LP POLYHARMONIC DIRICHLET PROBLEMS IN REGULAR DOMAINS II: THE UPPER HALF PLANE

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Abstract. In this article, we consider a class of Dirichlet problems with L^p boundary data for polyharmonic functions in the upper half plane. By introducing a sequence of new kernel functions called higher order Schwarz kernels, integral representation solutions of the problems are given.

1. Introduction

The best and most desirable way of solving a BVP (boundary value problem) for a partial differential equation is to obtain an explicit solution formula in terms of the given boundary data. When this is impossible or too hard to do, one turns to deal with the existence and estimates of the solutions. In recent years, the study of explicit solutions of BVPs has undergone a new phase of development ([1–8, 10, 11, 14–16, 20] and references therein). These include Dirichlet, Neumann, Schwarz and Robin problems for harmonic, biharmonic, polyharmonic and polyanalytic equations in regular domains (in the unit disc: [1, 2, 4, 7, 8, 11]; and in the upper-half plane: [3,5,6,10]) and in irregular domains (Lipschitz domains: [5,16,20]).

The purpose of this article is to solve the following polyharmonic Dirichlet problem (PHDP) for L^p data in the upper-half plane, $\mathbf{H}$, i.e.

$$\begin{cases} \Delta^n u = 0 \text{ in } \mathbf{H} \\ \Delta^j u = f_j \text{ on } \mathbb{R}, \end{cases} \tag{1.1}$$

where $\mathbb{R}$ is the real axis, $f_j \in L^p(\mathbb{R})$, $n \in \mathbb{N}$, $0 \leq j < n$, and $p \geq 1$. By introducing a series of new kernel functions, we will give potential solutions of the PHDPs (1.1). The kernel functions are higher order analogs of the classical Schwarz kernel for the upper half plane (see next section). To the authors' knowledge, this work is the first to give integral representations of the solutions of the BVPs for polyharmonic equations in the L^p setting. There have been studies on such BVPs ([1–8, 10, 14–16, 20] and references therein), those, however, do not present a complete and coherent integral representation theory except for some special cases ([1–4, 6–8, 10] and references therein).

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2. Sequence of Consecutive Schwarz kernels

Definition 2.1 ([7])**.** A sequence of real-valued functions of two variables $\{G_n(\cdot,\cdot)\}_{n=1}^{\infty}$ defined on $\mathbf{H}\times\mathbb{R}$ is called a sequence of *consecutive Schwarz kernels*, and, precisely, $G_n(\cdot,\cdot)$ is the *nth order Schwarz kernel*, if they satisfy the following conditions.
1. For all $n\in\mathbb{N}$, $G_n(\cdot,\cdot)\in C(\mathbf{H}\times\mathbb{R})$; $G_n(\cdot,t)\in C^{2n}(\mathbf{H})$ with any fixed $t\in\mathbb{R}$; and $G_n(z,\cdot)\in L^p(\mathbb{R}), p>1$, with any fixed $z\in\mathbf{H}$, and the non-tangential boundary value

$$\lim_{\substack{z\to s\\ z\in\mathbf{H}, s\in\mathbb{R}}} G_n(z,t)=G_n(s,t)$$

exists for all t and $s\neq t$;
2. $G_1(i,t)=\frac{1}{t^2+1}$ and $G_n(i,t)=0, n\geq 2$ and $t\in\mathbb{R}$, and for any $n\in\mathbb{N}$,

$$|G_n(z,t)|\leq \frac{M}{|t-z'|}$$

uniformly on $D_c\times\{t\in\mathbb{R}:|t|>T\}$ whenever $z'\in D_c$, where D_c is any compact set in $\overline{\mathbf{H}}$, M, T are positive constants depending only on D_c and n;
3. $(\partial_z\partial_{\overline{z}})G_1(z,t)=0$ and $(\partial_z\partial_{\overline{z}})G_n(z,t)=G_{n-1}(z,t)$ for $n>1$;
4. $\lim_{z\to s,\, z\in\mathbf{H}} \frac{1}{\pi}\int_{-\infty}^{+\infty} G_1(z,t)\gamma(t)\mathrm{d}t=\gamma(s)$, a.e., for any $\gamma\in L^p(\mathbb{R})$, $p\geq 1$;
5. $\lim_{z\to s,\, z\in\mathbf{H}} \frac{1}{\pi}\int_{-\infty}^{+\infty} G_n(z,t)\gamma(t)\mathrm{d}\tau=0$ for any $\gamma\in L^p(\mathbb{R})$, $p\geq 1$, $n\geq 2$,

where all limits are non-tangential [19].

Definition 2.2. Let D be a simply connected (bounded or unbounded) domain in the plane with smooth boundary ∂D, and $H(D)$ denote the set of all analytic functions in D. If f is a continuous function defined on $D\times\partial D$ satisfying $f(\cdot,t)\in H(D)$ for any fixed $t\in\partial D$, and $f(z,\cdot)\in L^p(\partial D), p\geq 1$, (or $C_0(\partial D)$ if ∂D is a boundless curve) for any fixed $z\in D$, then f is $H\times L^p$ $(H\times C_0)$ on $D\times\partial D$ and write $f\in(H\times L^p)(D\times\partial D)$ (or $(H\times C_0)(D\times\partial D)$). Likewise, $(H\times C)(D\times\partial D)$ may be similarly defined.

Lemma 2.3. *Let D be a simply connected unbounded domain in the plane with smooth boundary ∂D. If f is defined on $D\times\partial D$ that is analytic in D for any fixed $t\in\partial D$ and*

$$|f(z,t)|\leq M\frac{1}{|t-z'|} \tag{2.1}$$

uniformly on $D_c\times\{t\in\partial D:|t|>T\}$ whenever $z'\in D_c$, where D_c is any compact set in D, and M, T are positive constants depending only on D_c, then for any fixed $z_0\in D$, the primitive function

$$F(z,t)=\int_{z_0}^{z} f(\zeta,t)d\zeta,\ z\in D,\ t\in\partial D, \tag{2.2}$$

enjoys the same properties as f.

Proof. It is trivial to show the analyticity of $F(z,t)$ in $z\in D$ for each fixed $t\in\partial D$ ([18]). Then the inequality part of the Lemma follows from

$$|F(z,t)|\leq\int_{\gamma_{[z_0,z]}}|f(\zeta,t)||d\zeta| \tag{2.3}$$

since $\gamma_{[z_0,z]}\cap D_c$ is compact for any simple curve $\gamma_{[z_0,z]}$ in D that connects $z_0\in D$ with $z\in D_c$. □

Lemma 2.4. *Let D be a simply connected unbounded domain in the plane with smooth boundary ∂D. If $f \in (H \times L^p)(D \times \partial D)$ and*

$$|f(z,t)| \leq M \frac{1}{|t-z'|}$$

uniformly on $D_c \times \{t \in \partial D : |t| > T\}$ whenever $z' \in D_c$, where D_c is any compact set in D, and M, T are positive constants depending only on D_c, then for any fixed $z_0 \in D$, the primitive function

$$F(z,t) = \int_{z_0}^{z} f(\zeta,t)d\zeta \tag{2.4}$$

is also in $(H \times L^p)(D \times \partial D)$.

Proof. Since $f \in (H \times L^p)(D \times \partial D)$, from Lemma (2.3), $F(z,t)$ is analytic in $z \in D$ with any fixed $t \in \partial D$. By Minkowski's inequality for integrals [12, 17],

$$\begin{aligned} ||F(z,\cdot)||_p &= \left(\int_{\partial D} |F(z,t)|^p |dt|\right)^{1/p} \\ &\leq \int_{\gamma_{[z_0,z]}} \left| \int_{(\partial D)_T \cup \mathcal{C}(\partial D)_T} |f(\zeta,t)|^p \,|dt| \right|^{1/p} |d\zeta| \\ &\leq \int_{\gamma_{[z_0,z]}} \left| \int_{(\partial D)_T} |f(\zeta,t)|^p \,|dt| \right|^{1/p} |d\zeta| \\ &\quad + \int_{\gamma_{[z_0,z]}} \left| \int_{\mathcal{C}(\partial D)_T} \frac{M}{|t-z_0|^p} |dt| \right|^{1/p} |d\zeta| \\ &< +\infty \end{aligned} \tag{2.5}$$

where $\gamma_{[z_0,z]}$ is any simple curve from z_0 to z in D which is compact in D, $(\partial D)_T = \{t \in \partial D : |t| \leq T\}$ and $\mathcal{C}(\partial D)_T = \{t \in \partial D : |t| > T\}$, T and M are positive constants depending only on $\gamma_{[z_0,z]}$ by the condition (2.1) for f. □

Lemma 2.5. *Let D be a simply connected unbounded domain in the plane with smooth boundless boundary ∂D. If $f \in (H \times L^p)(D \times \partial D)$ and the nontangential boundary value*

$$\lim_{\substack{z \to s \\ z \in D, s \in \partial D}} f(z,t) = f(s,t) \tag{2.6}$$

exists on ∂D except $t \in \partial D$, and $f(s,\cdot) \in L^p(\partial D)$ for any fixed $s \in \partial D$. For any fixed $t \in \partial D$, $f(\cdot,t)$ can be continuously extended to $\overline{D} \setminus B(t,\delta)$, where $B(t,s)$ denotes the open disc of center t and radius δ for every $\delta > 0$. Moreover,

$$\lim_{\substack{z \to s \\ z \in \overline{D}, s \in \partial D}} |f(z,s)| = +\infty \quad \text{and} \quad \lim_{\substack{z \to s \\ z \in \overline{D}, s \in \partial D}} |(z-s)f(z,s)| = 0 \tag{2.7}$$

for any $s \in \partial D$, and

$$|f(z,t)| \leq M \frac{1}{|t-z'|}$$

uniformly on $D_c \times \{t \in \partial D : |t| > T\}$ whenever $z' \in D_c$ which is any compact set in $\overline{D}$, where M, T are positive constants depending only on D_c. For any fixed $z_0 \in D$, *set the primitive function*

$$F(z,t)=\int_{z_0}^{z} f(\zeta,t)d\zeta,\ z\in D,\ t\in\partial D,$$

then F enjoys the same properties as f does.

Proof. By the assumption, $f(\cdot,t)$ can be continuously extended to $\overline{D}\setminus\{t\}$ for any fixed $t\in\partial D$. Therefor, for any $z_0\in D$ and fixed $t\in\partial D$, we can define

$$F(s,t)=\int_{z_0}^{s} f(\zeta,t)d\zeta \text{ whenever } s\neq t. \tag{2.8}$$

Let $n_s=(x_s,y_s)$ be the unit normal vector at s of ∂D and $N_s=x_s+iy_s$. For any $0<\alpha<\pi/2$, denote

$$\vee_\alpha(s)=\{z\in D: \arccos\left(\frac{\Re\{(z-s)\overline{N}_s\}}{|z-s|}\right)<\alpha\} \tag{2.9}$$

as a pseudo-cone with vertex s and opening angle α in D. Due to the fact that $f(z,t)$ is continuous on $\vee_\alpha(s)\cup\{s\}, s\neq t$, it is clear that

$$|F(z,t)-F(s,t)|\leq\int_{\gamma[z,s]}|f(\zeta,t)|d\zeta\leq l\{\gamma[z,s]\}\max_{\zeta\in\gamma[z_0,s]}|f(\zeta,t)|, \tag{2.10}$$

where $z\in\vee_\alpha(s)$ and $\gamma[z,s]\subset\gamma[z_0,s]\subset\overline{D}$ for any $z_0\in D$ and $0<\alpha<\pi/2$. Thus $F(z,t)$ have the nontangential boundary value $F(s,t)$ given by (2.8).

From (2.7), it is easy to obtain

$$\lim_{\substack{z\to s\\ z\in\overline{D},s\in\partial D}}|F(z,s)|=+\infty. \tag{2.11}$$

For any fixed $z_0\in D$, define

$$L(z,s)=|z-s|\max_{\zeta\in\gamma[z_0,z]}|f(\zeta,s)| \tag{2.12}$$

where $z\in\overline{D}\setminus\{s\}$. Since by (2.7)

$$\begin{aligned} L(z,s)=|z-s||f(z^*,s)| &= \left|\frac{z-s}{z^*-s}\right||(z^*-s)f(z^*-s)| \\ &\leq |z^*-s||f(z^*,s)| \end{aligned} \tag{2.13}$$

in which $z^*\in\gamma[z_0,z]$ as z is sufficiently close to s, note that in this case z^* is also sufficiently close to s. Therefore,

$$\lim_{\substack{z\to s\\ z\in\overline{D},s\in\partial D}}L(z,s)=0. \tag{2.14}$$

Note that

$$\begin{aligned} |(z-s)F(z,s)| &= \left|(z-s)\int_{z_0}^{z}f(\zeta,s)d\zeta\right| \\ &\leq |z-s|\int_{\gamma[z_0,z]}|f(\zeta,s)|d\zeta| \\ &\leq l\{\gamma[z_0,z]\}|z-s|\max_{\zeta\in\gamma[z_0,z]}|f(\zeta,s)| \\ &\leq l\{\gamma[z_0,s]\}L(z,s). \end{aligned} \tag{2.15}$$

It immediately follows that

$$\lim_{\substack{z\to s\\ z\in\overline{D}, s\in\partial D}} |(z-s)F(z,s)| = 0. \tag{2.16}$$

Finally, by the assumptions, take the same arguments as in Lemmas 2.3-2.4, it is easy to prove that $F(s,\cdot)\in L^p(\partial D)$ for any fixed $s\in\partial D$ and

$$|F(z,t)| \le M\frac{1}{|t-z'|}$$

uniformly on $D_c\times\{t\in\partial D : |t|>T\}$ whenever $z'\in D_c$ which is any compact set in $\overline{D}$, where M, T are positive constants depending only on D_c. □

Lemma 2.6. *Let D be a simply connected unbounded domain in the plane with smooth boundless boundary ∂D. If $f\in(H\times L^q)(D\times\partial D)$ for any $q>1$, the nontangential boundary value*

$$\lim_{\substack{z\to s\\ z\in\overline{D}, s\in\partial D}} f(z,t) = f(s,t)$$

exists on ∂D except fixed $t\in\partial D$, and $f(s,\cdot)\in L^q(\partial D)$ for any fixed $s\in\partial D$. For any fixed $t\in\partial D$, $f(\cdot,t)$ can be continuously extended to $\overline{D}\setminus B(t,\delta)$, where $B(t,s)$ denotes the open disc of center t and radius δ for every $\delta>0$. Moreover,

$$\lim_{\substack{z\to s\\ z\in\overline{D}, s\in\partial D}} |f(z,s)| = +\infty \quad \text{and} \quad \lim_{\substack{z\to s\\ z\in\overline{D}, s\in\partial D}} |(z-s)f(z,s)| = 0$$

for any $s\in\partial D$ and

$$|f(z,t)| \le M\frac{1}{|t-z'|}$$

uniformly on $D_c\times\{t\in\partial D : |t|>T\}$ whenever $z'\in D_c$ which is any compact set in $\overline{D}$, where M, T are positive constants depending only on D_c. Then for any $\gamma\in L^p(\mathbb{R})$ with $p\ge 1$,

$$\lim_{\substack{z\to s\\ z\in\overline{D}, s\in\partial D}} \int_{\partial D} (z-s)f(z,t)\gamma(t)dt = 0. \tag{2.17}$$

Proof. For any fixed $z\in D$ and $s\in\partial D$, let $q=\frac{p}{p-1}$ as $p>1$, by Hölder's inequality, $f(z,\cdot)\gamma(\cdot)$, $f(s,\cdot)\gamma(\cdot)\in L^1(\mathbb{R})$. By the assumption, we know that $f(z,t)$, $f(s,t)\in C_0(\mathbb{R})$. Therefore, all the same $f(z,\cdot)\gamma(\cdot)$, $f(s,\cdot)\gamma(\cdot)\in L^1(\mathbb{R})$ when $p=1$.

For any $s\in\partial D$ and $0<\alpha<\pi/2$, define

$$U_{s,\alpha}(z,t) = \begin{cases} (z-s)f(z,t), & t\ne s;\\ 0, & t=s \end{cases} \tag{2.18}$$

where $z\in\vee_\alpha(s)$ and $t\in\partial D$. Note that $f\in C(D\times\partial D)$ and $f(\cdot,s)\in C(\overline{D}\setminus\{s\})$ for any fixed $s\in\partial D$, then from (2.7), $U_{s,\alpha}\in C(\vee_\alpha(s)\times\partial D)$. Then there exists $\eta>0$ such that

$$|U_{s,\alpha}(z,t)| < 1 \tag{2.19}$$

for any $z\in D$ and $t\in\partial D$ satisfying $|z-s|\le\eta$ and $|t-s|\le\eta$ respectively. Fix such η, by the assumption, we have that

$$|f(z,t)| \le \frac{M}{|t-z'|} \tag{2.20}$$

uniformly on $\bigtriangledown_{\alpha,\eta}(s)\times\{t\in\partial D: |t-s|\geq T\}$, where $z'\in\bigtriangledown_{\alpha,\eta}(s)=\{z\in\vee_\alpha(s): |z-s|\leq\eta\}$, M, T are positive constants depending only on $\bigtriangledown_{\alpha,\eta}(s)$. Split

$$\begin{aligned}\int_{\partial D}(z-s)f(z,t)\gamma(t)dt &= \int_{|t-s|\leq\eta,\ t\in\partial D}U_{s,\alpha}(z,t)\gamma(t)dt\\ &\quad+(z-s)\int_{\eta\leq|t-s|\leq T,\ t\in\partial D}f(z,t)\gamma(t)dt\\ &\quad+(z-s)\int_{|t-s|>T,\ t\in\partial D}f(z,t)\gamma(t)dt\end{aligned}\tag{2.21}$$

in which $z\in\bigtriangledown_{\alpha,\eta}(s)$. Moreover, from (2.6) and (2.7),

$$\lim_{\substack{z\to s\\ z\in D,s\in\partial D}}(z-s)f(z,t)\gamma(t)=0,\ \ a.e.\ t\in\partial D.\tag{2.22}$$

Thus, by dominated convergence theorem, (2.17) follows from (2.18)-(2.22) and the fact $f\in C(\bigtriangledown_{\alpha,\eta}(s)\times\{t\in\partial D:\eta\leq|t-s|\leq T\})$ in the sense of nontangential limits since α is arbitrary. □

Higher order Schwarz kernels are the key in our programme to solve the PHD problems (1.1). By the decomposition theorem of polyharmonic functions [8] and all above lemmas, we have the following

Theorem 2.7. *If $\{G_n(z,t)\}_{n=1}^\infty$ is a sequence of higher order Schwarz kernels defined on $\mathbf{H}\times\mathbb{R}$, i.e., $\{G_n(z,t)\}_{n=1}^\infty$ fulfills the aforementioned properties 1-5 in Definition 2.1, then, for $n>1$, there exist functions $G_{n,0}(z,t),G_{n,1}(z,t),\dots,G_{n,n-1}(z,t)$ defined on $\mathbf{H}\times\mathbb{R}$ such that*

$$G_n(z,t)=2\Re\Big\{\sum_{j=0}^{n-1}(\overline{z}+i)^jG_{n,j}(z,t)\Big\},\ z\in\mathbf{H},\ t\in\mathbb{R}\tag{2.23}$$

with

$$\partial_zG_{n,j}(z,t)=j^{-1}G_{n-1,j-1}(z,t)\tag{2.24}$$

for $1\leq j\leq n-1$ and

$$\partial_z^kG_{n,j}(i,t)=0\tag{2.25}$$

for $0\leq k\leq j-1$ with respect to $t\in\mathbb{R}$ as well as

$$G_{n,0}(z,t)=-\sum_{j=1}^{n-1}(z+i)^jG_{n,j}(z,t).\tag{2.26}$$

However,

$$G_1(z,t)=\frac{1}{2i}\Big[\frac{1}{t-z}-\frac{1}{t-\overline{z}}\Big]\tag{2.27}$$

is the classical Schwarz kernel for the upper half plane. All of above $G_{n,j}\in(H\times L^p)(\mathbf{H}\times\mathbb{R})$, the nontangential boundary value

$$\lim_{\substack{z\to s\\ z\in\mathbf{H},s\in\mathbb{R}}}G_{n,j}(z,t)=G_{n,j}(s,t)\tag{2.28}$$

exists on $\mathbb{R}$ except fixed $t\in\mathbb{R}$ and $G_{n,j}(s,t)\in L^p(\mathbb{R})$ regarding to t for any fixed $s\in\mathbb{R}$ and

$$|G_{n,j}(z,t)|\leq M\frac{1}{|t-z'|}\tag{2.29}$$

uniformly on $D_c \times \{t \in \mathbb{R} : |t| > T\}$ *whenever* $z' \in D_c$ *which is any compact set in* $\overline{\mathbf{H}}$*, where* M, T *are positive constants depending only on* D_c*. Moreover,*

$$\lim_{\substack{z \to s \\ z \in \overline{\mathbf{H}}, s \in \mathbb{R}}} |G_{n,j}(z,s)| = +\infty \quad \text{and} \quad \lim_{\substack{z \to s \\ z \in \overline{\mathbf{H}}, s \in \mathbb{R}}} |(z-s)G_{n,j}(z,s)| = 0 \tag{2.30}$$

for any $s \in \mathbb{R}$*.*

Proof. The property 3 indicates $G_n(z,t)$ is polyharmonic in $\mathbf{H}$. By the decomposition theorem of polyharmonic functions in [8], (2.23) holds with (2.24)-(2.26) where $G_{n,j}(z,t)$ defined on $\mathbf{H} \times \mathbb{R}$ is analytic in $\mathbf{H}$ regarding to z and has at least the jth order zero at $z = i$ for each $t \in \mathbb{R}$ as $n \geq 2$.

It is well known that $G_1(z,t) \in L^p(\mathbb{R})$ regarding to t for any fixed $z \in \mathbf{H}$ and $p \geq 1$, $G_1(i,t) = \frac{1}{t^2+1}$ and the property 4 in Definition 2.1 holds in the case that $G_1(z,t)$ is given by (2.27) [19]. Starting from $G_{1,0}(z,t) = \frac{1}{2i}\frac{1}{t-z}$ and using (2.24)-(2.26), all $G_{n,j}$ and G_n can be inductively obtained (see the following algorithm).

Note that $G_{1,0} \in (H \times L^p)(\mathbf{H} \times \mathbb{R})$ and the nontangential boundary value

$$\lim_{\substack{z \to s \\ z \in \mathbf{H}, s \in \mathbb{R}}} G_{1,0}(z,t) = G_{1,0}(s,t) \tag{2.31}$$

exists on $\mathbb{R}$ except fixed $t \in \mathbb{R}$ and $G_{1,0}(s,t) \in L^p(\mathbb{R})$ regarding to t for any fixed $s \in \mathbb{R}$ as well

$$\begin{aligned} |G_{1,0}(z,t)| &= \frac{1}{2|t-z|} \\ &= \frac{1}{2} \times \frac{\frac{1}{|t-z'|}}{1 - \frac{|z-z'|}{|t-z'|}} \\ &\leq M\frac{1}{|t-z'|} \end{aligned}$$

uniformly on $D_c \times \{t \in \mathbb{R} : |t| > T\}$ whenever $z' \in D_c$ which is any compact set in $\overline{\mathbf{H}}$, where M, T are positive constants depending only on D_c. Therefore, by Lemmas 2.3-2.5 and induction, for any $n \in \mathbb{N}$ and $0 \leq j \leq n-1$, $G_{n,j} \in (H \times L^p)(\mathbf{H} \times \mathbb{R})$, the nontangential boundary value

$$\lim_{\substack{z \to s \\ z \in \mathbf{H}, s \in \mathbb{R}}} G_{n,j}(z,t) = G_{n,j}(s,t)$$

exists on $\mathbb{R}$ except fixed $t \in \mathbb{R}$ and $G_{n,j}(s,t) \in L^p(\mathbb{R})$ regarding to t for any fixed $s \in \mathbb{R}$ and

$$|G_{n,j}(z,t)| \leq M\frac{1}{|t-z'|}$$

uniformly on $D_c \times \{t \in \mathbb{R} : |t| > T\}$ whenever $z' \in D_c$ which is any compact set in $\overline{\mathbf{H}}$, where M, T are positive constants depending only on D_c. Obviously, $G_{n,j}(z,t)$ has at least the jth order zero at $z = i$ for each $t \in \mathbb{R}$. Moreover,

$$\lim_{\substack{z \to s \\ z \in \overline{\mathbf{H}}, s \in \mathbb{R}}} |G_{n,j}(z,s)| = +\infty \quad \text{and} \quad \lim_{\substack{z \to s \\ z \in \overline{\mathbf{H}}, s \in \mathbb{R}}} |(z-s)G_{n,j}(z,s)| = 0$$

for any $s \in \partial D$ in terms of Lemma 2.5 since $G_{2,1}$ has the same properties by using straightforward calculations.

By (2.23) and (2.26),

$$\begin{aligned} G_n(z,t) &= 2\Re\Big\{(\overline{z}-z)\sum_{j=1}^{n-1}\sum_{l=0}^{j-1}(\overline{z}+i)^{j-1-l}(z+i)^l G_{n,j}(z,t)\Big\} \\ &= 2\Re\Big\{\sum_{j=1}^{n-1}\sum_{l=0}^{j-1}(\overline{z}+i)^{j-1-l}(z+i)^l\big[(\overline{z}-s)-(z-s)\big]G_{n,j}(z,t)\Big\} \end{aligned}$$

where $z \in \mathbf{H}$ and $t \in \mathbb{R}$ for any fixed $s \in \mathbb{R}$.

From the above facts, by Minkowski's inequality and Lemma 2.6, all G_n satisfy the properties 1, 2 and 5, i.e. the nontangential limits

$$\lim_{\substack{z\to s\\ z\in\mathbf{H}, s\in\mathbb{R}}}\int_{-\infty}^{+\infty} G_n(z,t)\gamma(t)dt = 0 \tag{2.32}$$

holds for any $n \geq 2$ and $\gamma \in L^p(\mathbb{R})$ with any $p \geq 1$. □

In fact, following from Theorem 2.7, we establish an algorithm to obtain all explicit expressions of higher order Schawrz kernels as follows.

For $n = 1$,

$$G_{1,0}(z,t) = \frac{1}{2i}\frac{1}{t-z}, \tag{2.33}$$

therefore

$$G_1(z,t) = 2\Re\{G_{1,0}(z,t)\} = \frac{1}{2i}\left(\frac{1}{t-z}-\frac{1}{t-\overline{z}}\right) = \frac{z-\overline{z}}{2i}\frac{1}{|t-z|^2} = \frac{y}{(x-t)^2+y^2}. \tag{2.34}$$

For $n = 2$,

$$G_{2,1}(z,t) = \int_i^z G_{1,0}(\zeta,t)d\zeta = \frac{1}{i}\int_i^z \frac{1}{t-\zeta}d\zeta = \frac{1}{2i}\log\frac{t-i}{t-z}, \tag{2.35}$$

and

$$\begin{aligned} G_{2,0}(z,t) &= -(z+i)G_{2,1}(z,t) = -\frac{z+i}{2i}\log\frac{t-i}{t-z} \\ &= -(z-x)G_{2,1}(z,t)-(x+i)G_{2,1}(z,t), \end{aligned} \tag{2.36}$$

therefore

$$\begin{aligned} G_2(z,t) &= 2\Re\{G_{2,0}(z,t)+(\overline{z}+i)G_{2,1}(z,t)\} \\ &= 2\Re\{-(z+i)G_{2,1}(z,t)+(\overline{z}+i)G_{2,1}(z,t)\} \\ &= 2\Re\{(\overline{z}-z)G_{2,1}(z,t)\} \\ &= 2\Re\left\{\frac{\overline{z}-z}{2i}\log\frac{t-i}{t-z}\right\} \\ &= \frac{z-\overline{z}}{2i}\log\left|\frac{t-z}{t-i}\right|^2 \\ &= y\log\frac{(x-t)^2+y^2}{t^2+1}. \end{aligned} \tag{2.37}$$

For $n = 3$,

$$
\begin{aligned}
(2.38)\qquad G_{3,2}(z,t) &= \frac{1}{2}\int_i^z G_{2,1}(\zeta,t)d\zeta = \frac{1}{4i}\int_i^z \log\frac{t-i}{t-\zeta}d\zeta \\
&= \frac{1}{4i}\left[(z-t)\log\frac{t-i}{t-z} + (z-i)\right] \\
&= \frac{1}{2}(z-t)G_{2,1}(z,t) + \frac{1}{4i}(z-i) \\
&= \frac{0!}{2!}(z-t)G_{2,1}(z,t) + \frac{1}{2!\times 1!\times 1\times 2i}(z-i) \\
&= \frac{1}{2!\times 2i}(z-t)\log\frac{t-i}{t-z} \\
&\quad + \frac{1}{2!\times 1!\times 1\times 2i}(z-i),
\end{aligned}
$$

$$
\begin{aligned}
(2.39)\; G_{3,1}(z,t) &= \int_i^z G_{2,0}(\zeta,t)d\zeta = -\int_i^z (\zeta+i)G_{2,1}(\zeta,t)d\zeta \\
&= -\int_i^z (\zeta-t)G_{2,1}(\zeta,t)d\zeta - (t+i)\int_i^z G_{2,1}(\zeta,t)d\zeta \\
&= -\int_i^z (\zeta-t)G_{2,1}(\zeta,t)d\zeta - 2(t+i)G_{3,2}(z,t) \\
&= -2\int_i^z (\zeta-t)\partial_\zeta G_{3,2}(\zeta,t)d\zeta - 2(t+i)G_{3,2}(z,t) \\
&= -2(z-t)G_{3,2}(z,t) + 2\int_i^z G_{3,2}(\zeta,t)d\zeta \\
&\quad -2(t+i)G_{3,2}(z,t) \\
&= -2(z-t)G_{3,2}(z,t) + 3!\times G_{4,3}(z,t) - 2(t+i)G_{3,2}(z,t) \\
&= -(z-t)G_{3,2}(\zeta,t) - 2(t+i)G_{3,2}(\zeta,t) \\
&\quad + \frac{1}{2!\times 1!\times 2\times 2i}(z-i)^2 \\
&= -(z+t+2i)G_{3,2}(\zeta,t) + \frac{1}{2!\times 1!\times 2\times 2i}(z-i)^2,
\end{aligned}
$$

and

$$
\begin{aligned}
(2.40)\quad G_{3,0}(z,t) &= -(z+i)G_{3,1}(z,t) - (z+i)^2 G_{3,2}(z,t) \\
&= -(z+i)\big[G_{3,1}(z,t) + (z+i)G_{3,2}(z,t)\big] \\
&= -(z+i)\big[G_{3,1}(z,t) + \big((z-t)+(t+i)\big)G_{3,2}(z,t)\big] \\
&= (z+i)\left[(t+i)G_{3,2}(z,t) - \frac{1}{2!\times 1!\times 2\times 2i}(z-i)^2\right] \\
&= (t+i)(z+i)G_{3,2}(z,t) \\
&\quad - \frac{1}{2!\times 1!\times 2\times 2i}(z+i)(z-i)^2,
\end{aligned}
$$

therefore

$$
\begin{aligned}
(2.41)\qquad G_3(z,t) &= 2\Re\Big\{(\overline{z}-z)\big[G_{3,1}(z,t)+(\overline{z}+z+2i)G_{3,2}(z,t)\big]\Big\}\\
&= 2\Re\Big\{(\overline{z}-z)\Big[(\overline{z}-t)G_{3,2}(z,t)+\frac{1}{2!\times 1!\times 2\times 2i}\\
&\quad\times(z-i)^2\Big]\Big\}\\
&= 2\Re\Big\{(\overline{z}-z)\Big[\frac{1}{2!}|z-t|^2G_{2,1}(z,t)+\frac{1}{2!\times 1!\times 1\times 2i}\\
&\quad\times(\overline{z}-t)(z-i)+\frac{1}{2!\times 1!\times 2\times 2i}(z-i)^2\Big]\Big\}\\
&= 2\Re\Big\{(\overline{z}-z)\Big[\frac{1}{2!\times 1!\times 2i}|z-t|^2\log\frac{t-i}{t-z}\\
&\quad+\frac{1}{2!\times 1!\times 1\times 2i}(\overline{z}-t)(z-i)\\
&\quad+\frac{1}{2!\times 1!\times 2\times 2i}(z-i)^2\Big]\Big\}\\
&= \frac{y}{2}\Big[\big[(x-t)^2+y^2\big]\log\frac{(x-t)^2+y^2}{t^2+1}\\
&\quad+2xt-3x^2-y^2+1\Big]
\end{aligned}
$$

For $n=4$,

$$
\begin{aligned}
(2.42)\ G_{4,3}(z,t) &= \frac{1}{3}\int_i^z G_{3,2}(\zeta,t)d\zeta\\
&= \frac{0!}{3!}\int_i^z(\zeta-t)G_{2,1}(\zeta,t)d\zeta+\frac{1}{3!\times 1!\times 1\times 2i}\\
&\quad\times\int_i^z(\zeta-i)d\zeta\\
&= \frac{0!\times 2}{3!}\int_i^z(\zeta-t)\partial_\zeta G_{3,2}(\zeta,t)d\zeta+\frac{1}{3!\times 2!\times 1\times 2i}\\
&\quad\times(z-i)^2\\
&= \frac{0!\times 2}{3!}(z-t)G_{3,2}(z,t)-\frac{1}{1\times 3}\int_i^z G_{3,2}(\zeta,t)d\zeta\\
&\quad+\frac{1}{3!\times 2!\times 1\times 2i}(z-i)^2\\
&= \frac{1!}{3!}(z-t)G_{3,2}(z,t)+\frac{1}{3!\times 2!\times 2\times 2i}(z-i)^2\\
&= \frac{1}{24i}(z-t)^2\log\frac{t-i}{t-z}+\frac{1}{24i}(z-t)(z-i)+\frac{1}{48i}(z-i)^2\\
&= \frac{1}{3!\times 2!\times 2i}(z-t)^2\log\frac{t-i}{t-z}\\
&\quad+\frac{1}{3!\times 2!\times 1\times 2i}(z-t)(z-i)\\
&\quad+\frac{1}{3!\times 2!\times 2\times 2i}(z-i)^2,
\end{aligned}
$$

$$(2.43)\quad \begin{aligned} G_{4,2}(z,t) &= \frac{1}{2}\int_i^z G_{3,1}(\zeta,t)d\zeta \\ &= -\frac{1}{2}\int_i^z (\zeta+t+2i)G_{3,2}(\zeta,t)d\zeta + \frac{1}{2!\times 2!\times 2\times 2i} \\ &\quad \times \int_i^z (\zeta-i)^2 d\zeta \\ &= -\frac{3}{2}\int_i^z (\zeta+t+2i)\partial_\zeta G_{4,3}(\zeta,t)d\zeta + \frac{1}{3!\times 2!\times 2\times 2i} \\ &\quad \times (z-i)^3 \\ &= -\frac{3}{2}(z+t+2i)G_{4,3}(z,t) + \frac{3}{2}\int_i^z G_{4,3}(\zeta,t)d\zeta \\ &\quad + \frac{1}{3!\times 2!\times 2\times 2i}(z-i)^3 \\ &= -\frac{3}{2}(z+t+2i)G_{4,3}(z,t) + \frac{3\times 4}{2}G_{5,4}(\zeta,t) \\ &\quad + \frac{1}{3!\times 2!\times 2\times 2i}(z-i)^3 \\ &= -\frac{3}{2}(z+t+2i)G_{4,3}(z,t) + \frac{1}{2}\times\frac{4!}{2!}G_{5,4}(\zeta,t) \\ &\quad + \frac{1}{3!\times 2!\times 2\times 2i}(z-i)^3 \\ &= -(z+2t+3i)G_{4,3}(z,t) \\ &\quad + \frac{2}{3!\times 2!\times 3\times 2i}(z-i)^3, \end{aligned}$$

$$(2.44)\quad \begin{aligned} G_{4,1}(z,t) &= \int_i^z G_{3,0}(\zeta,t)d\zeta \\ &= (t+i)\int_i^z (\zeta+i)G_{3,2}(\zeta,t)d\zeta - \frac{1}{2!\times 1!\times 2\times 2i} \\ &\quad \times \left[\int_i^z (\zeta+i)(\zeta-i)^2 d\zeta\right] \\ &= 3(t+i)\int_i^z (\zeta+i)\partial_\zeta G_{4,3}(\zeta,t)d\zeta - \frac{1}{2!\times 1!\times 2\times 2i} \end{aligned}$$

$$
\begin{aligned}
& \times\left[\frac{1}{3}(z+i)(z-i)^3-\frac{1}{12}(z-i)^4\right] \\
= \;& 3(t+i)\left[(z+i)G_{4,3}(z,t)-\int_i^z G_{4,3}(\zeta,t)d\zeta\right] \\
& -\frac{1}{2!\times 1!\times 2\times 2i}\times\left[\frac{1}{3}(z+i)(z-i)^3-\frac{1}{12}(z-i)^4\right] \\
= \;& (t+i)\left[3(z+i)G_{4,3}(z,t)-\frac{4!}{2!}G_{5,4}(z,t)\right] \\
& -\frac{1}{2!\times 1!\times 2\times 2i}\times\left[\frac{1}{3}(z+i)(z-i)^3-\frac{1}{12}(z-i)^4\right] \\
= \;& (t+i)(2z+t+3i)G_{4,3}(z,t) \\
& -\frac{1}{3!\times 2!\times 3\times 2i}(t+i)(z-i)^3 \\
& -\frac{3}{3!\times 2!\times 3\times 2i}(z+i)(z-i)^3 \\
& +\frac{1}{3!\times 2!\times 4\times 2i}(z-i)^4
\end{aligned}
$$

and

$$
\begin{aligned}
G_{4,0}(z,t) = \;& -(z+i)G_{4,1}(z,t)-(z+i)^2G_{4,2}(z,t) \\
& -(z+i)^3G_{4,3}(z,t) \\
= \;& -(z+i)\big[G_{4,1}(z,t)+(z+i)G_{4,2}(z,t) \\
& +(z+i)^2G_{4,3}(z,t)\big] \\
= \;& -(z+i)\Big[(t+i)^2G_{4,3}(z,t) \\
& -\frac{1}{3!\times 2!\times 3\times 2i}\big[(z+i)+(t+i)\big](z-i)^3 \\
& +\frac{1}{3!\times 2!\times 4\times 2i}(z-i)^4\Big],
\end{aligned}
\tag{2.45}
$$

therefore

$$
\begin{aligned}
G_4(z,t) = \;& 2\Re\Big\{(\overline{z}-z)\big[G_{4,1}(z,t)+(\overline{z}+z+2i)G_{4,2}(z,t) \\
& +\big[(\overline{z}+i)^2+(\overline{z}+i)(z+i)+(z+i)^2\big]G_{4,3}(z,t)\big]\Big\} \\
= \;& 2\Re\Big\{(\overline{z}-z)\Big[(\overline{z}-t)^2G_{4,3}(z,t) \\
& +\frac{1}{3!\times 2!\times 3\times 2i}\big[2(\overline{z}-z)+(z-t)\big](z-i)^3 \\
& +\frac{1}{3\times 2!\times 4\times 2i}(z-i)^4\Big]\Big\} \\
= \;& 2\Re\Big\{(\overline{z}-z)\Big[\frac{1}{3!\times 2!\times 2i}|z-t|^4\log\frac{t-i}{t-z} \\
& +\frac{1}{3!\times 2!\times 1\times 2i}(\overline{z}-t)^2(z-t)(z-i) \\
& +\frac{1}{3!\times 2!\times 2\times 2i}(\overline{z}-t)^2(z-i)^2
\end{aligned}
\tag{2.46}
$$

$$+\frac{1}{3!\times 2!\times 3\times 2i}\big[2(\overline{z}-z)+(z-t)\big](z-i)^3$$
$$+\frac{1}{3!\times 2!\times 4\times 2i}(z-i)^4\Big]\Big\}.$$

By similar calculations,

$$\begin{aligned}(2.47)\ G_5(z,t) &= 2\Re\Big\{(\overline{z}-z)\big[G_{5,1}(z,t)+(\overline{z}+z+2i)G_{5,2}(z,t)\\
&\quad +\big[(\overline{z}+i)^2+(\overline{z}+i)(z+i)+(z+i)^2\big]G_{5,3}(z,t)\big]\\
&\quad +\big[(\overline{z}+i)^3+(\overline{z}+i)^2(z+i)+(\overline{z}+i)(z+i)^2+(z+i)^3\big]\\
&\quad \times G_{5,4}(z,t)\big]\Big\}\\
&= 2\Re\Big\{(\overline{z}-z)\Big[(\overline{z}-t)^3G_{5,4}(z,t)\\
&\quad +\frac{1}{4!\times 3!\times 4\times 2i}\big[3(\overline{z}-z)(\overline{z}-t)+(z-t)^2\big](z-i)^4\\
&\quad +\frac{1}{4!\times 3!\times 5\times 2i}\big[3(\overline{z}-z)+(z-t)\big](z-i)^5\\
&\quad +\frac{1}{4!\times 3!\times 6\times 2i}(z-i)^6\Big]\Big\}\\
&= 2\Re\Big\{(\overline{z}-z)\Big[\frac{1}{4!\times 3!\times 2i}|z-t|^6\log\frac{t-i}{t-z}\\
&\quad +\frac{1}{4!\times 3!\times 1\times 2i}(\overline{z}-t)^3(z-t)^2(z-i)\\
&\quad +\frac{1}{4!\times 3!\times 2\times 2i}(\overline{z}-t)^3(z-t)(z-i)^2\\
&\quad +\frac{1}{4!\times 3!\times 3\times 2i}(\overline{z}-t)^3(z-i)^3\\
&\quad +\frac{1}{4!\times 3!\times 4\times 2i}\big[3(\overline{z}-z)(\overline{z}-t)+(z-t)^2\big](z-i)^4\\
&\quad +\frac{1}{4!\times 3!\times 5\times 2i}\big[3(\overline{z}-z)+(z-t)\big](z-i)^5\\
&\quad +\frac{1}{4!\times 3!\times 6\times 2i}(z-i)^6\Big]\Big\}\end{aligned}$$

and

$$\begin{aligned}(2.48) G_6(z,t) &= 2\Re\Big\{(\overline{z}-z)\big[G_{6,1}(z,t)+(\overline{z}+z+2i)G_{6,2}(z,t)\\
&\quad +\big[(\overline{z}+i)^2+(\overline{z}+i)(z+i)+(z+i)^2\big]G_{6,3}(z,t)\big]\\
&\quad +\big[(\overline{z}+i)^3+(\overline{z}+i)^2(z+i)+(\overline{z}+i)(z+i)^2+(z+i)^3\big]\\
&\quad \times G_{6,4}(z,t)+[(\overline{z}+i)^4+(\overline{z}+i)^3(z+i)+(\overline{z}+i)^2(z+i)^2\\
&\quad +(\overline{z}+i)(z+i)^3+(z+i)^4\big]G_{6,5}(z,t)\big]\Big\}\\
&= 2\Re\Big\{(\overline{z}-z)\Big[(\overline{z}-t)^4G_{6,5}(z,t)\\
&\quad +\frac{1}{5!\times 4!\times 5\times 2i}\big[2(\overline{z}-z)+(z-t)\big]\end{aligned}$$

$$
\begin{aligned}
&\times\big[2(\overline{z}-z)^2+2(\overline{z}-z)(z-t)+(z-t)^2\big](z-i)^5\\
&+\frac{1}{5!\times 4!\times 6\times 2i}\Big[\big[2(\overline{z}-z)+(z-t)\big]^2+2(z-t)^2\Big]\\
&\times(z-i)^6\\
&+\frac{1}{5!\times 4!\times 7\times 2i}\big[4(\overline{z}-z)+(z-t)\big](z-i)^5\\
&+\frac{1}{5\times 4!\times 8\times 2i}(z-i)^8\Big]\Big\}\\
= \quad & 2\Re\Big\{(\overline{z}-z)\Big[\frac{1}{5!\times 4!\times 2i}|z-t|^8\log\frac{t-i}{t-z}\\
&+\frac{1}{5!\times 4!\times 1\times 2i}(\overline{z}-t)^4(z-t)^3(z-i)\\
&+\frac{1}{5!\times 4!\times 2\times 2i}(\overline{z}-t)^4(z-t)^2(z-i)^2\\
&+\frac{1}{5!\times 4!\times 3\times 2i}(\overline{z}-t)^4(z-t)(z-i)^3\\
&+\frac{1}{5!\times 4!\times 4\times 2i}(\overline{z}-t)^4(z-i)^4\\
&+\frac{1}{5!\times 4!\times 5\times 2i}\big[2(\overline{z}-z)+(z-t)\big]\\
&\times\big[2(\overline{z}-z)^2+2(\overline{z}-z)(z-t)+(z-t)^2\big](z-i)^5\\
&+\frac{1}{5!\times 4!\times 6\times 2i}\Big[\big[2(\overline{z}-z)+(z-t)\big]^2+2(z-t)^2\Big]\\
&\times(z-i)^6\\
&+\frac{1}{5!\times 4!\times 7\times 2i}\big[4(\overline{z}-z)+(z-t)\big](z-i)^5\\
&+\frac{1}{5!\times 4!\times 8\times 2i}(z-i)^8\Big]\Big\}.
\end{aligned}
$$

In general, we have a unified expression of all $G_{n,n-1}$ and G_n as stated in the following theorem.

Theorem 2.8. *Let $G_{n,n-1}$ and G_n be stated as in Theorem 2.7, then for any $n>2$,*

$$
\begin{aligned}
G_{n,n-1}(z,t) &= \frac{(n-3)!}{(n-1)!}(z-t)G_{n-1,n-2}(z,t)\\
&\quad+\frac{1}{(n-1)!\times(n-2)!\times(n-2)\times 2i}(z-i)^{n-2}\\
&= \frac{1}{(n-1)!\times(n-2)!\times 2i}(z-t)^{n-2}\log\frac{t-i}{t-z}\\
&\quad+\sum_{j=1}^{n-2}\frac{1}{(n-1)!\times(n-2)!\times j\times 2i}(z-t)^{n-2-j}(z-i)^j
\end{aligned}
\tag{2.49}
$$

and

$$
\begin{aligned}
(2.50)\; G_n(z,t) &= 2\Re\Big\{(\overline{z}-z)\Big[\frac{1}{(n-1)!\times(n-2)!\times 2i}|z-t|^{2(n-2)}\log\frac{t-i}{t-z} \\
&+\sum_{j=1}^{n-2}\frac{1}{(n-1)!\times(n-2)!\times j\times 2i}(\overline{z}-t)^{n-2}(z-t)^{n-2-j} \\
&\times(z-i)^j \\
&+\sum_{j=n-1}^{2(n-2)}\sum_{l=0}^{n-2}\frac{1}{(n-1)!\times(n-2)!\times j\times 2i}C_{n-2}^{l}(\overline{z}-z)^l \\
&\times(z-t)^{2(n-2)-l-j}(z-i)^j\Big]\Big\} \\
&= 2\Re\Big\{(\overline{z}-z)\Big[\frac{1}{(n-1)!\times(n-2)!\times 2i}|z-t|^{2(n-2)}\log\frac{t-i}{t-z} \\
&+\sum_{j=1}^{n-2}\frac{1}{(n-1)!\times(n-2)!\times j\times 2i}(\overline{z}-t)^{n-2}(z-t)^{n-2-j} \\
&\times(z-i)^j \\
&+\sum_{j=n-1}^{2(n-2)}\sum_{l=j}^{2(n-2)}\frac{1}{(n-1)!\times(n-2)!\times j\times 2i}C_{n-2}^{l-j}(\overline{z}-z)^{l-j} \\
&\times(z-t)^{2(n-2)-l}(z-i)^j\Big]\Big\}
\end{aligned}
$$

where $z\in\mathbf{H}$ *and* $t\in\mathbb{R}$.

Proof. Obviously, (2.49) follows from (2.24) and (2.27) as $n=3$. Suppose that for $n>4$,

$$
\begin{aligned}
G_{n-1,n-2}(z,t) &= \frac{(n-4)!}{(n-2)!}(z-t)G_{n-2,n-3}(z,t) \\
&+\frac{1}{(n-2)!\times(n-3)!\times(n-3)\times 2i}(z-i)^{n-3} \\
&= \frac{1}{(n-2)!\times(n-3)!\times 2i}(z-t)^{n-3}\log\frac{t-i}{t-z} \\
&+\sum_{j=1}^{n-3}\frac{1}{(n-2)!\times(n-3)!\times j\times 2i}(z-t)^{n-3-j}(z-i)^j.
\end{aligned}
$$

Then

$$
\begin{aligned}
G_{n,n-1}(z,t) &= \frac{1}{n-1}\int_i^z G_{n-1,n-2}(\zeta,t)d\zeta \\
&= \frac{(n-4)!}{(n-1)!}\int_i^z(\zeta-t)G_{n-2,n-3}(\zeta,t)d\zeta
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{(n-1)!\times(n-2)!\times(n-3)\times 2i}(z-i)^{n-2}\\
=&\ \frac{(n-4)!\times(n-2)}{(n-1)!}\int_i^z(\zeta-t)\partial_\zeta G_{n-1,n-2}(\zeta,t)d\zeta\\
&+\frac{1}{(n-1)!\times(n-2)!\times(n-3)\times 2i}(z-i)^{n-2}\\
=&\ \frac{(n-4)!\times(n-2)}{(n-1)!}(z-t)G_{n-1,n-2}(z,t)\\
&-\frac{1}{(n-3)\times(n-1))}\int_i^z G_{n-1,n-2}(\zeta,t)d\zeta\\
&+\frac{1}{(n-1)!\times(n-2)!\times(n-3)\times 2i}(z-i)^{n-2}\\
=&\ \frac{(n-3)!}{(n-1)!}(z-t)G_{n-1,n-2}(z,t)\\
&+\frac{1}{(n-1)!\times(n-2)!\times(n-2)\times 2i}(z-i)^{n-2}\\
=&\ \frac{1}{(n-1)!\times(n-2)!\times 2i}(z-t)^{n-2}\log\frac{t-i}{t-z}\\
&+\sum_{j=1}^{n-2}\frac{1}{(n-1)!\times(n-2)!\times j\times 2i}(z-t)^{n-2-j}(z-i)^j.
\end{aligned}
$$

Therefore, by induction, (2.49) holds for any $n>2$.

Let

$$
H_n(z,t)=\frac{1}{(n-1)!\times(n-2)!\times 2i}|z-t|^{2(n-2)}\log\frac{t-i}{t-z} \tag{2.51}
$$

and

$$
\begin{aligned}
T_n(z,t) =&\ \sum_{j=1}^{n-2}\frac{1}{(n-1)!\times(n-2)!\times j\times 2i}(\overline{z}-t)^{n-2}(z-t)^{n-2-j}\\
&\times(z-i)^j\\
&+\sum_{j=n-1}^{2(n-2)}\sum_{l=j}^{2(n-2)}\frac{1}{(n-1)!\times(n-2)!\times j\times 2i}C_{n-2}^{l-j}(\overline{z}-z)^{l-j}\\
&\times(z-t)^{2(n-2)-l}(z-i)^j,
\end{aligned} \tag{2.52}
$$

therefore

$$
G_n(z,t)=2\Re[(\overline{z}-z)(H_n(z,t)+T_n(z,t))]. \tag{2.53}
$$

Thus

$$
\begin{aligned}
\partial_z\partial_{\overline{z}}G_n(z,t) =&\ 2\Re\{(\overline{z}-z)\partial_z\partial_{\overline{z}}H_n(z,t)+(\partial_z-\partial_{\overline{z}})H_n(z,t)\}\\
&+2\Re\{(\overline{z}-z)\partial_z\partial_{\overline{z}}T_n(z,t)+(\partial_z-\partial_{\overline{z}})T_n(z,t)\}.
\end{aligned} \tag{2.54}
$$

Since

$$(2.55)\quad \partial_z H_n(z,t) = \frac{1}{(n-1)! \times (n-3)! \times 2i}(\overline{z}-t)^{n-2}(z-t)^{n-3}\log\frac{t-i}{t-z} - \frac{1}{(n-1)! \times (n-2)! \times 2i}(\overline{z}-t)^{n-2}(z-t)^{n-3},$$

$$(2.56)\quad \partial_{\overline{z}} H_n(z,t) = \frac{1}{(n-1)! \times (n-3)! \times 2i}(\overline{z}-t)^{n-3}(z-t)^{n-2}\log\frac{t-i}{t-z},$$

then

$$(2.57)\quad (\partial_z - \partial_{\overline{z}})H_n(z,t) = \frac{1}{(n-1)! \times (n-3)! \times 2i}(\overline{z}-t)^{n-3}(z-t)^{n-3} \times(\overline{z}-z)\log\frac{t-i}{t-z} - \frac{1}{(n-1)! \times (n-2)! \times 2i} \times(\overline{z}-t)^{n-2}(z-t)^{n-3}$$

and

$$(2.58)\quad \partial_z\partial_{\overline{z}} H_n(z,t) = \frac{1}{(n-1)! \times (n-3)! \times 2i}(\overline{z}-t)^{n-3}(z-t)^{n-3} \times\left[(n-2)\log\frac{t-i}{t-z} - 1\right].$$

Thus

$$\begin{aligned}(2.59)\quad & (\overline{z}-z)\partial_z\partial_{\overline{z}} H_n(z,t) + (\partial_z - \partial_{\overline{z}})T_n(z,t) \\ = \; & (\overline{z}-z)\left[\frac{1}{(n-1)! \times (n-3)! \times 2i}(\overline{z}-t)^{n-3}(z-t)^{n-3}\log\frac{t-i}{t-z}\right] \\ & - \frac{1}{(n-1)! \times (n-3)! \times 2i}(\overline{z}-z)(\overline{z}-t)^{n-3}(z-t)^{n-3} \qquad (\spadesuit) \\ & - \frac{1}{(n-1)! \times (n-2)! \times 2i}(\overline{z}-t)^{n-2}(z-t)^{n-3}. \qquad (\clubsuit)\end{aligned}$$

On the other hand, since

$$\begin{aligned}(2.60)\quad \partial_z T_n(z,t) = \; & \sum_{j=1}^{n-3}\frac{1}{(n-1)! \times (n-3)! \times j \times 2i}(\overline{z}-t)^{n-2} \\ & \times(z-t)^{n-3-j}(z-i)^j \\ & + \frac{1}{(n-1)! \times (n-2)! \times 2i}(\overline{z}-t)^{n-2}(z-t)^{n-3} \\ & - \sum_{j=n-1}^{2(n-2)-1}\sum_{l=j+1}^{2(n-2)}\frac{1}{(n-1)! \times (n-2)! \times j \times 2i} \\ & \times(l-j)C_{n-2}^{l-j}(\overline{z}-z)^{l-j-1}(z-t)^{2(n-2)-l}(z-i)^j \\ & + \sum_{j=n-1}^{2(n-2)-1}\sum_{l=j}^{2(n-2)-1}\frac{1}{(n-1)! \times (n-2)! \times j \times 2i}\end{aligned}$$

$$
\begin{aligned}
&\times(2(n-2)-l)C_{n-2}^{l-j}(\overline{z}-z)^{l-j}(z-t)^{2(n-2)-l-1}(z-i)^{j}\\
&+\sum_{j=n-1}^{2(n-2)}\sum_{l=j}^{2(n-2)}\frac{1}{(n-1)!\times(n-2)!\times j\times 2i}\\
&\times jC_{n-2}^{l-j}(\overline{z}-z)^{l-j}(z-t)^{2(n-2)-l}(z-i)^{j-1}\\
= &\sum_{j=1}^{n-3}\frac{1}{(n-1)!\times(n-3)!\times j\times 2i}(\overline{z}-t)^{n-2}\\
&\times(z-t)^{n-3-j}(z-i)^{j}\\
&+\frac{1}{(n-1)!\times(n-2)!\times 2i}(\overline{z}-t)^{n-2}(z-t)^{n-3}\\
&-\sum_{j=n-1}^{2(n-2)-1}\sum_{l=j+1}^{2(n-2)}\frac{1}{(n-1)!\times(n-2)!\times j\times 2i}\\
&\times(l-j)C_{n-2}^{l-j}(\overline{z}-z)^{l-j-1}(z-t)^{2(n-2)-l}(z-i)^{j}\\
&+\sum_{j=n-1}^{2(n-2)-1}\sum_{l=j+1}^{2(n-2)}\frac{1}{(n-1)!\times(n-2)!\times j\times 2i}\\
&\times(2(n-2)-l+1)C_{n-2}^{l-j-1}(\overline{z}-z)^{l-j-1}(z-t)^{2(n-2)-l}\\
&\times(z-i)^{j}\\
&+\sum_{j=n-2}^{2(n-2)-1}\sum_{l=j+1}^{2(n-2)}\frac{1}{(n-1)!\times(n-2)!\times(j+1)\times 2i}\\
&\times(j+1)C_{n-2}^{l-j-1}(\overline{z}-z)^{l-j-1}(z-t)^{2(n-2)-l}(z-i)^{j}\\
= &\sum_{j=1}^{n-3}\frac{1}{(n-1)!\times(n-3)!\times j\times 2i}(\overline{z}-t)^{n-2}\\
&\times(z-t)^{n-3-j}(z-i)^{j}\\
&+\frac{1}{(n-1)!\times(n-2)!\times 2i}(\overline{z}-t)^{n-2}(z-t)^{n-3}\\
&+\sum_{j=n-1}^{2(n-2)-1}\sum_{l=j+1}^{2(n-2)}\frac{1}{(n-1)!\times(n-3)!\times j\times 2i}\\
&\times C_{n-2}^{l-j-1}(\overline{z}-z)^{l-j-1}(z-t)^{2(n-2)-l}(z-i)^{j}\\
&+\sum_{l=n-1}^{2(n-2)}\frac{1}{(n-1)!\times(n-2)!\times 2i}C_{n-2}^{l-n+1}\\
&\times(\overline{z}-z)^{l-n+1}(z-t)^{2(n-2)-l}(z-i)^{n-2},
\end{aligned}
$$

$$
\partial_{\overline{z}}T_n(z,t) = \sum_{j=1}^{n-2}\frac{1}{(n-1)!\times(n-3)!\times j\times 2i}(\overline{z}-t)^{n-3} \times(z-t)^{n-2-j}(z-i)^{j} \tag{2.61}
$$

$$
\begin{aligned}
& + \sum_{j=n-1}^{2(n-2)-1} \sum_{l=j+1}^{2(n-2)} \frac{1}{(n-1)! \times (n-2)! \times j \times 2i} \\
& \times (l-j) C_{n-2}^{l-j} (\overline{z}-z)^{l-j-1} (z-t)^{2(n-2)-l} (z-i)^j, \\
= & \sum_{j=1}^{n-2} \frac{1}{(n-1)! \times (n-3)! \times j \times 2i} (\overline{z}-t)^{n-3} \\
& \times (z-t)^{n-2-j} (z-i)^j \\
& + \sum_{j=n-1}^{2(n-2)-1} \sum_{l=j+1}^{2(n-2)} \frac{1}{(n-1)! \times (n-3)! \times j \times 2i} \\
& \times C_{n-3}^{l-j-1} (\overline{z}-z)^{l-j-1} (z-t)^{2(n-2)-l} (z-i)^j,
\end{aligned}
$$

therefore

$$
\begin{aligned}
(\partial_z - \partial_{\overline{z}}) T_n(z,t) = & \sum_{j=1}^{n-3} \frac{1}{(n-1)! \times (n-3)! \times j \times 2i} \\
& \times (\overline{z}-z)(\overline{z}-t)^{n-3} (z-t)^{n-3-j} (z-i)^j \\
& + \frac{1}{(n-1)! \times (n-2)! \times 2i} (\overline{z}-t)^{n-2} (z-t)^{n-3} \\
& - \frac{1}{(n-1)! \times (n-2)! \times 2i} (\overline{z}-t)^{n-3} (z-i)^{n-2} \\
& + \sum_{j=n-1}^{2(n-2)-1} \sum_{l=j+1}^{2(n-2)} \frac{1}{(n-1)! \times (n-3)! \times j \times 2i} \\
& \times \left[C_{n-2}^{l-j-1} - C_{n-3}^{l-j-1} \right] (\overline{z}-z)^{l-j-1} \\
& \times (z-t)^{2(n-2)-l} (z-i)^j \\
& + \sum_{l=n-1}^{2(n-2)} \frac{1}{(n-1)! \times (n-2)! \times 2i} C_{n-2}^{l-n+1} \\
& \times (\overline{z}-z)^{l-n+1} (z-t)^{2(n-2)-l} (z-i)^{n-2} \\
= & \sum_{j=1}^{n-3} \frac{1}{(n-1)! \times (n-3)! \times j \times 2i} \\
& \times (\overline{z}-z)(\overline{z}-t)^{n-3} (z-t)^{n-3-j} (z-i)^j \\
& + \frac{1}{(n-1)! \times (n-2)! \times 2i} (\overline{z}-t)^{n-2} (z-t)^{n-3} \\
& - \frac{1}{(n-1)! \times (n-2)! \times 2i} (\overline{z}-t)^{n-3} (z-i)^{n-2} \\
& + \sum_{j=n-1}^{2(n-2)-2} \sum_{l=j+2}^{2(n-2)} \frac{1}{(n-1)! \times (n-3)! \times j \times 2i} \\
& \times C_{n-3}^{l-j-2} (\overline{z}-z)^{l-j-1} (z-t)^{2(n-2)-l} (z-i)^j
\end{aligned}
\tag{2.62}
$$

$$+\sum_{l=n-1}^{2(n-2)}\frac{1}{(n-1)!\times(n-2)!\times 2i}C_{n-2}^{l-n+1}$$
$$\times(\overline{z}-z)^{l-n+1}(z-t)^{2(n-2)-l}(z-i)^{n-2}$$

and

$$\begin{aligned}(2.63)\ \partial_z\partial_{\overline{z}}T_n(z,t) &= \sum_{j=1}^{n-3}\frac{1}{(n-1)!\times(n-3)!\times j\times 2i}(n-2)(\overline{z}-t)^{n-3}\\ &\times(z-t)^{n-3-j}(z-i)^j\\ &+\frac{1}{(n-1)!\times(n-3)!\times 2i}(\overline{z}-t)^{n-3}(z-t)^{n-3}\\ &+\sum_{j=n-1}^{2(n-2)-2}\sum_{l=j+2}^{2(n-2)}\frac{1}{(n-1)!\times(n-3)!\times j\times 2i}\\ &\times(l-j-1)C_{n-2}^{l-j-1}(\overline{z}-z)^{l-j-2}(z-t)^{2(n-2)-l}(z-i)^j\\ &+\sum_{l=n}^{2(n-2)}\frac{1}{(n-1)!\times(n-2)!\times 2i}(l-n+1)C_{n-2}^{l-n+1}\\ &\times(\overline{z}-z)^{l-n}(z-t)^{2(n-2)-l}(z-i)^{n-2},\\ &= \sum_{j=1}^{n-3}\frac{1}{(n-1)!\times(n-3)!\times j\times 2i}(n-2)(\overline{z}-t)^{n-3}\\ &\times(z-t)^{n-3-j}(z-i)^j\\ &+\frac{1}{(n-1)!\times(n-3)!\times 2i}(\overline{z}-t)^{n-3}(z-t)^{n-3}\\ &+\sum_{j=n-1}^{2(n-2)-2}\sum_{l=j+2}^{2(n-2)}\frac{1}{(n-1)!\times(n-3)!\times j\times 2i}\\ &\times(n-2)C_{n-3}^{l-j-2}(\overline{z}-z)^{l-j-2}(z-t)^{2(n-2)-l}(z-i)^j\\ &+\sum_{l=n}^{2(n-2)}\frac{1}{(n-1)!\times(n-2)!\times 2i}(n-2)C_{n-3}^{l-n}\\ &\times(\overline{z}-z)^{l-n}(z-t)^{2(n-2)-l}(z-i)^{n-2}.\end{aligned}$$

Thus

$$\begin{aligned}(2.64)\qquad & (\overline{z}-z)\partial_z\partial_{\overline{z}}T_n(z,t)+(\partial_z-\partial_{\overline{z}})T_n(z,t)\\ = & \sum_{j=1}^{n-3}\frac{1}{(n-2)!\times(n-3)!\times j\times 2i}(\overline{z}-z)(\overline{z}-t)^{n-3}\\ &\times(z-t)^{n-3-j}(z-i)^j\\ &+\frac{1}{(n-1)!\times(n-3)!\times 2i}(\overline{z}-z)(\overline{z}-t)^{n-3}(z-t)^{n-3}\\ &+\frac{1}{(n-1)!\times(n-2)!\times 2i}(\overline{z}-t)^{n-2}(z-t)^{n-3}\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{(n-1)!\times(n-2)!\times 2i}(\overline{z}-t)^{n-3}(z-i)^{n-2}\\
&+\sum_{j=n-1}^{2(n-2)-2}\sum_{l=j+2}^{2(n-2)}\frac{1}{(n-1)!\times(n-3)!\times j\times 2i}\\
&\times C_{n-3}^{l-j-2}(\overline{z}-z)^{l-j-1}(z-t)^{2(n-2)-l}(z-i)^{j}\\
&+\sum_{l=n}^{2(n-2)}\frac{1}{(n-1)!\times(n-2)!\times 2i}\left[C_{n-2}^{l-n+1}+(n-2)C_{n-3}^{l-n}\right]\\
&\times(\overline{z}-z)^{l-n+1}(z-t)^{2(n-2)-l}(z-i)^{n-2}\\
&+\frac{1}{(n-1)!\times(n-2)!\times 2i}(z-t)^{n-3}(z-i)^{n-2}\\
=\ &\sum_{j=1}^{n-3}\frac{1}{(n-2)!\times(n-3)!\times j\times 2i}(\overline{z}-z)(\overline{z}-t)^{n-3}\\
&\times(z-t)^{n-3-j}(z-i)^{j}\\
&+\frac{1}{(n-1)!\times(n-3)!\times 2i}(\overline{z}-z)(\overline{z}-t)^{n-3}(z-t)^{n-3}\\
&+\frac{1}{(n-1)!\times(n-2)!\times 2i}(\overline{z}-t)^{n-2}(z-t)^{n-3}\\
&-\frac{1}{(n-1)!\times(n-2)!\times 2i}(\overline{z}-t)^{n-3}(z-i)^{n-2}\\
&+\sum_{j=n-1}^{2(n-2)-2}\sum_{l=j+2}^{2(n-2)}\frac{1}{(n-2)!\times(n-3)!\times j\times 2i}\\
&\times C_{n-3}^{l-j-2}(\overline{z}-z)^{l-j-1}(z-t)^{2(n-2)-l}(z-i)^{j}\\
&+\sum_{l=n}^{2(n-2)}\frac{1}{(n-2)!\times(n-3)!\times(n-2)\times 2i}C_{n-3}^{l-n}\\
&\times(\overline{z}-z)^{l-n+1}(z-t)^{2(n-2)-l}(z-i)^{n-2}\\
&+\sum_{l=n-1}^{2(n-2)}\frac{1}{(n-1)!\times(n-2)!\times 2i}C_{n-3}^{l-n+1}\\
&\times(\overline{z}-z)^{l-n+1}(z-t)^{2(n-2)-l}(z-i)^{n-2}\\
=\ &\sum_{j=1}^{n-3}\frac{1}{(n-2)!\times(n-3)!\times j\times 2i}(\overline{z}-z)(\overline{z}-t)^{n-3}\\
&\times(z-t)^{n-3-j}(z-i)^{j}\\
&+\frac{1}{(n-1)!\times(n-3)!\times 2i}(\overline{z}-z)(\overline{z}-t)^{n-3}(z-t)^{n-3}\\
&+\frac{1}{(n-1)!\times(n-2)!\times 2i}(\overline{z}-t)^{n-2}(z-t)^{n-3}\\
&-\frac{1}{(n-1)!\times(n-2)!\times 2i}(\overline{z}-t)^{n-3}(z-i)^{n-2} \qquad (\diamondsuit)
\end{aligned}
$$

$$
\begin{aligned}
& +\sum_{j=n-1}^{2(n-3)}\sum_{l=j}^{2(n-3)}\frac{1}{(n-2)!\times(n-3)!\times j\times 2i}\\
& \times C_{n-3}^{l-j}(\overline{z}-z)^{l-j+1}(z-t)^{2(n-3)-l}(z-i)^{j}\\
& +\sum_{l=n-2}^{2(n-3)}\frac{1}{(n-2)!\times(n-3)!\times(n-2)\times 2i}C_{n-3}^{l-(n-2)}\\
& \times(\overline{z}-z)^{l-(n-2)+1}(z-t)^{2(n-3)-l}(z-i)^{n-2}\\
& +\frac{1}{(n-1)!\times(n-2)!\times 2i}\left[\sum_{l=0}^{n-3}C_{n-3}^{l}(\overline{z}-z)^{l}(z-t)^{n-3-l}\right](z-i)^{n-2} \qquad (\diamondsuit)\\
= & \sum_{j=1}^{n-3}\frac{1}{(n-2)!\times(n-3)!\times j\times 2i}(\overline{z}-z)(\overline{z}-t)^{n-3}\\
& \times(z-t)^{n-3-j}(z-i)^{j}\\
& +\frac{1}{(n-1)!\times(n-3)!\times 2i}(\overline{z}-z)(\overline{z}-t)^{n-3}(z-t)^{n-3} \qquad (\spadesuit)\\
& +\frac{1}{(n-1)!\times(n-2)!\times 2i}(\overline{z}-t)^{n-2}(z-t)^{n-3} \qquad (\clubsuit)\\
& +\sum_{j=n-2}^{2(n-3)}\sum_{l=j}^{2(n-3)}\frac{1}{(n-2)!\times(n-3)!\times j\times 2i}\\
& \times C_{n-3}^{l-j}(\overline{z}-z)^{l-j+1}(z-t)^{2(n-3)-l}(z-i)^{j}.
\end{aligned}
$$

It is noteworthy that, in all of above calculations, we properly use the following elementary properties of binomial coefficients:

$$
C_n^k = C_n^{n-k},\ C_n^k = C_{n-1}^k + C_{n-1}^{k-1} \text{ and } kC_n^k = nC_{n-1}^{k-1} \tag{2.65}
$$

as $n \geq k \geq 1$.

Insert (2.59) and (2.64) into (2.54) and note that the terms marked by ($\spadesuit$) and ($\clubsuit$),

$$
\begin{aligned}
\partial_z\partial_{\overline{z}}G_n(z,t) = & 2\Re\Big\{(\overline{z}-z)\Big[\frac{1}{(n-2)!\times(n-3)!\times 2i}|z-t|^{2(n-3)}\log\frac{t-i}{t-z}\\
& +\sum_{j=1}^{n-3}\frac{1}{(n-2)!\times(n-3)!\times j\times 2i}(\overline{z}-t)^{n-3}(z-t)^{n-3-j}\\
& \times(z-i)^{j}\\
& +\sum_{j=n-2}^{2(n-3)}\sum_{l=j}^{2(n-3)}\frac{1}{(n-2)!\times(n-3)!\times j\times 2i}C_{n-3}^{l-j}(\overline{z}-z)^{l-j}\\
& \times(z-t)^{2(n-3)-l}(z-i)^{j}\Big]\Big\}\\
= & G_{n-1}(z,t).
\end{aligned}
$$

Then (2.50) holds by backward induction and we complete the theorem. □

3. Polyharmonic Dirichlet problems in the upper half plane

In this section, we solve the PHD problems (1.1), i.e.

$$\begin{cases} \Delta^n u = 0 \text{ in } \mathbf{H}, \\ \Delta^j u = f_j \text{ on } \mathbb{R} \end{cases}$$

where $\mathbf{H}$ is the upper half plane, $\mathbb{R}$ is the real axis, $f_j \in L^p(\mathbb{R})$, $n \in \mathbb{N}$, $0 \leq j < n$ and $p \geq 1$.

To do so, we need the following

Lemma 3.1. *Let D be a simply connected unbounded domain in the plane with smooth boundless boundary ∂D. If $f \in (H \times L^1)(D \times \partial D)$ and there exists $g \in L^p(\partial D)$, $p > 1$ such that*

$$|f(z,t)| \leq M \frac{g(t)}{|t - z_0|} \tag{3.1}$$

uniformly on $D_c \times \{t \in \partial D : |t| > T\}$ whenever $z_0 \in D_c$ which is any compact set in D, where M, T are positive constants depending only on D_c. Then

$$\frac{\partial}{\partial z}\left(\int_{\partial D} f(z,t)dt\right) = \int_{\partial D} \frac{\partial f}{\partial z}(z,t)dt. \tag{3.2}$$

Proof. Fix $z_0 \in D$, take any sequence $\{z_l\}_{l=1}^{+\infty}$ such that $\lim_{l\to+\infty} z_l = z_0$ and $z_l \neq z_0$ for all l. Since $f \in (H \times L^1)(D \times \partial D)$, denote

$$\begin{aligned} D_l(z_0,t) &= \frac{f(z_l,t) - f(z_0,t)}{z_l - z_0} \\ &= \frac{1}{2\pi i}\int_{|\zeta - z_0| = R} \frac{f(\zeta,t)}{(\zeta - z_l)(\zeta - z_0)} d\zeta, \end{aligned} \tag{3.3}$$

then by the assumptions,

$$\begin{aligned} |D_l(z_0,t)| &\leq \frac{1}{2\pi i}\int_{|\zeta - z_0| = R} \frac{|f(\zeta,t)|}{|\zeta - z_l|}\frac{d\zeta}{\zeta - z_0} \\ &\leq \frac{2M}{R}\frac{g(t)}{|t - z_0|} \end{aligned} \tag{3.4}$$

uniformly in $\{t \in \partial D : |t| > T\}$ whenever $z_l \in \{\zeta : |\zeta - z_0| < R/2\} \subset \{\zeta : |\zeta - z_0| < R\} \subset D$. Since

$$\lim_{l\to+\infty} D_l(z_0,t) = \frac{\partial f}{\partial z}(z_0,t),\ t \in \partial D, \tag{3.5}$$

by the continuity of f on compact set $\gamma_{[z_0,z]} \times (\partial D)_T$ (recall that $(\partial D)_T = \{t \in \partial D : |t| \leq T\}$) and dominated convergence theorem,

$$\lim_{l\to+\infty}\int_{\partial D} D_l(z_0,t)dt = \int_{\partial D}\frac{\partial f}{\partial z}(z,t)dt, \tag{3.6}$$

i.e.

$$\lim_{l\to+\infty}\frac{\int_{\partial D} f(z_l,t)dt - \int_{\partial D} f(z_0,t)dt}{z_l - z_0} = \int_{\partial D}\frac{\partial f}{\partial z}(z,t)dt. \tag{3.7}$$

Since the sequence $\{z_l\}_{l=1}^{+\infty}$ is arbitrary,

$$\frac{\partial}{\partial z}\left(\int_{\partial D} f(z,t)dt\right) = \int_{\partial D} \frac{\partial f}{\partial z}(z,t)dt. \qquad \square$$

From the above lemma, we obtain an important theorem concerning differentiability of integrals of higher order schwarz kernels as follows.

Theorem 3.2. *Let $\{\, G_n(z,t)\,\}_{n=1}^{\infty}$ is the sequence of higher order Schwarz kernels defined on $\mathbf{H}\times\mathbb{R}$, then for any $n>1$ and $\gamma\in L^p(\mathbb{R})$ with $p\geq 1$,*

$$\frac{\partial^2}{\partial z\partial\overline{z}}\left(\int_{-\infty}^{+\infty} G_n(z,t)\gamma(t)dt\right) = \int_{-\infty}^{+\infty} G_{n-1}(z,t)\gamma(t)dt. \tag{3.8}$$

Proof. By Theorem 2.8, for any $n>1$,

$$\begin{aligned} G_n(z,t) &= 2\Re\Big\{\sum_{j=0}^{n-1}(\overline{z}+i)^j G_{n,j}(z,t)\Big\} \\ &= 2\Re\Big\{\sum_{j=1}^{n-1}[(\overline{z}+i)^j-(z+i)^j]G_{n,j}(z,t)\Big\} \end{aligned} \tag{3.9}$$

where all $G_{n,j}(z,t)$ fulfill that

$$j\,\partial_z G_{n,j}(z,t) = G_{n-1,j-1}(z,t) \tag{3.10}$$

and

$$|G_{n,j}(z,t)|\leq M\frac{1}{|t-z'|}$$

uniformly on $D_c\times\{t\in\mathbb{R}:|t|>T\}$ whenever $z'\in D_c$ which is any compact set in $\overline{\mathbf{H}}$, where M, T are positive constants depending only on D_c. Hence

$$\begin{aligned} \int_{-\infty}^{+\infty} G_n(z,t)\gamma(t)dt &= 2\Re\Big\{\sum_{j=0}^{n-1}(\overline{z}+i)^j\int_{-\infty}^{+\infty} G_{n,j}(z,t)\gamma(t)dt\Big\} \\ &= 2\Re\Big\{\sum_{j=1}^{n-1}[(\overline{z}+i)^j-(z+i)^j] \\ &\quad\times\int_{-\infty}^{+\infty} G_{n,j}(z,t)\gamma(t)dt\Big\}. \end{aligned} \tag{3.11}$$

Similarly, by Lemma 3.1,

$$\begin{aligned} \frac{\partial}{\partial z}\left(\int_{-\infty}^{+\infty} G_n(z,t)\gamma(t)dt\right) &= \sum_{j=1}^{n-1}\Big\{[(\overline{z}+i)^j-(z+i)^j] \\ &\quad\times\int_{-\infty}^{+\infty}\partial_z G_{n,j}(z,t)\gamma(t)dt \\ &\quad -j\,(z+i)^{j-1}\int_{-\infty}^{+\infty} G_{n,j}(z,t)\gamma(t)dt \\ &\quad +j\,(z-i)^{j-1}\int_{-\infty}^{+\infty}\overline{G_{n,j}(z,t)}\,\overline{\gamma(t)}dt\Big\}. \end{aligned}$$

Further

$$\begin{aligned}
\frac{\partial^2}{\partial z\partial\overline{z}}\left(\int_{-\infty}^{+\infty} G_n(z,t)\gamma(t)dt\right) &= \sum_{j=1}^{n-1}\Big\{(\overline{z}+i)^{j-1}\int_{-\infty}^{+\infty}\Big(j\,\partial_z G_{n,j}(z,t)\Big)\gamma(t)dt \\
&\quad +(z-i)^{j-1}\int_{-\infty}^{+\infty}\overline{\Big(j\,\partial_z G_{n,j}(z,t)\Big)}\,\overline{\gamma(t)}dt\Big\} \\
&= 2\Re\Big\{\sum_{j=1}^{n-1}(\overline{z}+i)^j\int_{-\infty}^{+\infty} G_{n-1,j-1}(z,t)\gamma(t)dt\Big\} \\
&= 2\Re\Big\{\sum_{j=0}^{n-2}(\overline{z}+i)^j\int_{-\infty}^{+\infty} G_{n,j}(z,t)\gamma(t)dt\Big\} \\
&= \int_{-\infty}^{+\infty} G_{n-1}(z,t)\gamma(t)dt. \qquad \square
\end{aligned}$$

Now we give the main result for polyharmonic Dirichlet problems in the upper half plane as follows.

Theorem 3.3. *Let $\{\,G_n(z,t)\,\}_{n=1}^{\infty}$ is the sequence of higher order Schwarz kernels defined on $\mathbf{H}\times\mathbb{R}$, then for any $n>1$, the PHD problem (1.1) is solvable and its general solution is given by*

$$u(z)=\sum_{j=1}^{n}\frac{4^j}{\pi}\int_{-\infty}^{+\infty} G_j(z,t)f_{j-1}(t)dt+u_h(z) \tag{3.12}$$

where $u_h(z)$ denotes general solution of the accompanying homogeneous PHD problem

$$\begin{cases}\Delta^n u=0 \text{ in } \mathbf{H},\\ \Delta^j u=0 \text{ on } \mathbb{R}.\end{cases} \tag{3.13}$$

Proof. By Theorem 3.2 and the inductive porperty of higher order Schwarz kernels, take polyharmonic operators Δ^l, $1\le l\le n-1$ act on two sides of (3.12),

$$\Delta^l u(z)=\sum_{j=l+1}^{n}\frac{4^{j-l}}{\pi}\int_{-\infty}^{+\infty} G_{j-l}(z,t)f_{j-1}(t)dt+\Delta^l u_h(z) \tag{3.14}$$

since the Laplacian $\Delta=\frac{1}{4}\frac{\partial^2}{\partial z\partial\overline{z}}$. Thus the nontangential boundary value

$$\Delta^l u(s)=f_l(s),\ s\in\mathbb{R},\ 0\le l\le n-1 \tag{3.15}$$

follows from (2.21), $\Delta^l u_h=0$ on $\mathbb{R}$ and the nice property of G_1, i.e.

$$\lim_{\substack{z\to s\\ z\in\mathbf{H},s\in\mathbb{R}}}\frac{1}{\pi}\int_{-\infty}^{+\infty} G_1(z,t)\gamma(t)dt=\gamma(s) \tag{3.16}$$

for any $\gamma\in L^p(\mathbb{R})$ with any $p\ge 1$. In the same way, $\Delta^n u(z)=0$ for any $z\in\mathbf{H}$. So (3.12) is a solution of the PHD problem (1.1).

Denote

$$u^*(z)=\sum_{j=1}^{n}\frac{4^j}{\pi}\int_{-\infty}^{+\infty} G_j(z,t)f_{j-1}(t)dt. \tag{3.17}$$

Obviously, u^* is a special solution of the PHD problem (1.1). Since u_h is general solution of the PHD problem (3.13), then it is immediate from linear algebra that (3.12) is general solution of the PHD problem (1.1). □

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