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# The inverse Fueter mapping theorem for axially monogenic functions of degree $k$

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ABSTRACT

In this paper we first prove an important formula for the fractional Laplacian, and then we use it to invert the Fueter mapping theorem for axially monogenic functions of degree $k$. In fact, we prove that for every axially monogenic function of degree $k$

$$f(x) = [A(x_0, |\underline{x}|) + \frac{\underline{x}}{|\underline{x}|} B(x_0, |\underline{x}|)] P_k(\underline{x}), \quad x \in \mathbb{R}^{n+1},$$

there exists a holomorphic intrinsic function $f_k$ in $\mathbb{C}$ such that

$$f(x) = \tau_k(f_k)(x) := (-\Delta)^{k+(n-1)/2} \left( \vec{f}_k(x) P_k(\underline{x}) \right),$$

where $n$ can be any positive integer, $k$ can be any non-negative integer, $\vec{f}_k$ is the slice monogenic function induced by $f_k$, and $P_k(\underline{x})$ is an inner spherical monogenic polynomial of degree $k$. Using the maps $\tau_k$, $k = 0, 1, 2, \ldots$, we obtain a decomposition of a monogenic function for any value of the dimension $n$.


## 1. Introduction

The original version of the Fueter theorem provides a way to generate quaternionic valued monogenic functions from holomorphic functions defined in the upper half complex plane $\mathbb{C}^+$, see [7]. Later, the Fueter theorem was generalized to Euclidean space. In details, let $f_0(z) = u(s,t) + iv(s,t)$ be a holomorphic function defined on $O \subset \mathbb{C}^+$, where $u(s,t)$ and $v(s,t)$ are real-valued. Let

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$$\vec{f_0}(x) := u(x_0, |\underline{x}|) + \frac{\underline{x}}{|\underline{x}|} v(x_0, |\underline{x}|), \tag{1.1}$$

where

$$\vec{O} := \{x = x_0 + \underline{x} \in \mathbb{R}^{n+1} \mid (x_0, |\underline{x}|) \in O\}, \tag{1.2}$$

$x \in \vec{O}$, and $x_0$, $\underline{x} := x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n$ denote the scalar and the 1-vector part of $x$, respectively. Here, $\vec{f_0}$ is said to be the function induced by $f_0$, and $\vec{O}$ the set induced by $O$. Then, in $\vec{O}$, the function $\Delta^{(n-1)/2}\vec{f_0}$ is both left and right monogenic with respect to the generalized Cauchy-Riemann operator $D = \partial_{x_0} + \partial_{x_1}\mathbf{e}_1 + \partial_{x_2}\mathbf{e}_2 + \cdots + \partial_{x_n}\mathbf{e}_n$, where $\Delta^{(n-1)/2}$ denotes the fractional Laplace operator in the $n+1$ real variables $x_0, x_1, x_2, \cdots, x_n$.

When $n$ is an odd positive integer, the Fueter theorem was proved by Sce in 1957 ([13]). The proof of Sce's result is based on the computation of the pointwise differential operator $\Delta^{(n-1)/2}$, thus if $n$ is an even positive integer Sce's method of proof does not work.

Motivated by his study on the $H^\infty$-functional calculus of the Dirac operator on the sphere in general Euclidean spaces, Qian extended Fueter's theorem to all positive integers $n$ and, for an odd $n$, his result coincides with Sce's, see [9–11]. In [9] Qian realized that it is not necessary to consider holomorphic functions defined in an open set of the upper half complex plane. Instead, one can consider open sets in $\mathbb{C}$ which are symmetric with respect to the real line and holomorphic intrinsic functions defined on this set. These holomorphic functions $f_0$ are such that the induced function $\vec{f_0}$, defined as in (1.1), is what in modern terms is called a slice hyperholomorphic (or slice monogenic) function. If $f_0$ is intrinsic also $\vec{f_0}$ is intrinsic and $f_0$ and $\vec{f_0}$ are in one-to-one correspondence.

After that, a further generalization of Fueter's theorem appeared in [8,15], but see also the more recent [12] for an updated state-of-the-art of the theory. In the two papers [8,15] it is proved that

$$(-\Delta)^{k+\frac{n-1}{2}} \left(\vec{f_0}(x)P_k(\underline{x})\right)$$

is monogenic, where $n$ can be any positive integer, $k$ can be any non-negative integer and $P_k(\underline{x})$ is an inner spherical monogenic polynomial of degree $k$.

Thus, we introduce the so-called generalized Fueter mapping $\tau_k$. Let $f_0$ be a holomorphic intrinsic function defined in $\mathbb{C}$. We define the generalized Fueter mapping $\tau_k$ by

$$\tau_k(f_0) := (-\Delta)^{k+\frac{n-1}{2}} \left(\vec{f_0}(x)P_k(\underline{x})\right).$$

When $k$ is a positive integer, the mapping $\tau_k$ can be written in two steps:

$$\begin{aligned} f_0(z) = u(s,t) + iv(s,t) &\longrightarrow \vec{f_0}(x) = u(x_0, |\underline{x}|) + \frac{\underline{x}}{|\underline{x}|} v(x_0, |\underline{x}|) \\ &\longrightarrow (-\Delta)^{k+\frac{n-1}{2}} \left(\vec{f_0}(x)P_k(\underline{x})\right), \end{aligned}$$

where $\vec{f_0}$ is induced by $f_0$ and it is slice hyperholomorphic.

The generalized Fueter mapping $\tau_k$ maps a holomorphic intrinsic function defined in $\mathbb{C}$ to a monogenic function in $\mathbb{R}^{n+1}$.

It is natural to consider the inverse problem, i.e., whether is it is possible to relate a monogenic function $f$ with its so-called Fueter primitive, namely with a holomorphic intrinsic function which gives $f$ via Fueter's theorem. This problem was studied in [2,3] in the case $n \in \mathbb{N}$ being odd; and in [6] for any $n \in \mathbb{N}$ and $f$ being axially monogenic of degree $k = 0$.

Purpose of this paper is to generalize [3], to the case when $n$ is any natural number and when $f$ is any axially monogenic function of degree $k$ defined on $\mathbb{R}^{n+1}$. We will show that there exists a holomorphic intrinsic function $f_k$ in $\mathbb{C}$ such that $\tau_k(f_k) = f$. In other words, we shall prove that, for every axially monogenic function $f$ of degree $k$, there exists a holomorphic intrinsic function $f_k$ such that

$$f_k(z) \longrightarrow (-\Delta)^{k+\frac{n-1}{2}} \left(\vec{f}_k(x)P_k(\underline{x})\right) = f(x), \quad z \in \mathbb{C},\ x \in \mathbb{R}^{n+1},$$

for any $n \in \mathbb{N}$ and $P_k(\underline{x})$ is an inner spherical monogenic polynomial of degree $k \geq 0$. In [3], the authors prove that there exists a slice hyperholomorphic intrinsic function $\vec{f}_k$ such that $\Delta^{k+\frac{n-1}{2}} \left(\vec{f}_k(x)P_k(\underline{x})\right)$ is monogenic, where $n$ is any odd positive integer, $k$ is any non-negative integer. Since there is a one-to-one correspondence between the holomorphic intrinsic function $f_k$ and the induced slice hyperholomorphic intrinsic function $\vec{f}_k$, it is also immediate to show that the result in this paper generalizes the one proved in [3] for $n$ odd.

Since, as it is well known, see Lemma 5.1, or [5], all monogenic functions can be decomposed as a convergent series of axially monogenic functions of degree $k$, we shall also generalize the analog result in [3] by providing an inversion type theorem for general monogenic functions.

The structure of the paper is as follows. Section 2 contains some preliminary material on holomorphic intrinsic functions, Clifford analysis, and Fourier multipliers. Section 3 contains the proof of an important formula for the fractional Laplacian. In Section 4, we prove the inverse Fueter mapping theorem for axially monogenic functions of degree $k$: we consider an axially monogenic function $f$ of degree $k$ defined on $\mathbb{R}^{n+1}$, and we prove that there exists a holomorphic intrinsic function $f_k$ defined in $\mathbb{C}$ such that $\tau_k(f_k) = f$. Finally, in Section 5, we prove an inversion type theorem for every monogenic function by using its decomposition into axially monogenic functions of degree $k$.

## 2. Preliminary results

In this section we review some notations and basic facts useful in the sequel. We start by recalling the definition of Clifford algebra. Let $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$ be an orthonormal basis of Euclidean space $\mathbb{R}^n$, satisfying the relations $\mathbf{e}_i^2 = -1$ for $i = 1, 2, \cdots, n$ and $\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i = 0$ for $1 \leq i \neq j \leq n$. Then, the real Clifford algebra $\mathbb{R}_{0,n}$ is the real algebra generated by these elements, i.e.,

$$\mathbb{R}_{0,n} := \left\{ a = \sum_S a_S \mathbf{e}_S : \ a_S \in \mathbb{R},\ \mathbf{e}_S = \mathbf{e}_{j_1}\mathbf{e}_{j_2}\cdots\mathbf{e}_{j_k} \right\},$$

where $S := \{j_1, j_2, \cdots, j_k\} \subseteq \{1, 2, \cdots, n\}$ with $1 \leq j_1 < j_2 < \cdots < j_k \leq n$, or $S = \emptyset$, and $\mathbf{e}_\emptyset := 1$. As a real vector space $\mathbb{R}_{0,n}$ is $2^n$ dimensional. The set of elements of the form $x = x_0 + \underline{x} = x_0 + \sum_{j=1}^n$, the so-called paravectors, can be identified with the Euclidean space $\mathbb{R} \oplus \mathbb{R}^n$ via the map $x \mapsto (x_0, x_1, \ldots, x_n)$.

In the sequel, we will make use of the conjugate $\overline{x}$ of a paravector $x = x_0 + \underline{x}$ which is defined by $\overline{x} = x_0 - \underline{x}$. For any $x \in \mathbb{R}^{n+1}$, its norm in $\mathbb{R}^{n+1}$ is the Euclidean norm $|x| := \left(x_0^2 + x_1^2 + \cdots + x_n^2\right)^{1/2}$. Moreover, if $x \in \mathbb{R}^{n+1}\backslash\{0\}$, then the inverse $x^{-1}$ exists and $x^{-1} := \overline{x} \cdot |x|^{-2}$.

Finally, we recall that the Clifford algebra $\mathbb{C}_{0,n}$ is the complex algebra generated by $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$, i.e.

$$\mathbb{C}_{0,n} := \mathbb{C} \otimes \mathbb{R}_{0,n} = \mathbb{R}_{0,n} \oplus \mathrm{i}\mathbb{R}_{0,n},$$

where i is the imaginary unit of $\mathbb{C}$.

All the concepts introduced for $\mathbb{R}_{0,n}$ can be reformulated in the complex Clifford algebra, in particular for a paravector $x = x_0 + \sum_{j=1}^n x_j e_j$, $x_j \in \mathbb{C}$ we have $\overline{x} = \overline{x_0} - \sum_{j=1}^n \overline{x_j} e_j$.

We now turn to the concept of monogenic function which is crucial in Clifford analysis. By $Cl_{0,n}$ we mean either $\mathbb{R}_{0,n}$ or $\mathbb{C}_{0,n}$, and $C^1(\Omega, Cl_{0,n})$ (resp. $C^1(\underline{\Omega}, Cl_{0,n})$ denotes the set of continuously differentiable functions defined on an open set $\Omega \subset \mathbb{R}^{n+1}$ (resp. $\underline{\Omega} \subset \mathbb{R}^n$) and take values in the Clifford algebra $Cl_{0,n}$. Any $f \in C^1(\Omega, Cl_{0,n})$ can be written in the form $f = \sum_S f_S \mathbf{e}_S$, where the functions $f_S$ are $\mathbb{R}$-valued or $\mathbb{C}$-valued. Let $\mathbb{N}$ denote the set of all positive integers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Sometimes, for simplicity, we denote by $\partial_k$ the derivative with respect to $x_k$, i.e., $\partial_k := \partial_{x_k}$, where $x_k$ is the $k$-th variable of $x \in \mathbb{R}^{n+1}$.

The generalized Cauchy-Riemann operator is defined by

$$D := \partial_{x_0} + D_{\underline{x}} = \partial_{x_0} + \partial_{x_1}\mathbf{e}_1 + \partial_{x_2}\mathbf{e}_2 + \cdots + \partial_{x_n}\mathbf{e}_n.$$

We now introduce the definition of monogenic function:

**Definition 2.1** *(Monogenic function).* Let $f(x) \in C^1(\Omega, Cl_{0,n})$ (resp. $f(\underline{x}) \in C^1(\underline{\Omega}, Cl_{0,n})$. Then $f(x)$ (resp. $f(\underline{x})$) is called a (left) monogenic function if and only if

$$Df(x) = 0 \text{ (resp. } D_{\underline{x}} f(\underline{x}) = 0).$$

We note that the Cauchy kernel

$$E(x) := \frac{\overline{x}}{\omega_n |x|^{n+1}}, \ x \in \mathbb{R}^{n+1} \backslash \{0\},$$

plays a key role in Clifford analysis, where $\omega_n := 2\pi^{(n+1)/2}/\Gamma[(n+1)/2]$ is the surface area of the $n$-dimensional unit sphere in $\mathbb{R}^{n+1}$. Let $S$ be a region of $\mathbb{R}^{n+1}, S \subset \Omega$, and $\partial S$ be compact differentiable and oriented. If $f$ is left monogenic in $\Omega$, then its Cauchy integral formula is

$$\int_{\partial S} E(y-x) \mathrm{d}\sigma(y) f(y) = \begin{cases} f(x), & x \in S^o, \\ 0, & x \in \Omega \backslash S, \end{cases}$$

where $S^o$ denotes the interior of $S$, and the differential form $\mathrm{d}\sigma(y)$ is given by $\mathrm{d}\sigma(y) := \eta(y)\mathrm{d}S(y)$, $\eta(y)$ is the outer unit normal to $\partial S$ at the point $y$ and $\mathrm{d}S(y)$ is the surface measure of $\partial S$. For more details, one can see [1] and the references therein.

In the sequel we also need the following notations and definitions. By $\mathbb{S}^{n-1}$ we denote the $n-1$ dimensional unit sphere in $\mathbb{R}^n$

$$\mathbb{S}^{n-1} := \{\underline{x} \in \mathbb{R}^n : |\underline{x}|^2 = 1\}.$$

Since for every $\underline{\omega} \in \mathbb{S}^{n-1}$ we have that $\underline{\omega}^2 = -1$, we can consider

$$\mathbb{C}_{\underline{\omega}} := \mathbb{R} + \underline{\omega}\mathbb{R} := \{u + \underline{\omega} v : u, v \in \mathbb{R}, \ \ \underline{\omega} \in \mathbb{S}^{n-1}\}. \tag{2.1}$$

We note any $x \in \mathbb{R}^{n+1}$, which is no real corresponds $\underline{\omega}_x = \frac{\underline{x}}{|\underline{x}|}$. It is immediate that $x \in \mathbb{C}_{\underline{\omega}_x}$ and $x = x_0 + \underline{\omega}_x |\underline{x}|$. The notation

$$[x] := \{y \in \mathbb{R}^{n+1} \mid y = \mathrm{Re}(x) + \underline{\omega}|\underline{x}|, \forall \ \underline{\omega} \in \mathbb{S}^{n-1}\},$$

where $\mathrm{Re}(x)$ denotes the real part of $x$ denotes the $n-1$ dimensional sphere in $\mathbb{R}^{n+1}$ with radius $|\underline{x}|$ and centered at $\mathrm{Re}(x)$, see [4].

**Definition 2.2** *(Axially symmetric open set).* An open set $\Omega \subset \mathbb{R}^{n+1}$ is said to be axially symmetric if the $(n-1)$-sphere $[u + \underline{\omega} v]$ is contained in $\Omega$ whenever $u + \underline{\omega} v \in \Omega$ for some $u, v \in \mathbb{R}$, $\underline{\omega} \in \mathbb{S}^{n-1}$.

**Definition 2.3** *(Axially monogenic function of degree k).* Let $k \in \mathbb{N}_0$ and $\Omega$ be an axially symmetric open set. A function $f(x) \in C^1(\Omega, Cl_{0,n})$ is said to be axially monogenic of degree $k$ if it is monogenic and has the form

$$f(x) = \left( A(x_0, |\underline{x}|) + \frac{\underline{x}}{|\underline{x}|} B(x_0, |\underline{x}|) \right) P_k(\underline{x})$$

where $A(x_0, |\underline{x}|), B(x_0, |\underline{x}|)$ are real valued functions and $P_k(\underline{x})$ is an inner left spherical monogenic polynomial of degree $k$, namely a monogenic polynomial, homogeneous of degree $k$.

Precisely speaking, let $\mathbb{C}$ be the complex plane and $\mathbb{C}^+$ be the upper half complex plane, i.e.,

$$\mathbb{C}^+ := \{z \in \mathbb{C} \mid z = x_0 + \mathrm{i} y_0, y_0 > 0\}.$$

Let $O$ a non-empty open set in $\mathbb{C}^+$ and let $f_0(z) = u(x_0, y_0) + \mathrm{i} v(x_0, y_0)$ be a holomorphic function in $O$, where $u(x_0, y_0)$ and $v(x_0, y_0)$ are real valued functions. Then the set $O$ induces the axially symmetric open set $\vec{O} \subseteq \mathbb{R}^{n+1}$ in 1.2 and for $x \in \vec{O}$, we can define the induced function

$$\vec{f_0}(x) P_k(\underline{x}) := \left( u(x_0, |\underline{x}|) + \frac{\underline{x}}{|\underline{x}|} v(x_0, |\underline{x}|) \right) P_k(\underline{x}).$$

Let $k \in \mathbb{N}_0$ and let $n \in \mathbb{N}$ be odd. The generalization of Fueter's theorem in [15] asserts that the function $\Delta^{k+\frac{n-1}{2}} \left( \vec{f_0}(x) P_k(\underline{x}) \right)$, $x \in \vec{O}$ is monogenic.

The Fourier multiplier method used by Kou, Qian, Sommen in [8] is method used also in the present paper.

We recall that an open set $O \subset \mathbb{C}$ is said to be intrinsic if it is symmetric with respect to the real axis, i.e. if $z \in O$ then $\bar{z} \in O$. A holomorphic function $f_0(z)$ is called a holomorphic intrinsic function if it is defined in an intrinsic set $O$ and it satisfies $\overline{f_0(z)} = f_0(\overline{z})$. This last condition is equivalent to $u(x_0, y_0) = u(x_0, -y_0)$ and $v(x_0, y_0) = -v(x_0, -y_0)$. In particular, $v(x_0, 0) = 0$, i.e., $f_0$ is real valued if restricted to the real line in its domain. So, a characterization of a holomorphic intrinsic function is that the coefficients of its Laurent series expansion in any annulus centered at a real point and in its domain are all real. We note that the induced function $\vec{f_0}(x)$ is slice monogenic and intrinsic.

Finally, we denote by $\mathcal{S}(\mathbb{R}^{n+1})$ the Schwarz space and by $\mathcal{S}^*(\mathbb{R}^{n+1})$ the dual space of $\mathcal{S}(\mathbb{R}^{n+1})$. For any $\phi \in \mathcal{S}(\mathbb{R}^{n+1})$, the Fourier transform of $\phi$ is defined by

$$\widehat{\phi}(\xi) = \int_{\mathbb{R}^{n+1}} \phi(x) e^{-2\pi \mathrm{i} \langle x, \xi \rangle} \mathrm{d}x,$$

and the inverse Fourier transform of $\phi$ is defined by

$$\check{\phi}(x) = \int_{\mathbb{R}^{n+1}} \phi(\xi) e^{2\pi \mathrm{i} \langle \xi, x \rangle} \mathrm{d}\xi.$$

Then for every $f \in \mathcal{S}^*(\mathbb{R}^{n+1})$, its Fourier transform and inverse Fourier transform are defined by

$$\langle \mathcal{F}(f), \phi\rangle := \langle f, \widehat{\phi}\rangle, \ \forall\ \phi \in \mathcal{S}(\mathbb{R}^{n+1}),$$
$$\langle \mathcal{F}^{-1}(f), \phi\rangle := \langle f, \check{\phi}\rangle, \ \forall\ \phi \in \mathcal{S}(\mathbb{R}^{n+1}),$$

respectively.

We will use the Fourier multiplier operator induced by $g$:

$$M_g(f) := \mathcal{F}^{-1}[g\mathcal{F}(f)],$$

where the equality is in the tempered distribution sense. In particular, for $n \in \mathbb{N}$, $k \in \mathbb{N}_0$ and $x \in \mathbb{R}^{n+1}$, the fractional Laplace operator $(-\Delta)^{k+(n-1)/2}$ is defined via the Fourier multiplier operator $M_g$ with $g(x) := (2\pi|x|)^{2k+n-1}$. For more details about $(-\Delta)^{k+(n-1)/2}$, the reader may refer to page 117 in [16].

## 3. A important formula of the fractional Laplacian

In this section we prove an important formula for the fractional Laplacian, see Theorem 3.5. The following preliminary lemmas are needed. We start by recalling Lemma 3.1, which is crucial in the proof of Theorem 3.5:

**Lemma 3.1.** *(See [8, Lemma 2]) Let $\beta \in \mathbb{N}$, $-\beta < \alpha < \beta + n + 1$ and $P_\beta(x)$ be a homogeneous harmonic polynomial of degree $\beta$. Then*

$$\int_{\mathbb{R}^{n+1}} \frac{P_\beta(x)}{|x|^{\beta+n+1-\alpha}} \mathcal{F}(\varphi)(x)\mathrm{d}x = \gamma_{\beta,\alpha} \int_{\mathbb{R}^{n+1}} \frac{P_\beta(x)}{|x|^{\beta+\alpha}} \varphi(x)\mathrm{d}x \tag{3.1}$$

*for every $\varphi$ which is sufficiently rapidly decreasing at infinity and*

$$\gamma_{\beta,\alpha} := \mathrm{i}^\beta \pi^{(n+1)/2-\alpha}\Gamma(\beta/2 + \alpha/2)/\Gamma(\beta/2 + (n+1)/2 - \alpha/2), \tag{3.2}$$

*where* i *is an imaginary unit in the complex plane $\mathbb{C}$ and $\Gamma$ is the standard gamma function.*

**Remark 3.2.** Formula (3.1) implies, in the tempered distribution sense,

$$\mathcal{F}\left[\frac{P_\beta(x)}{|x|^{\beta+n+1-\alpha}}\right](\xi) = \gamma_{\beta,\alpha}\frac{P_\beta(\xi)}{|\xi|^{\beta+\alpha}}$$

or

$$\frac{P_\beta(x)}{|x|^{\beta+n+1-\alpha}} = \gamma_{\beta,\alpha}\mathcal{F}^{-1}\left[\frac{P_\beta(\xi)}{|\xi|^{\beta+\alpha}}\right](x),$$

where $\gamma_{\alpha,\beta}$ is as in (3.2).

Let $k \in \mathbb{N}_0$ and $n \in \mathbb{N}$, the following lemma states that the partial derivative $\partial_{x_0}$ commutes with the fractional Laplace operator $(-\Delta)^{k+(n-1)/2}$.

**Lemma 3.3.** *Let $k \in \mathbb{N}_0$, $n \in \mathbb{N}$ and $f$ be in the Schwarz class, then*

$$\partial_{x_0}\left[(-\Delta)^{k+\frac{n-1}{2}} f(x)\right] = (-\Delta)^{k+\frac{n-1}{2}}\left[\partial_{x_0} f(x)\right].$$

**Proof.** The proof of this lemma follows exactly the lines of the proof of Lemma 4.3 in [6], to which we refer the reader, so we do not repeat it here. □

**Remark 3.4.** The proof of Lemma 4.3 in [6] shows that the result is also true for the generalized Cauchy-Riemann operator $D = \partial_{x_0} + \partial_{\underline{x}}$, i.e.

$$D\left[(-\Delta)^{k+\frac{n-1}{2}} f(x)\right] = (-\Delta)^{k+\frac{n-1}{2}} \left[Df(x)\right].$$

Now we prove the main theorem of the section.

**Theorem 3.5.** *Let $n, l \in \mathbb{N}$, $k \in \mathbb{N}_0$ and $x \in \mathbb{R}^{n+1}\backslash\{0\}$. Then*

$$(-\Delta)^{k+\frac{n-1}{2}} \left[x^{-l} P_k(\underline{x})\right] = \frac{(-1)^{l-1}\lambda_{k,n}}{(l-1)!} \left((\partial_0)^{l-1} E_{2k}\right)(x) P_k\left(\underline{x}\right),$$

*where*

$$\lambda_{k,n} := (2\pi)^{n-1} \frac{\gamma_{k+1,k+n}}{\gamma_{k+1,k+1}}, \qquad E_{2k}(x) := \frac{\overline{x}}{|x|^{2k+n+1}}.$$

**Proof.** From the relation

$$x^{-l} = \left(\frac{\overline{x}}{|x|^2}\right)^l = \frac{(-1)^{l-1}}{(l-1)!} (\partial_{x_0})^{l-1} \left(\frac{\overline{x}}{|x|^2}\right),$$

and Lemma 3.3, we have

$$\begin{aligned}
&(-\Delta)^{k+\frac{n-1}{2}} \left[x^{-l} P_k(\underline{x})\right] \\
&= \frac{(-1)^{l-1}}{(l-1)!} (\partial_{x_0})^{l-1} \left[(-\Delta)^{k+\frac{n-1}{2}} \left(\frac{\overline{x} P_k(\underline{x})}{|x|^2}\right)\right] \\
&= \frac{(-1)^{l-1}}{(l-1)!} (\partial_{x_0})^{l-1} \left\{\mathcal{F}^{-1}\left[(2\pi|\xi|)^{2k+n-1} \mathcal{F}\left(\frac{\overline{x} P_k(\underline{x})}{|x|^2}\right)(\xi)\right](x)\right\} \\
&= \frac{(-1)^{l-1}(2\pi)^{2k+n-1}}{(l-1)!} (\partial_{x_0})^{l-1} \left\{\mathcal{F}^{-1}\left[|\xi|^{2k+n-1} \mathcal{F}\left(\frac{\overline{x} P_k(\underline{x})}{|x|^2}\right)(\xi)\right](x)\right\}.
\end{aligned}$$

We now apply Lemma 3.1 with $\beta = k+1, \alpha = k+n$, and we get

$$\begin{aligned}
&(-\Delta)^{k+\frac{n-1}{2}} \left[x^{-l} P_k(\underline{x})\right] \\
&= \frac{(-1)^{l-1}(2\pi)^{2k+n-1}}{(l-1)!} (\partial_{x_0})^{l-1} \left[\mathcal{F}^{-1}\left(\gamma_{k+1,k+n} |\xi|^{2k+n-1} \frac{\overline{\xi}}{|\xi|^{2k+n+1}} P_k\left(\underline{\xi}\right)\right)(x)\right] \\
&= \frac{(-1)^{l-1}(2\pi)^{2k+n-1}\gamma_{k+1,k+n}}{(l-1)!} (\partial_{x_0})^{l-1} \left[\mathcal{F}^{-1}\left(\frac{\overline{\xi} P_k(\underline{\xi})}{|\xi|^2}\right)(x)\right].
\end{aligned}$$

Using Lemma 3.1 for $\beta = k+1, \alpha = 1-k$, we obtain

$$(-\Delta)^{k+\frac{n-1}{2}} \left[x^{-l} P_k(\underline{x})\right] = \frac{(-1)^{l-1}(2\pi)^{2k+n-1}\gamma_{k+1,k+n}}{(l-1)!\gamma_{k+1,k+1}} (\partial_{x_0})^{l-1} \left(\frac{\overline{x} P_k(\underline{x})}{|x|^{2k+n+1}}\right).$$

Thus we have

$$(-\Delta)^{k+\frac{n-1}{2}} \left[x^{-l} P_k(\underline{x})\right] = \frac{(-1)^{l-1}\lambda_{k,n}}{(l-1)!} \left((\partial_0)^{l-1} E_{2k}\right)(x) P_k\left(\underline{x}\right),$$

where $E_{2k}(x) = \overline{x}/|x|^{2k+n+1}$ and $\lambda_{k,n} := (2\pi)^{2k+n-1}\gamma_{k+1,k+n}/\gamma_{k+1,k+1}$, and this concludes the proof. □

## 4. The inverse Fueter mapping theorem

As we mentioned in Section 2, if a holomorphic intrinsic function $f_0(z)$ is expanded at $z = 0$, its Laurent series expansion has the form $f_0(z) = \sum_{l\in\mathbb{Z}} a_l z^l$, where $a_l \in \mathbb{R}$. Then we have

$$\tau_k(f_0(z)) = \sum_{l\in\mathbb{Z}} a_l \tau_k(z^l) = \sum_{l\in\mathbb{Z}} a_l (-\Delta)^{k+\frac{n-1}{2}} \left(x^l P_k\left(\underline{x}\right)\right), \ x \in \mathbb{R}^{n+1}.$$

We know that $(-\Delta)^{k+(n-1)/2}\left(x^l P_k\left(\underline{x}\right)\right)$ are monogenic functions for $n \in \mathbb{N}$ and $l \in \mathbb{Z}$. Indeed, for odd values of $n$, the generalization of Fueter's theorem proved by Sommen in [15] shows that $(-\Delta)^{k+(n-1)/2}\left(x^l P_k\left(\underline{x}\right)\right)$ is axially monogenic. Later, Qian and his co-authors showed in [8] that $(-\Delta)^{k+(n-1)/2}\left(x^l P_k\left(\underline{x}\right)\right)$ is monogenic also for even indices $n$. Moreover, Theorem 3.5 gives the explicit expression of $(-\Delta)^{k+(n-1)/2}\left(x^l P_k\left(\underline{x}\right)\right)$ for $n \in \mathbb{N}$ and $l \in \mathbb{Z}\backslash\mathbb{N}_0$. Thus $\tau_k$ maps a holomorphic intrinsic function to a monogenic function. When $k = 0$, the mapping $\tau_0$ is the original Fueter mapping, so we would like to call the mapping $\tau_k$ the generalized Fueter mapping.

The rest of the section is devoted to prove that for every axially monogenic function $f$ of degree $k$, there exists a holomorphic intrinsic function $f_k$ such that $\tau_k(f_k) = f$. To this end, we need the functions defined below.

**Definition 4.1.** Let $n \in \mathbb{N}$, $k \in \mathbb{N}_0$ and $P_k(\underline{x})$ be an inner spherical monogenic polynomial of degree $k$. For any $x \in \mathbb{R}^{n+1}\backslash\mathbb{S}^{n-1}$, denote the kernels

$$\mathcal{K}^+_{k,n}(x) := \int_{\mathbb{S}^{n-1}} E(x-\underline{\omega})P_k(\underline{\omega})\mathrm{d}S(\underline{\omega})$$

and

$$\mathcal{K}^-_{k,n}(x) := \int_{\mathbb{S}^{n-1}} E(x-\underline{\omega})\underline{\omega}P_k(\underline{\omega})\mathrm{d}S(\underline{\omega})$$

where $E(x)$ is the Cauchy kernel and $\mathrm{d}S(\underline{\omega})$ is the surface measure on $\mathbb{S}^{n-1}$.

These two functions as well as the next two lemmas will play an important role in the proof of Theorem 4.7. Below, we denote by $M_k$ the set of all solid inner spherical monogenics of degree $k$.

**Lemma 4.2** *(see [14, Theorem 2.1]). Let $n \in \mathbb{N}$, $P_k(\underline{x}) \in M_k$ be fixed, and let $W_0(x_0)$ be a real analytic function in $\widetilde{\Omega} \subset \mathbb{R}$. Then there exists a unique sequence of analytic functions, $\{W_s(x_0)\}_{s>0}$, such that the series*

$$f(x_0, \underline{x}) = \sum_{s=0}^{\infty} \underline{x}^s W_s(x_0) P_k(\underline{x})$$

*is convergent in an open set $\Omega$ in $\mathbb{R}^{n+1}$ containing the set $\widetilde{\Omega}$, and its sum $f$ is monogenic in $\Omega$. The function $W_0(x_0)$ is determined by the relation*

$$P_k(\underline{\omega})W_0(x_0) = \lim_{|\underline{x}|\to 0} \frac{1}{|\underline{x}|^k} f(x_0, \underline{x}), \ \underline{\omega} = \frac{\underline{x}}{|\underline{x}|} \in \mathbb{S}^{n-1}.$$

*The series $f(x_0, \underline{x})$ is the generalized axial CK-extension of the function $W_0(x_0)$.*

**Lemma 4.3** *(see [3, Theorem 3.6]). Let $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$. For the given functions $\mathcal{K}^+_{k,n}(x)$ and $\mathcal{K}^-_{k,n}(x)$, there exist two functions $\mathcal{S}^+_{k,n}(x)$ and $\mathcal{S}^-_{k,n}(x)$ independent of $P_k(\underline{x})$ such that*

$$\mathcal{K}^+_{k,n}(x) = \mathcal{S}^+_{k,n}(x)P_k(\underline{x}), \quad \mathcal{K}^-_{k,n}(x) = \mathcal{S}^-_{k,n}(x)P_k(\underline{x}).$$

*Furthermore, for any $x_0 \in \mathbb{R}$ we have*

$$\lim_{|\underline{x}|\to 0} \mathcal{S}^+_{k,n}(x) = C_{k,n}\frac{x_0}{(1+x_0^2)^{k+(n+1)/2}},$$
$$\lim_{|\underline{x}|\to 0} \mathcal{S}^-_{k,n}(x) = -C_{k,n}\frac{1}{(1+x_0^2)^{k+(n+1)/2}},$$

*where*

$$C_{k,n} := \frac{(-1)^k\Gamma[k+(n+1)/2]}{\sqrt{\pi}\Gamma(k+n/2)}.$$

With the restrictions of $\mathcal{S}^+_{k,n}(x)$ and $\mathcal{S}^-_{k,n}(x)$ to the real line, i.e. $x = x_0$, we can construct two important holomorphic intrinsic functions denoted by $\mathcal{P}^+_{k,n}(z)$ and $\mathcal{P}^-_{k,n}(z)$, respectively. First, let $z \in \mathbb{C}\backslash\{\mathrm{i}, -\mathrm{i}\}$, replacing $x_0$ by $z$ in the restrictions of $\mathcal{S}^+_{k,n}(x)$ and $\mathcal{S}^-_{k,n}(x)$ to the real line we have the functions

$$C_{k,n}\frac{z}{(1+z^2)^{k+(n+1)/2}} \text{ and } -C_{k,n}\frac{1}{(1+z^2)^{k+(n+1)/2}}.$$

Then, let

$$\mathcal{P}^+_{k,n}(z) := \frac{C_{k,n}}{\lambda'_{k,n}} \cdot D_z^{-(2k+n-1)}\left\{\frac{z}{(1+z^2)^{k+(n+1)/2}}\right\},$$
$$\mathcal{P}^-_{k,n}(z) := -\frac{C_{k,n}}{\lambda'_{k,n}} \cdot D_z^{-(2k+n-1)}\left\{\frac{1}{(1+z^2)^{k+(n+1)/2}}\right\},$$

where $\lambda'_{k,n} = (-1)^{n-1}\lambda_{k,n}/(2k+n-1)!$ and $D_z^{-(2k+n-1)}$ stands for the $(2k+n-1)$-fold antiderivative operation with respect to variable $z$.

It is immediate to see that $\mathcal{P}^+_{k,n}(z)$ and $\mathcal{P}^-_{k,n}(z)$ are holomorphic intrinsic functions on $\mathbb{C}\backslash\{\mathrm{i}, -\mathrm{i}\}$. Further more, we have

**Lemma 4.4.** *Let $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$. Suppose that $\mathcal{P}^+_{k,n}(z)$ and $\mathcal{P}^-_{k,n}(z)$ are defined on $\mathbb{C}\backslash\{\mathrm{i}, -\mathrm{i}\}$. Then the power series expressings $\mathcal{P}^+_{k,n}(z)$ and $\mathcal{P}^-_{k,n}(z)$ converge absolutely and uniformly on the set $\in \{z : \ |z| \geq \rho,\ \rho > 1\}$.*

**Proof.** We only prove the case $\mathcal{P}^+_{k,n}(z)$, since $\mathcal{P}^-_{k,n}(z)$ can be proved with a similar method. Let $|z| > 1$, we have

$$\begin{aligned}\mathcal{P}^+_{k,n}(z) :=& \frac{C_{k,n}}{\lambda'_{k,n}} \cdot D_z^{-(2k+n-1)}\left\{\frac{z}{(1+z^2)^{k+(n+1)/2}}\right\}\\ =& \frac{C_{k,n}}{\lambda'_{k,n}} \cdot D_z^{-(2k+n-1)}\left\{zz^{-(2k+n+1)}\sum_{j=0}^{\infty}\binom{-\frac{2k+n+1}{2}}{j}z^{-2j}\right\}\\ =& \frac{C_{k,n}}{\lambda'_{k,n}} \cdot D_{x_0}^{-(2k+n-1)}\left\{\sum_{j=0}^{\infty}\binom{-\frac{2k+n+1}{2}}{j}z^{-(2j+2k+n)}\right\}\end{aligned}$$

$$=\frac{C_{k,n}}{\lambda'_{k,n}}\cdot\sum_{j=0}^{\infty}\binom{-\frac{2k+n+1}{2}}{j}\frac{(-1)^{n-1}(2j)!}{(2j+2k+n-1)!}z^{-(2j+1)}.$$

So, the power series expressing $\mathcal{P}^{+}_{k,n}(z)$ converges absolutely and uniformly on the set $\{z:\ |z|\geq\rho,\ \rho>1\}$. □

We now prove the following lemma.

**Lemma 4.5.** *Let $n\in\mathbb{N}$ and $k\in\mathbb{N}_0$. Suppose that $\mathcal{P}^{+}_{k,n}(z)$ and $\mathcal{P}^{-}_{k,n}(z)$ are defined on $\mathbb{C}\backslash\{\mathrm{i},-\mathrm{i}\}$. Then, for each $x\in\mathbb{R}^{n+1}\backslash\mathbb{S}^{n-1}$,*

$$\tau_k\left(\mathcal{P}^{+}_{k,n}\right)(x)=\mathcal{K}^{+}_{k,n}(x),$$
$$\tau_k\left(\mathcal{P}^{-}_{k,n}\right)(x)=\mathcal{K}^{-}_{k,n}(x).$$

**Proof.** We only prove

$$\tau_k\left(\mathcal{P}^{+}_{k,n}\right)(x)=\mathcal{K}^{+}_{k,n}(x),\quad x\in\mathbb{R}^{n+1}\backslash\mathbb{S}^{n-1},$$

since the equality

$$\tau_k\left(\mathcal{P}^{-}_{k,n}\right)(x)=\mathcal{K}^{-}_{k,n}(x),\quad x\in\mathbb{R}^{n+1}\backslash\mathbb{S}^{n-1}$$

can be proved with a similar method. The generalized Fueter mapping theorem states that $\tau_k\left(\mathcal{P}^{+}_{k,n}\right)(x)$ is monogenic in $x\in\mathbb{R}^{n+1}\backslash\mathbb{S}^{n-1}$. So the lemma is proved if we can show

$$\tau_k\left(\mathcal{P}^{+}_{k,n}\right)(x)=\mathcal{K}^{+}_{k,n}(x),$$

for $|x|>1$, in fact the equality would hold in $\mathbb{R}^{n+1}\backslash\mathbb{S}^{n-1}$ by the identity principle (see Corollary 10.7 in [1]).

By Lemma 4.4, we know the power series expressing $\mathcal{P}^{+}_{k,n}(z)$ converges absolutely and uniformly on the set $\{z:\ |z|\geq\rho,\ \rho>1\}$. Applying the operator $\tau_k$ to both sides of the power series expressing $\mathcal{P}^{+}_{k,n}(z)$, we obtain

$$\tau_k\left(\mathcal{P}^{+}_{k,n}\right)(x)=\frac{C_{k,n}}{\lambda'_{k,n}}\cdot\sum_{j=0}^{\infty}\binom{-\frac{2k+n+1}{2}}{j}\frac{(-1)^{n-1}(2j)!}{(2j+2k+n-1)!}\tau_k\left((\cdot)^{-(2j+1)}\right)(x).$$

Theorem 3.5 allows to compute $\tau_k\left((\cdot)^{-(2k+1)}\right)(x)$:

$$\begin{aligned}\tau_k\left((\cdot)^{-(2j+1)}\right)(x)&=(-\Delta)^{k+\frac{n-1}{2}}\left((\cdot)^{-(\vec{2}j+1)}\right)(x)\\&=\frac{\lambda_{k,n}}{(2j)!}\cdot\left((\partial_0)^{2j}E_{2k}\right)(x)P_k\left(\underline{x}\right).\end{aligned}$$

Thus, we obtain

$$\begin{aligned}\frac{1}{|\underline{x}|^k}\tau_k\left(\mathcal{P}^{+}_{k,n}\right)(x)=&\frac{C_{k,n}}{\lambda'_{k,n}}\cdot\sum_{j=0}^{\infty}\binom{-\frac{2k+n+1}{2}}{j}\frac{(-1)^{n-1}\lambda_{k,n}}{(2j+2k+n-1)!}\left((\partial_0)^{2j}E_{2k}\right)(x)P_k\left(\underline{\omega}\right)\\=:&\,\mathcal{R}^{+}_{2,k,n}(x)P_k\left(\underline{\omega}\right).\end{aligned}$$

To prove the equality $\tau_k\left(\mathcal{P}_{k,n}^+\right)(x) = \mathcal{K}_{k,n}^+(x)$, by Lemma 4.2 we just need to prove that

$$\lim_{|\underline{x}|\to 0} \mathcal{R}_{2,k,n}^+(x) = \lim_{|\underline{x}|\to 0} \mathcal{S}_{k,n}^+(x) = C_{k,n}\frac{x_0}{(1+x_0^2)^{k+(n+1)/2}}.$$

By taking the limit $|\underline{x}| \to 0$ on $\left((\partial_0)^{2j} E_{2k}\right)(x)$ we have

$$\lim_{|\underline{x}|\to 0} \left((\partial_0)^{2j} E_{2k}\right)(x) = \left((\partial_0)^{2j} E_{2k}\right)(x_0) = \frac{(2j+2k+n-1)!}{(2k+n-1)!} x_0^{-(2j+2k+n)},$$

where the first equality follows from the fact that $E_{2k}$ is a continuously differentiable function and the second equality is obtained by computing the partial derivative.

Setting $\lambda'_{k,n} = (-1)^{n-1}\lambda_{k,n}/(2k+n-1)!$, we have

$$\begin{aligned}
\lim_{|\underline{x}|\to 0} \mathcal{R}_{2,k,n}^+(x) &= \frac{C_{k,n}}{\lambda'_{k,n}} \cdot \sum_{j=0}^{\infty} \binom{-\frac{2k+n+1}{2}}{j} \frac{(-1)^{n-1}\lambda_{k,n}}{(2j+2k+n-1)!} \lim_{|\underline{x}|\to 0} \left((\partial_0)^{2j} E_{2k}\right)(x) \\
&= \frac{C_{k,n}}{\lambda'_{k,n}} \cdot \sum_{j=0}^{\infty} \binom{-\frac{2k+n+1}{2}}{j} \frac{(-1)^{n-1}\lambda_{k,n}}{(2k+n-1)!} x_0^{-(2j+2k+n)} \\
&= C_{k,n} x_0^{-(2k+n)} \cdot \sum_{j=0}^{\infty} \binom{-\frac{2k+n+1}{2}}{j} x_0^{-2j} \\
&= C_{k,n} \cdot \frac{x_0}{(1+x_0^2)^{k+(n+1)/2}} \\
&= \lim_{|\underline{x}|\to 0} \mathcal{S}_{k,n}^+(x).
\end{aligned}$$

So, we have

$$\tau_k\left(\mathcal{P}_{k,n}^+\right)(x) = \mathcal{K}_{k,n}^+(x), \quad |x| > 1,$$

and this concludes the proof. □

**Lemma 4.6.** *Let $n \in \mathbb{N}$, $y_0 \in \mathbb{R}$ and $r \in \mathbb{R}\backslash\{0\}$. Then*

$$|r|^{2k+n-1}\tau_k\left(\mathcal{P}_{k,n}^+\left(\frac{\cdot - y_0}{r}\right)\right)(x) = \tau_k\left(\mathcal{P}_{k,n}^+\right)(x'),$$
$$|r|^{2k+n-1}\tau_k\left(\mathcal{P}_{k,n}^-\left(\frac{\cdot - y_0}{r}\right)\right)(x) = \tau_k\left(\mathcal{P}_{k,n}^-\right)(x'),$$

*where $x' \in \mathbb{R}^{n+1}\backslash\mathbb{S}^{n-1}$ and $x' = (x - y_0)/r$.*

**Proof.** The proof of this result follows by direct computations and is similar to the proof of Theorem 3.5, so we omit it. □

The following theorem is the main theorem of the section. The proof follows the lines of the proof of Theorem 4.2 in [2] adapted as in the proof of Theorem 4.15 in [6]. Provided the importance of the result we repeat the proof here for the sake of completeness.

Note that we build the connection between the given axially monogenic function of degree $k$ with the corresponding holomorphic intrinsic function, while in [3] the author provide, in a basically equivalent way, the corresponding slice hyperholomorphic function but in the case when $n$ is odd.

**Theorem 4.7.** *Let $k \in \mathbb{N}_0$ and $n \in \mathbb{N}$. Let $\Omega \subset \mathbb{R}^{n+1}$ be an axially symmetric open set and $f(y) = f(y_0+\underline{\omega}r) = [A(y_0,r)+\underline{\omega}B(y_0,r)]\,P_k(\underline{y})$ be an axially monogenic function of degree $k$ defined on $\Omega$. Let $\Gamma_{\underline{\omega}}$ be the boundary of an open bounded set $\mathcal{V}_{\underline{\omega}} \subset \mathbb{R}+\underline{\omega}\mathbb{R}^+$ and $V := \cup_{\underline{\omega}\in\mathbb{S}^{n-1}}\mathcal{V}_{\underline{\omega}} \subset \Omega$. Furthermore, suppose that $\Gamma_{\underline{\omega}}$ is a regular curve whose parametric equations in the upper complex plane $\mathbb{C}^+_{\underline{\omega}} = \{y_0+\underline{\omega}r,\ y_0 \in \mathbb{R}, r \in \mathbb{R}^+\}$ are $y_0 = y_0(s), r = r(s)$ and are expressed in terms of the arc-length $s \in [0,L], L > 0$. Then, for all $x \in V$, there exists a holomorphic intrinsic function $f_k(z)$ defined on $\mathbb{C}\backslash\{\mathrm{i},-\mathrm{i}\}$ such that*

$$\tau_k\left(f_k\right)(x) = f(x),$$

*where*

$$\begin{aligned} f_k(z) := &\int_{\Gamma_{\underline{\omega}}} \mathcal{P}^-_{k,n}\left(\frac{z-y_0}{r}\right)\cdot r^{2k+n-2}[\mathrm{d}y_0 A(y_0,r) - \mathrm{d}r B(y_0,r)] \\ &- \int_{\Gamma_{\underline{\omega}}} \mathcal{P}^+_{k,n}\left(\frac{z-y_0}{r}\right)\cdot r^{2k+n-2}[\mathrm{d}y_0 B(y_0,r) + \mathrm{d}r A(y_0,r)]. \end{aligned}$$

**Proof.** The following notations will be used in the proof.

(1) $\Sigma$ is the manifold defined by

$$\Sigma := \{y_0+\underline{\omega}r \mid (y_0,r) \in \Gamma_{\underline{\omega}}, \underline{\omega} \in \mathbb{S}^{n-1}\}.$$

(2) $\mathrm{d}s$ is the infinitesimal arc-length, $\mathrm{d}S(\underline{\omega})$ is the infinitesimal element of surface area on $\mathbb{S}^{n-1}$.

(3) $\mathbf{t} = \frac{\mathrm{d}}{\mathrm{d}s}(y_0+\underline{\omega}r)$ is the unit tangent vector at a point of $\Gamma_{\underline{\omega}} \subset \mathbb{C}_{\underline{\omega}}$, while the normal unit vector $\mathbf{n}$ is given by

$$\mathbf{n} = -\underline{\omega}\mathbf{t} = \frac{\mathrm{d}}{\mathrm{d}s}[r(s) - \underline{\omega}y_0(s)].$$

(4) The scalar infinitesimal element of the manifold $\Sigma$, expressed in terms of $\mathrm{d}s$ and $\mathrm{d}S$, is given by

$$\mathrm{d}\Sigma = r^{n-1}\mathrm{d}s\mathrm{d}S(\underline{\omega}).$$

(5) The oriented infinitesimal element of manifold $\mathrm{d}\sigma(s,\underline{\omega})$ is given by

$$\begin{aligned} \mathrm{d}\sigma(s,\underline{\omega}) = \mathbf{n}\mathrm{d}\Sigma &= \frac{\mathrm{d}}{\mathrm{d}s}[r(s)-\underline{\omega}y_0(s)]r^{n-1}\mathrm{d}s\mathrm{d}S(\underline{\omega}) \\ &= [\mathrm{d}r(s) - \underline{\omega}\mathrm{d}y_0(s)]r^{n-1}\mathrm{d}S(\underline{\omega}). \end{aligned}$$

Because $f$ is monogenic, its Cauchy integral formula is

$$f(x_0+\underline{\nu}\rho) = \int_{\Gamma_{\underline{\omega}}}\int_{\mathbb{S}^{n-1}} E(y_0+\underline{\omega}r - x_0 - \underline{\nu}\rho)\mathrm{d}\sigma(s,\underline{\omega})f(y_0+\underline{\omega}r), \tag{4.1}$$

where $x = x_0 + \underline{\nu}\tau \in V$. We can split the formula (4.1) into two parts:

$$f(x_0+\underline{\nu}\rho) = -\int_{\Gamma_{\underline{\omega}}} \left[ \int_{\mathbb{S}^{n-1}} E(y_0+\underline{\omega}r - x_0 - \underline{\nu}\rho)\underline{\omega}P_k(\underline{\omega})\mathrm{d}S(\underline{\omega}) \right] \times r^{k+n-1}[\mathrm{d}y_0 A(y_0,r) - \mathrm{d}r B(y_0,r)] + \int_{\Gamma_{\underline{\omega}}} \left[ \int_{\mathbb{S}^{n-1}} E(y_0+\underline{\omega}r - x_0 - \underline{\nu}\rho)P_k(\underline{\omega})\mathrm{d}S(\underline{\omega}) \right] \times r^{k+n-1}[\mathrm{d}y_0 B(y_0,r) + \mathrm{d}r A(y_0,r)].$$

From the identity $E(tx) = t^{-n}E(x)$, for $t > 0$, we get

$$f(x_0+\underline{\nu}\rho) = \int_{\Gamma_{\underline{\omega}}} \left[ \int_{\mathbb{S}^{n-1}} r^{-n}E\left(\frac{x-y_0}{r} - \underline{\omega}\right)\underline{\omega}P_k(\underline{\omega})\mathrm{d}S(\underline{\omega}) \right] \times r^{k+n-1}[\mathrm{d}y_0 A(y_0,r) - \mathrm{d}r B(y_0,r)] - \int_{\Gamma_{\underline{\omega}}} \left[ \int_{\mathbb{S}^{n-1}} r^{-n}E\left(\frac{x-y_0}{r} - \underline{\omega}\right)P_k(\underline{\omega})\mathrm{d}S(\underline{\omega}) \right] \times r^{k+n-1}[\mathrm{d}y_0 B(y_0,r) + \mathrm{d}r A(y_0,r)].$$

By Definition 4.1, we can rewrite the above formula as

$$f(x) = \int_{\Gamma_{\underline{\omega}}} \mathcal{K}^{-}_{k,n}\left(\frac{x-y_0}{r}\right) r^{k-1}[\mathrm{d}y_0 A(y_0,r) - \mathrm{d}r B(y_0,r)] - \int_{\Gamma_{\underline{\omega}}} \mathcal{K}^{+}_{k,n}\left(\frac{x-y_0}{r}\right) r^{k-1}[\mathrm{d}y_0 B(y_0,r) + \mathrm{d}r A(y_0,r)].$$

Let $x' = (x-y_0)/r$, due to Lemma 4.5, we have $\tau_k\left(\mathcal{P}^{\pm}_{k,n}\right)(x') = \mathcal{K}^{\pm}_{k,n}(x')$. Note that Lemma 4.5 asserts that $\tau_k\left(\mathcal{P}^{\pm}_{k,n}\right)(x')$ may only be defined except the sphere $\mathbb{S}^{n-1}$. This restriction affects the integral below through the fixed $x$ but upon the related integral variable $s$ on the curve $\Gamma_{\underline{\omega}}$. The restriction, in fact, just excludes a set of Lebesgue measure zero on $\Gamma_{\underline{\omega}}$ and thus does not actually affect the value of the integral. So, we obtain

$$f(x) = \int_{\Gamma_{\underline{\omega}}} \tau_k\left(\mathcal{P}^{-}_{k,n}\right)(x')\, r^{k-1}[\mathrm{d}y_0 A(y_0,r) - \mathrm{d}r B(y_0,r)] - \int_{\Gamma_{\underline{\omega}}} \tau_k\left(\mathcal{P}^{+}_{k,n}\right)(x')\, r^{k-1}[\mathrm{d}y_0 B(y_0,r) + \mathrm{d}r A(y_0,r)].$$

By Lemma 4.6, we have

$$f(x) = \int_{\Gamma_{\underline{\omega}}} \tau_k\left(\mathcal{P}^{-}_{k,n}\left(\frac{z-y_0}{r}\right)\right)(x) r^{3k+n-2}[\mathrm{d}y_0 A(y_0,r) - \mathrm{d}r B(y_0,r)]$$

$$
\begin{aligned}
&- \int_{\Gamma_{\underline{\omega}}} \tau_k \left( \mathcal{P}^+_{k,n} \left( \frac{z - y_0}{r} \right) \right) (x) r^{3k+n-2} [\mathrm{d}y_0 B(y_0, r) + \mathrm{d}r A(y_0, r)] \\
&= \int_{\Gamma_{\underline{\omega}}} (-\Delta)^{k+\frac{n-1}{2}} \left[ \mathcal{P}^-_{k,n} \left( \frac{x - y_0}{r} \right) P_k \left( \frac{\underline{x} - y_0}{r} \right) \right] \\
&\qquad \times r^{3k+n-2} [\mathrm{d}y_0 A(y_0, r) - \mathrm{d}r B(y_0, r)] \\
&\quad - \int_{\Gamma_{\underline{\omega}}} (-\Delta)^{k+\frac{n-1}{2}} \left[ \mathcal{P}^+_{k,n} \left( \frac{x - y_0}{r} \right) P_k \left( \frac{\underline{x} - y_0}{r} \right) \right] \\
&\qquad \times r^{3k+n-2} [\mathrm{d}y_0 B(y_0, r) + \mathrm{d}r A(y_0, r)].
\end{aligned}
$$

Let $x \in \mathbb{R}^{n+1}$. For any $a \in \mathbb{R}$, it is immediate that the 1-vector part of $x \pm a$ and of $x$ are the same, i.e. $\underline{x \pm a} = \underline{x}$. So we have

$$
\begin{aligned}
f(x) = &\int_{\Gamma_{\underline{\omega}}} (-\Delta)^{k+\frac{n-1}{2}} \left[ \mathcal{P}^-_{k,n} \left( \frac{x - y_0}{r} \right) P_k \left( r^{-1} \underline{x} \right) \right] \\
&\qquad \times r^{3k+n-2} [\mathrm{d}y_0 A(y_0, r) - \mathrm{d}r B(y_0, r)] \\
&- \int_{\Gamma_{\underline{\omega}}} (-\Delta)^{k+\frac{n-1}{2}} \left[ \mathcal{P}^+_{k,n} \left( \frac{x - y_0}{r} \right) P_k \left( r^{-1} \underline{x} \right) \right] \\
&\qquad \times r^{3k+n-2} [\mathrm{d}y_0 B(y_0, r) + \mathrm{d}r A(y_0, r)].
\end{aligned}
$$

Besides, since none of the involved integrands have singularities, we can exchange the order of the integration and the mapping $(-\Delta)^{k+(n-1)/2}$.

$$
\begin{aligned}
f(x) = &(-\Delta)^{k+\frac{n-1}{2}} \left\{ \int_{\Gamma_{\underline{\omega}}} \mathcal{P}^-_{k,n} \left( \frac{x - y_0}{r} \right) P_k \left( r^{-1} \underline{x} \right) \right. \\
&\qquad \times r^{3k+n-2} [\mathrm{d}y_0 A(y_0, r) - \mathrm{d}r B(y_0, r)] \\
&- \int_{\Gamma_{\underline{\omega}}} \mathcal{P}^+_{k,n} \left( \frac{x - y_0}{r} \right) P_k \left( r^{-1} \underline{x} \right) \\
&\qquad \left. \times r^{3k+n-2} [\mathrm{d}y_0 B(y_0, r) + \mathrm{d}r A(y_0, r)] \right\}.
\end{aligned}
$$

Using the fact that $P_k(\underline{x})$ is homogeneous of degree $k$, i.e., $P_k(r^{-1}\underline{x}) = r^{-k} P_k(\underline{x})$, we deduce the following equality:

$$
\begin{aligned}
f(x) = &(-\Delta)^{k+\frac{n-1}{2}} \left\{ \int_{\Gamma_{\underline{\omega}}} \mathcal{P}^-_{k,n} \left( \frac{x - y_0}{r} \right) r^{-k} P_k \left( \underline{x} \right) \right. \\
&\qquad \times r^{3k+n-2} [\mathrm{d}y_0 A(y_0, r) - \mathrm{d}r B(y_0, r)] \\
&- \int_{\Gamma_{\underline{\omega}}} \mathcal{P}^+_{k,n} \left( \frac{x - y_0}{r} \right) r^{-k} P_k \left( \underline{x} \right) \\
&\qquad \left. \times r^{3k+n-2} [\mathrm{d}y_0 B(y_0, r) + \mathrm{d}r A(y_0, r)] \right\}.
\end{aligned}
$$

So we have

$$
\begin{aligned}
f(x) =&(-\Delta)^{k+\frac{n-1}{2}} \Bigg\{ \int_{\Gamma_{\underline{\omega}}} \mathcal{P}_{k,n}^{-}\left(\frac{x-y_0}{r}\right) P_k\left(\underline{x}\right) \\
&\times r^{2k+n-2}[\mathrm{d}y_0 A(y_0,r) - \mathrm{d}r B(y_0,r)] \\
&- \int_{\Gamma_{\underline{\omega}}} \mathcal{P}_{k,n}^{+}\left(\frac{x-y_0}{r}\right) P_k\left(\underline{x}\right) \\
&\times r^{2k+n-2}[\mathrm{d}y_0 B(y_0,r) + \mathrm{d}r A(y_0,r)] \Bigg\} \\
=&(-\Delta)^{k+\frac{n-1}{2}} \Bigg\{ \Bigg[ \int_{\Gamma_{\underline{\omega}}} \mathcal{P}_{k,n}^{-}\left(\frac{x-y_0}{r}\right) \cdot r^{2k+n-2}[\mathrm{d}y_0 A(y_0,r) - \mathrm{d}r B(y_0,r)] \\
&- \int_{\Gamma_{\underline{\omega}}} \mathcal{P}_{k,n}^{+}\left(\frac{x-y_0}{r}\right) \cdot r^{2k+n-2}[\mathrm{d}y_0 B(y_0,r) + \mathrm{d}r A(y_0,r)] \Bigg] P_k\left(\underline{x}\right) \Bigg\}.
\end{aligned}
$$

By setting

$$
\begin{aligned}
f_k(z) :=& \int_{\Gamma_{\underline{\omega}}} \mathcal{P}_{k,n}^{-}\left(\frac{z-y_0}{r}\right) \cdot r^{2k+n-2}[\mathrm{d}y_0 A(y_0,r) - \mathrm{d}r B(y_0,r)] \\
&- \int_{\Gamma_{\underline{\omega}}} \mathcal{P}_{k,n}^{+}\left(\frac{z-y_0}{r}\right) \cdot r^{2k+n-2}[\mathrm{d}y_0 B(y_0,r) + \mathrm{d}r A(y_0,r)],
\end{aligned}
$$

we have $f = \tau_k(f_k)$. The statement follows because $f_k(z)$ is a holomorphic intrinsic function on $\mathbb{C}\backslash\{\mathrm{i}, -\mathrm{i}\}$. □

## 5. The decomposition of monogenic functions

In this section, we prove a decomposition formula of monogenic functions using Theorem 4.7. A monogenic function $f$ defined on axially symmetric open set can be decomposed in terms of axially monogenic functions of degree $k$ as in the following result, see Theorem 1.4 of [3], or the corresponding material on page 189 of [5].

**Lemma 5.1.** *Let $\Omega \subset \mathbb{R}^{n+1}$ be an axially symmetric open set. Then every monogenic function $f : \Omega \to \mathbb{R}_{0,n}$ can be written in the form $f(x) = \sum_{k=0}^{\infty} f_k(x)$ with*

$$
f_k(x) = \sum_{j=1}^{m_k} \left[A_{k,j}(x_0,r) + \underline{\omega} B_{k,j}(x_0,r)\right] P_{k,j}(\underline{x}),
$$

*where $P_{k,j}(\underline{x})$ form a basis for the space of spherical monogenics of degree $k$ which has dimension $m_k$, $A_{k,j}(x_0,r)$ and $B_{k,j}(x_0,r)$ are suitable real-valued function.*

We now give a decomposition formula for monogenic functions.

**Theorem 5.2.** *Let $n \in \mathbb{N}$, $k \in \mathbb{N}_0$ and $\Omega \subseteq \mathbb{R}^{n+1}$ be an axially symmetric open set. Then for every monogenic function $f : \Omega \to \mathbb{R}_{0,n}$ has the following decomposition*

$$f(x) = \sum_{k\in\mathbb{N}_0} \sum_{j=1}^{m_k} \tau_{k,j}\left(f_{k,j}\right)(x),$$

*where $m_k$ is a non-negative integer number related to $k$ and the holomorphic intrinsic function $f_{k,j}$ is given by*

$$f_{k,j}(z) = \int_{\Gamma_{\underline{\omega}}} \mathcal{P}^{-}_{k,n}\left(\frac{z-y_0}{r}\right) \cdot r^{2k+n-2}[\mathrm{d}y_0 A_{k,j}(y_0,r) - \mathrm{d}r B_{k,j}(y_0,r)]$$
$$- \int_{\Gamma_{\underline{\omega}}} \mathcal{P}^{+}_{k,n}\left(\frac{z-y_0}{r}\right) \cdot r^{2k+n-2}[\mathrm{d}y_0 B_{k,j}(y_0,r) + \mathrm{d}r A_{k,j}(y_0,r)].$$

**Proof.** Lemma 5.1 yields the decomposition

$$f(x) = \sum_{k\in\mathbb{N}_0} f_k(x).$$

Moreover, each $f_k(x)$ is monogenic and has the following form

$$f_k(x) = \sum_{j=1}^{m_k} \left[A_{k,j}(x_0,r) + \underline{\omega} B_{k,j}(x_0,r)\right] P_{k,j}(\underline{x}),$$

where $P_{k,j}(\underline{x})$ form a basis for the space of spherical monogenic of degree $k$ (which has dimension $m_k$), and $A_{k,j}(x_0,r)$ and $B_{k,j}(x_0,r)$ are suitable real-valued function.

Then, for each fixed $j \in \mathbb{N}$, Theorem 4.7 states that for every axially monogenic function of degree $k$

$$[A_{k,j}(x_0,r) + \underline{\omega} B_{k,j}(x_0,r)]P_{k,j}(\underline{x})$$

there exists a holomorphic intrinsic function $f_{k,j}$ such that

$$\tau_{k,j}(f_{k,j})(x) = [A_{k,j}(x_0,r) + \underline{\omega} B_{k,j}(x_0,r)]P_{k,j}(\underline{x}),$$

where

$$f_{k,j}(z) = \int_{\Gamma_{\underline{\omega}}} \mathcal{P}^{-}_{k,n}\left(\frac{z-y_0}{r}\right) \cdot r^{2k+n-2}[\mathrm{d}y_0 A_{k,j}(y_0,r) - \mathrm{d}r B_{k,j}(y_0,r)]$$
$$- \int_{\Gamma_{\underline{\omega}}} \mathcal{P}^{+}_{k,n}\left(\frac{z-y_0}{r}\right) \cdot r^{2k+n-2}[\mathrm{d}y_0 B_{k,j}(y_0,r) + \mathrm{d}r A_{k,j}(y_0,r)].$$

So finally, we have

$$f(x) = \sum_{k\in\mathbb{N}_0} \sum_{j=1}^{m_k} \tau_{k,j}\left(f_{k,j}\right)(x),$$

and the statement follows. □

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