Some Remarks on the Boundary Behaviors of the Hardy Spaces

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In memory of Jaime Keller

Abstract. Some estimates and boundary properties for functions in the Hardy spaces are given.

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1. Introduction

Let $\mathbb{D} = \{ z = x + iy \in \mathbb{C} : |z| < 1 \}$ be the open unit disc. The holomorphic Hardy space $H^p(\mathbb{D})$ ($1 \leq p < \infty$) consists of all functions $f$ that are holomorphic in $\mathbb{D}$ and satisfy

$$\|f\|_p = \sup_{0 < r < 1} \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{1/p} < \infty.$$

Getting close to the boundary of $\mathbb{D}$, singularities may happen for functions in $H^p(\mathbb{D})$, where we have the well known estimate (cf. [3])

$$(1 - |z|)^{1/p} |f(z)| \leq C_p \|f\|_p \quad \text{for } 1 \leq p < \infty.$$

By using the density of the holomorphic polynomials (cf. [9]), or that of the Poisson integrals (cf. [5]), one can prove that

$$\lim_{|z| \to 1^-} (1 - |z|)^{1/p} |f(z)| = 0 \quad \text{for } 1 \leq p < \infty,$$

which is more precise than the previous inequality near the boundary.

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In the case $p = 2$, $\mathcal{H}^p(\mathbb{D})$ is of particular importance. It is a Hilbert space with the inner product

$$\langle f, g \rangle = \int_0^{2\pi} f(e^{i\theta})\overline{g(e^{i\theta})}d\theta, \quad f, g \in \mathcal{H}^2(\mathbb{D}).$$

In a number of practical applications as the underlying space $\mathcal{H}^2(\mathbb{D})$ plays an important role (e.g., in signal processing, image processing and coding theory). Observing that for any function $f \in \mathcal{H}^2(\mathbb{D})$, we have

$$\langle f, \phi_a \rangle = \sqrt{1 - |a|^2} f(a),$$

where $\phi_a(z) = \frac{\sqrt{1-|a|^2}}{1-\overline{a}z}$ is a unit vector of $\mathcal{H}^2(\mathbb{D})$ with the parameter $a \in \mathbb{D}$. By the aforementioned property, we get

$$\lim_{|a| \to 1^-} |\langle f, \phi_a \rangle| = 0,$$

which implies that there exists $a^* \in \mathbb{D}$ such that $|\langle f, \phi_{a^*} \rangle|$ attains the maximum value. This is crucial for the signal adaptive decomposition methods (as a variation and realization of greedy algorithm) introduced in [5, 8].

In this note, we give a generalization of the above result to higher dimensions, of which the special cases have been applied to the adaptive decompositions of functions of several variables ([6, 7]). Our method is a modification of the classic method (see [3, page 18]), which depends on some more delicate estimates. Before we state our main results, let us first have a quick review of some basic knowledge on Clifford algebra and Clifford analysis.

Let $e_1, \ldots, e_m$ be basic elements satisfying $e_i e_j + e_j e_i = -2\delta_{ij}, i, j = 1, \ldots, n$, where $\delta_{ij}$ equals 1 if $i = j$ and 0 otherwise. Let $\mathbb{R}^{m+1} = \{x = x_0 + x_1 e_1 + \cdots + x_m e_m : x_i \in \mathbb{R}, 0 \leq i \leq m\}$ be identified with the usual $(m+1)$-dimensional Euclidean space. The real Clifford algebra generated by $e_1, \ldots, e_m$, denoted by $\mathcal{A}_m$, is an associative algebra in which each element is of the form $x = \sum_T x_T e_T$, where $x_T \in \mathbb{R}, e_T = e_{i_1} e_{i_2} \cdots e_{i_l}$ and $T = \{1 \leq i_1 < i_2 < \cdots < i_l \leq m\}$ runs over all ordered subsets of $\{1, \ldots, m\}$ and $x_0 = x_0, e_0 = e_0 = 1$. The norm and the conjugate of $x$ are defined by $|x| = \left(\sum_T |x_T|^2\right)^{1/2}$ and $\overline{x} = \sum_T x_T \overline{e_T}$ respectively, where $\overline{e_T} = \overline{e_{i_1}} \cdots \overline{e_{i_l}} e_{i_1}$ and $\overline{e_i} = -e_i$ for $i \neq 0$, $\overline{e_0} = e_0$. We have for any $x, y, z \in \mathcal{A}_m$, $\overline{xy} = \overline{y} \overline{x}$, $(xy)z = x(yz)$ and $|xy| \leq 2m^2|x||y|$. A function $f(x) = \sum_T f_T(x) e_T \in C^1(\Omega, \mathcal{A}_m)$ is said to be left monogenic in the open set $\Omega \subset \mathbb{R}^{m+1}$ if and only if it satisfies the generalized Cauchy-Riemann equation

$$Df = \sum_{i=0}^{m} e_i \frac{\partial f}{\partial x_i} = 0,$$

where the Dirac operator $D$ is defined by $D = \frac{\partial}{\partial x_0} + \nabla = \sum_0^{m} e_i \frac{\partial}{\partial x_i}$. If $f$ is left monogenic, then each component of $f$ is a real-valued harmonic function. For more information about the monogenic function theory, see [2].

Let $B^m(x, \rho) = \{y \in \mathbb{R}^{m+1} : |y-x| < \rho\}$ be the open ball in $\mathbb{R}^{m+1}$, which is centered at $x$ and of radius $\rho$. For simplicity, we denote $B^m = B^m(0, 1)$. 

The monogenic Hardy space $H^p(\mathbb{R}^m) \ (1 \leq p < \infty)$, consists of all functions $f$ that are left monogenic in $\mathbb{R}^m$ and satisfy
\[
\|f\|_p = \sup_{0 < r < 1} \left( \int_{|\eta|=1} |f(r\eta)|^p dS \right)^{1/p} < \infty,
\] (1.1)
where $dS$ is the area element of $\partial \mathbb{R}^m$. We prove that

**Theorem 1.1.** If $f \in H^p(\mathbb{R}^m) \ (1 \leq p < \infty)$, then
\[
(1 - |x|)^{[\alpha]+\frac{m}{p}} |\partial^\alpha f(x)| \leq C_{m,p,|\alpha|}\|f\|_p,
\] (1.2)
where $\alpha = (l_0, l_1, \ldots, l_m)$, $|\alpha| = \sum_{i=0}^m l_i$ and $\partial^\alpha = \partial^{l_0}_{x_0} \partial^{l_1}_{x_1} \cdots \partial^{l_m}_{x_m}$. Write $x = |x|\xi = r\xi$, then we have
\[
\lim_{r \to 1^-} (1 - |x|)^{[\alpha]+\frac{m}{p}} |\partial^\alpha f(x)| = 0
\] (1.3)
uniformly in $|\xi| = 1$.

Corresponding to this, we also prove some propositions for the monogenic Hardy space $H^p(\mathbb{R}^{m+1}_+)$, which consists of all functions $f$ that are left monogenic on the half space $\mathbb{R}^{m+1}_+ = \{ x = x_0 + \bar{x} \in \mathbb{R}^{m+1} : x_0 > 0, \bar{x} = x_1e_1 + \cdots + x_me_m \in \mathbb{R}^m \}$ and satisfy
\[
\|f\|_p = \sup_{x_0 > 0} \left( \int_{\mathbb{R}^m} |f(x_0 + \bar{x})|^p d\bar{x} \right)^{1/p} < \infty,
\] (1.4)
where $d\bar{x} = dx_1 \cdots dx_m$. We note that for $f \in H^p(\mathbb{R}^{m+1}_+) \ (1 \leq p < \infty)$, the boundary values $f(\bar{x}) = \lim_{x_0 \to 0^+} f(x_0 + \bar{x})$ exist almost everywhere and comprise a function in $L^p(\mathbb{R}^m)$, of which the Poisson integral coincides with $f$ ([4]).

**Theorem 1.2.** Suppose $f \in H^p(\mathbb{R}^{m+1}_+) \ (1 \leq p < \infty)$, then
\[
x_0^{[\alpha]+\frac{m}{p}} |\partial^\alpha f(x)| \leq C_{m,p,|\alpha|}\|f\|_p;
\] (1.5)
moreover,
\[
\lim_{x_0 \to 0^+} x_0^{[\alpha]+\frac{m}{p}} |\partial^\alpha f(x_0 + \bar{x})| = \lim_{x_0 \to +\infty} x_0^{[\alpha]+\frac{m}{p}} |\partial^\alpha f(x_0 + \bar{x})| = 0
\] (1.6)
holds uniformly with respect to $\bar{x} \in \mathbb{R}^m$, and
\[
\lim_{|\bar{x}| \to +\infty} x_0^{[\alpha]+\frac{m}{p}} |\partial^\alpha f(x_0 + \bar{x})| = 0
\] (1.7)
holds uniformly in $x_0 > 0$.

**Remark 1.3.** Similar discussions as in Section 2 will show that Theorem 1.1 (resp. Theorem 1.2) holds for the harmonic Hardy space $H^p(\mathbb{B}^m)$ (resp. $H^p(\mathbb{B}^{m+1}_+)$) for $1 < p < \infty$, where by definition, a function $f$ lies in $H^p(\mathbb{B}^m)$ (resp. $H^p(\mathbb{B}^{m+1}_+)$) means that $f$ is harmonic in $\mathbb{B}^m$ (resp. $\mathbb{B}^{m+1}_+$) and (1.1) (resp. (1.4)) holds. But for the case $p = 1$, (1.3) (resp. (1.6) and (1.7)) may not hold for $H^1(\mathbb{B}^m)$ (resp. $H^1(\mathbb{B}^{m+1}_+)$). For example, $f(x_0, x_1) = \frac{x_0}{x_0^2 + x_1^2} \in H^1(\mathbb{B}^2_+)$, but $x_0f(x_0, x_1)$ does not uniformly tend to zero as $x_0 \to 0^+$. Some Remarks on the Boundary Behaviors 3
2. Proof of the Theorems

Proof of Theorem 1.1. From Cauchy’s estimate (cf. [1]) we know that
\[ |\partial^\alpha f(x)| \leq C_{m,|\alpha|}(1 - |x|)^{-|\alpha|} \max_{y \in \partial B^m(x, \frac{1-|x|}{2})} |f(y)|, \]
hence
\[ (1 - |x|)^{|\alpha| + \frac{m}{p}} |\partial^\alpha f(x)| \leq C_{m,|\alpha|} \max_{y \in \partial B^m(x, \frac{1-|x|}{2})} (1 - |y|)^{\frac{m}{p}} |f(y)|. \]

So, to prove (1.2) and (1.3), it is enough to show that
\[ (1 - |x|)^{\frac{m}{p}} |f(x)| \leq C_{m,p} \|f\|_p \quad (2.1) \]
and
\[ \lim_{r \to 1^-} (1 - |x|)^{\frac{m}{p}} |f(x)| = 0 \quad (2.2) \]
for \( 1 \leq p < \infty \).

Denote by \( V_r = C_m(1-r)^{m+1} \) the volume of the ball \( B^m(x, 1-r) \), write \( y = |y|\eta = \rho \eta \), note that
\[ |y - x| \geq ||y| - |x|| = |\rho - r|, \]
and
\[ |y - x| = |\rho \eta - r \xi| \]
\[ = |r(\eta - \xi) - (r - \rho)\eta| \]
\[ \geq r|\eta - \xi| - |r - \rho| \]
\[ \geq r|\eta - \xi| - |y - x|, \]
so \( y \in B^m(x, 1-r) \) implies
\[ \begin{cases} 2r - 1 < \rho < 1, \\ |\eta - \xi| < 2(1-r)/r. \end{cases} \]

Hence, for \( 1 \leq p < \infty \), we have
\[
\begin{align*}
|(1 - r)^{m/p} f(x)| &= (1 - r)^{m/p} \left| \int_{B^m(x,1-r)} f(y) dy \right| \\
&\leq (1 - r)^{m/p} \left( \int_{B^m(x,1-r)} |f(y)|^p dy \right)^{1/p} \\
&\leq (1 - r)^{m/p} \left( \int_{B^m(x,1-r)} \rho^m \int_{|\eta - \xi| < 2(1-r)/r} |f(\rho\eta)|^p dS d\rho \right)^{1/p} \\
&\leq (1 - r)^{m/p} \left( \int_{B^m(x,1-r)} \rho^m \int_{|\eta - \xi| < 2(1-r)/r} |f(\rho\eta)|^p dS \right)^{1/p} \\
&\leq (1 - r)^{m/p} \left( \int_{B^m(x,1-r)} \rho^m \int_{|\eta| = 1} |f(\rho\eta)|^p dS \right)^{1/p} \\
&= C_{m,p} \|f\|_p. \end{align*}
\]
(2.1) is now proved. On the other hand,
\[(2.3) \leq C_{m,p} \left( \int_{|\eta - \xi| < 2(1-r)/r} \sup_{0 < \rho < 1} |f(\rho \eta)|^p dS \right)^{1/p}.\]

Note that as a function of \(\eta\), \(\sup_{0 < \rho < 1} |f(\rho \eta)| \in L^p(\partial B^m)\), and the measure of the set \(\{\eta : |\eta - \xi| < 2(1-r)/r\}\) tends to zero as \(r \to 1^-\), (2.2) follows by the absolute continuity of the Lebesgue integral. \(\square\)

**Proof of Theorem 1.2.** By Cauchy’s estimate we have
\[|\partial^\alpha f(x)| \leq C_{m,|\alpha|} x_0^{-|\alpha|} \max_{y \in \partial B^m(x,x_0/2)} |f(y)|;\]

hence
\[x_0^{\alpha} \frac{m}{p} |\partial^\alpha f(x)| \leq C_{m,|\alpha|} \max_{y \in \partial B^m(x,x_0/2)} y_0^{m/p} |f(y)|.\]

So, the proof of (1.5) and (1.6) is now reduced to the proof of the following
\[(2.4) x_0^{m/p} |f(x)| \leq C_{m,p} \|f\|_p\]
and
\[\lim_{x_0 \to 0^+} x_0^{m/p} |f(x)| = \lim_{x_0 \to +\infty} x_0^{m/p} |f(x)| = 0 \quad (2.5)\]
for \(1 \leq p < \infty\). Once these have been proved, the proof of (1.7) will be reduced to the proof of
\[\lim_{|x| \to +\infty} |f(x_0 + x)| = 0 \quad (2.6)\]
uniformly with respect to \(x_0 \in [a, b] \subset (0, +\infty)\).

Denote by \(V_{x_0} = C_{m} x_0^{m+1}\) the volume of the ball \(B^m(x, x_0/2)\), then for \(1 \leq p < \infty\),
\[x_0^{m/p} |f(x)| = x_0^{m/p} \left| V_{x_0}^{-1} \int_{B^m(x, x_0/2)} f(y_0 + y) dy \right| \leq x_0^{m/p} \left( V_{x_0}^{-1} \int_{B^m(x, x_0/2)} |f(y_0 + y)|^p dy \right)^{1/p} \leq x_0^{m/p} \left( V_{x_0}^{-1} \int_{x_0/2}^{3x_0/2} \int_{B^m} |f(y_0 + y)|^p dydy_0 \right)^{1/p} \leq x_0^{m/p} \left( x_0 V_{x_0}^{-1} \sup_{y_0 > 0} \int_{B^m} |f(y_0 + y)|^p dy \right)^{1/p} = C_{m,p} \|f\|_p,
so (2.4) is verified.
On the other hand, when $x_0$ is small,

\[(2.7) \leq x_0^{m/p} \left( V^{-1} \int_{x_0}^{3x_0} \int_{|y-x| \leq \frac{x_0}{2}} |f(y_0 + y)|^p dy dy_0 \right)^{1/p} \]

\[\leq x_0^{m/p} \left( x_0 V^{-1} \sup_{y_0 \in \left( \frac{x_0}{2}, \frac{3x_0}{2} \right)} \int_{|y-x| \leq \frac{x_0}{2}} |f(y_0 + y)|^p dy \right)^{1/p} \]

\[\leq C_{m,p} \left( \int_{|y-x| \leq \frac{x_0}{2}} \sup_{y_0 \in \left( \frac{x_0}{2}, \frac{3x_0}{2} \right)} |f(y_0 + y)|^p dy \right)^{1/p} \]

Note that as a function of $y$, $\sup_{y_0 > 0} |f(y_0 + y)| \in L^p(\mathbb{R}^m)$ and the measure of the set $\{ y : |y - x| \leq \frac{x_0}{2} \}$ tends to zero as $x_0 \to 0^+$, by the absolute continuity of the Lebesgue integral we have

\[ \lim_{x_0 \to 0^+} x_0^{m/p} |f(x)| = 0. \]

When $x_0$ is large,

\[(2.8) \leq x_0^{m/p} \left( x_0 V_x^{-1} \sup_{y_0 \in \left( \frac{x_0}{2}, \frac{3x_0}{2} \right)} \int_{\mathbb{R}^m} |f(y_0 + y)|^p dy \right)^{1/p} \]

\[\leq C_{m,p} \left( \int_{\mathbb{R}^m} \sup_{y_0 \in \left( \frac{x_0}{2}, \frac{3x_0}{2} \right)} |f(y_0 + y)|^p dy \right)^{1/p} \]

holds uniformly with respect to $x \in \mathbb{R}^m$, and

\[ \sup_{y_0 \in \left( \frac{x_0}{2}, \frac{3x_0}{2} \right)} |f(y_0 + y)| \leq \sup_{y_0 > 0} |f(y_0 + y)| \in L^p(\mathbb{R}^m) \quad \text{for } 1 \leq p < \infty. \]

Also, from (2.4) we know that

\[ \sup_{y_0 \in \left( \frac{x_0}{2}, \frac{3x_0}{2} \right)} |f(y_0 + y)| \leq x_0^{-m/p} C_{m,p} \| f \|_p, \]

which implies

\[ \lim_{x_0 \to +\infty} \sup_{y_0 \in \left( \frac{x_0}{2}, \frac{3x_0}{2} \right)} |f(y_0 + y)| = 0 \]

holds uniformly with respect to $y \in \mathbb{R}^m$. By the Lebesgue’s dominated convergence theorem we have

\[ \lim_{x_0 \to +\infty} x_0^{m/p} |f(x)| = 0. \]
Now we proceed to prove (2.6). Since
\[ |f(x_0 + x)| = \frac{\Gamma\left(\frac{m+1}{2}\right)}{\pi^{\frac{m+1}{2}}} \int_{\mathbb{R}^m} \frac{x_0}{(|x - y|^2 + x_0^2)^{\frac{m+1}{2}}} f(y) dy \]
\[ \leq b \frac{\Gamma\left(\frac{m+1}{2}\right)}{\pi^{\frac{m+1}{2}}} \int_{\mathbb{R}^m} |f(y)| dy \frac{dy}{(|x - y|^2 + a^2)^{\frac{m+1}{2}}} \]
\[ = C_m \left( \int_{|y| > N} \frac{|f(y)| dy}{(|x - y|^2 + a^2)^{\frac{m+1}{2}}} + \int_{|y| \leq N} \frac{|f(y)| dy}{(|x - y|^2 + a^2)^{\frac{m+1}{2}}} \right) \]
\[ = C_m(I_1 + I_2), \]
by Hölder’s inequality,
\[ I_1 \leq \left( \int_{|y| > N} (|x - y|^2 + a^2)^{-\frac{m+1}{2}p'} dy \right)^{1/p} \left( \int_{|y| > N} |f(y)|^p dy \right)^{1/p} \]
\[ \leq \left( \int_{\mathbb{R}^m} (|y|^2 + a^2)^{-\frac{m+1}{2}p'} dy \right)^{1/p} \left( \int_{|y| > N} |f(y)|^p dy \right)^{1/p} \]
\[ \leq C_{m,p} \left( \int_{|y| > N} |f(y)|^p dy \right)^{1/p}, \]
where \( \frac{1}{p} + \frac{1}{p'} = 1 \). Because \( f(y) \in L^p(\mathbb{R}^m) \), \( I_1 \) is small provided \( N \) is large enough. With \( N \) fixed,
\[ I_2 \leq \frac{C_m}{|x|^{m+1}} \int_{|y| \leq N} |f(y)| dy \to 0 \quad (|x| \to +\infty), \]
due to \( f(y) \) is integrable on \( \{ y : |y| \leq N \} \), that proves (2.6).

The proof of Theorem 1.2 is now complete. \( \square \)

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