\underline{\omega}), \tag{26}$$

where

$$C^+_{m,k}(\underline{\xi}, \underline{\omega}) = \frac{m+k-2}{m-2} C_k^{(m-2)/2}(\langle \underline{\xi}, \underline{\omega} \rangle) + C_{k-1}^{m/2}(\langle \underline{\xi}, \underline{\omega} \rangle) \underline{\xi} \wedge \underline{\omega}, \tag{27}$$

where $C_k^{l/2}$ denotes the Gegenbauer polynomials with $C_{-1}^{m/2}(\cdot) = 0$ (see [7] or [5]). The right-hand-side of (27), in fact, is a function of $\underline{\omega}^{-1}\underline{x}$ [20, p. 392], and thus we may set

$$P^{(k)}(\underline{\omega}^{-1}\underline{x}) = r^k C^+_{m,k}(\underline{\xi}, \underline{\omega}), \quad k = 0, 1, 2, \ldots, \tag{28}$$

and, consequently,

$$M^+ f(\underline{x}) = \sum_{k=0}^{\infty} \frac{1}{\sigma_{m-1}} \int_{S^{m-1}} P^{(k)}(\underline{\omega}^{-1}\underline{x}) f(\underline{\omega}) d\sigma(\underline{\omega}). \tag{29}$$

Similarly to (29), we have

$$\begin{aligned} M^- f(\underline{x}) &= \sum_{k=-1}^{-\infty} \frac{|\underline{x}|^{-m+2-k}}{\sigma_{m-1}} \int_{S^{m-1}} C^-_{m,|k|-1}(\underline{\xi}, \underline{\omega}) f(\underline{\omega}) d\sigma(\underline{\omega}) \\ &= \sum_{k=-1}^{-\infty} \frac{1}{\sigma_{m-1}} \int_{S^{m-1}} P^{(k)}(\underline{\omega}^{-1}\underline{x}) f(\underline{\omega}) d\sigma(\underline{\omega}), \end{aligned} \tag{30}$$

where

$$C^-_{m,|k|-1}(\underline{\xi}, \underline{\omega}) = \frac{|k|}{m-2} C_{|k|}^{(m-2)/2}(\langle \underline{\xi}, \underline{\omega} \rangle) - C_{|k|-1}^{m/2}(\langle \underline{\xi}, \underline{\omega} \rangle) \underline{\xi} \wedge \underline{\omega}. \tag{31}$$

Set

$$P^{(k)}(\underline{\omega}^{-1}\underline{x}) = r^{-m+2-k} C^-_{m,|k|-1}(\underline{\xi}, \underline{\omega}), \quad k = -1, -2, \ldots. \tag{32}$$

For the Laplace-Fourier series expansion in the L^2 sense there holds

$$f(\underline{\xi}) = \sum_{k=-\infty}^{\infty} \frac{1}{\sigma_{m-1}} \int_{S^{m-1}} P^{(k)}(\underline{\omega}^{-1}\underline{\xi}) f(\underline{\omega}) d\sigma(\underline{\omega}). \tag{33}$$

This suggests that for scalar-valued functions f in the integral (33), and a general Clifford-valued function as well, the series

$$\mathrm{Sc}\left[\sum_{k=-\infty}^{\infty}\frac{1}{\sigma_{m-1}}P^{(k)}(\underline{\omega}^{-1}\underline{\xi})\right]$$

plays the role of the Dirac-δ function.

Theorem 4 *The Abel sum expansions of the inner Poisson kernel and its Cauchy-type harmonic conjugates are*, *respectively*,

$$P^{+}(\underline{x},\underline{\omega})=\frac{1}{\sigma_{m-1}}\sum_{-\infty}^{\infty}r^{|k|}P^{(k)}(\underline{\omega}^{-1}\underline{\xi}),\quad \underline{x}=r\underline{\xi},\ r<1, \tag{34}$$

and

$$Q^{+}(r\underline{\xi},\omega)=\frac{1}{\sigma_{m-1}}\left[\sum_{k=1}^{\infty}\frac{k}{m+k-2}r^{k}P^{(k)}(\underline{\omega}^{-1}\underline{\xi})\right.$$
$$\left.-\sum_{k=-\infty}^{-1}r^{|k|}P^{(k)}(\underline{\omega}^{-1}\underline{\xi})\right],\quad r<1. \tag{35}$$

Proof We set

$$A^{+}(r)=\frac{1}{\sigma_{m-1}}\left[\sum_{k=0}^{\infty}r^{k}P^{(k)}(\underline{\omega}^{-1}\underline{\xi})\right]. \tag{36}$$

From the analysis given above, we have

$$A^{+}(r)=\frac{1}{2}P^{+}(r\underline{\xi},\underline{\omega})+\frac{1}{2}S^{+}(r\underline{\xi},\underline{\omega})+\frac{1}{2}\tilde{S^{+}}(r\underline{\xi},\underline{\omega})$$
$$+\frac{1}{2}Q^{+}(r\underline{\xi},\underline{\omega}),\quad r<1, \tag{37}$$

where the last three entries are the harmonic representation of a half of the Cauchy singular integral of f, viz. $(1/2)\mathcal{C}f$. Similarly,

$$A^{-}(r)=\frac{1}{\sigma_{m-1}}\left[\sum_{-\infty}^{-1}P^{(k)}(\underline{\omega}^{-1}\underline{x})\right]$$
$$=\frac{1}{2}P^{-}(r\underline{\xi},\underline{\omega})-\frac{1}{2}S^{-}(r\underline{\xi},\underline{\omega})$$
$$-\frac{1}{2}\tilde{S^{-}}(r\underline{\xi},\underline{\omega})-\frac{1}{2}Q^{-}(r\underline{\xi},\underline{\omega}),\quad r>1. \tag{38}$$

We now adopt *Kelvin inversion* defined by $\mathcal{K}f(\underline{x}) = E(\underline{x})f(\frac{\underline{x}}{|\underline{x}|^2})$ that preserves harmonicity. It is easy to observe that the Kelvin inversion of A^-, denoted by $\mathcal{K}(A^-)$, satisfies the relation

$$\mathcal{K}(A^-)(r) = \frac{1}{2}P^+(r\underline{\xi},\underline{\omega}) - \frac{1}{2}S^+(r\underline{\xi},\underline{\omega}) - \frac{1}{2}\tilde{S^+}(r\underline{\xi},\underline{\omega}) - \frac{1}{2}Q^+(r\underline{\xi},\underline{\omega}), \quad r<1.$$

We thus arrive

$$P^+(\underline{x},\underline{\omega}) = A^+(r) + \mathcal{K}(A^-)(r).$$

Applying Kelvin inversion term by term to the series expansion of A^- in (38) (the first equality), and using (36), we obtain the Abel sum (34) expansion for the Poisson kernel.

Next we deduce the Abel sum formula of the conjugate Poisson kernel $Q^+(\underline{x},\omega)$. In fact, by (15) in Lemma 1, in (37) all the entries but except $(1/2)Q^+(r\underline{\xi},\underline{\omega})$ are already of the Abel sum form, therefore,

$$\begin{aligned}
\frac{1}{2}Q^+(r\underline{\xi},\omega) &= A^+(r) - \frac{1}{2}P^+(r\underline{\xi},\underline{\omega}) - \frac{1}{2}\frac{1}{\sigma_{m-1}}\left[\sum_{k=0}^{\infty} r^k \frac{m-2}{m+k-2}P^{(k)}(\underline{\omega}^{-1}\underline{\xi})\right] \\
&= \frac{1}{\sigma_{m-1}}\left[\sum_{k=0}^{\infty} r^k P^{(k)}(\underline{\omega}^{-1}\underline{\xi}) \right.\\
&\qquad \left. - \frac{1}{2}\sum_{-\infty}^{\infty} r^{|k|}P^{(k)}(\underline{\omega}^{-1}\underline{\xi}) - \frac{1}{2}\sum_{k=0}^{\infty} r^k\frac{m-2}{m+k-2}P^{(k)}(\underline{\omega}^{-1}\underline{\xi})\right] \\
&= \frac{1}{\sigma_{m-1}}\left[\frac{1}{2}\sum_{k=1}^{\infty}\frac{k}{m+k-2}r^kP^{(k)}(\underline{\omega}^{-1}\underline{\xi}) - \frac{1}{2}\sum_{k=-\infty}^{-1} r^{|k|}P^{(k)}(\underline{\omega}^{-1}\underline{\xi})\right].
\end{aligned}$$

We thus arrive (35). The proof is complete. □

Remark 2 The Abel sum expansion (34) is also a consequence of (29), (30) and the Plemelj-Sokhotzki Theorem. It can also be deduced from the formulas (27) and (31) where the scalar parts are separated from their 2-form parts. In fact, the effect of adding $\mathcal{K}(A^-)$ to A^+ is to cancel out the non-scalar part in (27). In the proof of Theorem 4 we preferred the systematic way of using Kelvin inversion.

Remark 3 As in the classical case the Poisson kernel formula (12) may also be directly obtained from this Kelvin inversion idea. Recall that

$$A^+ = E(\underline{\omega}-\underline{x})\underline{\omega}, \quad r<1; \quad \text{and} \quad A^- = E(\underline{\omega}-\underline{x})\overline{\underline{\omega}}, \quad r>1.$$

The Kelvin inversion of A^- has the form

$$K(A^-) = \frac{1}{r^{m-2}}E(\underline{\omega}-\underline{x}/r^2)\overline{\underline{\omega}}, \quad r<1.$$

Then $P^+(\underline{x},\underline{\omega})$ is given by the sum

$$\frac{1}{\sigma_{m-1}}\left(E(\underline{\omega}-\underline{x})\underline{\omega}+\frac{1}{r^{m-2}}E(\underline{\omega}-\frac{\underline{x}}{r^2})\overline{\underline{\omega}}\right)$$
$$=\frac{1}{\sigma_{m-1}}\left(\frac{(r\underline{\xi}-\underline{\omega})\underline{\omega}}{|\underline{\omega}-r\underline{\xi}|^m}+\frac{r^2}{r^m}\frac{(\frac{\underline{\xi}}{r}-\underline{\omega})(-\underline{\omega})}{|\underline{\omega}-\frac{\underline{\xi}}{r}|^m}\right)$$
$$=\frac{1}{\sigma_{m-1}}\left(\frac{r\underline{\xi}\underline{\omega}+1}{|\underline{\omega}-r\underline{\xi}|^m}+\frac{-r\underline{\xi}\underline{\omega}-r^2}{|r\omega-\underline{\xi}|^m}\right)$$
$$=\frac{1}{\sigma_{m-1}}\frac{1-r^2}{|\underline{\omega}-r\underline{\xi}|^m},\quad r<1. \tag{39}$$

Theorem 5 *The Abel sum expansions of the outer Poisson kernel and its Cauchy-type harmonic conjugates are, respectively,*

$$P^-(r\underline{\xi},\underline{\omega})=\frac{1}{\sigma_{m-1}}\sum_{-\infty}^{\infty}r^{-|k|-m+2}P^{(k)}(\underline{\omega}^{-1}\underline{\xi}),\quad r>1, \tag{40}$$

and

$$Q^-(r\underline{\xi},\underline{\omega})=\frac{1}{\sigma_{m-1}}\left[\sum_{k=1}^{\infty}\frac{1}{r^{m+k-2}}P^{(k)}(\underline{\omega}^{-1}\underline{\xi})\right.$$
$$\left.-\sum_{k=-\infty}^{-1}\frac{m+|k|-2}{|k|}\frac{1}{r^{m+|k|-2}}P^{(k)}(\underline{\omega}^{-1}\underline{\xi})\right]-\tilde{N}(r\underline{\xi},\underline{\omega}), \tag{41}$$

where $\tilde{N}$ is the Cauchy-type harmonic conjugate outside the unit ball of the Newton potential N, where

$$N(r\underline{\xi})=\frac{1}{\sigma_{m-1}}\frac{1}{r^{m-2}}$$

and

$$\tilde{N}(r\underline{\xi},\underline{\omega})=\frac{1}{\sigma_{m-1}}\frac{m-2}{r^{m-2}}\int_0^{\infty}\frac{\rho^{m-2}}{|\rho\underline{\xi}-\underline{\omega}|^m}d\rho\ \underline{\xi}\wedge\underline{\omega}. \tag{42}$$

Proof To obtain Abel sum expansion of the outer Poisson kernel we start from the series expansion (38). Applying Kelvin inversion to $A^+(r)$, we have

$$\mathcal{K}(A^+)(r)=\frac{1}{2}P^-(r\underline{\xi},\underline{\omega})+\frac{1}{2}S^-(r\underline{\xi},\underline{\omega})+\frac{1}{2}\tilde{S^-}(r\underline{\xi},\underline{\omega})+\frac{1}{2}Q^-(r\underline{\xi},\underline{\omega}),\quad r>1.$$

Consequently,

$$P^-(\underline{x},\underline{\omega})=A^-(r)+\mathcal{K}(A^+)(r).$$

By taking Kelvin inversion term by term to the series expression of A^+ in (36), with $\mathcal{K}(1) = \frac{1}{|\underline{x}|^{m-2}}$, and using the first equality in (38), we obtain the Abel sum formula (40).

Following the same principle, the Abel sum expansion of $Q^-(r\underline{\xi}, \underline{\omega})$ may be deduced based on (38). The relation (16) may be rewritten

$$\begin{aligned}\frac{1}{\sigma_{m-1}}\sum_{k=1}^{\infty}\frac{m-2}{k}\frac{1}{r^{m-2+k}}P^{(-k)}(\underline{\omega}^{-1}\underline{\xi})\\ =\left(S^-(r\underline{\xi},\underline{\omega})+\tilde{S^-}(r\underline{\xi})\right)-\left(N(r\underline{\xi})+\tilde{N}(r\underline{\xi},\underline{\omega})\right).\end{aligned}$$

This, together with the relations (38) and (40) gives

$$\begin{aligned}Q^-(r\underline{\xi},\underline{\omega}) &= -2A^-(r)+P^-(r\underline{\xi},\underline{\omega})-\left(S^-(r\underline{\xi},\underline{\omega})-\tilde{S^-}(r\underline{\xi},\underline{\omega})\right)\\ &=\frac{1}{\sigma_{m-1}}\left[-2\sum_{k=-\infty}^{-1}\frac{1}{r^{m+|k|-2}}P^{(k)}(\underline{\omega}^{-1}\underline{\xi})+\sum_{-\infty}^{\infty}\frac{1}{r^{m+|k|-2}}P^{(k)}(\underline{\omega}^{-1}\underline{\xi})\right]\\ &\quad-\left[\frac{1}{\sigma_{m-1}}\sum_{k=-\infty}^{-1}\frac{1}{r^{m+|k|-2}}\frac{m-2}{|k|}P^{(k)}(\underline{\omega}^{-1}\underline{\xi})+\left(N(r\underline{\xi})+\tilde{N}(r\underline{\xi},\underline{\omega})\right)\right]\\ &=\frac{1}{\sigma_{m-1}}\left[\sum_{k=1}^{\infty}\frac{1}{r^{m+k-2}}P^{(k)}(\underline{\omega}^{-1}\underline{\xi})\right.\\ &\quad\left.-\sum_{k=-\infty}^{-1}\frac{m+|k|-2}{|k|}\frac{1}{r^{m+|k|-2}}P^{(k)}(\underline{\omega}^{-1}\underline{\xi})\right]-\tilde{N}(r\underline{\xi},\underline{\omega}).\end{aligned}$$

Now we deduce the formula (43) for $\tilde{N}(r\underline{\xi},\underline{\omega})$. Owing to (16) and (25), we have

$$\begin{aligned}\frac{1}{\sigma_{m-1}}\frac{m-2}{r^{m-1}}\int_0^r\frac{\rho^{m-2}}{|\rho\underline{\xi}-\underline{\omega}|^m}d\rho\ \underline{x}\wedge\underline{\omega}-\tilde{N^-}(r\underline{\xi},\underline{\omega})\\ =-\frac{1}{\sigma_{m-1}}\frac{m-2}{r^{m-1}}\int_r^{\infty}\frac{\rho^{m-2}}{|\rho\underline{\xi}-\underline{\omega}|^m}d\rho\ \underline{x}\wedge\underline{\omega}.\end{aligned}$$

Therefore,

$$\tilde{N}(r\underline{\xi},\underline{\omega})=\frac{1}{\sigma_{m-1}}\frac{m-2}{r^{m-2}}\int_0^{\infty}\frac{\rho^{m-2}}{|\rho\underline{\xi}-\underline{\omega}|^m}d\rho\ \underline{\xi}\wedge\underline{\omega},\quad \text{a.e. } r>1. \tag{43}$$

The proof is complete. □

Remark 4 Similar to what is noted in Remark 2 and 3, the explicit formula of P^- and its Abel sum formula can also be obtained from the expansions in (27) and (31), and via the Kelvin inversion. In particular,

$$P^-(\underline{x},\underline{\omega}) = \frac{1}{\sigma_{m-1}}\left(E(\underline{\omega}-\underline{x})\overline{\underline{\omega}} + \frac{1}{r^{m-2}}E\left(\underline{\omega}-\frac{\underline{x}}{r^2}\right)\underline{\omega}\right) = \frac{1}{\sigma_{m-1}}\frac{r^2-1}{|\underline{\omega}-r\underline{\xi}|^m}. \quad (44)$$

Remark 5 The existing methods for computing harmonic conjugates cannot be directly used to obtain a harmonic conjugate of $N(r\underline{\xi})$, as the function does not fulfil the conditions required by the methods. For the method in terms of polar coordinates [6] the existence of a real-valued C^∞ function $C(\underline{\omega})$ on the sphere satisfying

$$\Delta^* C(\underline{\omega}) = -c^{m-1}(\partial_r u)_{r=c} \quad (45)$$

is required, where c is a value in a particular range associated with the starlike domain in which a harmonic conjugate is sought. When u is taken to be the Newton potential $N(r\underline{\xi})$, the right-hand-side of (45) is a non-zero constant. Such function $C(\underline{\omega})$ does not exist. Even in the $m=2$ case it already encounters difficulty. In the case $C(\underline{\omega})$ is a periodic function. With $\Delta^* = \partial^2_{\theta\theta}$, we should have $\Delta^* C(\underline{\omega}) = constant$, but it cannot hold [6]. In all the examples included in [6] the special values $c=0$ and $c=\infty$ have to be chosen in order to obtain the trivial function $C(\underline{\omega})=0$ as the solution. This may show some restrictions of the method.

5 Boundedness of The Hilbert Transformations

We give two proofs of the boundedness result.

Theorem 6 *The Hilbert transformations $H^\pm$ given by the L^p-limits*

$$H^\pm f(\underline{\xi}) = \lim_{r\to 1\mp}\int_{S^{m-1}} Q^\pm(r\underline{\xi},\underline{\omega})f(\underline{\omega})d\sigma(\underline{\omega})$$

are bounded from $L^p(S^{m-1})$ to $L^p(S^{m-1})$, $1<p<\infty$.

Proof 1 By comparing (13) and (25) we are reduced to showing the boundedness of

$$Hf(\underline{\xi}) = p.v.\int_{S^{m-1}}\frac{1}{\sigma_{m-1}}\left(\frac{2}{|\underline{\xi}-\underline{\omega}|^m} - (m-2)\int_0^1\frac{\rho^{m-2}}{|\rho\underline{\xi}-\underline{\omega}|^m}d\rho\right) \times (\underline{\xi}\wedge\underline{\omega})f(\underline{\omega})d\sigma(\underline{\omega}).$$

The operator given by the left hand side of (15) for $r=1$ and $m>2$, is a Fourier multiplier operator where the Fourier multiplier is given by the sequence $b_k = \frac{m-2}{m-2+k}$ that is the restriction to positive integers of the bounded holomorphic function $b(z) = \frac{m-2}{m-2+z}$ in $S_{\alpha,+}$, $\alpha>0$, the sector between the two rays $\rho e^{\pm i\alpha}$, $\rho>0$ (see [19] or [20]). Invoking the main result of [20], the operator is a member of the bounded holomorphic functional calculus of the spherical Dirac operator, and therefore is L^p bounded for the given range of p. As consequence its scalar part and the non-scalar part both are L^p bounded. What we use here is that its non-scalar part, the principal value singular integral operator induced by the kernel

$$\int_0^1 \frac{\rho^{m-2}}{|\rho\underline{\xi}-\underline{\omega}|^m} d\rho(\underline{\xi}\wedge\underline{\omega}),$$

is L^p bounded. On the other hand, the kernel

$$\frac{1}{\sigma_{m-1}}\frac{2}{|\underline{x}-\underline{\omega}|^m}(\underline{\xi}\wedge\underline{\omega}),$$

as the non-scalar part of the Cauchy kernel, the latter being L^p bounded, also induces a L^p bounded operator. The proof is complete. □

Proof 2 We are based on the relations in (9) and (10). On the unit sphere the double layer potential that gives rise to the scalar part $\mathrm{Sc}[\mathcal{C}]$ satisfies

$$|\langle E(\underline{\xi}-\underline{\omega}), n(\underline{\omega})\rangle| = \frac{1}{2|\underline{\xi}-\underline{\omega}|^{m-2}},$$

the operator $\mathrm{Sc}[\mathcal{C}]$ is then compact and the Fredholm theory may be used to show that the operator $\frac{1}{2}(I+\mathrm{Sc}[\mathcal{C}])^{-1}$ exists and is bounded in L^p for all $1<p<\infty$ (see [13], or [15] or [23]). Since the singular Cauchy integral operator $\mathcal{C}$ is bounded, and so is its non-scalar part $\mathrm{NSc}[\mathcal{C}]$, we thus conclude that

$$Hu = \mathrm{NSc}[\mathcal{C}](I+\mathrm{Sc}[\mathcal{C}])^{-1}u$$

is bounded in the L^p on the sphere. □

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