Pointwise Estimates for a Class of Singular Integrals and Higher Commutators

Qian Tao (钱涛)
Institute of Systems Science, Academia Sinica

Li Chun (李春)
Department of Mathematics, Peking University

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§1. Pointwise and Weak-type Estimates

Denote $M(R^k)$, $M(R^k \times R^k)$ as the space of Lebesque measurable functions defined on $R^k$, $R^k \times R^k$, respectively. Let $L:H \rightarrow M(R^k \times R^k)$ be a linear operator defined on $H$, which is a linear subspace of $M(R^k)$. There exist the following conditions on $L$ and $H$:

i) If $U$ is a convex open set in $R^k$, $x, y \in U$ and $a \in H$, then $\chi_U \cdot a \in H$, and

$$L(a)(x,y) = L(\chi_U \cdot a)(x,y),$$

where $\chi_U$ denotes the characteristic function of set $U$.

ii) There exists an operator $G: H \rightarrow M(R^k)$, such that for every open set $V \subset R^k$,

$$G(\chi_V a) = \chi_V G(a).$$

iii) Denote $Q$ as a cube in $R^k$, its sides are parallel to the axes, $A_r$ as the Hardy-Littlewood maximal function of $|f|^r$, $r \in [1, \infty)$, and $A_\infty(f) = \|f\|_\infty$. Let $2Q$ be the double of $Q$, and

$$A(a,x,y) = \sup_{Q \subset \Omega \subset \Omega^*} \frac{1}{|Q|} \int_{Q \setminus (x-t)} \frac{|x-y|}{|x-t|} |L(a)(x,y) - L(a)(t,y)| dt.$$  

$$B(a,x,y) = \sup_{Q \subset \Omega \subset \Omega^*} \frac{1}{|Q|} \int_{Q \setminus (x-t)} \frac{|x-y|}{|x-t|} |L(a)(x,y) - L(a)(y,t)| dt.$$  

Then for a certain $r \in (1, \infty]$, and every $a \in H$, $b = G(a),

$$\Lambda_r(L(a)(x,\cdot))(x) \leq C \Lambda_r(\Lambda_1(b))(x) \quad \text{a.e.} \quad (1.1)$$

$$\Lambda_r(L(a)(\cdot,y))(x) \leq C \Lambda_r(\Lambda_1(b))(x) \quad \text{a.e.} \quad (1.2)$$
\begin{align}
\Lambda(A(a, x))(x) &\leq C\Lambda(A(b))(x) \quad \text{a.e.} \tag{1.3} \\
\Lambda(B(a, x))(x) &\leq C\Lambda(A(b))(x) \quad \text{a.e.} \tag{1.4}
\end{align}

in which the constants \( C \) are independent of \( a \).

Let function \( K \in C^\infty(R^k \setminus \{0\}) \), we shall refer to the following inequalities as the standard estimates (on the kernel \( K \)):

iv) For every \( x \neq 0 \),

\[
|K(x)| \leq \frac{C}{|x|^k}, \quad |\nabla K(x)| \leq \frac{C}{|x|^{k+1}},
\]

where \( C \) are constants.

Denote

\[ T_\varepsilon(a, f)(x) = \int_{|x-y| > \varepsilon} L(a)(x, y)K(x-y)f(y)dy, \]

there exist the following conditions:

v) For a certain pair of \( p_1 \in (1, \infty), r_1 \in (1, \infty) \) such that \( q_1^{-1} = p_1^{-1} + r_1^{-1} \in (1, \infty) \), and for every \( f \in L^{p_1}(R^k) \), every \( a \in H, b = G(a), \varepsilon \in (0, \infty) \),

\[
\|T_\varepsilon(a, f)\|_{q_1} \leq C\|b\|_{r_1} \cdot \|f\|_{p_1},
\]

where the constant \( C \) is independent of \( \varepsilon \).

vi) With the notation as in v), the limite

\[ T(a, f)(x) = \lim_{\varepsilon \to 0} T_\varepsilon(a, f)(x) \]

exists a.e., and

\[
\|T(a, f)\|_{q_1} \leq C\|b\|_{r_1} \cdot \|f\|_{p_1},
\]

\( C \) is a constant.

Our main theorem is as follows.

**Theorem 1.** With the notation as above, there follow

1°. With the conditions i), ii), (1.1), (1.3), iv) and one of two conditions v) and vi) we have

\[
M(a, f)(x) = \sup_{\varepsilon > 0} |T_\varepsilon(a, f)(x)| \leq C(\Lambda_1(T(a, f))(x) + \Lambda_1(A(b))(x) + \Lambda_1(f)(x)), \quad \text{a.e.,}
\]

where \( p_1, q_1, r_1 \) are as in v) \( T(a, f) \) is as in vii) or is a weak-star accumulation point of the bounded family of continuous linear functionals \( \{T_\varepsilon(a, f)\}_{\varepsilon > 0} \subset (L^{q_1})^* \).

2°. Suppose the extra conditions (1.2), (1.4) are satisfied besides all the conditions in 1°, then for \( p_0, 1 = p_0^{-1} + r_1^{-1} \), \( r_1 \) is as in 1°, \( f \in L^{p_0}(R^k) \), we have

\[
\|\{x \in R^k: M(a, f)(x) > \lambda\}\| \leq C\|b\|_{r_1} \cdot \|f\|_{p_0}.
\]

3°. With \( r_1 = \infty \) in 1°, then for \( q \in (1, q_1), f \in L^q(R^k) \), there exists \( \|M(a, f)\|_q \leq C\|b\|_{r_1} \cdot \|f\|_q \).
the constants $C$ in $1^\circ$, $2^\circ$, $3^\circ$ depend only on the dimension $k$ and the constants appearing in iii) — vi).

Proof. Since $1^\circ$ implies the weak-type $(p_1, q_1)$ of $M(a,)$ see [2], remark 1), then $3^\circ$ follows from $1^\circ$, $2^\circ$ and Marcinkiewicz interpolation theorem. So we only need to prove $1^\circ$ and $2^\circ$.

Proof of $1^\circ$. Suppose that the condition v) is satisfied. If otherwise the condition vi) is satisfied, the proof is even simpler. Fix $x \in \mathbb{R}^k$, $\delta \in (0, \infty)$, denote $\chi_{\delta} = \chi_{S(x, \delta)}$, where $S(x, \delta)$ denotes the ball of center $x$ and radius $\delta$, then for $\varepsilon \in (0, \delta)$, we have

$$T_{\varepsilon}(a, f - \chi_{\delta}f) = T_{\delta}(a, f - \chi_{\delta}f). \quad (1.8)$$

By Banach - Alaoglu theorem, there is a sequence $\varepsilon_n$ such that $\lim \varepsilon_n = 0$, and for all $f \in L^{p_1}(\mathbb{R}^k)$, $T_{\varepsilon_n}(a, f)$ converges weak-star to a $T(a, f) \in L^{q_1}(\mathbb{R}^k)$, and

$$\|T(a, f)\|_{q_1} \leq C \|b\|_{r_1} \|f\|_{p_1}.$$ 

Passing to the limit $\varepsilon = \varepsilon_n \to 0$ in (1.8), we have

$$T(a, f - \chi_{\delta}f) = T_{\delta}(a, f - \chi_{\delta}f),$$

and therefore, for $t \in S(x, \delta/2)$, a.e. $x$,

$$T_{\delta}(a, f)(x) - T(a, f)(t) + T(a, \chi_{\delta}f)(t)$$

$$= \int_{|x - y| > \delta} f(y) \left(L(a)(x, y)K(x - y) - L(a)(t, y)K(t - y)\right) dy$$

$$= \int_{|x - y| > \delta} f(y) \sum_{i=1}^{2} \Delta_i(a, x, y, t) dy$$

where

$$\Delta_1 = L(a)(x, y)(K(x - y) - K(t - y)),$$

$$\Delta_2 = (L(a)(x, y) - L(a)(t, y))K(t - y).$$

Because for $|x - y| > \delta$, $|t - x| < \delta/2$ the condition iv) gives that

$$|K(x - y) - K(t - y)| \leq C\delta|x - y|^{-k-1}$$

we have

$$|\Delta_1| \leq C\delta|L(a)(x, y)||x - y|^{-k-1}.$$ 

Together with

$$\Delta_2 \leq C\delta \frac{|x - y|}{|x - t|} |L(a)(x, y) - L(a)(t, y)||x - y|^{-k-1}$$

there follows

$$|T_{\delta}(a, f)(x)| \leq |T(a, f)(t)| + |T(a, \chi_{\delta}f)(t)|$$

$$+ C \left( \int_{|x - y| > \delta} \frac{\delta|f(y)||L(a)(x, y)||x - y|^{-k-1}}{|x - y|^{k+1} |x - t|} dy + \int_{|x - y| > \delta} \frac{\delta|f(y)||x - y||L(a)(x, y) - L(a)(t, y)||x - y|^{k+1}}{|x - y|^{k+1} |x - t|} dy \right).$$
Integrating both sides of the last inequality in \( t \) over \( S(x, \delta/2) \) and dividing by \( |S(x, \delta/2)| \), we get
\[
|T_d(a, f)(x)| \leq \Lambda_1(T(a, f))(x) + C \left( \frac{1}{|S(x, \delta/2)|} \right) \int_{S(x, \delta/2)} |T(a, \chi_{\delta/2})(t)| dt
\]
\[
+ \int_{|x - y| > \delta} \frac{\delta |f(y)L(a)(x, y)|}{|x - y|^{k+1}} dy + \int_{|x - y| > \delta} \frac{\delta |f(y)|A(a, x, y)}{|x - y|^{k+1}} dy
\]
\[
= \Lambda_1(T(a, f))(x) + C \sum_{i=1}^{\delta} I_i.
\]

For \( I_2 \) we have, by using (1.1)
\[
I_2 = \sum_{i=0}^{\infty} \int_{2^{i+1}\delta < |x - y| < 2^{i+3}\delta} \frac{\delta |f(y)L(a)(x, y)|}{|x - y|^{k+1}} dy
\]
\[
\leq C \sum_{i=1}^{\infty} \left( \frac{\delta}{2^{i+1}\delta} \right)^k \left( \frac{1}{2^{i+1}\delta} \right)^k \int_{|x - y| < 2^{i+1}\delta} |f(y)L(a)(x, y)| dy
\]
\[
\leq C \sum_{i=1}^{\infty} \left( \frac{1}{2^{i+1}\delta} \right)^k \left( \frac{1}{2^{i+1}\delta} \right)^k \int_{|x - y| < 2^{i+1}\delta} |f(y)| dy \left( \int_{|x - y| < 2^{i+1}\delta} |L(a)(x, y)| dy \right)^{\frac{1}{q}}
\]
\[
\leq C \Lambda_{r_1}(\Lambda_{r_1}(b))(x)\Lambda_{r_4}(f)(x) \leq C \Lambda_{r_1}(\Lambda_{r_1}(b))(x)\Lambda_{r_1}(f)(x) \quad \text{a.e.} \quad (1.9)
\]

By the same method we can obtain the same estimate for \( I_3 \). To see \( I_1 \), from the condition i), \( \forall \varepsilon \in (0, \delta) \), we have
\[
T_\varepsilon(a, \chi_{\delta/2})(t) = \int_{|t - y| < \varepsilon} L(a)(t, y)K(t - y)f(y) dy
\]
\[
= \int_{|t - y| < \varepsilon} L(\chi_{\delta/2})(t, y)K(t - y)f(y) dy
\]
\[
= T_\varepsilon(\chi_{\delta/2}(a, \chi_{\delta/2})(t).
\]

Passing to the limite, there follows
\[
T(a, \chi_{\delta/2})(t) = T(\chi_{\delta/2}(a, \chi_{\delta/2})(t) \quad \text{for} \quad t \in S(x, \delta/2).
\]

By using Hölder inequality, we have
\[
I_1 \leq C \frac{1}{\delta^k} \int_{|t - x| < \delta/2} |T(a, \chi_{\delta/2})(t)| dt = \frac{1}{\delta^k} \int_{|t - x| < \delta/2} |T(\chi_{\delta/2}(a, \chi_{\delta/2})(t)| dt
\]
\[
\leq C \frac{\delta^{k+1}}{\delta^k} \|T(\chi_{\delta/2}(a, \chi_{\delta/2})\|_{q_1}
\]
\[
\leq C \delta^{-k+1} \|\chi_{\delta/2}\|_{r_1} \|\chi_{\delta/2}\|_{s_1} \leq C \Lambda_{r_1}(b)(x)\Lambda_{r_1}(f)(x) \quad \text{a.e.}
\]

Thus the proof of 1° is concluded.
Proof of $2^\circ$. We need the following lemma (for the proof see [2]).

**Lemma 1.** If $S$ is a sublinear operator of weak-type $(\rho_0, q_0)$, a sufficient condition that $S$ is also of weak-type $(p, q)$, where $p^{-1} - q^{-1} = \rho_0^{-1} - q_0^{-1}$, $\rho_0 > p \geq 1$, is that for every sequence of pairwise disjoint cubes $Q_i$, which satisfies the Whitney decomposition condition:

$$d(Q_i) \leq \text{dist}(Q_i, (\bigcup Q_i)') \leq 4d(Q_i) \quad \text{for every } i,$$

and for every function $h$ in $L^p(\mathbb{R}^k)$ having support in $\bigcup Q_i$ such that

$$\int_{Q_i} h(x)dx = 0 \quad \text{for every } i,$$

the following estimate holds

$$\{ x \in \mathbb{R}^k : Q_i : S[h](x) > \lambda \} \leq C(\|h\|_p/\lambda)^{\rho_0},$$

(1.10)

where $Q_i^* = 2Q_i$.

By applying Lemma 1 to the sublinear operator $M(a, f)$, which is known to be of weak-type $(\rho_1, q_1)$, $q_1 > 1$, $q_1^{-1} - p_1^{-1} = r_1^{-1}$, we need to show that the condition (1.10) is satisfied.

Let $\{Q_i\}, h \in L^{p_0}(\mathbb{R}^k)$ be as in Lemma 1, fix $x \in \mathbb{R}^k \setminus \{0\}$ and $\varepsilon > 0$, denote

$$I(x, \varepsilon) = \{ i : Q_i \cap S(x, \varepsilon) = \emptyset \},$$

$$J(x, \varepsilon) = \{ i : Q_i \cap S(x, \varepsilon) \neq \emptyset, \quad Q_i \setminus S(x, \varepsilon) \neq \emptyset \},$$

then

$$T_i(a, h)(x) = \sum_{i = 1}^{\infty} \int_{Q_i \setminus S(x, \varepsilon)} \cdots dy = \sum_{i \in I(x, \varepsilon)} \int_{Q_i} \cdots dy + \sum_{i \in J(x, \varepsilon)} \int_{Q_i \setminus S(x, \varepsilon)} \cdots dy,$$

where each of the integrands is $L(a)(x, y)K(x - y)h(y)$. By the property of Whitney decomposition there are constants $\alpha, \beta > 0$ such that for every $i \in J(x, \varepsilon)$, we have

$$Q_i \subset \{ y : \alpha \varepsilon < |y - x| < \beta \varepsilon \},$$

so that

$$\sum_{i \in J(x, \varepsilon)} \int_{Q_i \setminus S(x, \varepsilon)} |L(a)(x, y)K(x, y)h(y)|dy$$

$$\leq \int_{\alpha \varepsilon < |x - y| < \beta \varepsilon} |L(a)(x, y)|K(y)h(y)dy$$

$$\leq \int_{\alpha \varepsilon < |x - y| < \beta \varepsilon} |L(a)(x, y)|^{\rho_0} h(y)dy$$

$$\leq C \left( \frac{1}{\varepsilon^k} \int_{|x - y| < \beta \varepsilon} |L(a)(x, y)|^{\rho_0} dy \right)^{1/\rho_0} \left( \frac{1}{\varepsilon^k} \int_{|x - y| < \beta \varepsilon} |h(y)|^{\rho_0} dy \right)^{1/\rho_0}$$

$$\leq CA_{r_1}(A_1(b))(x)A_{p_0}(h)(x),$$

(1.11)
where we have used \( r_1 = p'_0 \).

For \( i \in I(x, \varepsilon) \) we will show that

\[
\left| \int_{Q_i} \cdots \, dy \right| = A_i(x) \leq C \delta_i \int_{Q_i} \frac{|L(a)(x, y)|}{|x - y|^{k+1}} |h(y)| \, dy + C \delta_i \int_{Q_i} \frac{E(a, x, y)}{|x - y|^{k+1}} |h(y)| \, dy \tag{1.12}
\]

where \( \delta_i = d(Q_i) \). Let \( t \in Q_i \), since \( \int_{Q_i} h(y) \, dy = 0 \), we have

\[
\int_{Q_i} L(a)(x, y)K(x - y)h(y)\,dy = \int_{Q_i} (L(a)(x, y)K(x - y) - L(a)(x, t)K(x - t))h(y)\,dy
\]

\[
= \int_{Q_i} h(y) \sum_{i=1}^{2} \Delta_i(a, x, y, t)\,dy \tag{1.13}
\]

where

\[
\Delta_1 = L(a)(x, y)(K(x - y) - K(x - t)),
\]

\[
\Delta_2 = (L(a)(x, y) - L(a)(x, t))K(x - t).
\]

Integrating in \( t \) over \( Q_i \), both sides of (1.13), dividing by \( |Q_i| \), as we did in the proof of 1° we get (1.12).

From (1.11), (1.12), there follows

\[
M(a, h)(x) \leq C \Lambda_{r_1} (\Lambda_1(b))(x) \Lambda_{p_0}(h)(x) + \sum_{i=1}^{\infty} A_i(x).
\]

The condition (1.10) will be satisfied if we show that

\[
|\{x \in \mathbb{R}^k : \Lambda_{r_1} (\Lambda_1(b))(x) \Lambda_{p_0}(h)(x) > \lambda\}| \leq C \frac{\|b\|_{r_1} \|h\|_{p_0}}{\lambda} \tag{1.14}
\]

and

\[
|\{x \in \mathbb{R}^k \setminus \bigcup Q_i^* : \sum_{i=1}^{\infty} A_i(x) > \lambda\}| \leq C \frac{\|b\|_{r_1} \|h\|_{p_0}}{\lambda} \tag{1.15}
\]

For (1.14) see [2] Remark 1, it remains to show (1.15) only. In fact we have

\[
\sum_{i=1}^{a} \int_{\mathbb{R}^k \setminus Q_i^*} A_i(x) \, dx \leq \sum_{i=1}^{\infty} \int_{\mathbb{R}^k \setminus Q_i^*} A_i(x) \, dx.
\]

There exists a constant \( \gamma \), which depends only on the dimension \( k \), such that if \( x \in Q_i^*, y \in Q_i \), then \( |x - y| > \gamma \delta_i \). Thus, according to (1.2), (1.4), using the same method in proving (1.9) we have
\[
\int_{\mathbb{R}^n \setminus \mathcal{Q}_1} A_i(x) \, dx \leq C \int_{\mathcal{Q}_1} \left( \int_{|x-y| > \gamma \delta_i} \frac{\delta_i |L(a)(x,y)|}{|x-y|^{k+1}} \, dx \right) |h(y)| \, dy \\
+ C \int_{\mathcal{Q}_1} \left( \int_{|x-y| > \gamma \delta_i} \frac{\delta_i L_i(b)(x,y)}{|x-y|^{k+1}} \, dx \right) |h(y)| \, dy \\
\leq C \int_{\mathcal{Q}_1} \Lambda_i(\Lambda_i(b))(y) |h(y)| \, dy.
\]

Therefore,
\[
\sum_{i=1}^{\infty} \int_{\mathbb{R}^n \setminus \mathcal{Q}_1} A_i(x) \, dx \leq C \int_{\mathbb{R}^n} \Lambda_i(\Lambda_i(b))(y) |h(y)| \, dy \leq C \|b\|_{r_i} \|h\|_{p_0}.
\]

The proof is thus finished.

Theorem 1 has the following extension:

**Theorem 2.** Suppose \(H_p, L_i, G_i\), and \(K\) are as in Th. 1, where \(i = 1, \ldots, n\). Denoting \(a = (a_1, \ldots, a_n)\), \(b = (G_1(a_1), \ldots, G_n(a_n))\) and

\[
L(a)(x,y) = \prod_{i=1}^{n} L_i(a_i)(x,y),
\]

\[
\|b\|_r = \prod_{i=1}^{n} \|b_i\|_{r_i},
\]

\[
\Lambda_i(\Lambda_i(b))(x) = \prod_{i=1}^{n} \Lambda_i(\Lambda_i(b_i))(x),
\]

\[
r = (r_1, \ldots, r_n),
\]

where \(r_i\)'s satisfy one of the following conditions:

1°. \(\forall \ i, \ r_i \in (1, \infty)\),

2°. \(\forall \ i, \ r_i = \infty\).

If for \(q: q^{-1} = p^{-1} + \sum_{i=1}^{n} r_i^{-1}\), for every \(i\) the conditions i) - iv) and one of v) and vi), which is with respect to \(p_i \in (1, \infty)\), are satisfied, then the conclusions of Th. 1 hold in the case of \(p = p_i\), \(q \in (1, \infty)\) for the conclusion 1°, \(q = 1\) for the conclusion 2° and \(q \in (1, q_1)\), \(r_i = \infty\) for the conclusion 3°, respectively.

The proof of Th. 2 is similar to the proof of Th. 1. We only point out following modification.

1°. To deal with the difference

\[
L(a)(x,y) - L(a)(t,y)
\]

we use the following formula:
\[
\prod_{i=1}^n b_i - \prod_{i=1}^n a_i = \sum_{j=1}^n \left( \prod_{i=1}^{j-1} a_i \right) (b_j - a_j) \left( \prod_{k=j+1}^n b_k \right) \text{ with } \prod_{i=1}^0 a_i = \prod_{k=n+1}^n b_k = 1.
\]

2°. Instead of using Hölder inequality to two factors we use Hölder inequality to \( n + 1 \) factors each time.

**Remark 1.** The condition iv) can be substituted by the following condition: \( K(x) = \frac{\Omega(x)}{|x|^k} \), where \( \Omega: R^k \rightarrow C \) satisfies the conditions: \( \Omega \) is homogeneous of degree 0, bounded, and

\[
\frac{1}{|S(x, \delta)|} \int_{S(x, \delta)} |\Omega(x - y) - \Omega(t - y)| dt \leq C \frac{\delta}{|x - y|}, \text{ for } |x - y| > 2\delta.
\]

§II. Application, Higher Commutators

**Theorem 3.** with the notation as in Th. 2, let \( H_t \) be the Space of the functions whose all derivatives of order \( m_i \) belong to \( L^r(R^n) \), \( G(a_1) = \sum_{|\beta| = n} |\partial^\beta a_1| \), \( L_t(a_1)(x, y) = \frac{P_m(a_1, x, y)}{|x - y|^m} \), where

\[
P_m(a_1, x, y) = a_1(x) - \sum_{|\beta| < \infty} \frac{(\partial^\beta a_1)(y)}{\beta!} (x - y)^\beta \text{ for } m \in \mathbb{Z}, \text{ which is the set of positive integers, and}
\]

\[P_0(a_1, x, y) = a_1(x).\] Here \( K(x) = \frac{\Omega(x)}{|x|^k} \), \( \Omega(x) \) satisfies the conditions mentioned in Remark 1, and satisfies 1°. \( \Omega(-x) = (-1)^{m+1} \Omega(x), |m| = \sum_{i=1}^n m_i, \text{ or } 2°. \int_{S^k-1} \Omega(x) x^\alpha d\sigma(x) = \text{ for } \forall \alpha \text{ such that } |\alpha| \leq |m|. \] Then the conclusions of Th. 2 hold.

Proof of Theorem 3. It is easy to see that for all i, conclusions i), ii) are satisfied. vi) follows from the main results of [4]. In order to use Th. 2 (exactly, Remark 1), we only need to examine iii). The following lemma is needed.

**Lemma 2.** \( \frac{|P_m(a_1, x, y)|}{|x - y|^m} \leq C(\Lambda_1(|\nabla^m a|)(x) + \Lambda_1(|\nabla^m a|)(y)), \) where \( m \in \mathbb{Z} \), and all the partial derivatives of order \( m \) of \( a \in M(R^n) \) are locally integrable.

Proof. The argument is similar to [5], Lemma 5. In fact, there we obtain that

\[
\frac{|P_m(a_1, x, y)|}{|x - y|^m} \leq I_1 + I_2,
\]

where

\[
I_1 \leq C \frac{1}{\varepsilon} \int_{|\xi| < \varepsilon} \frac{|\nabla^m a(y - \xi)|}{|\xi|^{m-1}} d\xi,
\]

\[
I_2 \leq C \frac{1}{\varepsilon^m} \int_{|u| < \varepsilon^m} |u|^{m-k} |\nabla^m a(x - u)| du.
\]
Using the method in proving (1.9), we obtain that

\[ I_1 \leq C \Lambda_1(|\nabla^m a|)(y), \quad I_2 \leq C \Lambda_1(|\nabla^m a|)(x). \]

By using the lemma, it is easy to prove (1.1), (1.2), so we only need to prove (1.3) and (1.4).

Proof of (1.3). For \( x, t \in Q, \ y \in 2Q \), we have

\[
\frac{1}{|Q|} \int_Q \frac{|x - y|}{|x - t|} \left| \frac{P_m(a, x, y) - P_m(a, t, y)}{|x - y|^m} \right| \, dt \\
\leq \frac{1}{|Q|} \int_Q \frac{|x - y|}{|x - t|} \left| \frac{1}{|x - y|^m} - \frac{1}{|t - y|^m} \right| |P_m(a, x, y)| \, dt \\
\quad + \frac{1}{|Q|} \int_Q \frac{|x - y|}{|x - t|} \left| \frac{1}{|t - y|^m} |P_m(a, x, y) - P_m(a, t, y)| \, dt \\
= I_1 + I_2,
\]

where

\[ I_1 \leq C(\Lambda_1(|\nabla^m a|)(x) + \Lambda_1(|\nabla^m a|)(y)). \]

To see \( I_2 \), using the formula

\[ P_m(a, x, y) - P_m(a, t, y) = \int_0^1 \nabla_x P_m(a, x - s(x - t), y) \cdot (x - t) \, ds \]

and

\[ \nabla_x P_m(a, x, y) = P_{m-1}(\nabla a, x, y), \]

which is a vector valued equality, we have

\[ I_2 \leq \int_0^1 ds \frac{1}{|Q_{x,s}|} \int_{Q_{x,s}} |P_{m-1}(\nabla a, x, y)| \, dz \]

where \( Q_{x,s} = x - s(x - Q), \ x \in Q_{x,s} \).

Therefore

\[ I_2 \leq C \int_0^1 ds \frac{1}{|Q_{x,s}|} \int_{Q_{x,s}} (\Lambda_1(|\nabla^m a|)(x) + \Lambda_1(|\nabla^m a|)(y)) \, dz \]

\[ \leq C(\Lambda_1(|\nabla^m a|)(x) + \Lambda_1(|\nabla^m a|)(y)), \]

and so \( \mathcal{A}(a, x, y) \) have the same estimate. Hence for all \( Q_1 \) and each \( x \in Q_1 \), we conclude

\[
\left( \frac{1}{|Q_1|} \int_{Q_1} \mathcal{A}(a, x, y)^p \, dy \right)^{\frac{1}{p}} \leq C \Lambda_p(\Lambda_1(|\nabla^m a|))(x).
\]
Proof of (1.4). Now we use the vector valued inequality

\[ \nabla_\xi P_m(a, x) = \frac{-1}{(m - 1)!} \left( \sum_{j=1}^k (x_j - \xi_j) \frac{\partial}{\partial \xi_j} \right)^{m-1} \nabla a(\xi), \]

there follows

\[ P_m(a, y, x) - P_m(a, y, t) = \int_0^1 \frac{-1}{(m - 1)!} \left( \sum_{j=1}^k (x_j - \xi_j) \frac{\partial}{\partial \xi_j} \right)^{m-1} \nabla a(\xi) \bigg|_{\xi = x - s(x - t)} \cdot (x - t) ds. \]

Therefore

\[ I_2 \leq C \int_0^1 ds \frac{1}{|Q|} \int_{\partial Q} |(\nabla^m a)(x - s(x - t))| dt \]
\[ \leq C \int_0^1 ds \frac{1}{|Q_{x,s}|} \int_{Q_{x,s}} |(\nabla^m a)(\xi)| d\xi \]
\[ \leq CA_1(|\nabla^m a|(x)). \]

The proof is thus finished.

Remark 2. In virtue of condition v), in Th. 3, the extra condition 1° or 2° upon \( K(x) \) can be substituted by some weaker conditions. For example, when \( n = 1 \), neither of the two conditions are necessary (see [3]).

We turn to the higher commutators of multiplier operators.

Let \( m = (m_1, \ldots, m_n) \in (Z \cup \{0\})^n \), \( \alpha = (\alpha_1, \ldots, \alpha_n) \in (R^k)^n \),

\[ R_{(a)}^{(m)} = R_{a_1}^{m_1} \cdots R_{a_n}^{m_n}, \]

\[ R_{a_n}^{m_n} g(\xi) = g(\xi - \alpha_n) - \sum_{|\beta| < m_n} \frac{\partial^\beta g(\xi)}{\beta!} (-\alpha_n)^\beta, \]

\[ R_{a_n}^{-m_n} g(\xi) = g(\xi - \alpha_n), \quad \forall \ i. \]

Denote

\[ M^l = \{ \omega \in C^\infty (R^k \setminus \{0\}); \quad \forall \ \beta, \exists \ C_\beta \text{ such that } |\partial^\beta \omega(\xi)| \leq C_\beta |\xi|^{-|\beta|} \}, \]

and for \( \alpha = (\alpha_1, \ldots, \alpha_n) \), \( a_i \in S(R^k) \), define

\[ T_{R_{(a)}^{(m)} a_0 \omega(\xi)}(a, f)(x) = \int_{R^{k\times l}} e^{ix\xi} R_{(a)}^{(m)} a_0(\xi) \hat{a}(\xi) \hat{f}(\xi - [x]) d\xi d\xi, \]

where \( \hat{a}(\alpha) = \prod_{i=1}^n \hat{a}(\alpha_i) \), \( [x] = \sum_{i=1}^n \alpha_i \), \( d\alpha = d\alpha_1 \cdots d\alpha_n \), and denote \( m + 1 = (m_1 + 1, \ldots, m_n + 1) \). We have

Theorem 4. If \( \omega \in M^l \), \( l = |m| = \sum_{i=1}^n m_i \), then
1. \[ \| T_{R^{[m]}}(a, f) \|_q \leq C \prod_{i=1}^n \| \nabla^m i a_i \|_{r_i} \| f \|_p, \]

where \( q^{-1} = p^{-1} + \sum_{i=1}^n r_i^{-1}, \) \( p, q, r_i \in (1, \infty), \forall i. \)

2. \[ \{ x : |T_{R^{[m]}}(a, f)(x)| > \lambda \} \leq C \frac{1}{\lambda} \prod_{i=1}^n \| \nabla^m i a_i \|_{r_i} \| f \|_p, \]

where \( 1 = p^{-1} + \prod_{i=1}^n r_i^{-1}, \) \( p, r_i \in (1, \infty), \forall i. \)

3. \[ \| T_{R^{[m]}}(a, f) \|_p \leq C \prod_{i=1}^n \| \nabla^m i a_i \|_{BMO} \| f \|_p, \]

where \( p \in (1, \infty), \) and in every case \( C \) is a constant independent of \( a, f. \)

Proof. 1° is a known result ([6], Th. 1). To prove 2° choose \( \phi \in C_0^\infty(\mathbb{R}^k) \) such that \( \text{supp} \ \phi \subset \{ 1/2 \leq |\xi| \leq 2 \}, \sum_{\xi \in \xi_0} \phi(2^{-j} \xi) = 1 \) for \( \xi \neq 0. \) Let \( \omega_N(\xi) = \sum_{\xi \in \xi_0} \omega(\xi) \phi(2^{-j} \xi), K_N = (\omega_N)^\vee, \) which denotes the inverse Fourier transformation of \( \omega_N. \) By a standard argument we get

\[ |K_N(x)| \leq \frac{C}{|x|^{k+1}}, \quad |\nabla K_N(x)| \leq \frac{C}{|x|^{k+2}}, \quad (2.1) \]

where \( C \) are independent of \( N. \)

Denote

\[ T_N^{[m]}(a, f) = T_{R^{[m]}}(a, f), \]

by a known result ([7], Th. 1)

\[ T_N^{[m]}(a, f)(x) = \int_{\mathbb{R}^k} \prod_{i=1}^n \frac{P_{m_i}(x, y)}{|x - y|^{s_i}} K_N(x - y)|x - y| f(y) dy. \]

From conclusion 1° of the theorem (1.7) holds for \( T_N^{[m]} \) and

\[ T_N^{[m]}(a, f)(x) = \lim_{r \to 0} (T_N^{[m]})_r(a, f)(x), \quad x \in \mathbb{R}^k, \quad (2.2) \]

so the condition vi) is satisfied. By using Th. 3 we obtain

\[ \{ x : |T_N^{[m]}(a, f)(x)| > \lambda \} \leq C \frac{1}{\lambda} \prod_{i=1}^n \| \nabla^m i a_i \|_{r_i} \| f \|_p, \quad (2.3) \]

where the constant \( C \) is independent of \( N. \)

From (2.2), we have

\[ \{ x : |T_N^{[m]}(a, f)(x)| > \lambda \} \subset \bigcup_{i=1}^{\infty} \bigcap_{N > i} \{ x : |T_N^{[m]}(a, f)(x)| > \lambda \}, \]

and thus the conclusion 2° holds.
To prove $3^o$, first, we have
\[ R^{(m,-)}_i \omega(\xi) = \sum_{\begin{array}{c} n \rightarrow m, 1, \ldots, m, 1, \ldots, 1 \\ \alpha = \ldots \alpha \\ i \in \ldots i \in i \in \\ \{0\} \{0\} \{0\} \\ \beta \in \ldots \beta^{(n)}, \beta \neq i_j \end{array}} C_{\alpha, \beta} R^{(m,-)}_i \omega(\xi)(-\alpha)^\beta + R^{(m,-)}_i \omega(\xi), \]
and then by using the induction on $n$ we conclude that (1.7) holds for $T^{(m+1)}_N$. So, from Th. 3 we get the weak-type estimate for the maximal operator of $T^{(m+1)}_N$, together with the property (2.1) of kernel $K(x)$. By using the same method as in [5], $3^o$ holds for $T^{(m+1)}_N$ with a constant independent of $N$, then by Fatou’s lemma we conclude $3^o$ for $T^{(m+1)}_N(a, f)$. 

A partial extension of Th. 4 is as follows.

**Theorem 5.** For $\omega \in M^t$ and $\gamma_i \in (\mathbb{Z} \setminus \{0\})^n$, $i = 1, 2$, such that $l + |\gamma_1| + |\gamma_2| = |\omega|, |\gamma_1| \leq \min_{1 \leq i \leq k} \{m_i\}$, then exist

1. $\|\partial^\gamma_1 T^{(m+1)}_N(a, \partial^\gamma_2 f)\|_q \leq C \prod_{i=1}^n \|\nabla^m_i a_i\|_r \cdot \|f\|_p$, where $p, q \in (1, \infty), \forall i, r_i \in (1, \infty)$ or

2. $\|\{x: |\partial^\gamma_1 T^{(m+1)}_N(a, \partial^\gamma_2 f)(x)| > \lambda\} \| \leq C \prod_{i=1}^n \|\nabla^m_i a_i\|_r \cdot \|f\|_p$, where $\lambda \in [1, \infty)$, \forall i, $r_i \in (1, \infty)$ or $\forall i, r_i \to \infty, 1 = p^{-1} + \sum_{i=1}^n \frac{1}{r_i}$. And in every case the constant $C$ is independent of $a, f$.

Proof. In the case of $r_i \in (1, \infty), \forall i$, the inequality in $1^o$ is a known result ([6], Th. 2). For the rest part of $1^o$, according to $3^o$ of Th. 4 and equation (2.4), we have the inequality in $\gamma_1 = \gamma_2 = 0$. By means of an induction on $(\gamma_1, \gamma_2)$ (see [6], Th. 2) we obtain the inequality in general case.

To prove $2^o$, as before, we use the induction for the first case of $r_i$ with the starting inequality in $\gamma_1 = \gamma_2 = 0$, which comes from $2^o$ of Th. 4. For the second case of $r_i$ when $\gamma_1 = \gamma_2 = 0$, using $1^o$ to $T^{(m+1)}_N(a, f)$, together with (2.1), we conclude that $T^{(m+1)}_N(a, f)$ is a Calderon-Zygmund operator, so the weak-type inequality holds with a constant $C$ independent of $N$. Passing to the limite $N \to \infty$, we get the conclusion. For the general case we use the induction too.

**References**


