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FIXED TRACE β-HERMITE ENSEMBLES: ASYMPTOTIC EIGENVALUE DENSITY AND THE EDGE OF THE DENSITY

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Abstract. In the present paper, fixed trace β-Hermite ensembles generalizing the fixed trace Gaussian Hermite ensemble are considered. For all β, we prove the Wigner semicircle law for these ensembles by using two different methods: one is the moment equivalence method with the help of the matrix model for general β, the other is to use asymptotic analysis tools. At the edge of the density, we prove that the edge scaling limit for β-HE implies the same limit for fixed trace β-Hermite ensembles. Consequently, explicit limit can be given for fixed trace GOE, GUE and GSE. Furthermore, for even β, analogous to β-Hermite ensembles, a multiple integral of the Konstevich type can be obtained.

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1. Introduction and main results

β-Hermite ensembles (β-HE) [7, 5] generalize the classical random matrix ensembles: Gaussian orthogonal, unitary and symplectic ensembles (denoted by GOE, GUE and GSE for short, which correspond to the Dyson index $\beta = 1, 2$ and 4) from the quantization index to the continuous exponents $\beta > 0$. These ensembles possess the joint probability density function (p.d.f.) of real eigenvalues $x_1, \ldots, x_N$ with the form

$$P_{\beta HE_N}(\mathbf{x}) = \frac{1}{Z_{\beta HE_N}} \prod_{1\le j<k\le N} |x_j - x_k|^\beta \prod_{i=1}^{N} e^{-x_i^2/2} \tag{1.1}$$

where the normalization constant $Z_{\beta HE_N}$ can be calculated with the help of the Selberg integrals [19]:

$$Z_{\beta HE_N} = (2\pi)^{\frac{N}{2}} \prod_{j=1}^{N} \frac{\Gamma(1+\frac{\beta j}{2})}{\Gamma(1+\frac{\beta}{2})}. \tag{1.2}$$

Recently, Dumitriu and Eldeman [7] have constructed a tri-diagonal real symmetric matrices of the form

$$H_\beta \sim \frac{1}{\sqrt{2}} \begin{pmatrix} \mathrm{N}[0,2] & \chi_{(N-1)\beta} & & & & \\ \chi_{(N-1)\beta} & \mathrm{N}[0,2] & \chi_{(N-2)\beta} & & & \\ & \chi_{(N-2)\beta} & \mathrm{N}[0,2] & \chi_{(N-3)\beta} & & \\ & \ddots & \ddots & \ddots & & \\ & & \chi_{2\beta} & \mathrm{N}[0,2] & \chi_\beta \\ & & & \chi_\beta & \mathrm{N}[0,2] \end{pmatrix} \tag{1.3}$$

where the N diagonal and $N-1$ subdiagonal elements are mutually independent, with standard normals on the diagonal, and $1/\sqrt{2}\,\chi_{k\beta}$ on the subdiagonal. It is worth mentioning that the p.d.f. of $1/\sqrt{2}\,\chi_{k\beta}$ is given by

$$\frac{2}{\Gamma(k\beta/2)} x^{k\beta-1} e^{-x^2}.$$

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Furthermore, they proved that the eigenvalues j.d.f. of H_β was given by (1.1).

Basing on the p.d.f. of eigenvalues in Eq. (1.1), the (level) density, or one-dimensional marginal eigenvalue density is defined as follows:

$$\rho_{\beta HE_N}(x_1) = \int_{\mathbb{R}^{N-1}} P_{\beta HE_N}(\mathbf{x})\, dx_2 \cdots dx_N. \tag{1.4}$$

One knows [6, 14] that the asymptotic eigenvalue density as $N \to \infty$ (density of states):

$$\lim_{N\to\infty} \sqrt{2\beta N}\rho_{\beta HE_N}(\sqrt{2\beta N}x) = \rho_{\mathrm{W}}(x) := \begin{cases} \frac{2}{\pi}\sqrt{1-x^2} & -1 < x < 1, \\ 0 & |x| \geq 1. \end{cases} \tag{1.5}$$

This result is referred to as the Wigner semicircle law. In terms of statistical physics, for any finite size N, we expect that most of the eigenvalues concentrate in the interval $(-\sqrt{2N}, \sqrt{2N})$, referred to the "bulk region" of mechanical problem, while the scaled density decreases rapidly in the vicinity of the spectrum edge $\approx \pm\sqrt{2N}$, referred to the "soft edge". At the edge of the spectrum, for Gaussian orthogonal, unitary and symplectic ensembles, or $\beta = 1, 2$ and 4, a classical result [9, 10, 11] claims that the edge scaling limit could be expressed in terms of Airy function. Explicitly, it says: for $\beta = 1, 2, 4$,

$$\lim_{N\to\infty} \frac{\sqrt{\beta}N^{5/6}}{\sqrt{2}} \rho_{\beta HE_N}\left(\sqrt{2\beta N}(1 + \frac{x}{2N^{2/3}})\right) = \mathbf{Ai}_\beta(x) \tag{1.6}$$

where

$$\mathbf{Ai}_\beta(x) = \begin{cases} (\mathrm{Ai}'(x))^2 - x(\mathrm{Ai}(x))^2 + \frac{1}{2}\mathrm{Ai}(x)\left(1 - \int_x^\infty \mathrm{Ai}(t)dt\right) & \beta = 1, \\ (\mathrm{Ai}'(x))^2 - x(\mathrm{Ai}(x))^2 & \beta = 2, \\ (\mathrm{Ai}'(2x))^2 - 2x(\mathrm{Ai}(2x))^2 - \mathrm{Ai}(2x)\int_x^\infty \mathrm{Ai}(2t)dt & \beta = 4. \end{cases} \tag{1.7}$$

Here, the Airy function of a real variable x can be defined as

$$\mathrm{Ai}(x) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{v^3/3 - xv} dv \tag{1.8}$$

satisfying the equation

$$\mathrm{Ai}''(x) = x\mathrm{Ai}(x). \tag{1.9}$$

Much finer correction terms to the large N asymptotic expansions of the eigenvalue density are considered in [10, 11], and (1.5) is also proved for $\beta = 1, 2, 4$. Recently, for every even β, Desrosiers and Forrester[5], by applying the steepest descent method, obtained a multiple integral of the Konstevich type at the edges which constitutes a β-deformation of the Airy function. That is, for even β,

$$\begin{aligned} \lim_{N\to\infty} \frac{\sqrt{\beta}N^{5/6}}{\sqrt{2}} \rho_{\beta HE_N}\left(\sqrt{2\beta N}(1 + \frac{x}{2N^{2/3}})\right) \\ = \frac{1}{2\pi}\left(\frac{4\pi}{\beta}\right)^{\beta/2} \frac{\Gamma(1+\beta/2)}{\prod_{j=2}^{\beta}\Gamma(1+2/\beta)^{-1}\Gamma(1+2j/\beta)} K_{\beta,\beta}(x) \end{aligned} \tag{1.10}$$

where $K_{\beta,\beta}(x)$ is a multiple integral of Konstevich type or multiple Airy integrals defined by [15]:

$$K_{n,\beta}(x) := -\frac{1}{(2\pi \mathrm{i})^n}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} dv_1 \cdots \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} dv_n \prod_{j=1}^{n} e^{v_j^3/3 - xv_j} \prod_{1\leq k<l\leq n} |v_k - v_l|^{4/\beta}. \tag{1.11}$$

Remark: When $\beta = 2, 4$, the right sides of (1.6) and (1.10) coincide. However, to stress the cases $\beta = 1, 2, 4$ and general β, we state them respectively.

In the present paper, we deal with fixed trace β-Hermite ensembles and extend the properties (1.5), (1.6) and (1.10) to these fixed trace ensembles. First, let us give a review [20, 19]. Proceeding from the analogy of a fixed energy in classical statistical mechanics, Rosenzweig defines [20] his "fixed trace" ensemble for a Gaussian real symmetric, Hermitian or self-dual matrix H by the requirement that the trace of H^2 be fixed to a number r^2 with no other constraint. The number r is called the strength of the ensemble. The joint probability density function for the matrix elements of H is therefore given by

$$P_r(H) = K_r^{-1}\,\delta\big(\frac{1}{r^2}\operatorname{tr} H^2 - 1\big)$$

where K_r is the normalization constant. Note that this probability density function is invariant under the conjugate action by orthogonal, unitary or symplectic groups, because of the invariance of the quantity $\operatorname{tr} H^2$. Now Rosenzweig's fixed trace ensemble has been extended to other ensembles, one of which is fixed trace β-HE. With the help of H_β in Eq. (1.3), G. LeCaër and R. Delannay [17] define the associated fixed trace β-HE as the ensemble of matrices:

$$F_\beta = \sqrt{N(N-1)/2}\, H_\beta \,/ \sqrt{\operatorname{tr} H_\beta^2}$$

satisfying $\operatorname{tr} F_\beta^2 = N(N-1)/2$. Its eigenvalue joint p.d.f has the form

$$P_{\beta FTE_N}(x_1, x_2, \cdots, x_N) = \frac{1}{Z_{\beta FTE_N}}\,\delta\,\big(\sum_{i=1}^{N} x_i^2 - \frac{N(N-1)}{2}\big) \prod_{1\le j<k\le N} |x_j - x_k|^\beta \tag{1.12}$$

where the normalization constant $Z_{\beta FTE_N}$ can be computed by virtue of variable substitution for the partition function $Z_{\beta HE_N}$:

$$Z_{\beta FTE_N} = \big(\frac{N(N-1)}{2}\big)^{\frac{N_\beta - 1}{2}} \frac{(2\pi)^{\frac{N}{2}} 2^{-\frac{N_\beta}{2}+1}}{\Gamma(\frac{N_\beta}{2})} \prod_{j=1}^{N} \frac{\Gamma(1+\frac{j\beta}{2})}{\Gamma(1+\frac{\beta}{2})} \tag{1.13}$$

where $N_\beta = N + \beta N(N-1)/2$. It is worth emphasizing that we have chosen the square of the strength $r^2 = N(N-1)/2$ since the expectation of $\operatorname{tr} H_\beta^2$ is $N(N-1)/2 + N/\beta \approx N(N-1)/2$ as $N \to \infty$, Ref.[19]. Notice the analogy: fixed trace β-HE bears the same relationship to β-HE that the microcanonical ensembles to the canonical ensemble in statistical physics [2]. Besides, G .Akemann et al [1] described further interesting physical features of fixed trace ensembles due to the interaction among eigenvalues introduced through a constraint.

As done usually for β-HE in Eq. (1.4), the density of fixed trace β-HE is written in the form

$$\rho_{\beta FTE_N}(x_1) = \int_{\Omega_{N-1}} P_{\beta FTE_N}(x_1, x_2, \cdots, x_N)\, d\,\sigma_{N-1} \tag{1.14}$$

where Ω_{N-1} denotes the sphere $x_2^2 + \cdots + x_N^2 = N(N-1)/2 - x_1^2$, and $d\,\sigma_{N-1}$ denotes the spherical measure.

The important thing to be noted about fixed trace GOE, GUE and GSE is their moment equivalence with the associated Gaussian ensembles of large dimensions (implying the semicircle law), see Mehta's book [19], **Sect**.27.1, p.488. At the end of this section, p.490, he writes:

> It is not very clear whether this moment equivalence implies that all local statistical properties of the eigenvalues in two sets of ensembles are identical. This is so because these local properties of eigenvalues may not be expressible only in terms of finite moments of the matrix elements.

In the Appendix of this paper, Combining Rosenzweig's method [19, 20] and Dumitriu and Edelman's matrix models (1.3), we present the moment equivalence between fixed trace β-HE and β-HE in the large N, which implies that the global density of fixed-trace β-HE also fits the semi-circle law for all β. We will derive the semicircle law using another quite different method, and prove that the property of the spectrum edge for β-HE implies the same property for fixed trace β-HE. Recently, Götze et al [12, 13] have proven universality of sine-kernel in the bulk for fixed GUE. In [18], asymptotic equivalence of local properties for correlation functions at zero and the edge of the spectrum between fixed trace β-HE and β-HE is proved, which implies universality of sine-kernel at zero and airy-kernel at the edge for fixed trace GOE, GUE and GSE. All these known results (to our knowledge), to some extent, answer this open problem.

Now we can state our main results. Let $C_c(\mathbb{R})$ be the set of all continuous functions on $\mathbb{R}$ with compact support. For fixed trace β-HE, the scaled eigenvalue density satisfies the Wiger semicircle law, i.e.,

Theorem 1. *Let $\rho_{\beta FTE_N}(x_1)$ be the eigenvalue density for fixed trace β-HE, defined by (1.14). If $f(x) \in C_c(\mathbb{R})$, then we have*

$$\lim_{N\to\infty} \int_{\mathbb{R}} f(x)\sqrt{2N}\rho_{\beta FTE_N}(\sqrt{2N}x) = \int_{\mathbb{R}} f(x)\rho_{\mathrm{W}}(x)dx$$

where

$$\rho_{\mathrm{W}}(x) := \begin{cases} \frac{2}{\pi}\sqrt{1-x^2} & -1 < x < 1, \\ 0 & |x| \geq 1. \end{cases}$$

At the edge of the spectrum we can prove that that the property of the spectrum edge for β-HE implies the same property for fixed trace β-HE, thus we extend Desrosiers and Forrester's result for even β to fixed trace case.

Theorem 2. *Let $\rho_{\beta HE_N}(x_1)$ and $\rho_{\beta FTE_N}(x_1)$ be the eigenvalue density of β-HE and that of fixed trace, respectively, defined by (1.4) and (1.14). Assume that $f(x) \in C_c(\mathbb{R})$. If $\forall\, h(t) \in C_c(\mathbb{R})$,*

$$\lim_{N\to\infty} \int_{\mathbb{R}} h(t)\frac{\sqrt{\beta}N^{5/6}}{\sqrt{2}}\rho_{\beta HE_N}\left(\sqrt{2\beta N}(1+\frac{t}{2N^{2/3}})\right) dt \tag{1.15}$$

exists, then

$$\lim_{N\to\infty} \int_{\mathbb{R}} f(x)\frac{N^{5/6}}{\sqrt{2}}\rho_{\beta FTE_N}\left(\sqrt{2N}(1+\frac{x}{2N^{2/3}})\right) dx = \lim_{N\to\infty} \int_{\mathbb{R}} f(t)\frac{\sqrt{\beta}N^{5/6}}{\sqrt{2}}\rho_{\beta HE_N}\left(\sqrt{2\beta N}(1+\frac{t}{2N^{2/3}})\right) dt. \tag{1.16}$$

It immediately follows from (1.6), (1.10) and Theorem 2 that

Corollary 3. *Let $\rho_{\beta FTE_N}(x_1)$ be the eigenvalue density for fixed trace β-HE, defined by (1.14). If $f(x) \in C_c(\mathbb{R})$, then at the edge of the spectrum one has*

$$\lim_{N\to\infty} \int_{\mathbb{R}} f(x)\frac{N^{5/6}}{\sqrt{2}}\rho_{\beta FTE_N}\left(\sqrt{2N}(1+\frac{x}{2N^{2/3}})\right) dx = \int_{\mathbb{R}} f(t)\mathbf{Ai}_{\beta}(t)dt \tag{1.17}$$

for $\beta = 1, 2, 4$ and

$$\lim_{N\to\infty} \int_{\mathbb{R}} f(x)\frac{N^{5/6}}{\sqrt{2}}\rho_{\beta FTE_N}\left(\sqrt{2N}(1+\frac{x}{2N^{2/3}})\right) dx = \frac{1}{2\pi}\left(\frac{4\pi}{\beta}\right)^{\beta/2} \frac{\Gamma(1+\beta/2)}{\prod_{j=2}^{\beta}\Gamma(1+2/\beta)^{-1}\Gamma(1+2j/\beta)} \int_{\mathbb{R}} f(t)K_{\beta,\beta}(t)dt$$

for even β. Here $\mathbf{Ai}_{\beta}(x)$ and $K_{\beta,\beta}(x)$ are defined by (1.7) and (1.11) respectively.

Theorem 1 will be proved in Sect. 3 and Theorem 2 in Sect. 4 after the preparatory work in Sect. 2.

2. An upper bound for the level density

In this section, we will give an estimation of the level density for fixed trace β-HE, with the help of the maximum of Vandermonde determinant on the sphere by Stieltjes [21]. For the convenience of our argument, let's first state Stieltjes's remarkable result as a lemma.

Lemma 4. *Let us consider a unit mass at each of the variable points $x_1, x_2, \cdots, x_N$ in the interval $[-\infty, +\infty]$ such that*

$$\sum_{i=1}^{N} x_i^2 \leq \frac{N(N-1)}{2},$$

then the maximal of

$$V(x_1, x_2, \cdots, x_N) = \prod_{1\leq j<k\leq N} |x_j - x_k|^2$$

is attained if and only if the x_j are the zeros of the Hermite polynomial $H_N(x)$, and the maximal is

$$\max_{\sum_{i=1}^N x_i^2 \le \frac{N(N-1)}{2}} V(x_1, x_2, \cdots, x_N) = 2^{-\frac{N(N-1)}{2}} \prod_{v=1}^N e^{v \ln v}. \tag{2.1}$$

Note that Stieltjes's result can be interpreted as a electrostatic problem and the maximum position corresponds to the condition of electrostatic equilibrium.

The following proposition gives an estimation of the level density, which gives a global control of the density for fixed trace β-HE.

Proposition 5. *For the level density $\rho_{\beta FTE_N}(x_1)$, rescaling*

$$x_1 = \sqrt{N(N-1)/2}\, x, \ -1 \le x \le 1,$$

then we have

$$\rho_{\beta FTE_N}\Big(\sqrt{\frac{N(N-1)}{2}}\, x\Big) \le e^{W_{N\beta} N} (1-x^2)^{\frac{N-2}{2}} \tag{2.2}$$

where $W_{N\beta} = \ln C_\beta + o(1)$ and

$$C_\beta = \exp\big(1 - \ln\sqrt{2\pi} + \frac{\beta}{2} - \frac{\beta}{2}\ln(\frac{\beta}{2})\big)\, \Gamma(1+\frac{\beta}{2}). \tag{2.3}$$

Proof. From (1.14) and (2.1), we see that

$$\rho_{\beta FTE_N}(x_1) \le \frac{1}{Z_{\beta FTE_N}} 2^{-\frac{\beta N(N-1)}{4}} \big(\prod_{v=1}^N e^{v\ln v}\big)^{\frac{\beta}{2}} \int_{x_2^2+\cdots+x_N^2 = \frac{N(N-1)}{2} - x_1^2} d\,\sigma_{N-1}, \tag{2.4}$$

where σ_{N-1} denotes $N-2$ dimensional spherical measure. By the formula for surface area of the sphere, a direct calculation shows that

$$\begin{aligned}
\rho_{\beta FTE_N}(x_1) &\le \frac{1}{Z_{\beta FTE_N}} 2^{-\frac{\beta N(N-1)}{4}} \big(\prod_{v=1}^N e^{v\ln v}\big)^{\frac{\beta}{2}} \frac{2\pi^{\frac{N-1}{2}}}{\Gamma(\frac{N-1}{2})} \big(\frac{N(N-1)}{2} - \frac{N(N-1)}{2}x^2\big)^{\frac{N-2}{2}} \\
&= \frac{2^{-\frac{\beta N(N-1)}{4}} \big(\prod_{v=1}^N e^{v\ln v}\big)^{\frac{\beta}{2}} 2\pi^{\frac{N-1}{2}} \big(\frac{N(N-1)}{2} - \frac{N(N-1)}{2}x^2\big)^{\frac{N-2}{2}} \Gamma(\frac{N_\beta}{2})}{\Gamma(\frac{N-1}{2}) (\frac{N(N-1)}{2})^{\frac{N_\beta - 1}{2}} (2\pi)^{\frac{N}{2}} 2^{-\frac{N_\beta}{2}+1}} \prod_{j=1}^N \frac{\Gamma(1+\frac{\beta}{2})}{\Gamma(1+\frac{j\beta}{2})} \\
&= \frac{(1-x^2)^{\frac{N-2}{2}} e^{\frac{\beta}{2}\sum_{v=1}^N v\ln v}}{\sqrt{\pi}\,\Gamma(\frac{N-1}{2})} \big(\frac{N(N-1)}{2}\big)^{\frac{N-2}{2}} \Big(\frac{\Gamma(\frac{N_\beta}{2})}{(\frac{N(N-1)}{2})^{\frac{N_\beta-1}{2}}}\Big) \prod_{j=1}^N \frac{\Gamma(1+\frac{\beta}{2})}{\Gamma(1+\frac{j\beta}{2})} &(2.5)\\
&\triangleq (1-x^2)^{\frac{N-2}{2}} g_{N\beta}. &(2.6)
\end{aligned}$$

It is sufficient to consider $\sqrt[N]{g_{N\beta}}$ as $N \to \infty$. Using Stirling's formula for the gamma function,

$$\Gamma(x) = (2\pi)^{1/2} e^{-x} x^{x-\frac{1}{2}} (1 + O(\frac{1}{x})) \tag{2.7}$$

for the large x. For the large N,

$$\Gamma(\frac{N-1}{2}) = (2\pi)^{1/2} e^{-\frac{N-1}{2}} (\frac{N-1}{2})^{\frac{N}{2}-1} (1 + O(\frac{1}{N})), \tag{2.8}$$

$$\Gamma(\frac{N_\beta}{2}) = (2\pi)^{1/2} e^{-N_\beta/2} (\frac{N_\beta}{2})^{\frac{N_\beta - 1}{2}} (1 + O(\frac{1}{N})). \tag{2.9}$$

Note that

$$\Big(\frac{N_\beta}{N(N-1)}\Big)^{\frac{N_\beta-1}{2}} = (\frac{\beta}{2})^{\frac{N_\beta-1}{2}} \Big(1 + \frac{2}{\beta(N-1)}\Big)^{\frac{N_\beta-1}{2}},$$

thus $g_{N\beta}$ can be rewritten as

$$g_{N\beta} = \frac{1}{\sqrt{\pi}} e^{\frac{N-1}{2}} \big(\Gamma(1+\frac{\beta}{2})\big)^N \Big(1 + \frac{2}{\beta(N-1)}\Big)^{\frac{N_\beta-1}{2}} (1 + O(\frac{1}{N}))\, \tilde{g}_{N\beta} \tag{2.10}$$

where

$$\tilde{g}_{N\beta} = \frac{\exp(\frac{\beta}{2}\sum_{v=1}^{N} v\ln v)}{\prod_{j=1}^{N}\Gamma(1+\frac{j\beta}{2})}\, N^{\frac{N-2}{2}} e^{-\frac{N_\beta}{2}} \,(\frac{\beta}{2})^{\frac{N_\beta-1}{2}}.$$

Observe that

$$\lim_{N\to\infty}\left(\frac{1}{\sqrt{\pi}}\,e^{\frac{N-1}{2}}\big(\Gamma(1+\frac{\beta}{2})\big)^N\Big(1+\frac{2}{\beta(N-1)}\Big)^{\frac{N_\beta-1}{2}}(1+O(\frac{1}{N}))\right)^{1/N} = e\,\Gamma(1+\frac{\beta}{2}). \tag{2.11}$$

On the other hand, using Stolz's rule,

$$\begin{aligned}
&\lim_{N\to\infty}\frac{1}{N}\ln\tilde{g}_{N\beta}\\
&=\lim_{N\to\infty}\frac{1}{N}\left(\frac{\beta}{2}\sum_{v=1}^{N}v\ln v+\frac{N-2}{2}\ln N-\frac{N_\beta}{2}-\sum_{j=1}^{N}\ln\Gamma(1+\frac{j\beta}{2})+\frac{N_\beta-1}{2}\ln(\frac{\beta}{2})\right)\\
&=\lim_{N\to\infty}\left(\frac{\beta}{2}N\ln N+\frac{1}{2}\ln N-\frac{N-3}{2}\ln(1-\frac{1}{N})-\ln\Gamma(1+\frac{N\beta}{2})+\frac{1}{2}(\ln\frac{\beta}{2}-1)(1-\beta+\beta N)\right)\\
&=-\ln\sqrt{2\pi}+\frac{\beta}{2}-\frac{\beta}{2}\ln(\frac{\beta}{2}).
\end{aligned}\tag{2.12}$$

In the above calculation, we make use of the following asymptotic expansion:

$$\ln\Gamma(1+\frac{N}{2}\beta)=\ln\sqrt{2\pi}-\frac{N}{2}\beta+(\frac{N}{2}\beta+\frac{1}{2})\ln(\frac{N}{2}\beta)+O(\frac{1}{N}). \tag{2.13}$$

Combining (2.11), (2.12) and (2.6), this completes the proof of Proposition 5. □

3. Proof of Theorem 1

To prove Theorem 1, we will first prove the following Lemma 6, which means that the level density of β-HE defined by (1.4) and that of fixed trace β-HE given by (1.14) are *almost equivalent*. Our arguments depend on the following integral equation

$$\rho_{\beta HE_N}(x_1)=\frac{1}{C_{N\beta}}\int_{|x_1|}^{+\infty}e^{-r^2/2}r^{N_\beta-2}\rho_{\beta FTE_N,1}(\frac{x_1}{r})dr, \tag{3.1}$$

obtained in [17, 4] where $C_{N\beta}=\Gamma(N_\beta/2)2^{N_\beta/2-1}$. Here $\rho_{\beta FTE_N,1}(x_1)$ denoting the level density of fixed trace β-HE whose strength is 1, is defined by

$$\rho_{\beta FTE_N,1}(x_1)=\frac{1}{Z_{\beta FTE_N,1}}\int_{\mathbb{R}^{N-1}}\delta(\sum_{i=1}^{N}x_i^2-1)\prod_{1\le j<k\le N}|x_j-x_k|^\beta dx_2\ldots dx_N \tag{3.2}$$

where the partition function

$$Z_{\beta FTE_N,1}=\frac{(2\pi)^{\frac{N}{2}}2^{-\frac{N_\beta}{2}+1}}{\Gamma(\frac{N_\beta}{2})}\prod_{j=1}^{N}\frac{\Gamma(1+\frac{j\beta}{2})}{\Gamma(1+\frac{\beta}{2})}. \tag{3.3}$$

A direct calculation shows

$$\rho_{\beta FTE_N,1}(x)=\sqrt{\frac{N(N-1)}{2}}\rho_{\beta FTE_N}(\sqrt{\frac{N(N-1)}{2}}x). \tag{3.4}$$

It follows from with Proposition 5 that

$$\rho_{\beta FTE_N,1}(x)\le\sqrt{\frac{N(N-1)}{2}}e^{W_{N\beta}N}(1-x^2)^{\frac{N-2}{2}} \tag{3.5}$$

for any $-1\le x\le 1$. Now we are ready to state the following *almost equivalent* lemma about the two ensembles for all $\beta>0$, which has been obtained in [25] for $\beta=2$ without rigorous arguments.

Lemma 6. *Let $-1 \le x \le 1$ be fixed. For the level density of fixed trace β-HE defined by (1.4) and that of β-HE given by (1.14), we have*

$$\rho_{\beta HE_N}(\sqrt{2N\beta}x) = \big(\frac{1}{\sqrt{\beta}} + O(\frac{1}{N})\big)\,\rho_{\beta FTE_N}\big(\sqrt{2N}x(1+O(\alpha_N))\big) + O(e^{-\beta N^{2(1-\theta)}(1+o(1))}) \tag{3.6}$$

for large N where $\alpha_N = \frac{1}{N^\theta}$, $0<\theta<0.5$.

Proof. Dividing the right hand side of the integral equation (3.1) into three parts, then

$$\begin{aligned}&\rho_{\beta HE_N}(x_1)\\ &= \frac{1}{C_{N\beta}}\Big(\int_{|x_1|}^{\sqrt{\beta}N(\frac{1}{\sqrt{2}}-\alpha_N)} + \int_{\sqrt{\beta}N(\frac{1}{\sqrt{2}}+\alpha_N)}^{+\infty} + \int_{\sqrt{\beta}N(\frac{1}{\sqrt{2}}-\alpha_N)}^{\sqrt{\beta}N(\frac{1}{\sqrt{2}}+\alpha_N)}\Big)\, e^{-r^2/2}r^{N_\beta-2}\,\rho_{\beta FTE_N,1}(\frac{x_1}{r})dr\\ &= I + II + III. \end{aligned}\tag{3.7}$$

Next we will estimate I and II respectively. For large N, the function $e^{-r^2/2}r^{N_\beta-2}$ attains its maximum at $\sqrt{N_\beta-2}$, which satisfying

$$\sqrt{N_\beta - 2} > \sqrt{\beta}N(\frac{1}{\sqrt{2}} - \alpha_N). \tag{3.8}$$

Together with (2.7) and (3.5), I can be dominated by

$$\begin{aligned} I &\le \frac{1}{C_{N\beta}}\int_{|x_1|}^{\sqrt{\beta}N(\frac{1}{\sqrt{2}}-\alpha_N)} e^{-r^2/2}r^{N_\beta-2}dre^{W_{N\beta}N}\sqrt{\frac{N(N-1)}{2}}\\ &\le \frac{e^{-(\sqrt{\beta}N(\frac{1}{\sqrt{2}}-\alpha_N))^2/2}(\sqrt{\beta}N)^{N_\beta-2}(\frac{1}{\sqrt{2}}-\alpha_N)^{N_\beta-2}}{\Gamma(\frac{N_\beta}{2})2^{\frac{N_\beta}{2}-1}}(\sqrt{\beta}N(\frac{1}{\sqrt{2}}-\alpha_N)-|x_1|)e^{W_{N\beta}N}\sqrt{\frac{N(N-1)}{2}}\\ &\le \frac{e^{-(\sqrt{\beta}N(\frac{1}{\sqrt{2}}-\alpha_N))^2/2}(\sqrt{\beta}N)^{N_\beta-2}(\frac{1}{\sqrt{2}}-\alpha_N)^{N_\beta-2}}{\sqrt{2\pi}e^{-\frac{N_\beta}{2}}(N_\beta/2)^{\frac{N_\beta-1}{2}}(1+O(1/N^2))2^{\frac{N_\beta}{2}-1}}(\sqrt{\beta}N(\frac{1}{\sqrt{2}}-\alpha_N)-|x_1|)e^{W_{N\beta}N}\sqrt{\frac{N(N-1)}{2}}\\ &\le C'\frac{e^{-(\sqrt{\beta}N(\frac{1}{\sqrt{2}}-\alpha_N))^2/2}(\sqrt{\beta}N)^{N_\beta}(\frac{1}{\sqrt{2}}-\alpha_N)^{N_\beta}}{e^{-N_\beta/2}N_\beta^{\frac{N_\beta}{2}}}Ne^{W_{N\beta}N} \end{aligned}\tag{3.9}$$

where we have used the fact that

$$\sqrt{\frac{N(N-1)}{2}}(\frac{1}{\sqrt{2}}-\alpha_N)^{-2}(\sqrt{\beta}N)^{-2}\Big(\sqrt{\beta}N(\frac{1}{\sqrt{2}}-\alpha_N)-|x_1|\Big)\,\frac{1}{2\sqrt{\pi}(N_\beta)^{-1/2}} \le C'N \tag{3.10}$$

for large N. Here C' is a constant only depending on β. We will estimate the right hand side of (3.9). Expanding

$$\ln(N_\beta^{N_\beta/2}) = \frac{N_\beta}{2}\big(\ln(\frac{\beta}{2}N^2) + \ln(1+(\frac{2}{\beta}-1)\frac{1}{N})\big)$$

and by a direct calculation, one obtains

$$\frac{(\sqrt{\beta}N)^{N_\beta}}{N_\beta^{N_\beta/2}} = \exp(\frac{N_\beta}{2}\ln 2 - \frac{N}{2} + \frac{\beta N}{4} + O(1)). \tag{3.11}$$

Again, expanding

$$\ln(\frac{1}{\sqrt{2}} - \alpha_N) = -\ln\sqrt{2} - \sqrt{2}\alpha_N - \alpha_N^2 + O(\alpha_N^3).$$

therefore I can be dominated by

$$\begin{aligned} I \le C'\exp\Big(&-\frac{\beta}{2}(\frac{1}{\sqrt{2}}-\alpha_N)^2N^2 - \big(\ln\sqrt{2}+\sqrt{2}\alpha_N+\alpha_N^2+O(\alpha_N^3)\big)N_\beta + \frac{1}{2}N_\beta + \frac{\ln 2}{2}N_\beta\\ &- \frac{N}{2} + \frac{\beta N}{4} + O(1) + W_{N\beta}N + \ln N\Big) \end{aligned}$$

$$
\begin{aligned}
&= C' \exp\Big(-\frac{\beta}{2}(\frac{1}{\sqrt{2}}-\alpha_N)^2 N^2 - \frac{\beta}{2}\big(\sqrt{2}\alpha_N + \alpha_N^2 + O(\alpha_N^3)\big)N^2 + \frac{\beta}{4}N^2 + W_{N\beta}N + \ln N + O(1)\Big) \\
&= C' \exp\big(-\beta\alpha_N^2 N^2 + \beta N^2 O(\alpha_N^3) + W_{N\beta}N + \ln N + O(1)\big)
\end{aligned}
$$

(3.12)

$$
= C' \exp\big(-\beta N^{2-2\theta}(1 + O(N^{-\theta}) + O(N^{2\theta-2}) + O(N^{2\theta-1}))\big) = O(e^{-\beta N^{2(1-\theta)}(1+o(1))}).
$$

Here we should take $\theta \in (0, 0.5)$. For the convenience of our argument, write

$$
O\Big(\exp\big(-\beta N^{2-2\theta}(1 + O(N^{-\theta}) + O(N^{2-2\theta}) + O(N^{2\theta-1}))\big)\Big) \triangleq \Xi_N.
$$

Similarly , we can estimate II. For large N, the function $e^{-r^2/2}r^{N_\beta}$ attains its maximum at $\sqrt{N_\beta}$ with the condition

$$
\sqrt{N_\beta} < \sqrt{\beta}N(\frac{1}{\sqrt{2}} + \alpha_N). \tag{3.13}
$$

This shows that II can be dominated by

$$
\begin{aligned}
II &< \frac{1}{C_{N\beta}} \int_{\sqrt{\beta}N(\frac{1}{\sqrt{2}}+\alpha_N)}^{+\infty} e^{-r^2/2} r^{N_\beta} r^{-2} dr e^{W_{N\beta}N} \sqrt{\frac{N(N-1)}{2}} \\
&\le \frac{e^{-(\sqrt{\beta}N(\frac{1}{\sqrt{2}}+\alpha_N))^2/2}(\sqrt{\beta}N)^{N_\beta}(\frac{1}{\sqrt{2}}+\alpha_N)^{N_\beta}}{\Gamma(\frac{N_\beta}{2})2^{\frac{N_\beta}{2}-1}} \int_{\sqrt{\beta}N(\frac{1}{\sqrt{2}}+\alpha_N)}^{+\infty} r^{-2} dr e^{W_{N\beta}N} \sqrt{\frac{N(N-1)}{2}} \\
&\le C'' \exp\big(-\beta N^{2-2\theta}(1 + O(N^{2\theta-2}) + O(N^{-\theta}) + O(N^{2\theta-1}))\big) = \Xi_N. \qquad (3.14)
\end{aligned}
$$

Here C'' is a constant only depending on β. By (3.12) and (3.14), the identity (3.7) can be reduced to

$$
\rho_{\beta HE_N}(x_1) = \frac{1}{C_{N\beta}} \int_{\sqrt{\beta}N(\frac{1}{\sqrt{2}}-\alpha_N)}^{\sqrt{\beta}N(\frac{1}{\sqrt{2}}+\alpha_N)} e^{-r^2/2} r^{N_\beta-2} \rho_{\beta FTE_N,1}(\frac{x_1}{r}) dr + \Xi_N. \tag{3.15}
$$

Using the intermediate value theorem of integral,

$$
\rho_{\beta HE_N}(x_1) = \rho_{\beta FTE_N,1}(\frac{x_1}{\xi_N(x_1)}) \frac{1}{C_{N\beta}} \int_{\sqrt{\beta}N(\frac{1}{\sqrt{2}}-\alpha_N)}^{\sqrt{\beta}N(\frac{1}{\sqrt{2}}+\alpha_N)} e^{-r^2/2} r^{N_\beta-2} dr + \Xi_N \tag{3.16}
$$

where $\sqrt{\beta}N(1/\sqrt{2}-\alpha_N) \le \xi_N(x_1) \le \sqrt{\beta}N(1/\sqrt{2}+\alpha_N)$. If we repeat the procedure of obtaining the estimate of I and II, then

$$
\frac{1}{C_{N\beta}} \int_0^{\sqrt{\beta}N(\frac{1}{\sqrt{2}}-\alpha_N)} e^{-\frac{r^2}{2}} r^{N_\beta-2} dr = O(e^{-\beta N^{2(1-\theta)}(1+o(1))}), \tag{3.17}
$$

$$
\frac{1}{C_{N\beta}} \int_{\sqrt{\beta}N(\frac{1}{\sqrt{2}}+\alpha_N)}^{+\infty} e^{-\frac{r^2}{2}} r^{N_\beta-2} dr = O(e^{-\beta N^{2(1-\theta)}(1+o(1))}) \tag{3.18}
$$

for $0 < \theta < 1$. Actually, the process of the argument of (3.9) tells us that the term $\sqrt{(N(N-1))/2}e^{W_{N\beta}}$ will disappear in that of obtaining (3.17). Hence the left hand side of (3.17) can be dominated by

$$
C_1 \frac{e^{-(\sqrt{\beta}N(\frac{1}{\sqrt{2}}-\alpha_N))^2/2}(\sqrt{\beta}N)^{N_\beta}(\frac{1}{\sqrt{2}}-\alpha_N)^{N_\beta}}{e^{-N_\beta/2}N_\beta^{\frac{N_\beta}{2}}} \tag{3.19}
$$

where

$$
C_1 = (\frac{1}{\sqrt{2}}-\alpha_N)^{-2}(\sqrt{\beta}N)^{-2}\Big(\sqrt{\beta}N(\frac{1}{\sqrt{2}}-\alpha_N)\Big)\frac{1}{2\sqrt{\pi}(N_\beta)^{-1/2}}. \tag{3.20}
$$

Here it is easy to see that C_1 is bounded for large N. Repeated arguments similar with (3.12) show that (3.19) can be dominated by $\exp\big(-\beta N^{2-2\theta}(1 + O(N^{-\theta}) + O(N^{2\theta-2}))\big)$. The same method can be used to obtain (3.18). Hence, for $0<\theta<1$,

$$\frac{1}{C_{N\beta}}\int_{\sqrt{\beta}N(\frac{1}{\sqrt{2}}-\alpha_N)}^{\sqrt{\beta}N(\frac{1}{\sqrt{2}}+\alpha_N)} e^{-\frac{r^2}{2}}r^{N_\beta-2}dr=\frac{1}{C_{N\beta}}\Big(\int_0^{+\infty}-\int_0^{\sqrt{\beta}N(\frac{1}{\sqrt{2}}-\alpha_N)}-\int_{\sqrt{\beta}N(\frac{1}{\sqrt{2}}+\alpha_N)}^{+\infty}\Big)e^{-\frac{r^2}{2}}r^{N_\beta-2}dr$$

$$=\frac{1}{C_{N\beta}}\int_0^{+\infty}e^{-\frac{r^2}{2}}r^{N_\beta-2}dr+O(e^{-\beta N^{2(1-\theta)}(1+o(1))})$$

(3.21)

$$=\frac{\Gamma(\frac{N_\beta-1}{2})}{\sqrt{2}\Gamma(\frac{N_\beta}{2})}+O(e^{-\beta N^{2(1-\theta)}(1+o(1))})=\frac{\sqrt{2}}{N\sqrt{\beta}}+O(\frac{1}{N^2}).$$

It is worth emphasizing that the range of θ in Eqs. (3.17) and (3.18) plays a vital role in the proof of Theorem 2.

Hence Eq.(3.16) can be reduced to

$$\rho_{\beta HE_N}(x_1)=\rho_{\beta FTE_N,1}(\frac{x_1}{\xi_N(x_1)})(\frac{\sqrt{2}}{N\sqrt{\beta}}+O(\frac{1}{N^2}))+\Xi_N. \tag{3.22}$$

If we make the change of variables $x_1=\sqrt{2N\beta}x$, the relation (3.4) implies that

$$\rho_{\beta HE_N}(\sqrt{2N\beta}x)=\rho_{\beta FTE_N,1}\left(\frac{\sqrt{2N\beta}x}{\xi_N(x)}\right)\left(\frac{\sqrt{2}}{N\sqrt{\beta}}+O(\frac{1}{N^2})\right)+\Xi_N.$$

$$=\rho_{\beta FTE_N}\left(\sqrt{\frac{N(N-1)}{2}}\frac{\sqrt{2N\beta}x}{\xi_N(x)}\right)\left(\frac{\sqrt{2}}{N\sqrt{\beta}}+O(\frac{1}{N^2})\right)\sqrt{\frac{N(N-1)}{2}}+\Xi_N$$

$$=\big(\frac{1}{\sqrt{\beta}}+O(\frac{1}{N})\big)\,\rho_{\beta FTE_N}\big(\sqrt{2N}x(1+O(\alpha_N))\big)+\Xi_N. \tag{3.23}$$

Here we make use of the fact that for the large N,

$$\sqrt{\frac{N(N-1)}{2}}\frac{\sqrt{2N\beta}x}{\xi_N(x)}=(1+O(\alpha_N))\sqrt{2N}x,$$

$$\left(\frac{\sqrt{2}}{N\sqrt{\beta}}+O(\frac{1}{N^2})\right)\sqrt{\frac{N(N-1)}{2}}=\frac{1}{\sqrt{\beta}}+O(\frac{1}{N}).$$

Note that when $0<\theta<0.5$ Ξ_N can be rewritten by

$$\Xi_N=O(e^{-\beta N^{2(1-\theta)}(1+o(1))}),$$

thus we conclude the lemma. □

Now let us turn to the proof of Theorem 1.

Proof of Theorem 1. Let $f(x)\in C_c(\mathbb{R})$. Since f is bounded, using Lemma 6, for fixed $0<\theta<0.5$ we have

$$\int_{\mathbb{R}}f(x)\sqrt{2N\beta}\rho_{\beta HE_N}(\sqrt{2N\beta}x)dx$$

$$=\int_{\mathbb{R}}f(x)\sqrt{2N\beta}(\frac{1}{\sqrt{\beta}}+O(\frac{1}{N}))\rho_{\beta FTE_N}(\sqrt{2N}x(1+O(N^{-\theta})))dx+O(e^{-\beta N^{2(1-\theta)}(1+o(1))})$$

$$=(1+O(N^{-\theta}))\int_{\mathbb{R}}f(y(1+O(N^{-\theta})))\sqrt{2N}\rho_{\beta FTE_N}(\sqrt{2N}\,y)dy+O(e^{-\beta N^{2(1-\theta)}(1+o(1))}).$$

The function $f(x)\in C_c(\mathbb{R})$ means that for any $\epsilon>0$, there exists some $\delta(\epsilon)>0$ such that $|f(x)-f(y)|<\epsilon$ if $|x-y|<\delta$. Hence, there exists N_1 such that for any $y\in supp(f)$, $|y(1+O(N^{-\theta}))-y|<\delta$ for $N>N_1$, then $|f(y(1+O(N^{-\theta}))-1)-f(y)|<\epsilon$. This shows that

$$\Big|(1+O(N^{-\theta}))\int_{\mathbb{R}}[f(y(1+O(N^{-\theta})))-f(y)]\sqrt{2N}\rho_{\beta FTE_N}(\sqrt{2N}y)dy\Big|$$
$$\leq 2\epsilon\int_{\mathbb{R}}\sqrt{2N}\rho_{\beta FTE_N}(\sqrt{2N}y)dy=2\epsilon.$$

Therefore,

$$\int_{\mathbb{R}}f(x)\sqrt{2N\beta}\rho_{\beta HE_N}(\sqrt{2N\beta}x)dx$$
$$=(1+O(N^{-\theta}))\int_{\mathbb{R}}f(y)\sqrt{2N}\rho_{\beta FTE_N}(\sqrt{2N}\,y)d\,y+O(e^{-\beta N^{2(1-\theta)}(1+o(1))})+2\epsilon.$$

Since by (1.5)

$$\lim_{N\to\infty}\sqrt{2\beta N}\rho_{\beta HE_N}(\sqrt{2\beta N}x)=\rho_{\mathrm{W}}(x),$$

this completes this proof. □

4. Proof of Theorem 2

The classic result [9] claims that the order of scaling at the spectrum edge is $O(N^{-2/3})$. It follows from Lemma 6 that if $\alpha_N=N^{-\theta}$, $2/3<\theta<1$, then

$$(1+\frac{x}{2N^{2/3}})(1+O(\alpha_N))=1+\frac{x}{2N^{2/3}}+O(N^{-\theta}).$$

The term $O(N^{-\theta})$, comparing with $O(N^{-2/3})$, is a small perturbation. Therefore the edge scaling limit of fixed trace β-HE could be expected. But Lemma 6 is established for $0<\theta<0.5$. The main difficulty results from Proposition 5. In order to avoid it, the edge scaling limit will be proved in the weak sense. Note that the asymptotic results (3.17) and (3.18) will be frequently used for any $2/3<\theta<1$ in the subsequent proof. We now turn to the proof of Theorem 2.

Proof of Theorem 2. Following the similar arguments of Theorem 1, by the basic relation (3.1) between two ensembles, we have

$$\frac{\sqrt{\beta}N^{5/6}}{\sqrt{2}}\int_{\mathbb{R}}f(x)\rho_{\beta HE_N}\left(\sqrt{2N\beta}(1+\frac{x}{2N^{2/3}})\right)dx$$
$$=\frac{1}{C_{N\beta}}\frac{\sqrt{\beta}N^{5/6}}{\sqrt{2}}\int_{\mathbb{R}}f(x)\int_{\sqrt{2\beta N}(1+\frac{x}{2N^{2/3}})}^{\infty}e^{-r^2/2}r^{N_\beta-2}\rho_{\beta FTE,1}\Big(\frac{\sqrt{2N\beta}(1+\frac{x}{2N^{2/3}})}{r}\Big)drdx$$
$$=\frac{1}{C_{N\beta}}\frac{\sqrt{\beta}N^{5/6}}{\sqrt{2}}\int_{\mathbb{R}}f(x)\Big(\int_{\sqrt{2\beta N}(1+\frac{x}{2N^{2/3}})}^{\sqrt{\beta}N(1/\sqrt{2}-\alpha_N)}+\int_{\sqrt{\beta}N(1/\sqrt{2}-\alpha_N)}^{\sqrt{\beta}N(1/\sqrt{2}+\alpha_N)}+\int_{\sqrt{\beta}N(1/\sqrt{2}+\alpha_N)}^{\infty}\Big)$$
$$\times e^{-r^2/2}r^{N_\beta-2}\rho_{\beta FTE,1}\Big(\frac{\sqrt{2N\beta}(1+\frac{x}{2N^{2/3}})}{r}\Big)drdx$$
$$=I_1+I_2+I_3. \tag{4.1}$$

The first step is to estimate I_1. Making the change of variables

$$x=2N^{2/3}(r(1+\frac{y}{2N^{2/3}})-1) \tag{4.2}$$

and assuming $|f(x)|\leq M$, I_1 can be dominated by

$$I_1=\frac{1}{C_{N\beta}}\frac{\sqrt{\beta}N^{5/6}}{\sqrt{2}}\int_{\mathbb{R}}\int_0^{\sqrt{\beta}N(1/\sqrt{2}-\alpha_N)}1_{\sqrt{2\beta N}r(1+y/2N^{2/3})\leq r\leq\sqrt{\beta}N(1/\sqrt{2}-\alpha_N)}(r)$$
$$\times f(2N^{2/3}(r(1+\frac{y}{2N^{2/3}})-1))e^{-r^2/2}r^{N_\beta-1}\rho_{\beta FTE,1}\Big(\sqrt{2N\beta}(1+\frac{y}{2N^{2/3}})\Big)dr\,dy$$

$$
\begin{aligned}
&\le M\frac{1}{C_{N\beta}}\frac{\sqrt{\beta}N^{5/6}}{\sqrt{2}}\int_{\mathbb{R}}\int_0^{\sqrt{\beta}N(1/\sqrt{2}-\alpha_N)} e^{-r^2/2}r^{N_\beta-1}\rho_{\beta FTE,1}\Big(\sqrt{2N\beta}(1+\frac{y}{2N^{2/3}})\Big)drdy\\
&= M\frac{1}{C_{N\beta}}\int_0^{\sqrt{\beta}N(1/\sqrt{2}-\alpha_N)} e^{-r^2/2}r^{N_\beta-1}dr\frac{\sqrt{\beta}N^{5/6}}{\sqrt{2}}\int_{\mathbb{R}}\rho_{\beta FTE,1}\Big(\sqrt{2N\beta}(1+\frac{y}{2N^{2/3}})\Big)dy\\
&= M\,O(e^{-\beta N^{2(1-\theta)}(1+o(1))})\frac{\sqrt{\beta}N^{5/6}}{\sqrt{2}}\frac{\sqrt{2}N^{1/6}}{\sqrt{\beta}}\int_{\mathbb{R}}\rho_{\beta FTE,1}(t)dt\\
&= O(N\,e^{-\beta N^{2(1-\theta)}(1+o(1))}). \qquad (4.3)
\end{aligned}
$$

Here we apply (3.17) and $\int_{\mathbb{R}}\rho_{\beta FTE,1}(y)dy = 1$ to obtain the above result. Similarly, making the same variable substitution as I_1, I_3 can be dominated by

$$
\begin{aligned}
I_3 = \frac{1}{C_{N\beta}}\frac{\sqrt{\beta}N^{5/6}}{\sqrt{2}}\int_{\mathbb{R}}\int_{\sqrt{\beta}N(1/\sqrt{2}+\alpha_N)}^{\infty} & f(2N^{2/3}(r(1+\frac{y}{2N^{2/3}})-1))\\
&\times e^{-r^2/2}r^{N_\beta-1}\rho_{\beta FTE,1}\Big(\sqrt{2N\beta}(1+\frac{y}{2N^{2/3}})\Big)dr\,dx
\end{aligned}
$$

$$
\begin{aligned}
&\le M\frac{1}{C_{N\beta}}\frac{\sqrt{\beta}N^{5/6}}{\sqrt{2}}\int_{\mathbb{R}}\int_{\sqrt{\beta}N(1/\sqrt{2}+\alpha_N)}^{\infty} e^{-r^2/2}r^{N_\beta-1}\rho_{\beta FTE,1}\Big(\sqrt{2N\beta}(1+\frac{y}{2N^{2/3}})\Big)drdx\\
&= M\frac{1}{C_{N\beta}}\int_{\sqrt{\beta}N(1/\sqrt{2}+\alpha_N)}^{\infty} e^{-r^2/2}r^{N_\beta-1}dr\frac{\sqrt{\beta}N^{5/6}}{\sqrt{2}}\int_{\mathbb{R}}\rho_{\beta FTE,1}\Big(\sqrt{2N\beta}(1+\frac{y}{2N^{2/3}})\Big)dy\\
&= O(N\,e^{-\beta N^{2(1-\theta)}(1+o(1))}). \qquad (4.4)
\end{aligned}
$$

The key step is to deal with I_2. Applying (3.4) to I_2, we find

$$
I_2 = \frac{1}{C_{N\beta}\,a_{N\beta}}\frac{N^{5/6}}{\sqrt{2}}\int_{\mathbb{R}}\int_{\sqrt{\beta}N(\frac{1}{\sqrt{2}}-\alpha_N)}^{\sqrt{\beta}N(\frac{1}{\sqrt{2}}+\alpha_N)} f(x)e^{-r^2/2}r^{N_\beta-2}\rho_{\beta FTE_N}\Big(\frac{\sqrt{2N}(1+\frac{x}{2N^{2/3}})}{a_{N\beta}r}\Big)drdx \qquad (4.5)
$$

where

$$
a_{N\beta} = \sqrt{\frac{2}{\beta N(N-1)}}\,. \qquad (4.6)
$$

Making the change of variables $x = 2N^{2/3}(a_{N\beta}r(1+y/(2N^{2/3}))-1)$,

$$
\begin{aligned}
I_2 = \frac{1}{C_{N\beta}}\frac{N^{5/6}}{\sqrt{2}}\int_{\mathbb{R}}\int_{\sqrt{\beta}N(\frac{1}{\sqrt{2}}-\alpha_N)}^{\sqrt{\beta}N(\frac{1}{\sqrt{2}}+\alpha_N)} & f(2N^{2/3}(a_{N\beta}r(1+\frac{y}{2N^{2/3}})-1))\\
& e^{-r^2/2}r^{N_\beta-1}\rho_{\beta FTE_N}\Big(\sqrt{2N}(1+\frac{y}{2N^{2/3}})\Big)dr\,dy. \qquad (4.7)
\end{aligned}
$$

Basing on (3.21), it is not difficult to see that for large N,

$$
\begin{aligned}
&\frac{1}{C_{N\beta}}\frac{N^{5/6}}{\sqrt{2}}\int_{\mathbb{R}}\int_{\sqrt{\beta}N(\frac{1}{\sqrt{2}}-\alpha_N)}^{\sqrt{\beta}N(\frac{1}{\sqrt{2}}+\alpha_N)} f(y)e^{-r^2/2}r^{N_\beta-1}\rho_{\beta FTE_N}\Big(\sqrt{2N}(1+\frac{y}{2N^{2/3}})\Big)drdy \qquad (4.8)\\
&= \frac{1}{C_{N\beta}}\int_{\sqrt{\beta}N(\frac{1}{\sqrt{2}}-\alpha_N)}^{\sqrt{\beta}N(\frac{1}{\sqrt{2}}+\alpha_N)} e^{-r^2/2}r^{N_\beta-1}dr\int_{\mathbb{R}}f(y)\frac{N^{5/6}}{\sqrt{2}}\rho_{\beta FTE_N}\Big(\sqrt{2N}(1+\frac{y}{2N^{2/3}})\Big)\,dy\\
&= (1+O(\frac{1}{N}))\int_{\mathbb{R}}f(y)\frac{N^{5/6}}{\sqrt{2}}\rho_{\beta FTE_N}\Big(\sqrt{2N}(1+\frac{y}{2N^{2/3}})\Big)\,dy
\end{aligned}
$$

Next, we will prove that the limit of the difference between I_2 and (4.8) is zero as N goes to infinity. Note that $f(x)\in C_c(\mathbb{R})$. For any $\epsilon > 0$, there exists some $\delta(\epsilon) > 0$ such that if $|x-y| < \delta$, then

$$
|f(x)-f(y)| < \epsilon.
$$

Since

$$\sqrt{\beta}N(\frac{1}{\sqrt{2}}-\alpha_N)\leq r\leq\sqrt{\beta}N(\frac{1}{\sqrt{2}}+\alpha_N),$$

one has $a_{N\beta}r=1+O(\alpha_N)$.

Hence, for any $y\in supp(f)$, there exists N_1 not depending on y such that

$$|2N^{2/3}(a_{N\beta}r(1+\frac{y}{2N^{2/3}})-1)-y|=|O(N^{\frac{2}{3}-\theta})-y\,O(N^{-\theta})|<\delta$$

for $N>N_1$. Thus,

$$|f\big(2N^{2/3}(a_{N\beta}r(1+\frac{y}{2N^{2/3}})-1)\big)-f(y)|<\epsilon.$$

Pick an interval $[-K,K]$ such that

$$supp(f)\subset[-K,K],\ \{2N^{2/3}(a_{N\beta}r(1+\frac{y}{2N^{2/3}})-1)|\,y\in supp(f)\}\subset[-K,K].$$

It follows from Lemma 7 below that the difference between (4.7) and (4.8) can be dominated by

$$\begin{aligned}|I_2-(4.8)|&\leq\frac{\epsilon}{C_{N\beta}}\frac{N^{5/6}}{\sqrt{2}}\int_{-K}^{K}\int_{\sqrt{\beta}N(\frac{1}{\sqrt{2}}-\alpha_N)}^{\sqrt{\beta}N(\frac{1}{\sqrt{2}}+\alpha_N)}e^{-r^2/2}r^{N_\beta-1}\rho_{\beta FTE_N}\Big(\sqrt{2N}(1+\frac{y}{2N^{2/3}})\Big)drdy\\&=\epsilon\frac{N^{5/6}}{\sqrt{2}}\int_{-K}^{K}\rho_{\beta FTE_N}\left(\sqrt{2N}(1+\frac{y}{2N^{2/3}})\right)dy\,\frac{1}{C_{N\beta}}\int_{\sqrt{\beta}N(\frac{1}{\sqrt{2}}-\alpha_N)}^{\sqrt{\beta}N(\frac{1}{\sqrt{2}}+\alpha_N)}e^{-r^2/2}r^{N_\beta-1}dr\\&\leq\epsilon(1+O(\frac{1}{N}))\frac{N^{5/6}}{\sqrt{2}}\int_{-K}^{K}\rho_{\beta FTE_N}\left(\sqrt{2N}(1+\frac{y}{2N^{2/3}})\right)dy\qquad(4.9)\\&\leq 2\epsilon M_K\end{aligned}$$

where the constant M_K only depends on K. It is worth mentioning that we have applied the followng Lemma 7 to (4.9). With the requirement of the continuity of our argument, Lemma 7 will be proved after completing this proof. Now we have proved that

$$\lim_{N\to\infty}|I_2-(4.8)|=0.$$

Combining (4.1), (4.3), (4.4), (4.7) and (4.8),

$$\begin{aligned}&\int_{\mathbb{R}}f(x)\frac{\sqrt{\beta}N^{5/6}}{\sqrt{2}}\rho_{\beta HE_N}\left(\sqrt{2N\beta}(1+\frac{x}{2N^{2/3}})\right)dx\\&\qquad=(1+O(\frac{1}{N}))\int_{\mathbb{R}}f(y)\frac{N^{5/6}}{\sqrt{2}}\rho_{\beta FTE_N}\left(\sqrt{2N}(1+\frac{y}{2N^{2/3}})\right)dy+o(1).\end{aligned}$$

It immediately follows from the assumption of Theorem 2 that

$$\begin{aligned}&\lim_{N\to\infty}\int_{\mathbb{R}}f(x)\frac{N^{5/6}}{\sqrt{2}}\rho_{\beta FTE_N}\left(\sqrt{2N}(1+\frac{x}{2N^{2/3}})\right)dx\\&\qquad=\lim_{N\to\infty}\int_{\mathbb{R}}f(x)\frac{\sqrt{\beta}N^{5/6}}{\sqrt{2}}\rho_{\beta HE_N}\left(\sqrt{2N\beta}(1+\frac{x}{2N^{2/3}})\right)dx.\end{aligned}$$

This completes the proof. □

The following lemma will be proved by using a similar argument from Lemma 4 in [12].

Lemma 7. *If* $\forall\, h(x)\in C_c(\mathbb{R})$,

$$\lim_{N\to\infty}\int_{\mathbb{R}}h(x)\frac{\sqrt{\beta}N^{5/6}}{\sqrt{2}}\rho_{\beta HE_N}\left(\sqrt{2N\beta}(1+\frac{x}{2N^{2/3}})\right)dx\qquad(4.10)$$

exists, then for any fixed R,

$$\frac{N^{5/6}}{\sqrt{2}}\int_{-R}^{R}\rho_{\beta FTE_N}\left(\sqrt{2N}(1+\frac{y}{2N^{2/3}})\right)dy\leq M_R.\qquad(4.11)$$

where M_R *is a constant only depending on* R.

Proof. If $\sqrt{\beta}N(\frac{1}{\sqrt{2}} - \alpha_N) \le u \le \sqrt{\beta}N(\frac{1}{\sqrt{2}} + \alpha_N)$, then there exists N_0 such that for $N > N_0$,

$$|2N^{2/3}(a_{N\beta}u(1+\frac{y}{2N^{2/3}}) - 1)| \le R + 1 \tag{4.12}$$

where $a_{N\beta}$ is given by (4.6). Let $\eta \in (0,1)$ be a real number and let $\phi(t)$ be a smooth decreasing function on $[0, R+1)$ such that $\phi(t) = 1$ for $t \in [0, R+1)$ and $\phi(t) = 0$ for $t \ge (1+\eta)(R+1)$. Therefore, we have

$$\begin{aligned} &\frac{N^{5/6}}{\sqrt{2}} \int_{-R}^{R} \rho_{\beta FTE_N}\left(\sqrt{2N}(1+\frac{y}{2N^{2/3}})\right) dy \\ &\le \frac{N^{5/6}}{\sqrt{2}} \int_{\mathbb{R}} \phi(2N^{2/3}(a_{N\beta}u(1+\frac{y}{2N^{2/3}}) - 1))\rho_{\beta FTE_N}\left(\sqrt{2N}(1+\frac{y}{2N^{2/3}})\right) dy. \end{aligned} \tag{4.13}$$

Multiplying both sides by

$$\frac{1}{C_{N\beta}} e^{-u^2/2} u^{N_\beta - 1}$$

then integrating about u on

$$[\sqrt{\beta}N(\frac{1}{\sqrt{2}} - \alpha_N), \sqrt{\beta}N(\frac{1}{\sqrt{2}} + \alpha_N)],$$

one obtains

$$\frac{1}{C_{N\beta}} \int_{\sqrt{\beta}N(\frac{1}{\sqrt{2}}-\alpha_N)}^{\sqrt{\beta}N(\frac{1}{\sqrt{2}}+\alpha_N)} e^{-u^2/2} u^{N_\beta-1} du \, \frac{N^{5/6}}{\sqrt{2}} \int_{-R}^{R} \rho_{\beta FTE_N}\left(\sqrt{2N}(1+\frac{y}{2N^{2/3}})\right)$$

$$\begin{aligned} \le \frac{1}{C_{N\beta}} \frac{N^{5/6}}{\sqrt{2}} \int_{\mathbb{R}} \int_{\sqrt{\beta}N(\frac{1}{\sqrt{2}}-\alpha_N)}^{\sqrt{\beta}N(\frac{1}{\sqrt{2}}+\alpha_N)} &\phi(2N^{2/3}(a_{N\beta}u(1+\frac{y}{2N^{2/3}}) - 1)) \\ &\times e^{-u^2/2} u^{N_\beta-1} \rho_{\beta FTE_N}\left(\sqrt{2N}(1+\frac{y}{2N^{2/3}})\right) du\, dy. \end{aligned}$$

On the one hand, combining (3.21), it is easy to observe that the left hand side of the above inequality equals

$$(1 + O(\frac{1}{N}))\frac{N^{5/6}}{\sqrt{2}} \int_{-R}^{R} \rho_{\beta FTE_N}\left(\sqrt{2N}(1+\frac{y}{2N^{2/3}})\right) d\, y. \tag{4.14}$$

On the other hand, it follows from (4.1), (4.3), (4.4) and (4.7) that the right hand side of the above inequality equals

$$\int_{\mathbb{R}} \phi(y) \frac{\sqrt{\beta}N^{5/6}}{\sqrt{2}} \rho_{\beta HE_N}\left(\sqrt{2N\beta}(1+\frac{y}{2N^{2/3}})\right) dy + O(N\, e^{-\beta N^{2(1-\theta)}(1+o(1))}). \tag{4.15}$$

The existence of the limit of (4.10) proves this lemma. □

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Appendix: equivalence of moments

This appendix presents their moment equivalence between fixed trace β-Hermite ensembles and β-Hermite ensembles in the large N.

Recall that Dumitriu and Edelman's β-Hermite tri-diagonal matrix models

$$H_\beta = \begin{bmatrix} a_N & b_{N-1} & & \\ b_{N-1} & a_{N-1} & \ddots & \\ & \ddots & \ddots & b_1 \\ & & b_1 & a_1 \end{bmatrix}$$

where $a_j \sim N(0,1),\ j=1,\cdots,N$ and $\sqrt{2}b_j \sim \chi_{j\beta},\ j=1,\cdots,N-1$. Then proceeding from the analogy of a fixed energy in classical statistical mechanics, one can define a 'fixed trace' ensemble by the requirement that the trace of H_β^2 be fixed to a number r^2 with no other constraint. The number r is called the strength of the ensemble. The joint probability density function for the matrix elements of H_β is therefore given by

$$P_r(H_\beta) = K_r^{-1}\,\delta\big(\frac{1}{r^2}\mathrm{tr}\,H_\beta^2 - 1\big)\prod_{j=1}^{N-1} b_j^{j\beta-1}$$

with

$$K_r = \int\cdots\int \delta\big(\frac{1}{r^2}\mathrm{tr}\,H_\beta^2 - 1\big)\prod_{j=1}^{N-1} b_j^{j\beta-1} d\,a\,d\,b$$

where $d\,a = \prod_{j=1}^N d\,a_j$ and $d\,b = \prod_{j=1}^{N-1} d\,b_j$.

Note that for the β-Hermite ensemble the joint probability function

$$P(H_\beta) = K^{-1}\exp\big(-\frac{1}{2}\mathrm{tr}\,H_\beta^2\big)\prod_{j=1}^{N-1} b_j^{j\beta-1}$$

with

$$K = \int\cdots\int \exp\big(-\frac{1}{2}\mathrm{tr}\,H_\beta^2\big)\prod_{j=1}^{N-1} b_j^{j\beta-1} d\,a\,d\,b.$$

When we choose the number r^2 as the average of H_β^2, i.e.,

$$\langle \mathrm{tr}\,H_\beta^2\rangle = K^{-1}\int\cdots\int \mathrm{tr}\,H_\beta^2 \exp\big(-\frac{1}{2}\mathrm{tr}\,H_\beta^2\big)\prod_{j=1}^{N-1} b_j^{j\beta-1} d\,a\,d\,b = r^2,$$

then for any fixed value of the sum

$$s = \sum_{j=1}^{N}\eta_j^{(a)} + \sum_{j=1}^{N-1}\eta_j^{(b)}, \quad \eta_j^{(a)},\ \eta_j^{(b)} \ge 0,$$

the ratio the moments

$$M_r(N,\eta) = \Big\langle \prod_{j=1}^{N}(a_j)^{\eta_j^{(a)}}\prod_{j=1}^{N-1}(b_j)^{\eta_j^{(b)}}\Big\rangle_r$$

and

$$M(N,\eta) = \Big\langle \prod_{j=1}^{N}(a_j)^{\eta_j^{(a)}}\prod_{j=1}^{N-1}(b_j)^{\eta_j^{(b)}}\Big\rangle$$

tends to unity as the number of dimensions N tends to infinity. The subscript r and non-subscript denote that the average is taken in the fixed trace and β-Hermite ensembles, respectively.

First, let's calculate $r^2 = \langle \mathrm{tr}\,H_\beta^2\rangle$ with a basic manipulation in statistical mechanics. Write

$$g_\beta(\lambda) = c_H^\beta\,|\Delta(\lambda)|^\beta \exp\big(-t\sum_{j=1}^N \lambda_j^2\big)$$

where

$$c_H^\beta = (2t)^{N/2+\beta N(N-1)/4}\,(2\pi)^{-N/2}\prod_{j=1}^{N}\frac{\Gamma(1+\beta/2)}{\Gamma(1+j\beta/2)}.$$

Note that $g_\beta(\lambda)$ with $t=1/2$ corresponds to the joint probability distribution of eigenvalues for β-Hermite ensembles [7]. A partial differentiation with respect to t and setting $t=1/2$ gives

$$\langle \mathrm{tr}\,H_\beta^2\rangle = 2(N/2 + \beta N(N-1)/4).$$

Next, to calculate $M_r(N,\eta)$, substitute $(2\xi)^{-1/2}\,r\,a_j$ for a_j and $(2\xi)^{-1/2}\,r\,b_j$ for b_j where ξ is a parameter. This gives

$$M_r(N,\eta)\big(\frac{2\xi}{r^2}\big)^{N/2+\beta N(N-1)/4+s/2} = K_r^{-1}\int\cdots\int \delta\big(\frac{1}{2\xi}\mathrm{tr}\,H_\beta^2-1\big)\prod_{j=1}^{N}(a_j)^{\eta_j^{(a)}}\prod_{j=1}^{N-1}(b_j)^{\eta_j^{(b)}}\prod_{j=1}^{N-1}b_j^{j\beta-1}d\,a\,d\,b.$$

Multiplying both sides by $e^{-\xi}$ and integrating on ξ from 0 to ∞, we get

$$M_r(N,\eta)\Gamma(L+s/2+1)\,L^{-L-s/2} = K_r^{-1}\int\cdots\int \exp\big(-\frac{1}{2}\mathrm{tr}\,H_\beta^2\big)\prod_{j=1}^{N}(a_j)^{\eta_j^{(a)}}\prod_{j=1}^{N-1}(b_j)^{\eta_j^{(b)}}\prod_{j=1}^{N-1}b_j^{j\beta-1}d\,a\,d\,b$$

where we have put

$$L=\frac{1}{2}r^2=N/2+\beta N(N-1)/4.$$

or

$$M_r(N,\eta)=\frac{L^{L+s/2}}{\Gamma(L+s/2+1)}\frac{K}{K_r}M(N,\eta).$$

Setting $\eta_j^{(a)}=\eta_j^{(b)}=0$ in the above and using the normalization condition $M_r(N,0)=M(N,0)=1$, we get the ratio of the constants K and K_r. Substituting this ratio we then obtain

$$M_r(N,\eta)=\frac{L^{s/2}\Gamma(L+1)}{\Gamma(L+s/2+1)}M(N,\eta).$$

As $N\to\infty, L\to\infty$, and we can use Stirling's formula for the gamma function for the large x

$$\Gamma(x+1)=x^{-x}\,e^{-x}\sqrt{2\pi x}\,[1+O(1/x)],$$

to prove the asymptotic equality of all the finite moments $s\ll N$.

In sum, the result of moment equivalence can be stated as follows:

Theorem 8. *With the above notation $M_r(N,\eta)$and $M(N,\eta)$, we have*

$$\lim_{N\to\infty}\frac{M_r(N,\eta)}{M(N,\eta)}=1.$$

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