Results Math 71 (2017), 801–823

1422-6383/17/030801-23
published online July 16, 2016
DOI 10.1007/s00025-016-0575-2

Results in Mathematics

Schwarz Problems for Poly-Hardy Space on the Unit Ball

Uwe Kähler, Min Ku and Tao Qian

Abstract. In this paper we study the Schwarz boundary value problems (for short BVP) for the poly-Hardy space defined on the unit ball of higher dimensional Euclidean space $\mathbb{R}^n$. We first discuss the boundary behavior of functions belonging to the poly-Hardy class. Then we construct the Schwarz kernel function, and describe the boundary properties of the Schwarz-type integrable operator. Finally, we study the Schwarz BVPs for the Hardy class and the poly-Hardy class on the unit ball of higher dimensional Euclidean space $\mathbb{R}^n$, and obtain the expressions of solutions, explicitly. As an application, the monogenic signals considered for the Hardy spaces defined on the unit sphere are reconstructed when the scalar- and sub-algebra-valued data are given, which is the extension of the analytic signals for the Hardy spaces on the unit circle of the complex plane.

Mathematics Subject Classification. Primary: 44A15; Secondary: 42A38, 42C40.

Keywords. Hardy space, Schwarz problems, Schwarz kernel, monogenic signals.

1. Introduction

One of the fundamental boundary value problems in the classical complex analysis is the Schwarz boundary value problem, for short, the Schwarz problem. When a real valued continuous function on the unit circle of the complex plane is given, an analytic function is found on the unit disc, satisfying that

This work was supported by Portuguese funds through the CIDMA-Center for Research and Development in Mathematics and Applications, and the Portuguese Foundation for Science and Technology ("FCT-Fundação para a Ciência e a Tecnologia"), within project UID/MAT/0416/2013, by University of Macau MYRG115(Y1-L4)-FST13-QT, and by the Postdoctoral Foundation from FCT (Portugal) under Grant No. SFRH/BPD/74581/2010.

the boundary value of the real part of it on the unit circle coincides with the prescribed function. This problem is as a particular and simplest case of the Riemann–Hilbert problem first proposed by Riemann in 1851, where an analytic function is sought on the unit disc of the complex plane obtaining a given linear combination of its real and imaginary parts on the boundary. In 1872, Schwarz solved this particular problem long before the general Riemann–Hilbert problem was successfully dealt with, see Refs. e.g. [1,2]. He made a modification of the classical Cauchy kernel, and the real part of his kernel, nowadays known as the Schwarz kernel, turns out to coincide with the classical Poisson kernel for the harmonic functions [3–5]. Later on, Refs. [6,7] studied the Schwarz problems for the Hardy spaces and the poly-Hardy spaces on the unit disc of the complex plane, and got the explicit integrable solutions. Hereby, what interests us lies in the observation that the solutions to the Schwarz BVPs in the case of the Hardy spaces on the unit disc or on the half plane is equivalent to the reconstruction of the analytic signals (The analytic signals are regarded as the non-tangential boundary values of functions belonging to the Hardy spaces on the unit disc or on the real line [8,9].) for the Hardy spaces on the unit circle or on the real line.

The Riemann–Hilbert BVPs including the Schwarz problem as a special case, for the monogenic functions and the poly-monogenic functions defined in the sub-domains of higher dimensional Euclidean space were discussed in Refs. e.g. [10–12], making full use of Clifford analysis, which is an elegant generalization of the theory of complex analysis into higher dimensional Euclidean space (seen in Refs. e.g. [13–15]). However, no study of the Riemann–Hilbert BVPs has been done to link to the monogenic signals defined on the sphere of higher dimensional Euclidean space $\mathbb{R}^n$ (Here, the monogenic signals could be regarded as the non-tangential limits of monogenic functions on the boundary [8,9].). On the other hand, although the monogenic signals in three dimensions were studied in Refs. [16–18], utilizing the Clifford algebra valued Hilbert transforms (see Refs. e.g. [8,13,15,19]), to the authors' knowledge, only special ones have been explicitly presented. Thus, the natural questions arise as what the expressions of all of them would look like and how they link to the Riemann–Hilbert BVPs for the poly-Hardy spaces in higher dimensional Euclidean space $\mathbb{R}^n$. These will be not only purely theoretical questions, since such problems are closely linked to physical applications like signal processing [8,16] or problems in fluid mechanics [14,15]. Furthermore, to solve these questions gives impetus to consider respective discrete problems [20,21]. Moreover, in Refs. [8,22], the unique vector-valued monogenic signal in four dimensions was constructed under the additional Cauchy-type harmonic conjugate condition. Motivated by these considerations, we will extend the connection between analytic signals and the Schwarz BVPs for Hardy spaces to the case of higher dimensional Euclidean space $\mathbb{R}^n$, and use them to reconstruct all of the monogenic signals considered on the unit sphere of higher dimensional Euclidean space $\mathbb{R}^n$ in explicit form. Referring to the case of Hilbert-type BVPs for the poly-Hardy spaces on the unit circle [7], we will mainly study the Schwarz BVPs for the poly-Hardy spaces defined on the unit ball of higher dimensional Euclidean space $\mathbb{R}^n$. We first introduce the poly-Hardy class on the unit ball of higher dimensional Euclidean space $\mathbb{R}^n$, characterize the boundary behavior of functions in the poly-Hardy class, construct the Schwarz kernel function in the higher dimensions, and describe the boundary properties of the Schwarz integrable operator. Then making full use of them, we solve the Schwarz BVPs for the Hardy class on the unit ball and for the poly-Hardy class on the unit ball, of higher dimensional Euclidean space $\mathbb{R}^n$, respectively. Finally, we get the explicit expressions of solutions. As an application, all of the monogenic signals considered on the unit sphere of higher dimensional Euclidean space $\mathbb{R}^n$ are reconstructed when the scalar- and sub-algebra-valued boundary data are given.

The paper is organized as follows. In Sect. 2 we simply recall some basic facts about Clifford analysis which will be needed in the sequel. In Sect. 3, we will introduce the poly-Hardy space on the unit ball of higher dimensional Euclidean space $\mathbb{R}^n$, derive the decomposition theorem and characterize the boundary behavior of the functions in the poly-Hardy class. In Sect. 4, we introduce the theory of the Schwarz boundary value problem for the Hardy class on the unit ball of higher dimensional Euclidean space $\mathbb{R}^n$ with $\mathbb{L}_p\big(1 < p < +\infty\big)$-integrable boundary data, and construct the Schwarz kernel function to solve it. In the last section we will discuss the Schwarz boundary value problems for the poly-Hardy class on the unit ball of higher dimensional Euclidean space $\mathbb{R}^n$ with boundary data belonging to $\mathbb{L}_p\big(1 < p < +\infty\big)$ space. For this class of Schwarz boundary value problem, based on the decomposition theorem, we get the explicit expressions of their solutions.

2. Preliminaries

In this section we simply review some basic facts about Clifford algebras which will be needed in the sequel. More details could be seen in Refs. e.g. [13–15] or monographs elsewhere.

Let the Euclidean space $\mathbb{R}^n\,(n \in \mathbb{N}, n \geq 2)$ possess an orthogonal basis $\big\{e_1, e_2, \ldots, e_n\big\}$ satisfying $e_i^2 = -1$ for $i = 1, 2, \ldots, n$, $e_ie_j + e_je_i = 0$ for $1 \leq i < j \leq n$, and $e_{h_1}e_{h_2}\ldots e_{h_r} = e_{h_1h_2\ldots h_r}$ for $1 \leq h_1 < h_2 < \cdots < h_r \leq n$. Thus leads to the 2^n-dimensional real Clifford algebra $\mathbb{R}_n$, having the basis $\{e_{\mathcal{A}} : \mathcal{A} = \{h_1, \ldots, h_r\} \in \mathcal{PN}\}$, where $\mathcal{N}$ stands for the set $\big\{1, 2, \ldots, n\big\}$ and $\mathcal{PN}$ denotes the family of all order-preserving subsets of $\mathcal{N}$. In particular, we denote $e_\emptyset$ as e_0, which is the identity element now written as 1. All of the scalars, 1-vectors and 2-vectors in $\mathbb{R}_n$ are denoted by $\mathbb{R}_0, \mathbb{R}_1(\cong \mathbb{R}^n)$ and $\mathbb{R}_2$, respectively. An arbitrary element of $\mathbb{R}_1$ is denoted by $x = \sum_{j=1}^n x_je_j \triangleq \underline{x} + x_ne_n, x_j \in \mathbb{R}_0\big(j = 1, 2, \ldots, n\big)$. Let $\mathbb{R}_{n-1}$ be a sub-algebra of $\mathbb{R}_n$ constructed by $\big\{e_1, e_2, \ldots, e_{n-1}\big\}$, then $\mathbb{R}_n = \mathbb{R}_{n-1} \oplus e_n\mathbb{R}_{n-1}$. Given a $\lambda = \lambda_1 + e_n\lambda_2 \in \mathbb{R}_n$ with $\lambda_1, \lambda_2 \in \mathbb{R}_{n-1}$, we define $X^{(n)}(\lambda) = \lambda_1$ and $Y^{(n)}(\lambda) = \lambda_2$, and $\mathrm{Sc}(\lambda) = \lambda_0, \lambda_0 \in \mathbb{R}_0$

being the scalar part of λ. The conjugation is defined by $\bar{a} = \sum_{\mathcal{A}} a_{\mathcal{A}}\bar{e}_{\mathcal{A}}, \bar{e}_{\mathcal{A}} = (-1)^{\frac{k(k+1)}{2}} e_{\mathcal{A}}, N(\mathcal{A}) = k, a_{\mathcal{A}} \in \mathbb{R}_0$. The inner product $\langle\cdot,\cdot\rangle$ in $\mathbb{R}_n$ is defined by putting $\langle a, \bar{b}\rangle = [ab]_0$ for arbitrary $a, b \in \mathbb{R}_n$. The Clifford product of $x, y \in \mathbb{R}_1$ is defined by $xy = -\langle x, y\rangle + x \wedge y, x \wedge y = \sum_{1\leq i<j\leq n} e_i e_j (x_i y_j - x_j y_i)$. It is easy to derive the norm on $\mathbb{R}_n$ by $|a| = (\sum_{\mathcal{A}} |a_{\mathcal{A}}|^2)^{\frac{1}{2}} = \sqrt{\langle a, \overline{a}\rangle}$. Especially, we have $|x|^2 = (\sum\limits_{j=1}^{n} x_j^2)^{\frac{1}{2}} = \langle x, \overline{x}\rangle, x = \sum_{j=1}^n e_j x_j \in \mathbb{R}_1$. Moreover, $x^{-1} \triangleq \overline{x}|x|^{-2}$ is the inverse of $x \in \mathbb{R}_1 \backslash \{0\}$, i.e., $xx^{-1} = x^{-1}x = 1$. Up to the conjugation it corresponds to the Kelvin inverse in real analysis.

We introduce the Dirac operator $\mathcal{D} = \sum_{j=1}^n e_j \partial_{x_j}$, where $\partial_{x_j}, j = 1, 2, \ldots, n$, denotes the partial differential operator $\frac{\partial}{\partial_{x_j}}$. It results in the decomposition of the negative Laplacian in $\mathbb{R}_1$, i.e., $\mathcal{D}^2 = -\sum_{j=1}^n \partial_{x_j}^2$.

Let $\mathbb{B} = \{x \in \mathbb{R}_1 : |x| < 1\}$ be the unit ball centered at the origin with its boundary $\partial\mathbb{B} = \{x \in \mathbb{R}_1 : |x| = 1\}$. The $\mathbb{L}_p(1 < p < +\infty)$-integrability, continuity, differentiability, and so on of a function $\phi(x) = \sum_{\mathcal{A}} \phi_{\mathcal{A}}(x)e_{\mathcal{A}} : \mathbb{B} \to \mathbb{R}_n$, are ascribed to each component $\phi_{\mathcal{A}}(x) : \mathbb{B} \to \mathbb{R}_0$, which is continuous, differentiable, and so on, respectively. Let $\mathcal{C}^k(\mathbb{B}, \mathbb{R}_n)(k \geq 1, k \in \mathbb{N})$ denote the space of all k-times continuously differentiable functions defined in $\mathbb{B}$. The null solutions to the higher order Dirac operator $\mathcal{D}^k$, i.e., $\mathcal{D}^k\phi(x) = 0$, are the so-called poly-monogenic functions, particularly monogenic when $k = 1$. The set of all these functions is denoted by $\mathbb{H}_k(\mathbb{B}) = \{\phi : \mathbb{B} \to \mathbb{R}_n \big| \mathcal{D}^k\phi = 0\}$.

3. Poly-Hardy Spaces

In this section we will introduce the poly-Hardy space on the unit ball of higher dimensional Euclidean space $\mathbb{R}^n$, get a decomposition theorem for it, and study its boundary behavior.

Let ϕ be a function defined on the unit ball $\mathbb{B}$ of higher dimensional Euclidean space $\mathbb{R}^n$, the monogenic Hardy space on the unit ball is defined as

$$\mathbb{H}^p(\mathbb{B}) = \Big\{\phi \in \mathbb{H}_1(\mathbb{B}) : |\phi|_{1,p} < +\infty\Big\} \tag{1}$$

where $|\phi|_{1,p} = \sup_{0\leq r<1} |\phi_r|_p$, with

$$|\phi_r|_p = \left(\int_{\partial\mathbb{B}} \big|\phi_r(\eta)\big|^p dS_\eta\right)^{\frac{1}{p}},\ \phi_r(\eta) = \phi(r\eta), 0 \leq r < 1, \eta \in \partial\mathbb{B}, 1 < p < +\infty. \tag{2}$$

Then the space $\mathbb{H}^p(\mathbb{B})$ is a Banach space under the norm of (2). In the special case of $p = 2$, $\mathbb{H}^2(\mathbb{B})$ is a Hilbert space under the inner product $(f, g) = \int_{\partial\mathbb{B}} \overline{f}(\eta)g(\eta)dS_\eta, f, g \in \mathbb{H}^2(\mathbb{B})$.

Let $\mathbb{H}_k(\mathbb{B})\big(k \geq 1, k \in \mathbb{N}\big)$ be as in Sect. 2. For arbitrary function $\phi \in \mathbb{H}_k(\mathbb{B})$, we define functions as follows

$$\phi^j = \mathcal{D}^j \phi, \quad x \in \mathbb{B}, \quad j = 0, 1, 2, \ldots, k-1, \tag{3}$$

and, hereby, $\phi^0 = \phi, x \in \mathbb{B}$.

Definition 3.1. For arbitrary $1 < p < +\infty$ and $k \geq 1, k \in \mathbb{N}$, the set of all functions defined on the unit ball $\mathbb{B}$ of higher dimensional Euclidean space $\mathbb{R}^n$, satisfying

$$\left\{\phi \in \mathbb{H}_k(\mathbb{B}) : \left|\phi^j\right|_{1,p} < +\infty, \quad j = 0, 1, 2, \ldots, k-1\right\}, \tag{4}$$

is the so-called poly-Hardy space of order k defined on the unit ball, where the norm $|.|_{1,p}$ is defined as (2) and $\phi^j\big(j = 0, 1, 2, \ldots, k-1\big)$ given in (3). Therefore, such a poly-Hardy space defined on the unit ball of higher dimensional Euclidean space $\mathbb{R}^n$ will be denoted as $\mathbb{H}_k^p(\mathbb{B})$.

It is obvious that the space $\mathbb{H}_k^p(\mathbb{B})\big(1 < p < +\infty\big)$ is linear. Define

$$|\phi|_{k,p} = \sum_{j=0}^{k-1} \left|\phi^j\right|_{1,p}, \ \phi \in \mathbb{H}_k^p(\mathbb{B}), \tag{5}$$

where the norm $|.|_{1,p}$ is given in (2). In the following context we assume $\mathbb{H}_1^p(\mathbb{B}) = \mathbb{H}^p(\mathbb{B})$ without confusion and for brevity. In particular, when $k = 1$, the norm $|.|_{k,p}$ in (5) reduces to the case of (2). When $k = 2$, the space $\mathbb{H}_2^p(\mathbb{B})\big(1 < p < +\infty\big)$ is actually the harmonic Hardy space defined on the unit ball $\mathbb{B}$ of higher dimensional Euclidean space $\mathbb{R}^n$. Generally, when $k = 2m, m \in \mathbb{N}$, the space $\mathbb{H}_k^p(\mathbb{B})\big(1 < p < +\infty\big)$ is the poly-harmonic Hardy space defined on the unit ball $\mathbb{B}$ of higher dimensional Euclidean space $\mathbb{R}^n$. From Definition 4.1, we get $\mathbb{H}_k^p(\mathbb{B}) = \{\phi \in \mathbb{H}_k(\mathbb{B}) : |\phi|_{k,p} < +\infty\}$. In order to avoid the trivial case, we assume $k > 1, k \in \mathbb{N}$ without explanation.

Before characterizing the boundary behavior of the poly-Hardy space $\mathbb{H}_k^p(\mathbb{B})$, we need several lemmas as follows.

Lemma 3.2 (see Refs. [23–27]). *The shifted Euler operator defined on the space $\mathcal{C}^1\big(\mathbb{B}, \mathbb{R}_n\big)$ is given by*

$$E_s = sI + \sum_{j=1}^{n} x_j \partial_{x_j}\big(s > 0\big), \tag{6}$$

with I being the identity operator defined on the space $\mathcal{C}^1\big(\mathbb{B}, \mathbb{R}_n\big)$. The operator $I_s : \mathcal{C}\big(\mathbb{B}, \mathbb{R}_n\big) \to \mathcal{C}\big(\mathbb{B}, \mathbb{R}_n\big)$ is defined by

$$I_s\phi = \int_0^1 \phi(tx)t^{s-1}dt(s>0), \quad x \in \mathbb{B}. \tag{7}$$

Then on the space $\mathcal{C}^1(\mathbb{B}, \mathbb{R}_n)$ *we have*

(i) $E_sI_s = I_sE_s = I$,

(ii) $\mathcal{D}E_s\phi = E_{s+1}\mathcal{D}\phi$ *and* $E_sx\phi = xE_{s+1}\phi$.

Furthermore, if $\phi \in \mathcal{C}^k(\mathbb{B}, \mathbb{R}_n), k \in \mathbb{N}$, *is a solution to* $\mathcal{D}^k\phi = 0$, *then* $E_s\phi$ *and* $I_s\phi$ *are both solutions to* $\mathcal{D}^k\phi = 0$, *where* $\mathcal{D}^k\phi \triangleq \mathcal{D}^{k-1}(\mathcal{D}\phi)$.

Lemma 3.3 (see Refs. [23,26,27]). *Let* $j \in \mathbb{N}$ *be arbitrary. If* $\phi \in \mathcal{C}^1(\mathbb{B}, \mathbb{R}_n)$ *is monogenic, then*

$$\mathcal{D}\left(x^j\phi\right) = \begin{cases} -2mx^{2m-1}\phi, & \text{if } j = 2m, \\ -2x^{2(m-1)}E_{\frac{n+1}{2}+[\frac{j}{2}]-1}\phi, & \text{if } j = 2m-1, \end{cases} \quad x \in \mathbb{B},$$

where $m \in \mathbb{N}$. *Moreover, for* $l, p \in \mathbb{N}$ *and* $2 \le l \le j$, *one gets*

$$\mathcal{D}^l x^j\phi = C_{l,j}x^{j-l}E_{\frac{n+1}{2}+[\frac{j-l}{2}]}\dots E_{\frac{n+1}{2}+[\frac{j}{2}]-1}\phi, \tag{8}$$

with

$$C_{l,j} = \begin{cases} 2^l m(m-1)\dots(m-p+1), & \text{if } j = 2m, l = 2p, \\ -2^l m(m-1)\dots(m-p), & \text{if } j = 2m, l = 2p+1, \\ 2^l(m-1)\dots(m-p), & \text{if } j = 2m-1, l = 2p, \\ -2^l(m-1)\dots(m-p+1), & \text{if } j = 2m-1, l = 2p-1. \end{cases}$$

Particularly, for $l = j$, *we derive*

$$\mathcal{D}^jx^j\phi = C_{j,j}E_{\frac{n+1}{2}}\dots E_{\frac{n+1}{2}+m-1}\phi \quad \text{with} \quad C_{j,j} = \begin{cases} 2^jm!, & \text{if } j = 2m, \\ -2^j(m-1)!, & \text{if } j = 2m-1. \end{cases}$$

Lemma 3.4. (i) *If* $\phi \in \mathbb{H}^p(\mathbb{B})$, *and* $E_s\,(s>0)$ *defined by* (6), *then* $E_s\phi \in \mathbb{H}^p(\mathbb{B})$. (ii) *If* $\phi \in \mathbb{H}^p(\mathbb{B})$, *and* $I_s\,(s>1)$ *defined by* (7), *then* $I_s\phi \in \mathbb{H}^p(\mathbb{B})$.

Proof. (i) Since $\phi \in \mathbb{H}^p(\mathbb{B})$, due to Lemma 3.2, then $\mathcal{D}E_s\phi = 0, x \in \mathbb{B}$. From (5), $|\phi|_{1,p} < +\infty$. Therefore, using the Minkowski's inequality, we get $E_s\phi \in \mathbb{H}^p(\mathbb{B})$.

(ii) Since $\phi \in \mathbb{H}^p(\mathbb{B})$, then $|\phi|_{1,p} < +\infty$, that is, $\sup_{0\le r<1}\left(\int_{\partial\mathbb{B}}|\phi(r\eta)|^p d\eta\right)^{\frac{1}{p}} < +\infty$.

As $I_s\phi(x) = \int_0^1 \phi(tx)t^{s-1}dt\,(s>1), x \in \mathbb{B}$, then applying the Hölder inequality, we have

$$\begin{aligned}\int_{\partial\mathbb{B}}|I_s\phi(x)|^p\,dx &\le \int_{\partial\mathbb{B}}\int_0^1 |\phi(tx)|^p\,t^{p(s-1)}dtdx = \int_0^1\int_{t\partial\mathbb{B}}|\phi(u)|^p\,dut^{p(s-1)-1}dt \\ &\le |\phi|_{1,p}^p\int_0^1 t^{p(s-1)-1}dt = \frac{1}{p(s-1)}|\phi|_{1,p}^p < +\infty. \end{aligned} \tag{9}$$

Therefore, $|I_s\phi(x)|_{1,p} < +\infty$, that is, $I_s\phi \in \mathbb{H}^p(\mathbb{B})$. □

Now, let us derive the following theorem for the poly-Hardy space $\mathbb{H}_k^p(\mathbb{B})$.

Theorem 3.5. *Let $\mathbb{H}_k^p(\mathbb{B}), 1 < p < +\infty$, be the poly-Hardy space of order $k\big(k > 1\big)$ defined on the unit ball of higher dimensional Euclidean space $\mathbb{R}^n$, then*

$$\mathbb{H}_k^p(\mathbb{B}) = \mathbb{H}_1^p(\mathbb{B}) \oplus x\mathbb{H}_1^p(\mathbb{B}) \oplus \cdots \oplus x^{k-1}\mathbb{H}_1^p(\mathbb{B}), \tag{10}$$

where $x^j\mathbb{H}_1^p(\mathbb{B}) = \{x^j\phi(x) : \phi \in \mathbb{H}_1^p(\mathbb{B})\}, j = 0, 1, 2, \ldots, k-1$.

Proof. First of all, there needs to prove that

$$\mathbb{H}_k^p(\mathbb{B}) \subset \mathbb{H}_1^p(\mathbb{B}) \oplus x\mathbb{H}_1^p(\mathbb{B}) \oplus \cdots \oplus x^{k-1}\mathbb{H}_1^p(\mathbb{B}). \tag{11}$$

In fact, if $\phi \in \mathbb{H}_k^p(\mathbb{B})$, applying Definition 4.1 and Lemma 3.3 in [23] or [27], one has the unique decomposition

$$\phi = \sum_{j=0}^{k-1} x^j\phi_j, \quad x \in \mathbb{B}, \tag{12}$$

where each ϕ_j is monogenic in $\mathbb{B}\big(j = 0, 1, 2, \ldots, k-1\big)$ and given by

$$\begin{cases} \phi_0 = \big(I - x\mathcal{Q}_1\mathcal{D}\big)\big(I - x^2\mathcal{Q}_2\mathcal{D}^2\big)\ldots\big(I - x^{k-1}\mathcal{Q}_{k-1}\mathcal{D}^{k-1}\big)\phi, \\ \phi_1 = \mathcal{Q}_1\mathcal{D}\big(I - x^2\mathcal{Q}_2\mathcal{D}^2\big)\ldots\big(I - x^{k-1}\mathcal{Q}_{k-1}\mathcal{D}^{k-1}\big)\phi, \\ \qquad\vdots \qquad\qquad\qquad\qquad \vdots \\ \phi_{k-2} = \mathcal{Q}_{k-2}\mathcal{D}^{k-2}\big(I - x^{k-1}\mathcal{Q}_{k-1}\mathcal{D}^{k-1}\big)\phi, \\ \phi_{k-1} = \mathcal{Q}_{k-1}\mathcal{D}^{k-1}\phi, \end{cases}$$

with $\mathcal{Q}_j = \frac{1}{a_j} I_{\frac{n}{2}} I_{\frac{n}{2}+1} \ldots I_{\frac{n}{2}+[\frac{j-1}{2}]}, a_j = (-2)^k[\frac{j}{2}]!$ for $j = 1, 2, \ldots, k-1$, and

$$[s] = \begin{cases} q, & \text{if } q \in \mathbb{N}, \\ q+1, & \text{if } s = q+t, q \in \mathbb{N}, 0 < t < 1. \end{cases}$$

Moreover, by Lemma 3.3, we get

$$\mathcal{D}\phi = \sum_{j=1}^{k-1} \mathcal{D}\left(x^j\phi_j\right) = \begin{cases} -\sum\limits_{j=1}^{k-1} 2mx^{j-1}\phi_j, & \text{if } j = 2m, \\ -\sum\limits_{j=1}^{k-1} 2x^{j-1}E_{\frac{n+1}{2}+[\frac{j}{2}]-1}\phi_j, & \text{if } j = 2m-1, \end{cases} \tag{13}$$

and for $l \in \mathbb{N}$ and $2 \le l \le j$,

$$\mathcal{D}^l\phi = \sum_{j=1}^{k-1} \mathcal{D}^l\left(x^j\phi_j\right) = \sum_{j=1}^{k-1} C_{l,j}x^{j-l}E_{\frac{n+1}{2}+[\frac{j-l}{2}]}\ldots E_{\frac{n+1}{2}+[\frac{j}{2}]-1}\phi_j, \quad x \in \mathbb{B}. \tag{14}$$

This implies

$$\phi_{k-1} = C_{k-1,k-1}^{-1} I_{\frac{n+1}{2}+m-1}\ldots I_{\frac{n+1}{2}}\phi^{k-1}, \quad x \in \mathbb{B} \text{ with}$$

$$C_{k-1,k-1} = \begin{cases} 2^{k-1}m!, & \text{if } k-1 = 2m, \\ -2^{k-1}\big(m-1\big)!, & \text{if } k-1 = 2m-1. \end{cases}$$

Since $\phi \in \mathbb{H}_k^p(\mathbb{B})$, then $\left|\phi^{k-1}\right|_{1,p} < +\infty$. Therefore, associating with Lemma 3.4, we have $|\phi_{k-1}|_{1,p} < +\infty$. Secondly, we also get

$$\begin{aligned}\phi_{k-2} = &C_{k-2,k-2}^{-1} I_{\frac{n+1}{2}+m-1} \cdots I_{\frac{n+1}{2}} \\ &\times \left(\phi^{k-2} - C_{k-2,k-1} x E_{\frac{n+1}{2}+[\frac{1}{2}]} \cdots E_{\frac{n+1}{2}+[\frac{k-2}{2}]-1} \phi_{k-1}\right), \quad x \in \mathbb{B}. \end{aligned} \tag{15}$$

From (12), we know $\mathcal{D}(\phi^{k-2} - C_{k-2,k-1} x E_{\frac{n+1}{2}+[\frac{1}{2}]} \cdots E_{\frac{n+1}{2}+[\frac{k-2}{2}]-1} \phi_{k-1}) = 0, x \in \mathbb{B}$. Making full use of Lemmas 3.2 and 3.4, we have $|x E_{\frac{n+1}{2}+[\frac{1}{2}]} \cdots E_{\frac{n+1}{2}+[\frac{k-2}{2}]-1} \phi_{k-1}|_{1,p} < +\infty$. Associating with $|\phi_{k-1}|_{1,p} < +\infty$ and $|\phi^{k-2}|_{1,p} < +\infty$, we get $|\phi_{k-2}|_{1,p} < +\infty$.

Applying (14) and Lemma 3.4, by iteration of the same procedure, one has $|\phi_j|_{1,p} < +\infty, j = 0, 1, \ldots, k-3$.

This implies $\phi_j \in \mathbb{H}_1^p(\mathbb{B}), j = 0, 1, 2, \ldots, k-1$, which gives the validity of (11). Thus, we get

$$\mathbb{H}_k^p(\mathbb{B}) = \mathbb{H}_1^p(\mathbb{B}) + x\mathbb{H}_1^p(\mathbb{B}) + \cdots + x^{k-1}\mathbb{H}_1^p(\mathbb{B}). \tag{16}$$

Finally, let $0 = \sum_{j=0}^{k-1} x^j \phi_j(x)$ with $\phi_j \in \mathbb{H}_1^p(\mathbb{B}), j = 0, 1, 2, \ldots, k-1$. Starting with (3), one gets that $\phi_j \equiv 0, j = 0, 1, 2, \ldots, k-1, x \in \mathbb{B}$. The proof of the result is completed. □

Due to Theorem 3.5, we might characterize the boundary behavior of functions belonging to the poly-Hardy space $\mathbb{H}_k^p(\mathbb{B})$.

Theorem 3.6. *If $\phi \in \mathbb{H}_k^p(\mathbb{B})\big(1 < p < +\infty\big)$, then ϕ has the non-tangential limit ϕ^+ almost everywhere on the sphere $\partial\mathbb{B}$ under the $|\cdot|_p$-norm given by (2), and*

$$\lim_{r\to 1} \left|\phi^+ - \phi_r\right|_p = 0, \tag{17}$$

where ϕ_r is same with (2).

Proof. Since $\phi \in \mathbb{H}_k^p(\mathbb{B})\big(1 < p < +\infty\big)$, making full use of Theorem 3.5, we have

$$\phi(x) = \sum_{j=0}^{k-1} x^j \phi_j(x), \phi_j \in \mathbb{H}_1^p(\mathbb{B}), \quad j = 0, 1, 2, \ldots, k-1. \tag{18}$$

Applying the boundary behavior of the monogenic Hardy space $\mathbb{H}_1^p(\mathbb{B})$(see Theorem 7.9 of Chapter 2 in Ref. [15]), $\phi_j\big(j = 0, 1, 2, \ldots, k-1\big)$ has non-tangential limit $\phi_j^+(t)$ almost everywhere on the sphere $\partial\mathbb{B}$, and

$$\lim_{r\to 1} \left|\phi_j^+ - (\phi_j)_r\right|_p = 0, \tag{19}$$

where $(\phi_j)_r\big(j=0,1,2,\ldots,k-1\big)$ is defined similarly to (2). Hence, associating with the decomposition (18), we obtain

$$\phi^+(t)=\sum_{j=0}^{k-1} t^j\phi_j^+(t),\quad \text{a.e. } t\in\partial\mathbb{B}. \tag{20}$$

This implies that ϕ has non-tangential limit ϕ^+ almost everywhere on the sphere $\partial\mathbb{B}$. Applying (19) and (20), one has

$$\left|\phi^+-\phi_r\right|_p\le\sum_{j=0}^{k-1}\left[\left|\phi_j^+-(\phi_j)_r\right|_p+\left(1-r^j\right)\left|(\phi_j)_r\right|_p\right], \tag{21}$$

which leads to the validity of (17) by (19). It follows the result. □

Corollary 3.7. *If $\phi\in\mathbb{H}_k^p(\mathbb{B})\big(1<p<+\infty\big)$, then $\mathcal{D}^j\phi$ have the non-tangential limit $\left(\mathcal{D}^j\phi\right)^+, j=1,2,\ldots,k-1$, almost everywhere on the sphere $\partial\mathbb{B}$ under the $|\cdot|_p$-norm given by (2). This implies that the non-tangential limits of the right-hand sides of (13) and (14) all exit under the $|\cdot|_p$-norm given by (2).*

Proof. Since $\phi\in\mathbb{H}_k^p(\mathbb{B})\big(1<p<+\infty\big)$, associating with (5), then $\mathcal{D}^j\phi\in\mathbb{H}_{k-j}^p(\mathbb{B})\big(1<p<+\infty\big), j=1,2,\ldots,k-1$. According to Theorem 3.6, it follows the result. □

Remark 3.8. Theorem 3.6 gives the characterization of boundary behavior of the poly-Hardy space $\mathbb{H}_k^p(\mathbb{B})\big(1<p<+\infty\big)$. The boundary value ϕ^+ seen in (20) is the so-called non-tangential boundary value of ϕ. In what follows, the symbol ϕ^+ will be understood as the non-tangential boundary value of ϕ if no explanation. Moreover, if $k=1$, when the dimension of the space considered is $n=4$, it reduces to that in [8]. When the dimension of the space considered is $n=2$, it reduces to that in [7].

4. Schwarz BVPs for Hardy Space

In this section, we study the non-homogeneous Schwarz BVPs for the Hardy class on the unit ball of higher dimensional Euclidean space $\mathbb{R}^n$ with $\mathbb{L}_p\,(1<p<+\infty)$-integrable boundary data. For this kind of Schwarz BVPs, we obtain the explicit expressions of the solutions.
First, let us give a theorem about the harmonic conjugate of a monogenic function defined on the unit ball of higher dimensional Euclidean space $\mathbb{R}^n$.

Definition 4.1. Let Ω be a sub-domain of $\mathbb{R}^n$ $(\mathbb{R}^n\cong\mathbb{R}_1)$, and $\phi,\psi\in\mathcal{C}^2\left(\Omega,\mathbb{R}_{n-1}\right)$ are harmonic. If the function $\phi+e_n\psi$ is monogenic in Ω, then ϕ is called harmonic conjugate to ψ.

Obviously, then also ψ is harmonic conjugate to ϕ. According to Definition 4.1, for the function

$$P(x,y)=\frac{1-|x|^2}{\omega_n|y-x|^n}\big(n\geq 2,n\in\mathbb{N}\big),\quad x\in\mathbb{B},y\in\partial\mathbb{B},\tag{22}$$

where ω_n denotes the area of unit sphere $\partial\mathbb{B}$, a harmonic conjugate function $Q(x,y)$ of $P(x,y)$ could be given by

$$\begin{aligned}Q(x,y)=\frac{1}{\omega_n}\Bigg[&\frac{(2-n)\left(\underline{x}-\underline{y}\right)}{\left|\underline{y}-\underline{x}\right|^{n-1}}+\frac{(n-1)\left(\underline{x}-\underline{y}\right)\left\langle\underline{x}-\underline{y},2\underline{x}\right\rangle}{\left|\underline{y}-\underline{x}\right|^{n+1}}\\&-\frac{2\underline{x}}{\left|\underline{y}-\underline{x}\right|^{n-1}}F\left(\frac{n}{2},\frac{x_n-y_n}{\left|\underline{y}-\underline{x}\right|^{n}}\right)\\&+\frac{2\left(\underline{x}-\underline{y}\right)}{\left|y-x\right|^{n}}\left(\frac{\left(x_n-y_n\right)\left\langle\underline{x}-\underline{y},\underline{x}\right\rangle}{\left|\underline{y}-\underline{x}\right|^{2}}-x_n\right)\Bigg],\end{aligned}\tag{23}$$

where

$$F(\alpha,t)=\begin{cases}\frac{1}{2\alpha-1}\frac{t}{(1+t^2)^{\alpha-1}}+\frac{2\alpha-3}{2\alpha-2}F\big(\alpha-1,t\big), & 2\alpha\in\mathbb{N}+2,\alpha\neq 1,\\ \frac{1}{2\alpha-2}\frac{t}{(1+t^2)^{\alpha-1}}F\big(\frac{3}{2}-\alpha,1;2-\alpha;1+t^2\big), & \alpha\in\mathbb{N}+\frac{3}{2},\\ \frac{1}{2\alpha-2}\frac{t}{(1+t^2)^{\alpha-1}}\left(\sum\limits_{k=0}^{\alpha-2}\frac{\left(\frac{3}{2}-\alpha\right)_k}{(2-\alpha)_k}\left(1+t^2\right)^k\right) & \alpha\in\mathbb{N}+1,\\ \quad+\frac{1}{2\alpha-2}\frac{(\frac{3}{2}-\alpha)_{\alpha-2}}{(2-\alpha)_{\alpha-2}}\arctan t, & \end{cases}\tag{24}$$

with

$$(a)_k\triangleq\begin{cases}1, & k=0,\\ a(a+1)\dots(a+k-1), & k\in\mathbb{N},\end{cases}\tag{25}$$

and $F\big(a,b;c;t\big)$ stands for the hyper-geometric function, see Refs. e.g. [12,28]. Similar results could be also seen in Refs. e.g. [28–33].

Theorem 4.2. *Let $P\big(x,y\big),Q\big(x,y\big)$ be as (22) and (23), respectively. Introduce the function*

$$K\big(x,y\big)=P\big(x,y\big)+e_nQ\big(x,y\big),\quad x\in\mathbb{B},y\in\partial\mathbb{B}.\tag{26}$$

If $f\in\mathbb{L}_p\big(\partial\mathbb{B},\mathbb{R}_{n-1}\big)$, then

$$p.v.\int_{\partial\mathbb{B}}Q\left(x,y\right)f(y)dS_y,\quad x\in\partial\mathbb{B}\tag{27}$$

is well defined, where p.v. is short for the Cauchy principle value. Moreover,

$$Qf(x)\triangleq p.v.\int_{\partial\mathbb{B}}Q\left(x,y\right)f(y)dS_y\in\mathbb{L}_p\big(\partial\mathbb{B},\mathbb{R}_{n-1}\big).\tag{28}$$

Proof. Observing that $Q(x,y)$ of the singular integrable operator (27) behaves like the Cauchy kernel $G_1(y-x)$, which is given by

$$G_1(y-x) = \frac{1}{\omega_n}\frac{\overline{y-x}}{|y-x|^n}, \quad y \neq x, \tag{29}$$

at the singular point when we consider the Cauchy principle value of the singular integrable operator (27), we get (28). □

Furthermore, Theorem 4.2 leads to

$$\text{p.v.} \int_{\partial\mathbb{B}} K(x,y) f(y) dS_y \in \mathbb{L}_p\big(\partial\mathbb{B}, \mathbb{R}_n\big), \tag{30}$$

and

$$\begin{aligned}
&\lim_{\mathbb{B}\ni x\to y\in\partial\mathbb{B}} \int_{\partial\mathbb{B}} K(x,y) f(y) dS_y \\
&\quad = f(y) + e_n \text{ p.v.} \int_{\partial\mathbb{B}} Q(x,y) f(y) dS_y \in \mathbb{L}_p\big(\partial\mathbb{B}, \mathbb{R}_n\big).
\end{aligned} \tag{31}$$

Remark 4.3. Also, we could give the definition of the harmonic conjugate, starting with a scalar-valued function. Let Ω be a sub-domain of $\mathbb{R}^n$ $(\mathbb{R}^n \cong \mathbb{R}_1)$ with Lipschitz boundary $\partial\Omega$. If $\phi \in \mathcal{C}^2(\Omega, \mathbb{R}_n)$ is a scalar-valued harmonic, and $\psi \in \mathcal{C}^2(\Omega, \mathbb{R}_n)$ is also harmonic whose scalar part is zero, satisfying

$$\phi+\psi = \frac{1}{\omega_n}\int_{\partial\Omega} G_1(y-x)\, d\sigma_y f(y) \text{ for a scalar-valued } f, \quad x \in \mathbb{R}^n\backslash\partial\Omega, \tag{32}$$

where ω_n is the area of unit sphere of $\mathbb{R}^n$ $(\mathbb{R}^n \cong \mathbb{R}_1)$ and $G_1(y-x)$ is given by (29), then ϕ is so-called to be the Cauchy-type harmonic conjugate of ψ. In particular, for the function

$$P(x,y) = \frac{1}{\omega_n}\frac{1-|x|^2}{|y-x|^n}, \tag{33}$$

its unique Cauchy-type harmonic conjugate is

$$\widetilde{Q}(x,y) = \frac{1}{\omega_n}\left(\frac{2}{|y-x|^n} - \frac{m-2}{r^{m-1}}\int_0^r \frac{\rho^{m-1}}{|\rho\zeta - y|^n} d\rho\right) x \wedge y, \tag{34}$$

where $x = r\zeta, y, \zeta \in \partial\mathbb{B}, 0 \le r < 1$. Results similar to Theorem 4.2 could be seen in Ref. [22]. Both of Theorem 4.2 and Remark 4.3 allow us to define a Hilbert transform $\mathcal{H}(f) = Qf$ on the sphere $\partial\mathbb{B}$.

Next, let us consider our Schwarz boundary value problem for Hardy space.

Problem I. Given the boundary data $f \in \mathbb{L}_p\big(\partial\mathbb{B}, \mathbb{R}_{n-1}\big)$, find a function $\phi \in \mathbb{H}^p(\mathbb{B})$, satisfying the Schwarz boundary value condition

$$X^{(n)}\big\{\lambda\phi\big\} = f, \quad \text{a.e. } x \in \partial\mathbb{B}, \tag{35}$$

where $\lambda \in \mathbb{R}_{n-1}$ is an arbitrary invertible constant with its inverse λ^{-1}.

Theorem 4.4. *For the given function $f \in \mathbb{L}_p\big(\partial\mathbb{B}, \mathbb{R}_{n-1}\big)$, Problem (35) has the general solution and its explicit form is given by*

$$\phi(x) = \int_{\partial\mathbb{B}} K\big(x,y\big)\lambda^{-1} f(y) dS_y + \lambda^{-1} e_n \sum_{s=0}^{+\infty} V_{l_1,\ldots,l_s}(\underline{x}) a_{l_1,\ldots,l_s}, x \in \mathbb{B}, \quad (36)$$

where $K\big(x,y\big)$ is given by (26), and

$$V_{l_1,\ldots,l_s}(\underline{x}) = \frac{1}{s!} \sum_{\pi(l_1,\ldots,l_s)} z_{l_1} z_{l_2} \ldots z_{l_s}$$

with $z_s = x_s e_0 - x_1 e_1 e_s \,(s = 1, 2, \ldots, n-1)\,, a_{l_1,\ldots,l_s} \in \mathbb{R}_{n-1}$ being constants, respectively.

Proof. From Theorem 4.2, we know that

$$\psi_1(x) = \int_{\partial\mathbb{B}} K\big(x,y\big)\lambda^{-1} f(y) dS_y \in \mathbb{H}^p(\mathbb{B}) \quad (37)$$

satisfies the condition of Problem (35). Hence, we only need to consider the homogeneous Schwarz problem

$$X^{(n)}\big\{\Phi\big\} = 0, \quad \text{a.e. } x \in \partial\mathbb{B}. \quad (38)$$

In what follows, we will first prove that if there exists a solution to the homogeneous case of Problem (38), then it could be given explicitly.

In fact, Let Φ is a solution to the homogeneous case of Problem (35), then we have $\mathcal{D}\Phi = 0, x \in \mathbb{B}$ and

$$\begin{cases} \Delta\big(X^{(n)}\Phi\big) = 0, & x \in \mathbb{B}, \\ \big(X^{(n)}\Phi\big) = 0, & \text{a.e. } x \in \partial\mathbb{B}. \end{cases} \quad (39)$$

Therefore, $X^{(n)}\Phi \equiv 0, x \in \mathbb{B}$, that is, $\Phi = e_n Y^{(n)}\Phi, x \in \mathbb{B}$.

Secondly, beginning with Definition 4.1, we get that the solution to Problem (38) is presented by

$$Y^{(n)}\Phi(x) = \sum_{s=0}^{+\infty} V_{l_1,\ldots,l_s}(\underline{x}) a_{l_1,\ldots,l_s}, a_{l_1,\ldots,l_s} \in \mathbb{R}_{n-1}, \quad (40)$$

where $V_{l_1,\ldots,l_s}(\underline{x}) = \frac{1}{s!}\sum_{\pi(l_1,\ldots,l_s)} z_{l_1} z_{l_2} \ldots z_{l_s}, z_s = x_s e_0 - x_1 e_1 e_s \,(s = 1, 2, \ldots, n-1)\,, a_{l_1,\ldots,l_s} \in \mathbb{R}_{n-1}$.

Thus the result follows. □

Remark 4.5. When $p = 2, \lambda = 1$, Problem (35) changes into the case

$$\begin{cases} \phi \in \mathbb{H}^2(\mathbb{B}), \\ X^{(n)}\big\{\phi\big\} = f, \quad \text{a.e. } x \in \partial\mathbb{B}, \end{cases} \quad (41)$$

where $f \in \mathbb{L}_2\big(\partial\mathbb{B}, \mathbb{R}_{n-1}\big)$. Its solving actually corresponds to reconstruct the real algebra $\mathbb{R}_n$-valued monogenic signals of the Hardy space defined on $\mathbb{B}$

when a real sub-algebra valued $\mathbb{R}_{n-1}$ boundary datum is given on the sphere $\partial\mathbb{B}$.

Similarly, when a scalar-valued initial datum is given on the sphere $\partial\mathbb{B}$, to find a paravector valued monogenic signal is equivalent to solve a Schwarz BVP as follows.

Problem II. Find a function $\phi : \mathbb{B} \to \mathbb{R}_0 \oplus \mathbb{R}_2$, satisfying

$$\begin{cases} \phi \in \mathbb{H}^2(\mathbb{B}), \\ \mathrm{Sc}\{\phi\} = f, \quad \text{a.e. } x \in \partial\mathbb{B}, \end{cases} \tag{42}$$

where $f \in \mathbb{L}_2\big(\partial\mathbb{B}, \mathbb{R}_0\big)$. Its unique solution is presented by

$$\phi(x) = \int_{\partial\mathbb{B}} \widetilde{K}\big(x, y\big) f(y) dS_y, \quad x \in \mathbb{B}, \tag{43}$$

where

$$\widetilde{K}\big(x, y\big) = P(x, y) + \widetilde{Q}(x, y), \quad x \in \mathbb{B}, y \in \partial\mathbb{B}, \tag{44}$$

with $\widetilde{Q}(x, y)$ seen in (34).

Remark 4.6. When the dimension of the considered space is equal to 2, Problems (35) and (42) are both changed into the classical Schwarz BVP [7]: to find a function $\phi : \mathbb{D} \to \mathbb{C}$, where $\mathbb{C} \cong \{x_0 + e_1x_1 : x_0, x_1 \in \mathbb{R}_0\}$ or $\mathbb{R}_2$, $\mathbb{D} = \{|x| < 1 : x = x_0 + e_1x_1, x_0, x_1 \in \mathbb{R}_0\}$ or $\mathbb{D} = \{|x| < 1 : x = x_1e_1 + e_2x_2, x_1, x_2 \in \mathbb{R}_0\}$ with its boundary $\partial\mathbb{D}$, satisfying

$$\begin{cases} \phi \in \mathbb{H}^p(\mathbb{D}, \mathbb{C}), \\ \mathrm{Re}\{\phi\} = f, \quad \text{a.e. } x \in \partial\mathbb{D}, \end{cases} \tag{45}$$

where $\mathrm{Re}(x) = x_0$ or $x_1, x \in \mathbb{C}$ with $\mathbb{C}$ as above, which corresponds to the real part of a complex number in the planar complex analysis. After direct observation, $K\big(x, y\big)$ and $\widetilde{K}\big(x, y\big)$ defined by (26) and (44) play the role analogous to the classical Schwarz kernel function in Problems (35) and (42), respectively. Therefore, hereby, they are the so-called Schwarz kernel functions in the higher dimensions, for short, still the Schwarz kernel functions. Thus, Theorem 4.2 gives the characterization of the boundary properties of the Schwarz integrable operator, which corresponds to the Plemelj–Sokhotsky formula in complex analysis.

5. Schwarz BVPs for Poly-Hardy Space

In this section, we extend the results obtained in the previous section. We study the Schwarz BVP for the poly-Hardy class on the unit ball with $\mathbb{L}_p\,(1 < p < +\infty)$-integrable boundary data, and derive the explicit expressions of the solutions.

Problem III. Given the boundary data $f_j \in \mathbb{L}_p\big(\partial\mathbb{B}, \mathbb{R}_{n-1}\big), j = 0, 1, 2, \ldots, k-1$, find a function $\phi : \mathbb{B} \to \mathbb{R}_n$ satisfying the Schwarz boundary value conditions

$$\begin{cases} \phi \in \mathbb{H}^p_k(\mathbb{B}), \quad 1 < p < +\infty, \\ X^{(n)}\big\{\lambda\phi\big\} = f_0, \quad \text{a.e. } x \in \partial\mathbb{B}, \\ \quad\vdots \qquad\qquad\qquad \vdots \\ X^{(n)}\big\{\lambda\mathcal{D}^{k-1}\phi\big\} = f_{k-1}, \quad \text{a.e. } x \in \partial\mathbb{B}, \end{cases} \tag{46}$$

where $\lambda \in \mathbb{R}_{n-1}$ is a constant with its inverse λ^{-1}.

Theorem 5.1. *For the given function $f_j \in \mathbb{L}_p\big(\partial\mathbb{B}, \mathbb{R}_{n-1}\big), j = 0, 1, 2, \ldots, k-1$, Problem (46) has the general solution and its explicit form is given by*

$$\phi(x) = \sum_{j=0}^{k-1} x^j \phi_j(x), x \in \mathbb{B}, \tag{47}$$

with

$$\phi_j(x) = \begin{cases} \int_{\partial\mathbb{B}} K(x,y)\lambda^{-1}\widetilde{f}_0(y)dS_y + \lambda^{-1}e_n \sum\limits_{s=0}^{+\infty} V_{l_1,\ldots,l_s}(\underline{x})a^{(0)}_{l_1,\ldots,l_s}, \quad \text{if } j = 0, \\ C^{-1}_{1,1} I_{\frac{n+1}{2}} \left(\int_{\partial\mathbb{B}} K(x,y)\lambda^{-1}\widetilde{f}_1(y)dS_y + \lambda^{-1}e_n \sum\limits_{s=0}^{+\infty} V_{l_1,\ldots,l_s}(\underline{x})a^{(1)}_{l_1,\ldots,l_s}\right), \quad \text{if } j = 1, \\ C^{-1}_{j,j} I_{\frac{n+1}{2}+[\frac{j}{2}]-1} \cdots I_{\frac{n+1}{2}} \left(\int_{\partial\mathbb{B}} K(x,y)\lambda^{-1}\widetilde{f}_j(y)dS_y + \lambda^{-1}e_n \sum\limits_{s=0}^{+\infty} V_{l_1,\ldots,l_s}(\underline{x})a^{(j)}_{l_1,\ldots,l_s}\right), \\ \quad \text{if } 2 \le j \le k-1, \end{cases} \tag{48}$$

where $K\big(x,y\big)$ as (26), $a^{(j)}_{l_1,\ldots,l_s} \in \mathbb{R}_{n-1}, j = 0, 1, 2, \ldots, k-1$ are all constants, and for arbitrary $x \in \partial\mathbb{B}$,

$$\widetilde{f}_j(x) = \begin{cases} f_0(x) - \sum\limits_{l=1}^{k-1} C^{-1}_{l,l} X^{(n)} \left\{x^l \left(\int_{\partial\mathbb{B}} K(x,y)\lambda^{-1}\widetilde{f}_l(y)dS_y + \lambda^{-1}e_n \sum\limits_{s=0}^{+\infty} V_{l_1,\ldots,l_s}(\underline{x})a^{(l)}_{l_1,\ldots,l_s}\right)\right\}, \quad \text{if } j = 0, \\ f_1(x) - \sum\limits_{l=2}^{k-1} C_{1,l}C^{-1}_{l,l} X^{(n)} \left\{x^l I_{\frac{n+1}{2}+[\frac{l}{2}]-1} \cdots I_{\frac{n+1}{2}} \left(\int_{\partial\mathbb{B}} K(x,y)\lambda^{-1}\widetilde{f}_l(y)dS_y\right.\right. \\ \left.\left. + \lambda^{-1}e_n \sum\limits_{s=0}^{+\infty} V_{l_1,\ldots,l_s}(\underline{x})a^{(l)}_{l_1,\ldots,l_s}\right)\right\}, \quad \text{if } j = 1, k \text{ odd}, \\ f_1(x) - \sum\limits_{l=2}^{k-1} C_{1,l}C^{-1}_{l,l} X^{(n)} \left\{x^l I_{\frac{n+1}{2}+[\frac{l}{2}]} \cdots I_{\frac{n+1}{2}} \left(\int_{\partial\mathbb{B}} K(x,y)\lambda^{-1}\widetilde{f}_l(y)dS_y\right.\right. \\ \left.\left. + \lambda^{-1}e_n \sum\limits_{s=0}^{+\infty} V_{l_1,\ldots,l_s}(\underline{x})a^{(l)}_{l_1,\ldots,l_s}\right)\right\}, \quad \text{if } j = 1, k \text{ even}, \\ f_j(x) - \sum\limits_{l=j+1}^{k-1} C_{k-1,l}C^{-1}_{l,l} X^{(n)} \left\{x^l I_{\frac{n+1}{2}+[\frac{l-j}{2}]} \cdots I_{\frac{n+1}{2}} \left(\int_{\partial\mathbb{B}} K(x,y)\lambda^{-1}\widetilde{f}_l(y)dS_y\right.\right. \\ \left.\left. + \lambda^{-1}e_n \sum\limits_{s=0}^{+\infty} V_{l_1,\ldots,l_s}(\underline{x})a^{(l)}_{l_1,\ldots,l_s}\right)\right\}, \quad \text{if } 2 \le j \le k-1. \end{cases} \tag{49}$$

Proof. Since $\phi \in \mathbb{H}^p_k(\mathbb{B}), 1 < p < +\infty$, in virtue of Theorem 3.5, there exists unique functions ϕ_j satisfying $\phi_j \in \mathbb{H}^p(\mathbb{B}), j = 0, 1, 2, \ldots, k-1$ and $\phi = \sum_{j=0}^{k-1} x^j \phi_j$.

Using Lemma 3.2 for arbitrary $l \in \mathbb{N}, l \le j$, we have

$$\mathcal{D}^l\phi = \sum_{j=0}^{k-1} \mathcal{D}^l\big(x^j\phi_j\big) = \sum_{j=l}^{k-1} C_{l,j}x^{j-l}\Big(E_{\frac{n+1}{2}+[\frac{j-l}{2}]} \cdots E_{\frac{n+1}{2}+[\frac{j}{2}]-1}\phi_j\Big). \tag{50}$$

Then, making use of Theorem 3.6 and Corollary 3.7, Problem (46) is equivalent to the case

$$\begin{cases} \phi_j \in \mathbb{H}^p(\mathbb{B}), & j=0,1,2,\dots,k-1,\\ X^{(n)}\left\{\lambda x^j \phi_j\right\} = f_0, & x\in\partial\mathbb{B},\\ X^{(n)}\left\{\lambda \sum\limits_{j=1}^{k-1} C_{1,j} x^{j-1} E_{\frac{n+1}{2}+[\frac{j-1}{2}]}\dots E_{\frac{n+1}{2}+[\frac{j}{2}]-1}\phi_j\right\} = f_1, & x\in\partial\mathbb{B},\\ \quad\vdots \qquad\qquad\qquad \vdots & \\ X^{(n)}\left\{\lambda \sum\limits_{j=l}^{k-1} C_{l,j} x^{j-l} E_{\frac{n+1}{2}+[\frac{j-l}{2}]}\dots E_{\frac{n+1}{2}+[\frac{j}{2}]-1}\phi_j\right\} = f_l, & x\in\partial\mathbb{B},\\ \quad\vdots \qquad\qquad\qquad \vdots & \\ X^{(n)}\left\{\lambda C_{k-1,k-1} E_{\frac{n+1}{2}}\dots E_{\frac{n+1}{2}+[\frac{k-1}{2}]-1}\phi_{k-1}\right\} = f_{k-1}, & x\in\partial\mathbb{B}. \end{cases} \tag{51}$$

We now proceed by induction. First, we consider

$$\begin{cases} \phi_{k-1} \in \mathbb{H}^p(\mathbb{B}),\\ X^{(n)}\left\{\lambda C_{k-1,k-1} E_{\frac{n+1}{2}}\dots E_{\frac{n+1}{2}+[\frac{k-1}{2}]-1}\phi_{k-1}\right\} = f_{k-1}, \quad x\in\partial\mathbb{B}. \end{cases} \tag{52}$$

By Theorem 4.4, Problem (52) has the explicit solution

$$\begin{aligned} \phi_{k-1}(x) &= C_{k-1,k-1}^{-1} I_{\frac{n+1}{2}+[\frac{k-1}{2}]-1}\dots I_{\frac{n+1}{2}} \\ &\quad\times\left(\int_{\partial\mathbb{B}} K\big(x,y\big)\lambda^{-1} f_{k-1}(y)dS_y + \lambda^{-1} e_n \sum_{s=0}^{+\infty} V_{l_1,\dots,l_s}(\underline{x}) a_{l_1,\dots,l_s}^{(k-1)}\right), \quad x\in\mathbb{B}. \end{aligned} \tag{53}$$

Applying Lemma 3.4, and (37), we get that

$$\begin{aligned} & I_{\frac{n+1}{2}+[\frac{k-1}{2}]-1}\dots I_{\frac{n+1}{2}}\left(\int_{\partial\mathbb{B}} K\big(t,y\big)\lambda^{-1} f_{k-1}(y)dS_y + \lambda^{-1} e_n \sum_{s=0}^{+\infty} V_{l_1,\dots,l_s}(\underline{t}) a_{l_1,\dots,l_s}^{(k-1)}\right) \\ &= \lim_{\mathbb{B}\ni x\to t\in\partial\mathbb{B}}\Bigg\{ I_{\frac{n+1}{2}+[\frac{k-1}{2}]-1}\dots I_{\frac{n+1}{2}}\Bigg(\int_{\partial\mathbb{B}} K\big(x,y\big)\lambda^{-1} f_{k-1}(y)dS_y \\ &\quad + \lambda^{-1} e_n \sum_{s=0}^{+\infty} V_{l_1,\dots,l_s}(\underline{x}) a_{l_1,\dots,l_s}^{(k-1)}\Bigg)\Bigg\}, \end{aligned}$$

belongs to $\mathbb{L}_p\big(\partial\mathbb{B},\mathbb{R}_n\big)$. Therefore, $\phi_{k-1}\in\mathbb{L}_p\big(\partial\mathbb{B},\mathbb{R}_n\big)$, thus leading to $t\phi_{k-1}\in\mathbb{L}_p\big(\partial\mathbb{B},\mathbb{R}_n\big)$.

Then, we solve the second boundary value problem

$$\begin{cases} \phi_{k-2} \in \mathbb{H}^p(\mathbb{B}),\\ X^{(n)}\left\{\lambda C_{k-2,k-2}\left(E_{\frac{n+1}{2}}\dots E_{\frac{n+1}{2}+[\frac{k-2}{2}]-1}\phi_{k-2}\right)\right\} = \widetilde{f}_{k-2}, \quad x\in\partial\mathbb{B}. \end{cases} \tag{54}$$

where

$$\widetilde{f}_{k-2}(x) = f_{k-2}(x) - C_{k-1,k-2}C^{-1}_{k-1,k-1}X^{(n)}\left\{xI_{\frac{n+1}{2}}\left(\int_{\partial\mathbb{B}}K\big(x,y\big)\lambda^{-1}f_{k-1}(y)dS_y\right.\right.$$
$$\left.\left. + \lambda^{-1}e_n\sum_{s=0}^{+\infty}V_{l_1,\ldots,l_s}(\underline{x})a^{(k-1)}_{l_1,\ldots,l_s}\right)\right\} \tag{55}$$

on $\partial\mathbb{B}$. Hence, Problem (54) has the explicit solution

$$\phi_{k-2}(x) = C^{-1}_{k-2,k-2}I_{\frac{n+1}{2}+[\frac{k-2}{2}]-1}\ldots I_{\frac{n+1}{2}}$$
$$\times\left(\int_{\partial\mathbb{B}}K\big(x,y\big)\lambda^{-1}\widetilde{f}_{k-2}(y)dS_y + \lambda^{-1}e_n\sum_{s=0}^{+\infty}V_{l_1,\ldots,l_s}(\underline{x})a^{(k-2)}_{l_1,\ldots,l_s}\right), \quad x\in\mathbb{B}. \tag{56}$$

Inductively on $2\leq l\leq k-2$, the following boundary value problem

$$\begin{cases}\phi_l\in\mathbb{H}^p(\mathbb{B}),\\ X^{(n)}\left\{\lambda C_{l,l}E_{\frac{n+1}{2}}\ldots E_{\frac{n+1}{2}+[\frac{l}{2}]-1}\phi\right\} = \widetilde{f}_l, \quad x\in\partial\mathbb{B},\end{cases} \tag{57}$$

where

$$\widetilde{f}_l(x) = f_l(x) - \sum_{j=l+1}^{k-1}C_{k-1,j}C^{-1}_{j,j}X^{(n)}\left\{x^jI_{\frac{n+1}{2}+[\frac{j-l}{2}]}\ldots I_{\frac{n+1}{2}}\left(\int_{\partial\mathbb{B}}K\big(x,y\big)\lambda^{-1}\widetilde{f}_j(y)dS_y\right.\right.$$
$$\left.\left.+\lambda^{-1}e_n\sum_{s=0}^{+\infty}V_{l_1,\ldots,l_s}(\underline{x})a^{(j)}_{l_1,\ldots,l_s}\right)\right\} \tag{58}$$

on $\partial\mathbb{B}$, has the explicit solution

$$\phi_l(x) = C^{-1}_{l,l}I_{\frac{n+1}{2}+[\frac{l}{2}]-1}\ldots I_{\frac{n+1}{2}}$$
$$\times\left(\int_{\partial\mathbb{B}}K\big(x,y\big)\lambda^{-1}\widetilde{f}_l(y)dS_y + \lambda^{-1}e_n\sum_{s=0}^{+\infty}V_{l_1,\ldots,l_s}(\underline{x})a^{(l)}_{l_1,\ldots,l_s}\right), \quad x\in\mathbb{B}. \tag{59}$$

Now, the remaining two cases when $l=0,1$ are treated as follows. For $l=1$, the boundary value problem

$$\begin{cases}\phi_1\in\mathbb{H}^p(\mathbb{B}),\\ X^{(n)}\left\{\lambda C_{1,1}E_{\frac{n+1}{2}}\phi_1\right\} = \widetilde{f}_1, \quad x\in\partial\mathbb{B},\end{cases}$$

where

$$\widetilde{f}_1(x) = \begin{cases} f_1(x) - \sum\limits_{j=2}^{k-1}C_{1,j}C^{-1}_{j,j}X^{(n)}\left\{x^jI_{\frac{n+1}{2}+[\frac{j}{2}]-1}\ldots I_{\frac{n+1}{2}}\left(\int_{\partial\mathbb{B}}K(x,y)\lambda^{-1}\widetilde{f}_j(y)dS_y\right.\right.\\ \left.\left.+\ \lambda^{-1}e_n\sum\limits_{s=0}^{+\infty}V_{l_1,\ldots,l_s}(\underline{x})a^{(j)}_{l_1,\ldots,l_s}\right)\right\}, \quad \text{if } k \text{ odd},\\ f_1(x) - \sum\limits_{j=2}^{k-1}C_{1,j}C^{-1}_{j,j}X^{(n)}\left\{x^jI_{\frac{n+1}{2}+[\frac{j}{2}]}\ldots I_{\frac{n+1}{2}}\left(\int_{\partial\mathbb{B}}K(x,y)\lambda^{-1}\widetilde{f}_j(y)dS_y\right.\right.\\ \left.\left.+\ \lambda^{-1}e_n\sum\limits_{s=0}^{+\infty}V_{l_1,\ldots,l_s}(\underline{x})a^{(j)}_{l_1,\ldots,l_s}\right)\right\}, \quad \text{if } k \text{ even},\end{cases} \quad x\in\partial\mathbb{B}, \tag{60}$$

has the explicit solution

$$\phi_1(x) = C_{1,1}^{-1} I_{\frac{n+1}{2}} \left(\int_{\partial\mathbb{B}} K\big(x,y\big)\lambda^{-1}\widetilde{f}_1(y)dS_y + \lambda^{-1}e_n \sum_{s=0}^{+\infty} V_{l_1,\ldots,l_s}(\underline{x}) a_{l_1,\ldots,l_s}^{(1)} \right), \quad x \in \mathbb{B}. \tag{61}$$

For $l = 0$, the boundary value problem

$$\begin{cases} \phi_0 \in \mathbb{H}^p(\mathbb{B}), \\ X^{(n)}\{\lambda\phi_0\} = \widetilde{f}_0, \quad x \in \partial\mathbb{B}, \end{cases} \tag{62}$$

where

$$\widetilde{f}_0(x) = f_0(x) - \sum_{j=1}^{k-1} C_{j,j}^{-1} X^{(n)} \times \left\{ x^j \left(\int_{\partial\mathbb{B}} K\big(x,y\big)\lambda^{-1}\widetilde{f}_j(y)dS_y + \lambda^{-1}e_n \sum_{s=0}^{+\infty} V_{l_1,\ldots,l_k}(\underline{x}) a_{l_1,\ldots,l_s}^{(j)} \right) \right\}, \quad x \in \partial\mathbb{B}, \tag{63}$$

has the explicit solution

$$\phi_0(x) = \int_{\partial\mathbb{B}} K\big(x,y\big)\lambda^{-1}\widetilde{f}_0(y)dS_y + \lambda^{-1}e_n \sum_{s=0}^{+\infty} V_{l_1,\ldots,l_s}(\underline{x}) a_{l_1,\ldots,l_s}^{(0)}, \quad x \in \mathbb{B}. \tag{64}$$

Associating with (53)–(64), we obtain that Problem (46) has the explicit solution $\phi(x) = \sum_{j=0}^{k-1} x^j \phi_j$. It follows the result. □

Corollary 5.2. *When $p = 2, \lambda_j = 1, j = 0, 1, \ldots, k-1$, Problem (46) reduces into the case*

$$\begin{cases} \phi \in \mathbb{H}_k^2(\mathbb{B}), \quad 1 < p < +\infty, \\ X^{(n)}\big\{\phi\big\} = f_0, \quad a.e.\ x \in \partial\mathbb{B}, \\ X^{(n)}\big\{\mathcal{D}\phi\big\} = f_1, \quad a.e.\ x \in \partial\mathbb{B}, \\ \quad \vdots \qquad\qquad \vdots \\ X^{(n)}\big\{\mathcal{D}^{k-1}\phi\big\} = f_{k-1}, \quad a.e.\ x \in \partial\mathbb{B}. \end{cases} \tag{65}$$

Its solutions actually correspond to reconstruct the real algebra $\mathbb{R}_n$-valued monognic signals of the Hardy space defined on $\mathbb{B}$ when a group of real subalgebra valued $\mathbb{R}_{n-1}$ initial datum is given on the sphere $\partial\mathbb{B}$ of higher dimensional Euclidean space $\mathbb{R}^n$. Similar results also refer to [28].

Moreover, we can consider the following problem, analogous to Theorem 5.1.

Problem IV. Given the boundary data $f_j \in \mathbb{L}_p\big(\partial\mathbb{B}, \mathbb{R}_{n-1}\big), j = 0,1,2,\ldots,k-1$, find a function $\phi : \mathbb{B} \to \mathbb{R}_n$ satisfying the Schwarz boundary value conditions

$$\begin{cases} \phi \in \mathbb{H}_k^p(\mathbb{B}), & 1 < p < +\infty, \\ X^{(n)}\big\{\lambda_0 \phi\big\} = f_0, & \text{a.e. } x \in \partial\mathbb{B}, \\ \quad\vdots & \quad\vdots \\ X^{(n)}\big\{\lambda_{k-1}\mathcal{D}^{k-1}\phi\big\} = f_{k-1}, & \text{a.e. } x \in \partial\mathbb{B}, \end{cases} \tag{66}$$

where $\lambda_j \in \mathbb{R}_{n-1}$ is a constant with its inverse $\lambda_j^{-1}, j = 0,1,2,\ldots k-1$.

Theorem 5.3. *For the given function $f_j \in \mathbb{L}_p\big(\partial\mathbb{B}, \mathbb{R}_{n-1}\big), j = 0,1,2,\ldots,k-1$, Problem (46) has the general solution and its explicit form is given by*

$$\phi(x) = \sum_{j=0}^{k-1} x^j \phi_j(x), \quad x \in \mathbb{B}, \tag{67}$$

with

$$\phi_j(x) = \begin{cases} \int_{\partial\mathbb{B}} K(x,y)\lambda_0^{-1}\widetilde{f}_0(y)dS_y + \lambda_0^{-1}e_n \sum\limits_{s=0}^{+\infty} V_{l_1,\ldots,l_s}(\underline{x})a^{(0)}_{l_1,\ldots,l_s}, \quad \text{if } j = 0, \\ C_{1,1}^{-1} I_{\frac{n+1}{2}} \left(\int_{\partial\mathbb{B}} K(x,y)\lambda_1^{-1}\widetilde{f}_1(y)dS_y + \lambda_1^{-1}e_n \sum\limits_{s=0}^{+\infty} V_{l_1,\ldots,l_s}(\underline{x})a^{(1)}_{l_1,\ldots,l_s}\right), \quad \text{if } j = 1, \\ C_{j,j}^{-1} I_{\frac{n+1}{2}+[\frac{j}{2}]-1} \cdots I_{\frac{n+1}{2}} \left(\int_{\partial\mathbb{B}} K(x,y)\lambda_j^{-1}\widetilde{f}_j(y)dS_y + \lambda_j^{-1}e_n \sum\limits_{s=0}^{+\infty} V_{l_1,\ldots,l_s}(\underline{x})a^{(j)}_{l_1,\ldots,l_s}\right), \\ \quad \text{if } 2 \le j \le k-1, \end{cases} \tag{68}$$

where $K(x,y)$ as (26), $a^{(j)}_{l_1,\ldots,l_s} \in \mathbb{R}_{n-1}, j = 0,1,2,\ldots,k-1$ are all constants, and for arbitrary $x \in \partial\mathbb{B}$,

$$\widetilde{f}_j(x) = \begin{cases} f_0(x) - \sum\limits_{l=1}^{k-1} C_{l,l}^{-1} X^{(n)}\left\{x^l \left(\int_{\partial\mathbb{B}} K(x,y)\lambda_l^{-1}\widetilde{f}_l(y)dS_y + \lambda_l^{-1}e_n \sum\limits_{s=0}^{+\infty} V_{l_1,\ldots,l_s}(\underline{x})a^{(l)}_{l_1,\ldots,l_s}\right)\right\}, \\ \quad \text{if } j = 0, \\ f_1(x) - \sum\limits_{l=2}^{k-1} C_{1,l}C_{l,l}^{-1} X^{(n)}\left\{x^l I_{\frac{n+1}{2}+[\frac{l}{2}]-1} \cdots I_{\frac{n+1}{2}} \left(\int_{\partial\mathbb{B}} K(x,y)\lambda_l^{-1}\widetilde{f}_l(y)dS_y \right.\right. \\ \left.\left. + \lambda_l^{-1}e_n \sum\limits_{s=0}^{+\infty} V_{l_1,\ldots,l_s}(\underline{x})a^{(l)}_{l_1,\ldots,l_s}\right)\right\}, \quad \text{if } j = 1, k \text{ odd}, \\ f_1(x) - \sum\limits_{l=2}^{k-1} C_{1,l}C_{l,l}^{-1} X^{(n)}\left\{x^l I_{\frac{n+1}{2}+[\frac{l}{2}]} \cdots I_{\frac{n+1}{2}} \left(\int_{\partial\mathbb{B}} K(x,y)\lambda_l^{-1}\widetilde{f}_l(y)dS_y \right.\right. \\ \left.\left. + \lambda_l^{-1}e_n \sum\limits_{s=0}^{+\infty} V_{l_1,\ldots,l_s}(\underline{x})a^{(l)}_{l_1,\ldots,l_s}\right)\right\}, \quad \text{if } j = 1, k \text{ even}, \\ f_j(x) - \sum\limits_{l=j+1}^{k-1} C_{k-1,l}C_{l,l}^{-1} X^{(n)}\left\{x^l I_{\frac{n+1}{2}+[\frac{l-j}{2}]} \cdots I_{\frac{n+1}{2}} \left(\int_{\partial\mathbb{B}} K(x,y)\lambda_l^{-1}\widetilde{f}_l(y)dS_y \right.\right. \\ \left.\left. + \lambda_l^{-1}e_n \sum\limits_{s=0}^{+\infty} V_{l_1,\ldots,l_s}(\underline{x})a^{(l)}_{l_1,\ldots,l_s}\right)\right\}, \quad \text{if } 2 \le j \le k-1. \end{cases} \tag{69}$$

Similarly, we solve the Schwarz BVP as follows.

Problem V. Given the boundary data $f_j \in \mathbb{L}_p\left(\partial\mathbb{B}, \mathbb{R}_0\right), j = 0,1,2,\ldots,k-1$, find a function $\phi : \mathbb{B} \to \mathbb{R}_0 \oplus \mathbb{R}_2$, satisfying the Schwarz boundary value conditions

$$\begin{cases} \phi \in \mathbb{H}_k^p(\mathbb{B}), \quad 1 < p < +\infty, \\ \mathrm{Sc}\big\{\lambda\phi\big\} = f_0, \ \text{a.e. } x \in \partial\mathbb{B}, \\ \quad\vdots \qquad\qquad \vdots \\ \mathrm{Sc}\big\{\lambda\mathcal{D}^{k-1}\phi\big\} = f_{k-1}, \quad \text{a.e. } x \in \partial\mathbb{B}, \end{cases} \tag{70}$$

where $\lambda \in \mathbb{R}_0 \backslash \{0\}$ is a constant with its inverse λ^{-1}.

Theorem 5.4. *For the given function $f_j \in \mathbb{L}_p(\partial\mathbb{B}, \mathbb{R}_0), j = 0, 1, 2, \ldots, k-1$, Problem (70) has the general solution and its unique form is given by*

$$\phi(x) = \sum_{j=0}^{k-1} x^j \phi_j(x), \quad x \in \mathbb{B}, \tag{71}$$

with

$$\phi_j(x) = \begin{cases} \int_{\partial\mathbb{B}} \widetilde{K}(x,y)\lambda^{-1}\widetilde{f}_0(y)dS_y, & \text{if } j = 0, \\ C_{1,1}^{-1} I_{\frac{n+1}{2}} \left(\int_{\partial\mathbb{B}} \widetilde{K}(x,y)\lambda^{-1}\widetilde{f}_1(y)dS_y \right), & \text{if } j = 1, \\ C_{j,j}^{-1} I_{\frac{n+1}{2}+[\frac{j}{2}]-1} \ldots I_{\frac{n+1}{2}} \left(\int_{\partial\mathbb{B}} \widetilde{K}(x,y)\lambda^{-1}\widetilde{f}_j(y)dS_y \right), & \text{if } 2 \le j \le k-1, \end{cases} \tag{72}$$

where $\widetilde{K}(x,y)$ as (44), and for arbitrary $x \in \partial\mathbb{B}$,

$$\widetilde{f}_j(x) = \begin{cases} f_0(x) - \sum\limits_{l=1}^{k-1} C_{l,l}^{-1} Sc\left\{ x^l \left(\int_{\partial\mathbb{B}} \widetilde{K}(x,y)\lambda^{-1}\widetilde{f}_l(y)dS_y \right) \right\}, \\ \quad \text{if } j = 0, \\ f_1(x) - \sum\limits_{l=2}^{k-1} C_{1,l} C_{l,l}^{-1} Sc\left\{ x^l I_{\frac{n+1}{2}+[\frac{l}{2}]-1} \ldots I_{\frac{n+1}{2}} \left(\int_{\partial\mathbb{B}} \widetilde{K}(x,y)\lambda^{-1}\widetilde{f}_l(y)dS_y \right) \right\}, \\ \quad \text{if } j = 1, k \text{ odd}, \\ f_1(x) - \sum\limits_{l=2}^{k-1} C_{1,l} C_{l,l}^{-1} Sc\left\{ x^l I_{\frac{n+1}{2}+[\frac{l}{2}]} \ldots I_{\frac{n+1}{2}} \left(\int_{\partial\mathbb{B}} \widetilde{K}(x,y)\lambda^{-1}\widetilde{f}_l(y)dS_y \right) \right\}, \\ \quad \text{if } j = 1, k \text{ even}, \\ f_j(x) - \sum\limits_{l=j+1}^{k-1} C_{k-1,l} C_{l,l}^{-1} Sc\left\{ x^l I_{\frac{n+1}{2}+[\frac{l-j}{2}]} \ldots I_{\frac{n+1}{2}} \left(\int_{\partial\mathbb{B}} \widetilde{K}(x,y)\lambda^{-1}\widetilde{f}_l(y)dS_y \right) \right\}, \\ \quad \text{if } 2 \le j \le k-1. \end{cases} \tag{73}$$

Furthermore, we solve a boundary value problem as follows.

Problem VI. Given the boundary data $f_j \in \mathbb{L}_p(\partial\mathbb{B}, \mathbb{R}_0), j = 0, 1, 2, \ldots, k-1$, find a function $\phi : \mathbb{B} \to \mathbb{R}_0 \oplus \mathbb{R}_2$, satisfying the Schwarz boundary value conditions

$$\begin{cases} \phi \in \mathbb{H}_k^p(\mathbb{B}), \quad 1 < p < +\infty, \\ \mathrm{Sc}\{\lambda_0 \phi\} = f_0, \quad \text{a.e. } x \in \partial\mathbb{B}, \\ \mathrm{Sc}\{\lambda_1 \mathcal{D}\phi\} = f_1, \ \text{a.e. } x \in \partial\mathbb{B}, \\ \qquad \vdots \qquad\qquad\qquad \vdots \\ \mathrm{Sc}\{\lambda_{k-1}\mathcal{D}^{k-1}\phi\} = f_{k-1}, \ \text{a.e. } x \in \partial\mathbb{B}, \end{cases} \tag{74}$$

where $\lambda_j \in \mathbb{R}_0 \backslash \{0\}$ is a constant with its inverse $\lambda_j^{-1}, j = 0, 1, 2, \ldots, k-1$.

Theorem 5.5. *For the given function $f_j \in \mathbb{L}_p(\partial\mathbb{B}, \mathbb{R}_0), j = 0, 1, 2, \ldots, k-1$, Problem (70) has the general solution and its unique form is given by*

$$\phi(x) = \sum_{j=0}^{k-1} x^j \phi_j(x), \quad x \in \mathbb{B}, \tag{75}$$

with

$$\phi_j(x) = \begin{cases} \int_{\partial\mathbb{B}} \widetilde{K}\big(x,y\big)\lambda_0^{-1}\widetilde{f}_0(y)dS_y, & \text{if } j=0, \\ C_{1,1}^{-1} I_{\frac{n+1}{2}} \left(\int_{\partial\mathbb{B}} \widetilde{K}\big(x,y\big)\lambda_1^{-1}\widetilde{f}_1(y)dS_y\right), & \text{if } j=1, \\ C_{j,j}^{-1} I_{\frac{n+1}{2}+[\frac{j}{2}]-1} \cdots I_{\frac{n+1}{2}} \left(\int_{\partial\mathbb{B}} \widetilde{K}\big(x,y\big)\lambda_j^{-1}\widetilde{f}_j(y)dS_y\right), & \text{if } 2 \le j \le k-1, \end{cases} \tag{76}$$

where $\widetilde{K}\big(x,y\big)$ *as* (44)*, and for arbitrary* $x \in \partial\mathbb{B}$*,*

$$\widetilde{f}_j(x) = \begin{cases} f_0(x) - \sum_{l=1}^{k-1} C_{l,l}^{-1} Sc\left\{x^l \left(\int_{\partial\mathbb{B}} \widetilde{K}(x,y)\lambda_0^{-1}\widetilde{f}_l(y)dS_y\right)\right\}, \quad \text{if } j=0, \\ f_1(x) - \sum_{l=2}^{k-1} C_{1,l}C_{l,l}^{-1} Sc\left\{x^l I_{\frac{n+1}{2}+[\frac{l}{2}]-1} \cdots I_{\frac{n+1}{2}} \left(\int_{\partial\mathbb{B}} \widetilde{K}(x,y)\lambda_l^{-1}\widetilde{f}_l(y)dS_y\right)\right\}, \\ \quad \text{if } j=1, k \text{ odd}, \\ f_1(x) - \sum_{l=2}^{k-1} C_{1,l}C_{l,l}^{-1} Sc\left\{x^l I_{\frac{n+1}{2}+[\frac{l}{2}]} \cdots I_{\frac{n+1}{2}} \left(\int_{\partial\mathbb{B}} \widetilde{K}(x,y)\lambda_l^{-1}\widetilde{f}_l(y)dS_y\right)\right\}, \\ \quad \text{if } j=1, k \text{ even}, \\ f_j(x) - \sum_{l=j+1}^{k-1} C_{k-1,l}C_{l,l}^{-1} Sc\left\{x^l I_{\frac{n+1}{2}+[\frac{l-j}{2}]} \cdots I_{\frac{n+1}{2}} \left(\int_{\partial\mathbb{B}} \widetilde{K}(x,y)\lambda_l^{-1}\widetilde{f}_l(y)dS_y\right)\right\}, \\ \quad \text{if } 2 \le j \le k-1. \end{cases} \tag{77}$$

Proof. Applying Theorem 5.4, the proof of the result is complete. □

Remark 5.6. When $p = 2, \lambda_j = 1, j = 0, 1, \ldots, k-1$, Problem (74) turns into the case

$$\begin{cases} \phi \in \mathbb{H}_k^2(\mathbb{B}), & 1 < p < +\infty, \\ \mathrm{Sc}\big\{\phi\big\} = f_0, & \text{a.e. } x \in \partial\mathbb{B}, \\ \mathrm{Sc}\big\{\mathcal{D}\phi\big\} = f_1, & \text{a.e. } x \in \partial\mathbb{B}, \\ \quad\vdots & \quad\vdots \\ \mathrm{Sc}\big\{\mathcal{D}^{k-1}\phi\big\} = f_{k-1}, & \text{a.e. } x \in \partial\mathbb{B}. \end{cases} \tag{78}$$

This could be understood as the reconstruction of the monogenic signals considered on the unit sphere when a group of scalar valued initial data is given, which is appearing in engineering applications.

Remark 5.7. When $k = 2m, m \in \mathbb{N}$, Problems (46), (66), (70) and (74) change into the Schwarz BVPs for the poly-harmonic Hardy spaces defined on the unit ball $\mathbb{B}$ of higher dimensional Euclidean space $\mathbb{R}^n$, respectively. This means that the solutions to Problems (46), (66), (70) and (74) derive the solutions to the corresponding Schwarz BVPs for the poly-harmonic Hardy spaces defined on the unit ball $\mathbb{B}$ of higher dimensional Euclidean space $\mathbb{R}^n$. Moreover, when the dimension of the space considered is $n = 2$, Problems (46), (70) and (74) reduce to those discussed in Refs. e.g. [5,7], respectively.

Remark 5.8. In this context, when the given boundary data take Clifford subalgebra values and scalar values, respectively, two different kinds of Schwarz

BVPs for the poly-Hardy spaces on the unit ball of higher dimensional Euclidean space $\mathbb{R}^n$ are considered. Compared to the classical Schwarz problems in complex analysis (see, Refs. e.g. [5,7]), the explicit solutions to these two kinds of Schwarz BVPs are gotten while the uniqueness of them are different. One contains the Schwarz BVPs (70) and (74), and the general solutions to it are unique. Another contains the Schwarz BVPs (35), (46) and (66). Although the general solutions to it could be also presented explicitly, they are not unique. This is because they contain infinitely many arbitrary constants overdetermined. If one wants to determine the arbitrary constants involved in the solutions to the Schwarz BVPs considered, there needs to impose more constraints to the Schwarz BVPs. This is not within the scope of the present article, and will be further discussed in a forthcoming paper.

References

[1] Muskhelishvili, N.I.: Singular integrable equations, 2nd edn. Noordhoff, Groningen (1968)

[2] Vekua, I.N.: Generalized analytic function. Pergamon Press, Oxford (1962)

[3] Garnett, J.B.: Bounded analytic functions, Revised first edition. Springer (2007, GTM 236)

[4] Balk, M.B.: On poly-analytic functions. Akademie Verlag, Berlin (1991)

[5] Begehr, H., Schmersau, D.: The Schwarz problem for poly-analytic functions. ZAA **24**(2), 341–351 (2005)

[6] Monakhov, V.N.: Boundary value problems with free boundaries for elliptic systems of equations, Translations of Mathematical Monographs, vol. 57. AMS, Providence (1983)

[7] Wang, Y.F.: On Hilbert-type boundary-value problem of poly-Hardy class on the unit disc. Complex Var. Elliptic Equ. **58**(4), 497–509 (2013)

[8] Qian, T., Sprößig, W., Wang, J.X.: Adaptive Fourier decomposition of functions in quaternionic Hardy spaces. Math. Methods Appl. Sci. **35**(1), 43–64 (2012)

[9] Cerejeiras, P., Kähler, U.: Monogenic signal theory. In: Alpay, D. (ed.) Operator theory, pp. 1–22 (2015). doi:10.1007/978-3-0348-0692-3_14-1

[10] Min, K., Kähler, U., Wang, D.S.: Riemann boundary value problems on the sphere in Clifford analysis. Adv. Appl. Clifford Alg. **22**(2), 365–390 (2012)

[11] Cerejeiras, P., Kähler, U., Min, K.: On the Riemann boundary value problem for null solutions to iterated generalized Cauchy-Riemann operator in Clifford analysis. Results Math. **63**(3–4), 1375–1394 (2013)

[12] Xu, Z.Y., Chen, J., Zhang, W.G.: A harmonic conjugate of the Poisson kernel and a boundary value problem for monogenic functions in the unit ball of $\mathbb{R}^n (n \geq 2)$. Simon Stevin **64**(2), 187–201 (1990)

[13] Brackx, F., Delanghe, R., Sommen, F.: Clifford analysis, Res. Notes Math., vol. 76. Pitman Publ., Boston-London-Melbourne (1982)

[14] Gürlebeck, K., Sprößig, W.: Quaternionic and Clifford calculus for physicists and engineers. Wiley, Chichester New York (1997)

[15] Gilbert, J., Murray, M.: Clifford algebras and Dirac operators in harmonic analysis. Cambridge University Press, Cambridge (1991)

[16] Wietzke, L., Sommer, G.: The conformal monogenic signal. In: Rigoll G. (ed.) DAGM pattern recognition, LNCS, vol. 5096, pp. 527–536. Springer, Berlin (2008)

[17] Bülow, T., Sommer, G.: Hypercomplex signals-a novel extension of the analytic signal to the multidimensional case. IEEE Trans. Signal Process **49**(11), 2844–2852 (2001)

[18] Felsberg, M., Sommer, G.: The monogenic signal. IEEE Trans. Signal Process **49**(12), 3136–3144 (2001)

[19] Min, K., Wang, D.S.: Half Dirichlet problem for matrix functions on the unit ball in Hermitian Clifford analysis. J. Math. Anal. Appl. **374**, 442–457 (2011)

[20] Cerejeiras, P., Kähler, U., Min, K., Sommen, F.: Discrete Hardy spaces. J. Fourier Anal. Appl. **20**, 715–750 (2014)

[21] Cerejeiras, P., Kähler, U., Min, K.: Discrete Hilbert boundary value problems on half lattices. J. Diff. Equ. Appl. **21**(12), 1277–1304 (2015)

[22] Qian, T., Yang, Y.: Hilbert transforms on the sphere with the Clifford algebra setting. J. Fourier Anal. Appl. **15**, 753–774 (2009)

[23] Min, K., Yinxiong, F., Kähler, U., Cerejeiras, P.: Riemann boundary value problems for iterated Dirac operator on the ball in Clifford analysis. Complex Anal. Oper. Theory **7**, 673–693 (2013)

[24] Min, K., Wang, D.S.: Solutions to polynomial Dirac equations on unbounded domains in Clifford analysis. Math. Methods Appl. Sci. **34**, 418–427 (2011)

[25] Min, K., Wang, D.S., Dong, L.: Solutions to polynomial generalized Bers-Vekua equations in Clifford analysis. Complex Anal. Oper. Theory **6**(2), 407–424 (2012)

[26] Min, K., Kähler, U., Cerejeiras, P.: Dirichlet-type problems for the iterated Dirac operator on the unit ball in Clifford analysis. In: Gentili, G. et al. (eds.) Advances in hyper-complex analysis, Springer INdAM Series 1, Springer-Verlag Italian (2013). doi:10.1007/978-88-470-2445-8_5

[27] Malonek, H.R., Ren G., B.: Almansi-type theorems in Clifford analysis. Math. Methods Appl. Sci. **25**, 1541–1552 (2006)

[28] Fuli, H., Min, K., Pei, D., Kähler, U.: Riemann-Hilbert problems for poly-Hardy spaces on the unit ball. Complex Var. Elliptic Equ. **61**(6), 772–790 (2016)

[29] Brackx, F., Van Acker, N.: A conjugate Poisson kernel in Euclidean space. Simon Stevin **67**(1-2), 3–14 (1993)

[30] Brackx, F., De Schepper, H.: Conjugate harmonic functions in Euclidean space: a spherical approach. Comput. Methods Funct. Theory **6**(1), 165–182 (2006)

[31] Brackx, F., De Knock, B., De Schepper, H., Eelbode, D.: On the interplay between the Hilbert transform and conjugate harmonic functions. Math. Methods Appl. Sci. **29**, 1435–1450 (2006)

[32] Constales, D.: A conjugate harmonic to the Poisson kernel in the unit ball of $\mathbb{R}^n$. Simon Stevin **62**, 289–291 (1988)

[33] Delanghe, R., Brackx, F.: Hypercomplex function theory and Hilbert modules with reproducing kernel. Proc. Lond. Math. Soc. **37**(3), 545–576 (1978)

Min Ku and Uwe Kähler
CIDMA
Department of Mathematics
University of Aveiro
Aveiro
Portugal
e-mail: kumin0844@163.com; ukaehler@ua.pt

Tao Qian
Faculty of Science and Technology
University of Macau
Macao
China
e-mail: fsttq@umac.mo

Received: October 29, 2015.
Accepted: June 30, 2016.