Journal of Geometry and Physics 124 (2018) 225–232

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/geomphys

Clifford coherent state transforms on spheres

Pei Dang [a], José Mourão [b], João P. Nunes [b,*], Tao Qian [c]

[a] Faculty of Information Technology, Macau University of Science and Technology, Macao
[b] Department of Mathematics and Center for Mathematical Analysis, Geometry and Dynamical Systems, Instituto Superior Técnico, University of Lisbon, Portugal
[c] Department of Mathematics, Faculty of Science and Technology, University of Macau, Macao

ARTICLE INFO

Article history:
Received 20 January 2017
Accepted 5 November 2017
Available online 14 November 2017

Keywords:
Clifford analysis
Coherent state transforms
Cauchy–Kowalewski extension

ABSTRACT

We introduce a one-parameter family of transforms, $U^t_{(m)}$, $t > 0$, from the Hilbert space of Clifford algebra valued square integrable functions on the m–dimensional sphere, $L^2(\mathbb{S}^m, d\sigma_m)\otimes\mathbb{C}_{m+1}$, to the Hilbert spaces, $\mathcal{M}L^2(\mathbb{R}^{m+1}\setminus\{0\}, d\mu_t)$, of solutions of the Euclidean Dirac equation on $\mathbb{R}^{m+1}\setminus\{0\}$ which are square integrable with respect to appropriate measures, $d\mu_t$. We prove that these transforms are unitary isomorphisms of the Hilbert spaces and are extensions of the Segal–Bargman coherent state transform, $U_{(1)} : L^2(\mathbb{S}^1, d\sigma_1) \longrightarrow \mathcal{H}L^2(\mathbb{C}\setminus\{0\}, d\mu)$, to higher dimensional spheres in the context of Clifford analysis. In Clifford analysis it is natural to replace the analytic continuation from $\mathbb{S}^m$ to $\mathbb{S}^m_{\mathbb{C}}$ as in (Hall, 1994; Stenzel, 1999; Hall and Mitchell, 2002) by the Cauchy–Kowalewski extension from $\mathbb{S}^m$ to $\mathbb{R}^{m+1}\setminus\{0\}$. One then obtains a unitary isomorphism from an L^2–Hilbert space to a Hilbert space of solutions of the Dirac equation, that is to a Hilbert space of monogenic functions.

1. Introduction

In this work, we continue to explore the extensions of coherent state transforms to the context of Clifford analysis started in [1–4]. In [3], an extension of the coherent state transform (CST) to unitary maps from the spaces of L^2 functions on $M = \mathbb{R}^m$ and on the m–dimensional torus, $M = \mathbb{T}^m$, to the spaces of square integrable monogenic functions on $\mathbb{R} \times M$ was studied.

We consider the cases when M is an m–dimensional sphere, $M = S^m$, equipped with the $SO(m+1, \mathbb{R})$–invariant metric of unit radius. These cases are a priori more complicated than those studied before as the transform uses (for $m > 1$) the Laplacian and the Dirac operators for the non–flat metrics on the spheres. We show that there is a unique $SO(m+1, \mathbb{R})$ invariant measure on $\mathbb{R} \times \mathbb{S}^m \cong \mathbb{R}^{m+1} \setminus \{0\}$ such that the natural Clifford CST (CCST) is unitary. This transform is factorized into a contraction operator given by heat operator evolution at time $t = 1$ followed by Cauchy–Kowalewski (CK) extension, which exactly compensates the contraction for our choice of measure on $\mathbb{R}^{m+1} \setminus \{0\}$. In the usual coherent state Segal–Bargmann transforms [5–10], instead of the CK extension to a manifold with one more real dimension, one considers the analytic continuation to a complexification of the initial manifold (playing the role of phase space of the system). The CCST is of interest in Quantum Field Theory as it establishes natural unitary isomorphisms between Hilbert spaces of solutions of the Dirac equation and one-particle Hilbert spaces in the Schrödinger representation. The standard CST, on the other hand, studies the unitary equivalence of the Schrödinger representation with special Kähler representations with the wave functions defined on the phase space.

In the Section 3.2 we consider a one-parameter family of CCST, using heat operator evolution at time $t > 0$ followed by CK extension, and we show that, by changing the measure on $\mathbb{R}^{m+1} \setminus \{0\}$ to a new Gaussian (in the coordinate $\log(|\underline{x}|)$)

* Corresponding author.
E-mail address: jpnunes@math.tecnico.ulisboa.pt (J.P. Nunes).

https://doi.org/10.1016/j.geomphys.2017.11.002
0393-0440/

measure $d\mu_t$, these transforms are unitary. As t approaches 0 (so that the first factor in the transform is contracting less than for higher values of t) the measures $d\mu_t$ become more concentrated around the radius $|\underline{x}| = 1$ sphere and as $t \to 0$, the measure $d\mu_t$ converges to the measure

$$\delta(y)\, dy\, d\sigma_m\,,$$

where $y = \log(|\underline{x}|)$, supported on $\mathbb{S}^m$.

2. Clifford analysis

Let us briefly recall from [11–18], some definitions and results from Clifford analysis. Let $\mathbb{R}_{m+1}$ denote the real Clifford algebra with $(m+1)$ generators, $e_j, j = 1, \ldots, m+1$, identified with the canonical basis of $\mathbb{R}^{m+1} \subset \mathbb{R}_{m+1}$ and satisfying the relations $e_i e_j + e_j\, e_i = -2\delta_{ij}$. Let $\mathbb{C}_{m+1} = \mathbb{R}_{m+1} \otimes \mathbb{C}$. We have that $\mathbb{R}_{m+1} = \oplus_{k=1}^{m+1} \mathbb{R}_{m+1}^k$, where $\mathbb{R}_{m+1}^k$ denotes the space of k-vectors, defined by $\mathbb{R}_{m+1}^0 = \mathbb{R}$ and $\mathbb{R}_{m+1}^k = \mathrm{span}_{\mathbb{R}}\{e_A \,:\, A \subset \{1, \ldots, m+1\}, |A| = k\}$, where $e_{i_1 \ldots i_k} = e_{i_1} \ldots e_{i_k}$.

Notice also that $\mathbb{R}_1 \cong \mathbb{C}$ and $\mathbb{R}_2 \cong \mathbb{H}$. The inner product in $\mathbb{R}_{m+1}$ is defined by

$$\langle u, v\rangle = \left(\sum_A u_A e_A, \sum_B v_B e_B\right) = \sum_A u_A v_A.$$

The Dirac operator is defined as

$$\underline{D} = \sum_{j=1}^{m+1} e_j\, \partial_{x_j}.$$

We have that $\underline{D}^2 = -\Delta_{m+1}$.

Consider the subspace of $\mathbb{R}_{m+1}$ of 1-vectors

$$\{\underline{x} = \sum_{j=1}^{m+1} x_j e_j :\ x = (x_1, \ldots, x_m) \in \mathbb{R}^{m+1}\} \cong \mathbb{R}^{m+1},$$

which we identify with $\mathbb{R}^{m+1}$. Note that $\underline{x}^2 = -|\underline{x}|^2 = -(x, x)$.

Recall that a continuously differentiable function f on an open domain $\mathcal{O} \subset \mathbb{R}^{m+1}$, with values on $\mathbb{C}_{m+1}$, is called (left) monogenic on $\mathcal{O}$ if it satisfies the Dirac equation (see, for example, [11,12,15])

$$\underline{D}f(x) = \sum_{j=1}^{m+1} e_j\, \partial_{x_j} f(x) = 0.$$

For $m = 1$, monogenic functions on $\mathbb{R}^2$ correspond to holomorphic functions of the complex variable $x_1 + e_1 e_2\, x_2$.

The Cauchy kernel,

$$E(x) = \frac{\overline{x}}{|x|^{m+1}},$$

is a monogenic function on $\mathbb{R}^{m+1} \setminus \{0\}$. In the spherical coordinates, $r = e^y = |\underline{x}|$, $\xi = \frac{x}{|\underline{x}|}$, the Dirac operator reads

$$\underline{D} = \frac{1}{r}\,\underline{\xi}\left(r\partial_r + \Gamma_{\underline{\xi}}\right) = e^{-y}\underline{\xi}\left(\partial_y + \Gamma_{\underline{\xi}}\right), \tag{2.1}$$

where $\Gamma_{\underline{\xi}}$ is the spherical Dirac operator,

$$\Gamma_{\underline{\xi}} = -\underline{\xi}\,\partial_{\underline{\xi}} = -\sum_{i<j} e_{ij}\left(x_i \partial_{x_j} - x_j \partial_{x_i}\right).$$

We see from (2.1) that the equation for monogenic functions in the spherical coordinates is, on $\mathbb{R}^{m+1} \setminus \{0\}$, equivalent to

$$\underline{D}(f) = 0 \Leftrightarrow \partial_y f = -\Gamma_{\underline{\xi}}(f)\,, \qquad r > 0. \tag{2.2}$$

The Laplacian Δ_x has the form

$$\Delta_x = \partial_r^2 + \frac{m}{r}\partial_r + \frac{1}{r^2}\Delta_{\underline{\xi}},$$

where $\Delta_{\underline{\xi}}$ is the Laplacian on the sphere (for the invariant metric). The relation between the spherical Dirac operator and the spherical Laplace operator is (see e.g. [12], (0.16) and section II.1)

$$\Delta_{\underline{\xi}} = \left((m-1)I - \Gamma_{\underline{\xi}}\right)\, \Gamma_{\underline{\xi}}. \tag{2.3}$$

Let $\mathcal{H}(m+1,k)$ denote the space of ($\mathbb{C}_{m+1}$–valued) spherical harmonics of degree k. These are the eigenspaces of the self-adjoint spherical Laplacian, $\Delta_{\underline{\xi}}$,

$$\begin{aligned} f &\in \mathcal{H}(m+1,k) \\ \Delta_{\underline{\xi}}(f) &= -k(k+m-1)f. \end{aligned} \tag{2.4}$$

The spaces $\mathcal{H}(m+1,k)$ are a direct sum of eigenspaces of the self-adjoint spherical Dirac operator

$$\begin{aligned} \mathcal{H}(m+1,k) &= \mathcal{M}^+(m+1,k) \oplus \mathcal{M}^-(m+1,k-1) \\ \Gamma_{\underline{\xi}}(P_k(f)) &= -kP_k(f) \\ \Gamma_{\underline{\xi}}(Q_l(f)) &= (l+m)Q_l(f), \quad f \in L^2(\mathbb{S}^m, d\sigma_m) \otimes \mathbb{C}_{m+1}, \end{aligned} \tag{2.5}$$

where P_k, Q_l, denote the orthogonal projections on the subspaces $\mathcal{M}^+(m+1,k)$ and $\mathcal{M}^-(m+1,l)$ of $L^2(\mathbb{S}^m, d\sigma_m) \otimes \mathbb{C}_{m+1}$. The functions in $\mathcal{M}^+(m+1,k)$ and $\mathcal{M}^-(m+1,l)$ are in fact the restriction to $\mathbb{S}^m$ of (unique) monogenic functions

$$\begin{aligned} \tilde{P}_k(f)(\underline{x}) &= r^k\, P_k(f)\left(\frac{\underline{x}}{|\underline{x}|}\right) \\ \tilde{Q}_l(f)(\underline{x}) &= r^{-(l+m)}\, Q_l(f)\left(\frac{\underline{x}}{|\underline{x}|}\right), \quad f \in L^2(\mathbb{S}^m, d\sigma_m) \otimes \mathbb{C}_{m+1}, k, l \in \mathbb{Z}_{\geq 0}, \end{aligned} \tag{2.6}$$

where, for all $f \in L^2(\mathbb{S}^m, d\sigma_m) \otimes \mathbb{C}_{m+1}$, $\tilde{P}_k(f)$ are monogenic homogeneous polynomials of degree k and $\tilde{Q}_l(f)$ are monogenic functions on $\mathbb{R}^{m+1} \setminus \{0\}$, homogeneous of degree $-(l+m)$.

3. Clifford coherent state transforms on spheres

3.1. CCST on spheres and its unitarity

Definition 3.1. Let $\mathcal{A}(\mathbb{S}^m)$ be the space of analytic $\mathbb{C}_{m+1}$–valued functions on $\mathbb{S}^m$ with monogenic continuation to the whole of $\mathbb{R}^{m+1} \setminus \{0\}$.

Remark 3.2. Let V denote the space of finite linear combinations of spherical monogenics,

$$V = \mathrm{span}_{\mathbb{C}} \left\{P_k(f), Q_l(f), \ \ k, l \in \mathbb{Z}_{\geq 0}, f \in L^2(\mathbb{S}^m, d\sigma_m) \otimes \mathbb{C}_{m+1}\right\}.$$

We see from (2.6) that $V \subset \mathcal{A}(\mathbb{S}^m)$. We will denote by $\tilde{V}$ the space of CK extensions of elements of V to $\mathbb{R}^{m+1} \setminus \{0\}$ (see (2.6)),

$$\tilde{V} = \mathrm{span}_{\mathbb{C}} \left\{\tilde{P}_k(f), \tilde{Q}_l(f), \ \ k, l \in \mathbb{Z}_{\geq 0}, f \in L^2(\mathbb{S}^m, d\sigma_m) \otimes \mathbb{C}_{m+1}\right\}. \quad \diamond \tag{3.1}$$

In analogy with the case $m = 1$ and also with the "usual CST on spheres", introduced in [9,10], we will introduce the CCST

$$\begin{aligned} U_{(m)} \ &: \ L^2(\mathbb{S}^m, d\sigma_m) \otimes \mathbb{C}_{m+1} \longrightarrow \mathcal{M}L^2(\mathbb{R}^{m+1} \setminus \{0\}, \tilde{\rho}_m\, d^{m+1}x) \\ U_{(m)} &= CK_{\mathbb{S}^m} \circ e^{\Delta_{\underline{\xi}}/2} = e^{-y\Gamma_{\underline{\xi}}} \circ e^{\Delta_{\underline{\xi}}/2} \\ U_{(m)}(f)(\underline{x}) &= \int_{\mathbb{S}^m} \tilde{K}_1(\underline{x}, \underline{\xi}) f(\underline{\xi})\, d\sigma_m\,, \end{aligned} \tag{3.2}$$

where $CK_{\mathbb{S}^m} : \mathcal{A}(\mathbb{S}^m) \longrightarrow \mathcal{M}(\mathbb{R}^{m+1} \setminus \{0\})$ denotes the CK extension, K_1 is the heat kernel on $\mathbb{S}^m$ at time $t = 1$ and $\tilde{K}_1(\cdot, \underline{\xi})$ denotes the CK extension of K_1 to $\mathbb{R}^{m+1} \setminus \{0\}$ in its first variable (see Lemma 3.5, (3.8) and (3.9)). Our goal is to find (whether there exists) a function $\tilde{\rho}_m$ on $\mathbb{R}^{m+1} \setminus \{0\}$,

$$\tilde{\rho}_m(\underline{x}) = \rho_m(y), \qquad y = \log(|\underline{x}|)$$

which makes the (well defined) map in (3.2) unitary. For $m = 1$ there is a unique positive answer to the above question given by

$$\rho_1(y) = \frac{1}{\sqrt{\pi}}\, e^{-y^2-2y}$$

so that

$$\tilde{\rho}_1(\underline{x}) = \frac{1}{\sqrt{\pi}}\, e^{-\log^2(|\underline{x}|)-2\log(|\underline{x}|)}.$$

Our main result in the present paper is the following.

Theorem 3.3. *The map $U_{(m)}$ in (3.2) is a unitary isomorphism for*

$$\tilde{\rho}_m(\underline{x}) = \frac{e^{-\frac{(m-1)^2}{4}}}{\sqrt{\pi}} e^{-\log^2(|\underline{x}|)-2\log(|\underline{x}|)}. \tag{3.3}$$

Remark 3.4. It is remarkable that the only dependence on m of the corresponding function $\rho_m(y)$ is in the constant multiplicative factor, $e^{-\frac{(m-1)^2}{4}}$. $\diamondsuit$

Given the factorized form of $U_{(m)}$ in (3.2) we have the diagram

$$\begin{array}{ccc} & & \mathcal{M}L^2(\mathbb{R}^{m+1}\setminus\{0\}, \tilde{\rho}_m\, d^{m+1}x) \\ & \overset{U_{(m)}}{\nearrow} & \uparrow CK_{\mathbb{S}^m} = e^{-y\Gamma_{\underline{\xi}}} \\ L^2(\mathbb{S}^m, d\sigma_m)\otimes\mathbb{C}_{m+1} & \xrightarrow[e^{\Delta_{\underline{\xi}}/2}]{} & \mathcal{A}(\mathbb{S}^m), \end{array} \tag{3.4}$$

We divide the proof of Theorem 3.3 into several lemmas.

Lemma 3.5. *Let $f \in \mathcal{A}(\mathbb{S}^m)$ and consider its Dirac operator spectral decomposition or, equivalently, its decomposition in spherical monogenics,*

$$f = \sum_{k\geq 0} P_k(f) + \sum_{k\geq 0} Q_k(f). \tag{3.5}$$

Then its CK extension is given by

$$\begin{aligned} CK_{\mathbb{S}^m}(f)(\underline{x}) &= \sum_{k\geq 0} \tilde{P}_k(f)(\underline{x}) + \sum_{k\geq 0} \tilde{Q}_k(f)(\underline{x}) \\ &= \sum_{k\geq 0} |\underline{x}|^k P_k(f)\left(\frac{\underline{x}}{|\underline{x}|}\right) + \sum_{k\geq 0} |\underline{x}|^{-(k+m)} Q_k(f)\left(\frac{\underline{x}}{|\underline{x}|}\right) \\ &= e^{-y\Gamma_{\underline{\xi}}}(f) = |\underline{x}|^{-\Gamma_{\underline{\xi}}}(f). \end{aligned} \tag{3.6}$$

Proof. Since $f \in \mathcal{A}(\mathbb{S}^m)$ the two first lines in the right hand side of (3.6) are the Laurent expansion of $CK_{\mathbb{S}^m}(f)(\underline{x})$ in spherical monogenics (see [12], Theorem 1, p. 189), uniformly convergent on compact subsets of $\mathbb{R}^{m+1}\setminus\{0\}$. The third line in the right hand side follows from (2.5) and the fact that $\Gamma_{\underline{\xi}}$ is a self-adjoint operator. ■

Remark 3.6. We thus see that, for $f \in \mathcal{A}(\mathbb{S}^m)$, the operator of CK extension to $\mathbb{R}^{m+1}\setminus\{0\}$ is

$$CK_{\mathbb{S}^m} = e^{-y\Gamma_{\underline{\xi}}},$$

in agreement with (2.2) and (3.2). $\diamondsuit$

Lemma 3.7. *Let $f \in L^2(\mathbb{S}^m, d\sigma_m)\otimes\mathbb{C}_{m+1}$ and consider its decomposition in spherical monogenics,*

$$f = \sum_{k\geq 0} P_k(f) + \sum_{k\geq 0} Q_k(f).$$

Then the map

$$\begin{aligned} U_{(m)} &: L^2(\mathbb{S}^m, d\sigma_m)\otimes\mathbb{C}_{m+1} \longrightarrow \mathcal{M}(\mathbb{R}^{m+1}\setminus\{0\}) \\ U_{(m)} &= CK_{\mathbb{S}^m}\circ e^{\Delta_{\underline{\xi}}/2} = e^{-y\Gamma_{\underline{\xi}}}\circ e^{\Delta_{\underline{\xi}}/2}, \end{aligned}$$

where $\mathcal{M}(\Omega)$ denotes the space of monogenic functions on the open set $\Omega \subset \mathbb{R}^{m+1}$, is well defined and

$$\begin{aligned} U_{(m)}(f)(\underline{x}) &= e^{-y\Gamma_{\underline{\xi}}}\circ e^{\Delta_{\underline{\xi}}/2}(f)(\underline{x}) \\ &= \sum_{k\geq 0} e^{-k(k+m-1)/2}|x|^k P_k(f)\left(\frac{x}{|x|}\right) + \sum_{k\geq 0} e^{-(k+1)(k+m)/2}|x|^{-(k+m)} Q_k(f)\left(\frac{x}{|x|}\right) \\ &= \int_{\mathbb{S}^m} \tilde{K}_1(\underline{x}, \underline{\xi}) f(\xi)\, d\sigma_m(\xi), \end{aligned} \tag{3.7}$$

where K_1 denotes the heat kernel on $\mathbb{S}^m$ at time $t = 1$ and $\tilde{K}_1$ is the CK extension to $\mathbb{R}^{m+1}\setminus\{0\}$ of K_1 in its first variable.

Proof. From (2.3), (2.4) and (2.1) we have

$$\begin{aligned} e^{\Delta_{\underline{\xi}}/2}(f)(\underline{\eta}) &= \sum_{k\geq 0} e^{-k(k+m-1)/2} P_k(f)(\underline{\eta}) + \sum_{k\geq 0} e^{-(k+1)(k+m)/2} Q_k(f)(\underline{\eta}) \\ &= \int_{\mathbb{S}^m} K_1(\underline{\eta},\underline{\xi}) f(\underline{\xi})\, d\sigma_m(\underline{\xi}). \end{aligned}$$

From [12,18] we obtain

$$K_1(\underline{\eta},\underline{\xi}) = \sum_{k\geq 0} e^{-k(k+m-1)/2} \left(C^+_{m+1,k}(\underline{\eta},\underline{\xi}) + C^-_{m+1,k-1}(\underline{\eta},\underline{\xi}) \right), \tag{3.8}$$

where $C^-_{m+1,-1} = 0$,

$$\begin{aligned} C^+_{m+1,k}(\underline{\eta},\underline{\xi}) &= \frac{1}{1-m}\left[-(m+k-1)\, C_k^{(m-1)/2}(\langle \underline{\eta},\underline{\xi}\rangle) + (1-m)\, C_{k-1}^{(m+1)/2}(\langle \underline{\eta},\underline{\xi}\rangle)\, \underline{\eta}\wedge\underline{\xi} \right], \\ C^-_{m+1,k-1}(\underline{\eta},\underline{\xi}) &= \frac{1}{m-1}\left[k\, C_k^{(m-1)/2}(\langle \underline{\eta},\underline{\xi}\rangle) + (1-m)\, C_{k-1}^{(m+1)/2}(\langle \underline{\eta},\underline{\xi}\rangle)\, \underline{\eta}\wedge\underline{\xi} \right], \quad k\geq 1, \end{aligned}$$

$\underline{\eta}\wedge\underline{\xi} = \sum_{i<j}(\eta_i\xi_j - \eta_j\xi_i)e_{ij}$ and C_k^{ν} denotes the Gegenbauer polynomial of degree k associated with ν.

Now we prove that $K_1(\cdot,\underline{\xi}) \in \mathcal{A}(\mathbb{S}^m)$ for every $\xi \in \mathbb{S}^{m+1}$. From Lemma 3.5 and (3.8) we conclude that if $K_1(\cdot,\underline{\xi})$ has a CK extension then its Laurent series is given by

$$\begin{aligned} \tilde{K}_1(\underline{x},\underline{\xi}) &= \tilde{K}_1^+(\underline{x},\underline{\xi}) + \tilde{K}_1^-(\underline{x},\underline{\xi}) \\ &= \sum_{k\geq 0} e^{-k(k+m-1)/2}\, |x|^k\, C^+_{m+1,k}\left(\frac{\underline{x}}{|x|},\underline{\xi}\right) + \sum_{k\geq 1} e^{-k(k+m-1)/2}\, |x|^{-(k+m-1)}\, C^-_{m+1,k-1}\left(\frac{\underline{x}}{|x|},\underline{\xi}\right). \end{aligned} \tag{3.9}$$

Let us now show that this series is uniformly convergent in all compact subsets of $\mathbb{R}^{m+1}\setminus\{0\}$. From the explicit expressions for the degree k Gegenbauer polynomials (see e.g. [12], p. 182)

$$C_k^{m/2}(\langle \underline{\eta},\underline{\xi}\rangle) = \sum_{j=0}^{[k/2]} \frac{(-1)^j 2^{k-2j}(m/2)_{k-j}}{j!(k-2j)!}\, \langle \underline{\eta},\underline{\xi}\rangle^{k-2j},$$

where $(a)_j = a(a+1)\cdots(a+j-1)$. We see that

$$|C_k^{m/2}(\langle \underline{\eta},\underline{\xi}\rangle)| \leq \frac{(m+2k)!!}{(m-1)!!}\sum_{j=0}^{[k/2]} \frac{2^{-j}}{j!(k-2j)!} \leq \frac{(2k+m)!}{(m-1)!}, \quad \forall \eta,\xi\in\mathbb{S}^m.$$

Therefore we obtain that

$$\begin{aligned} |C^+_{m+1,k}(\underline{\eta},\underline{\xi})| &\leq \frac{(2k+m-1)!}{(m-1)!}(k+m-1) + \frac{(2k+m-1)!}{m!}\, m(m-1) \\ &= \frac{(2k+m-1)!}{(m-1)!}(k+2m-2), \quad \forall \eta,\xi\in\mathbb{S}^m. \end{aligned} \tag{3.10}$$

Let $s\in(0,1)$. From the Stirling formula and (3.10) we conclude that there exists $k_0\in\mathbb{N}$ such that

$$|C^+_{m+1,k}(\underline{\eta},\underline{\xi})| \leq e^{sk(k+m-1)/2}, \quad \forall \eta,\xi\in\mathbb{S}^m, \forall k > k_0,$$

and therefore

$$e^{-k(k+m-1)/2}\, |C^+_{m+1,k}\left(\frac{\underline{x}}{|x|},\underline{\xi}\right)| \leq e^{-(1-s)k(k+m-1)/2}, \quad \forall \eta,\xi\in\mathbb{S}^m, \forall k > k_0.$$

Then the series,

$$\tilde{K}_1^+(\underline{x},\underline{\xi}) = \sum_{k\geq 0} e^{-k(k+m-1)/2}\, |x|^k\, C^+_{m+1,k}\left(\frac{\underline{x}}{|x|},\underline{\xi}\right),$$

is uniformly convergent on all compact subsets of $\mathbb{R}^{m+1}$ and therefore its sum is monogenic on $\mathbb{R}^{m+1}$ in the first variable. To prove that the second series in (3.9) is uniformly convergent in compact subsets of $\mathbb{R}^{m+1}\setminus\{0\}$ we use the fact that the inversion is an isomorphism between $\mathcal{M}(\mathbb{R}^{m+1})$ and $\mathcal{M}_0(\mathbb{R}^{m+1}\setminus\{0\})$ (see section 1.6.5 of [12])

$$f \mapsto If,\ If(\underline{x}) = \frac{\underline{x}}{|x|^{m+1}} f\left(\frac{\underline{x}}{|x|^2}\right).$$

It is then equivalent to prove that the series

$$\left((I\otimes \mathrm{Id})(\tilde{K}_1^-)\right)(\underline{x},\underline{\xi}) = \frac{\underline{x}}{|x|^2}\sum_{k\geq 1} e^{-k(k+m-1)/2}\,|x|^k\, C^-_{m+1,k-1}\left(\frac{\underline{x}}{|x|},\underline{\xi}\right),$$

is uniformly convergent on compact subsets of $\mathbb{R}^{m+1}$. But this is a direct consequence of the following inequalities for $|C^-_{m+1,k-1}(\underline{\eta},\underline{\xi})|$, similar to the inequalities (3.10) for $|C^+_{m+1,k}(\underline{\eta},\underline{\xi})|$,

$$|C^-_{m+1,k-1}(\underline{\eta},\underline{\xi})| \leq \frac{(2k+m-1)!}{(m-1)!}\,(k+m-1)\,,\quad \forall \eta,\xi\in\mathbb{S}^m\,. \tag{3.11}$$

We have thus established that $\tilde{K}_1(\cdot,\xi)\in\mathcal{M}(\mathbb{R}^{m+1}\setminus\{0\})$, $\forall\xi\in\mathbb{S}^m$ with Laurent series given by (3.9). Analogously we can show that $\tilde{K}_1(\cdot,\cdot)\in C^\infty(\mathbb{R}^{m+1}\setminus\{0\}\times\mathbb{S}^m)\otimes\mathbb{C}_{m+1}$.

From (3.10) and (3.11), we also obtain,

$$\begin{aligned}
|P_k(f)(\eta)| &= \left|\int_{\mathbb{S}^m} C^+_{m+1,k}(\underline{\eta},\underline{\xi})f(\xi)\,d\sigma_m\right| \leq \frac{(2k+m-1)!}{(m-1)!}\,(k+2m-2)\,\|f\|,\\
|Q_{k-1}(f)(\eta)| &= \left|\int_{\mathbb{S}^m} C^-_{m+1,k-1}(\underline{\eta},\underline{\xi})f(\xi)\,d\sigma_m\right| \leq \frac{(2k+m-1)!}{(m-1)!}\,(k+m-1)\,\|f\|,\qquad \forall\eta\in\mathbb{S}^m\,.
\end{aligned}$$

As in the case of $\tilde{K}_1(\cdot,\xi)$, these inequalities imply that, for every $f\in L^2(\mathbb{S}^m,d\sigma_m)\otimes\mathbb{C}_{m+1}$, the Laurent series for $U_m(f)$ in (3.7) is uniformly convergent on compact subsets of $\mathbb{R}^{m+1}\setminus\{0\}$. ■

Lemma 3.8. *The map $U_{(m)}$ in (3.2) and (3.7) is an isometry for the measure factor $\tilde{\rho}_m$ given by (3.3).*

Proof. Given the $SO(m+1,\mathbb{R})$–invariance of the measures on $\mathbb{S}^m$ and on $\mathbb{R}^{m+1}\setminus\{0\}$ in (3.2) and (3.3), so that (3.5) is an orthogonal decomposition and so is (3.7), we see that to prove isometricity of $U_{(m)}$ it is sufficient to prove

$$\begin{aligned}
\|U_{(m)}(P_k(f))\| &= \|P_k(f)\|,\\
\|U_{(m)}(Q_k(f))\| &= \|Q_k(f)\|,
\end{aligned} \tag{3.12}$$

for all $k\in\mathbb{Z}_{\geq 0}$ and $f\in L^2(\mathbb{S}^m,d\sigma_m)\otimes\mathbb{C}_{m+1}$. We have

$$\begin{aligned}
\|U_{(m)}(P_k(f))\|^2 &= e^{-k(k+m-1)}\int_0^\infty r^{2k}\rho_m(\log(r))r^m dr\,\|P_k(f)\|^2,\\
\|U_{(m)}(Q_{k-1}(f))\|^2 &= e^{-k(k+m-1)}\int_0^\infty r^{-2(k-1+m)}\rho_m(\log(r))r^m dr\,\|Q_{k-1}(f)\|^2\,,
\end{aligned}$$

and therefore isometricity is equivalent to the following two infinite systems of equations setting constraints on the Laplace transform of the function $\rho_m(y)$. The system coming from the P_k is

$$\int_{\mathbb{R}}\rho_m(y)\,e^{y(2k+m+1)}\,dy = e^{k(k+m-1)}\,,\qquad k\in\mathbb{Z}_{\geq 0}, \tag{3.13}$$

and the system coming from the Q_k is

$$\int_{\mathbb{R}}\rho_m(y)\,e^{-y(2k+m-3)}\,dy = e^{k(k+m-1)}\,,\qquad k\in\mathbb{Z}_{\geq 0}. \tag{3.14}$$

It is easy to verify that the function ρ_m corresponding to $\tilde{\rho}_m$ in (3.3)

$$\rho_m(y) = \frac{e^{-\frac{(m-1)^2}{4}}}{\sqrt{\pi}}\,e^{-y^2-2y}$$

satisfies both (3.13) and (3.14). ■

Remark 3.9. Notice that each of the two systems (3.13) and (3.14) determines ρ_m uniquely so that it is remarkable that they both give the same solution. ◊

Proof. (of Theorem 3.3). From Lemmas 3.5, 3.7 and 3.8 we see that the only missing part is the surjectivity of $U_{(m)}$. But this follows from the fact that the space $\tilde{V}$ in (3.1) is dense, with respect to uniform convergence on compact subsets, in the space of monogenic functions on $\mathbb{R}^{m+1}\setminus\{0\}$ and therefore is also dense on $\mathcal{M}L^2(\mathbb{R}^{m+1}\setminus\{0\},\tilde{\rho}_m\,d^{m+1}x)$ since this has finite measure. Since the image of an isometric map is closed and the image of $U_{(m)}$ contains $\tilde{V}$ we conclude that $U_{(m)}$ is surjective. ■

As we mentioned in the introduction the mechanism for the unitarity of the CST, $U_{(m)}$, was its factorization into a contraction given by heat operator evolution at time $t=1$ followed by Cauchy–Kowalewski (CK) extension, which exactly compensates the contraction, given our choice of measure on $\mathbb{R}^{m+1}\setminus\{0\}$.

3.2. One-parameter family of unitary transforms

In the present section we will consider a one-parameter family of transforms, using heat operator evolution at time $t > 0$ followed by CK extension. We show that, by changing the measure on $\mathbb{R}^{m+1} \setminus \{0\}$ to a new Gaussian (in the coordinate $\log(|x|)$) measure

$$d\mu_t = \tilde{\rho}_m^t \, d^{m+1}x,$$

these transforms are unitary. Thus we consider the transforms

$$\begin{aligned} U_{(m)}^t \quad &: \quad L^2(\mathbb{S}^m, d\sigma_m) \otimes \mathbb{C}_{m+1} \longrightarrow \mathcal{M}L^2(\mathbb{R}^{m+1} \setminus \{0\}, \tilde{\rho}_m^t \, d^{m+1}x) \\ U_{(m)}^t &= CK_{\mathbb{S}^m} \circ e^{t\Delta_{\underline{\xi}}/2} = e^{-y\Gamma_{\underline{\xi}}} \circ e^{t\Delta_{\underline{\xi}}/2} \\ U_{(m)}^t(f)(\underline{x}) &= \int_{\mathbb{S}^m} \tilde{K}_t(\underline{x}, \underline{\xi}) f(\xi) \, d\sigma_m \,, \end{aligned} \tag{3.15}$$

where $\tilde{K}_t(\cdot, \xi)$ denotes the CK extension of K_t to $\mathbb{R}^{m+1} \setminus \{0\}$ in its first variable.

Our goal is to find (whether there exist), for every $t > 0$, a function $\tilde{\rho}_m^t$ on $\mathbb{R}^{m+1} \setminus \{0\}$,

$$\tilde{\rho}_m^t(\underline{x}) = \rho_m^t(y),$$

which makes the (well defined) map in (3.15) unitary. Again, for $m = 1$, there is a unique positive answer to the above question given by

$$\rho_1^t(y) = \frac{1}{\sqrt{t\pi}} \, e^{-\frac{y^2}{t} - 2y}$$

so that

$$\tilde{\rho}_1^t(\underline{x}) = \frac{1}{\sqrt{t\pi}} \, e^{-\frac{1}{t}\log^2(|x|) - 2\log(|x|)}.$$

We then have

Theorem 3.10. *The map $U_{(m)}^t$ in (3.15) is a unitary isomorphism for*

$$\tilde{\rho}_m^t(\underline{x}) = \frac{e^{-\frac{t(m-1)^2}{4}}}{\sqrt{t\pi}} \, e^{-\frac{1}{t}\log^2(|x|) - 2\log(|x|)}. \tag{3.16}$$

Given the factorized form of $U_{(m)}^t$ in (3.15) we have the diagram

$$\begin{array}{ccc} & & \mathcal{M}L^2(\mathbb{R}^{m+1} \setminus \{0\}, \tilde{\rho}_m^t \, d^{m+1}x) \\ & \overset{U_{(m)}^t}{\nearrow} & \uparrow {\scriptstyle CK_{\mathbb{S}^m} = e^{-y\Gamma_{\underline{\xi}}}} \\ L^2(\mathbb{S}^m, d\sigma_m) \otimes \mathbb{C}_{m+1} & \xhookrightarrow[e^{t\Delta_{\underline{\xi}}/2}]{} & \mathcal{A}(\mathbb{S}^m), \end{array} \tag{3.17}$$

Again we divide the proof of Theorem 3.10 into several lemmas. Notice, however, that Lemma 3.5 remains unchanged.

Lemma 3.11. *Let $f \in L^2(\mathbb{S}^m, d\sigma_m) \otimes \mathbb{C}_{m+1}$ and consider its decomposition in spherical monogenics,*

$$f = \sum_{k \geq 0} P_k(f) + \sum_{k \geq 0} Q_k(f).$$

Then the map

$$\begin{aligned} U_{(m)}^t \quad &: \quad L^2(\mathbb{S}^m, d\sigma_m) \otimes \mathbb{C}_{m+1} \longrightarrow \mathcal{M}(\mathbb{R}^{m+1} \setminus \{0\}) \\ U_{(m)}^t &= CK_{\mathbb{S}^m} \circ e^{t\Delta_{\underline{\xi}}/2} = e^{-y\Gamma_{\underline{\xi}}} \circ e^{t\Delta_{\underline{\xi}}/2} \,, \end{aligned}$$

is well defined and

$$\begin{aligned} U_{(m)}^t(f)(\underline{x}) &= e^{-y\Gamma_{\underline{\xi}}} \circ e^{t\Delta_{\underline{\xi}}/2}(f)(\underline{x}) \\ &= \sum_{k \geq 0} e^{-tk(k+m-1)/2} \, |x|^k \, P_k(f)\left(\frac{\underline{x}}{|x|}\right) + \sum_{k \geq 0} e^{-t(k+1)(k+m)/2} \, |x|^{-(k+m)} \, Q_k(f)\left(\frac{\underline{x}}{|x|}\right) \\ &= \int_{\mathbb{S}^m} \tilde{K}_t(\underline{x}, \underline{\xi}) f(\xi) \, d\sigma_m(\xi), \end{aligned} \tag{3.18}$$

where $\tilde{K}_t$ is the CK extension to $\mathbb{R}^{m+1} \setminus \{0\}$ of K_t in its first variable.

Proof. The proof is identical to the proof of Lemma 3.7. The Gaussian form (in k) of the coefficients coming from $e^{t\Delta_{\underline{\xi}}/2}$ and the inequalities (3.10), (3.11) again imply that $\tilde{K}_t(\cdot, \xi)$ and $U^t_{(m)}(f)$ are monogenic on $\mathbb{R}^{m+1} \setminus \{0\}$ and their Laurent series are given by

$$\begin{aligned}\tilde{K}_t(\underline{x}, \underline{\xi}) &= \tilde{K}^+_t(\underline{x}, \underline{\xi}) + \tilde{K}^-_t(\underline{x}, \underline{\xi}) \qquad (3.19)\\ &= \sum_{k\geq 0} e^{-tk(k+m-1)/2}\, |x|^k\, C^+_{m+1,k}\left(\frac{x}{|x|}, \underline{\xi}\right) + \sum_{k\geq 1} e^{-tk(k+m-1)/2}\, |x|^{-(k+m-1)}\, C^-_{m+1,k-1}\left(\frac{x}{|x|}, \underline{\xi}\right),\end{aligned}$$

and by (3.18). ■

Lemma 3.12. *The map $U^t_{(m)}$ in (3.15) and (3.18) is an isometry for the measure factor $\tilde{\rho}^t_m$ given by (3.16).*

Proof. Given the $SO(m+1, \mathbb{R})$–invariance of the measures on $\mathbb{S}^m$ and on $\mathbb{R}^{m+1} \setminus \{0\}$ in (3.15) and (3.16) we see that to prove isometricity of $U^t_{(m)}$ it is sufficient to prove

$$\begin{aligned}\|U^t_{(m)}(P_k(f))\| &= \|P_k(f)\|,\\ \|U^t_{(m)}(Q_k(f))\| &= \|Q_k(f)\|, \qquad (3.20)\end{aligned}$$

for all $k \in \mathbb{Z}_{\geq 0}$ and $f \in L^2(\mathbb{S}^m, d\sigma_m) \otimes \mathbb{C}_{m+1}$. Again, isometricity is equivalent to the following two infinite systems of equations setting constraints on the Laplace transform of the functions $\rho^t_m(y)$. The system coming from the P_k is

$$\int_{\mathbb{R}} \rho^t_m(y)\, e^{y(2k+m+1)}\, dy = e^{tk(k+m-1)}, \qquad k \in \mathbb{Z}_{\geq 0}, \qquad (3.21)$$

and the system coming from the Q_k is

$$\int_{\mathbb{R}} \rho^t_m(y)\, e^{-y(2k+m-3)}\, dy = e^{tk(k+m-1)}, \qquad k \in \mathbb{Z}_{\geq 0}. \qquad (3.22)$$

It is easy to verify that the function ρ^t_m corresponding to $\tilde{\rho}^t_m$ in (3.16) satisfies both (3.21) and (3.22). ■

Proof. The proof of Theorem 3.10 is completed exactly as the proof of Theorem 3.3 so that we omit it here. ■

Acknowledgments

The authors were partially supported by Macau Government FDCT through the project 099/2014/A2, *Two related topics in Clifford analysis*, by the Macao Science and Technology Development Fund, MSAR, Ref. 045/2015/A2 and by the University of Macau Research Grant MYRG115 (Y1-L4)-FST13-QT. The authors JM and JPN were also partly supported by FCT/Portugal through the projects UID/MAT/04459/2013 and PTDC/MAT-GEO/3319/2014.

References

[1] W.D. Kirwin, J. Mourão, J.P. Nunes, T. Qian, Extending coherent state transforms to Clifford analysis, J. Math. Phys. 57 (2016) 103505.
[2] K. Diki, A. Ghanmi, A quaternionic analogue of the Segal-Bargmann transform, Complex Anal. Oper. Theory 11 (2017) 457. http://dx.doi.org/10.1007/s11785-016-0609-5.
[3] J. Mourão, J.P. Nunes, T. Qian, Coherent state transforms and the Weyl equation in Clifford analysis, J. Math. Phys. 58 (2017) 013503.
[4] D. Peña Peña, I. Sabadini, F. Sommen, Segal–Bargmann–Fock modules of monogenic functions, J. Math. Phys. 58 (2017) 103507. http://dx.doi.org/10.1063/1.5008651.
[5] V. Bargmann, On a Hilbert space of analytic functions and an associated integral transform, Part I, Comm. Pure Appl. Math. 14 (1961) 187–214.
[6] I. Segal, Mathematical characterization of the physical vacuum for a linear Bose–Einstein field, Illinois J. Math. 6 (1962) 500–523.
[7] B.C. Hall, The Segal-Bargmann "coherent-state" transform for Lie groups, J. Funct. Anal. 122 (1994) 103–151.
[8] B.C. Hall, Holomorphic methods in analysis and mathematical physics, in: First Summer School in Analysis and Mathematical Physics (Cuernavaca Morelos, 1998), in: Contemp. Math., vol. 260, 2000, pp. 1–59.
[9] M. Stenzel, The Segal–Bargmann transform on a symmetric space of compact type, J. Funct. Anal. 165 (1999) 44–58.
[10] B.C. Hall, J. Mitchell, Coherent states on spheres, J. Math. Phys. 43 (2002) 1211–1236.
[11] F. Brackx, R. Delanghe, F. Sommen, Clifford Analysis, in: Research Notes in Mathematics, vol. 76, Pitman, Boston, 1982.
[12] R. Delanghe, F. Sommen, V. Soucek, Clifford Algebra and Spinor–Valued Functions, in: Mathematics and its Applications, vol. 53, Kluwer, 1992.
[13] N. De Schepper, F. Sommen, Cauchy–Kowalevski extensions and monogenic plane waves using spherical monogenics, Bull. Braz. Math. Soc. 44 (2013) 321–350.
[14] C. Li, A. McIntosh, T. Qian, Clifford algebras, Fourier transforms and singular convolution operators on Lipschitz surfaces, Rev. Mat. Iberoam. 19 (1994) 665–721.
[15] V. Souček, Generalized C-R equations on manifolds, in: J.S.R. Chisholm, A.K. Common (Eds.), Clifford Algebras and their Applications in Mathematical Physics, in: NATO ASI Series C, vol. 183, D. Reidel Pub. Company, 1986.
[16] M. Fei, T. Qian, Pointwise convergence for expansions in spherical monogenics, Acta Math. Sci. 29B (2009) 1241–1250.
[17] D. Peña Peña, T. Qian, F. Sommen, An alternative proof of Fueter's theorem, Complex Var. Elliptic Equ. 51 (2006) 913–922.
[18] P. Dang, T. Qian, Q. Chen, Uncertainty principle and phase–amplitude analysis of signals on the unit sphere, Adv. Appl. Clifford Algebras 27 (2017) 2985. http://dx.doi.org/10.1007/s00006-017-0808-9.