# SINGULAR INTEGRALS ALONG LIPSCHITZ CURVES WITH HOLOMORPHIC KERNELS

A. McIntosh
(*Macquarie University, Australia*)
*and*
T. Qian
(*Academia Sinica, China*)
Received Nov.14, 1989

*Abstract*

*If* $\gamma(x)=x+iA(x)$, $\tan^{-1}\|A'\|_\infty<\omega<\pi/2$ ,

$$S^o_\omega=\{z\in\mathbb{C}\mid |\arg z|<\omega,\ \text{or},\ |\arg(-z)|<\omega\}.$$

*We have proved that if* $\varphi$ *is a holomorphic function in* $S^o_\omega$ *and* $|\varphi(z)|\leqslant\frac{C}{|z|}$, *denoting*

$$Tf(z)=\int\varphi(z-\zeta)f(\zeta)d\zeta,\ \forall f\in C_o(\gamma),\quad \forall z\bar{\in}\text{supp}f,$$

*where* $C_c(\gamma)$ *denotes the class of continuous functions with compact supports, then the following two conditions are equivalent:*

1° *T can be extended to be a bounded operator on* $L^2(\gamma)$;

2° *there exists a function* $\varphi_1\in H^\infty(S^o_\omega)$ *such that* $\varphi'_1(z)=\varphi(z)+\varphi(-z)$, $\forall z\in S^o_\omega$.

## § 1 Introduction

The following formula between the Hibert transform and its multiplier representation is well known:

$$\text{p.v}\frac{i}{\pi}\int_{\mathbb{R}}\frac{1}{x-y}f(y)dy=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ix\xi}\text{sgn}\xi\hat{f}(\xi)d\xi,\quad \forall f\in S(\mathbb{K}). \tag{1}$$

In [1], [3], [4] the authors developed a generalized Fourier transform theory related to Lipschitz curves. In the theory it was proved that with respect to any Lipschitz curve the multiplier associated to the Cauchy integral along this curve is still the signum function. Exactly there exists

$$\text{p.v}\frac{i}{\pi}\int_{\gamma}\frac{1}{z-\zeta}f(\zeta)d\zeta=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{iz\xi}\text{sgn}\xi\hat{f}(\xi)d\xi \tag{2}$$

for all functions $f$ belonging to a nice subclass of $L^2(\gamma)$, where $\gamma=\{x+iA(x)|-\infty<x<\infty\}$, $A(x)$ is a bounded Lipschitz function, i.e., $\|A'\|_\infty=M<\infty$, $\|A\|_\infty=N<\infty$ and

$$\hat{f}(\xi)=\int_{\gamma}e^{-i\xi z}F(z)dz. \tag{3}$$

If $\gamma=\mathbf{R}$, then formula (2) reduces to formula (1). In the following we denote

$L^2(\gamma)=\{f:\gamma\to\mathbb{C}\,|f$ is measurable w. r. t. $d\zeta$ and $\|f\|_2=(\int_{\gamma}|f(\zeta)|^2|d\zeta|)^{\frac{1}{2}}<\infty\}$,

$$S^o_{\mu,\pm}=\{z\in\mathbb{C}\,||\arg(\pm z)|<\mu\},\quad 0<\mu<\pi/2,$$

$$S^o_\mu=S^o_{\mu,+}\bigcup S^o_{\mu,-}$$

and

$$C^o_{\mu,\pm}=\{z\in\mathbb{C}\,|\pm\text{Im}z>0\}\bigcup S^o_\mu.$$

For any open sets $P$ and $Q$ in the complex plane $\mathbb{C}$, denote

$$H^\infty(P)=\{b:P\to\mathbb{C}\mid b\text{ is holomorphic and bounded}\}$$

and

$$K(Q)=\{\varphi:Q\to\mathbb{C}\mid\varphi\text{ is holomorphic and }\exists\,C\text{ s.t.}|\varphi(z)|\leqslant\frac{C}{|z|},z\in Q\}.$$

Later we'll use $P=S^o_{\mu,\pm}$ or $S^o_\mu$ and $Q=C^o_{\mu,\pm},S^o_\mu$ respectively.

For $b\in H^\infty(S^o_{\mu,+})$ and $z\in C^o_{\mu,+}$, define

$$\mathscr{G}(b)(z)=\varphi(z)=\frac{1}{2\pi}\int_{\rho\theta}e^{iz\zeta}b(\zeta)d\zeta,$$

where $-\mu<-\theta<\arg(z)<\pi-\theta<\pi+\mu$ and $\rho\theta=\{se^{i\theta}:0<S<\infty\}$. The defintion is independent of $\theta$. When $z\in S^o_\mu$ we define

$$\mathscr{G}_1(b)(z)=\varphi_1(z)=\int_{\delta(z)}\varphi(\zeta)d\zeta,$$

where the integral is along a contour $\delta(z)$, from $-z$ to $z$ in $C^o_{\mu,+}$.

Similarly, for $b\in H^\infty(S^o_{\mu,-})$ and $z\in C^o_{\mu,-}$, define

$$\mathscr{G}(b)(z) = \varphi(z) = \frac{-1}{2\pi}\int_{\rho\theta} e^{iz\zeta} b(\zeta)d\zeta,$$

where $-\pi-\mu < -\theta < \arg(z) < \pi-\theta < \mu$ and $\rho\theta = \{se^{i\theta}: 0 < s < \infty\}$. The definition is independent of $\theta$. When $z \in S^{o}_{\mu}$, define

$$\mathscr{G}_1(b)(z) = \varphi_1(z) = \int_{\delta(z)} \varphi(\zeta)d\zeta,$$

where this time the contour is in $C^{o}_{\mu,-}$, from $-z$ to $z$.

Now assume that $\tan^{-1}M < \omega < \pi/2$. For any function $b \in H^{\infty}(S^{o}_{\omega})$ one can associate it with a $L^2(\gamma)$ bounded operator defined formally by

$$b(D_\gamma)f(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iz\xi} b(\xi)\hat{f}(\xi)d\xi,$$

where $\hat{f}$ is defined by (3) and we use notation $b(D_\gamma)$ in a similar way as we use $b(D)$ for the Classical Fourier multiplier operator while $D = \frac{1}{i}\frac{d}{dx}$. But here $D_\gamma$ is the differential operator along curve $\gamma: D_\gamma = (1+iA'(x))^{-1}D$.

In this paper, however, we'll use the definition by A. McIntosh[2] which is equivalent to the above mentioned one. By this approach one first define $b(D_\gamma)$ for the following subclass of $H^{\infty}(S^{o}_{\omega,\pm})$:

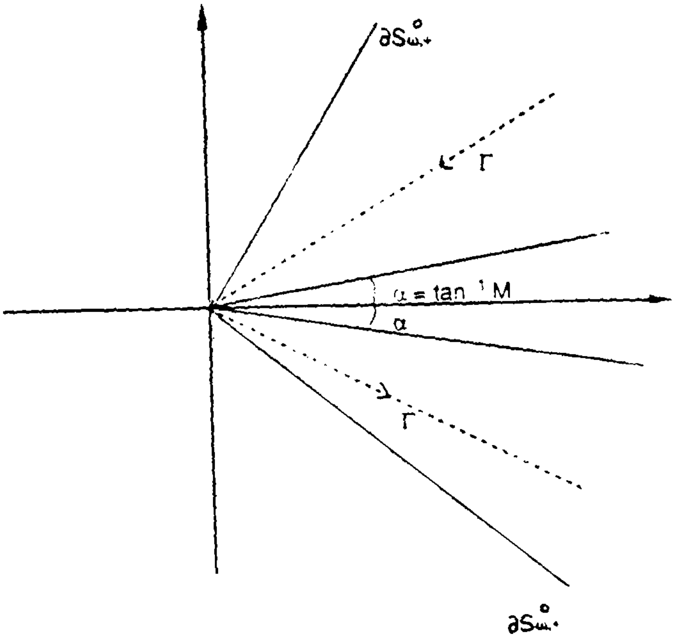
$$\Psi(S^{o}_{\omega,\pm}) = \left\{ b \in H^{\infty}(S^{o}_{\omega,\pm}) | \exists C,\ s>0 \text{ s.t. } |b(\zeta)| \leqslant \frac{C|\zeta|^{s}}{1+|\zeta|^{2s}},\ \zeta \in S^{o}_{\omega,\pm} \right\}$$

and for $b \in \Psi(S^{o}_{\omega,\pm})$, define

$$b(D_\gamma)f = \frac{1}{2\pi i}\int_{\Gamma} b(\zeta)(\zeta I - D_\gamma)^{-1} f d\zeta, \quad f \in L^2(\gamma), \tag{4}$$

where the integral contour $\Gamma$ is as shown in the picture(A).

The integral in the right hand side in the formula (4) is absolutely convergent in the operator norm and independent of $\Gamma$. Having done this then we can extend to define $b(D_\gamma)$ for $b$ in a bigger class of functions including $H^{\infty}(S^{o}_{\omega,+})$. And, since $D_\gamma$ satisfies the quadratic estimates, we can conclude that $b(D_\gamma)$ are bounded operators for all $b \in H^{\infty}(S^{o}_{\omega,+})$ (See [2]). The same is ture for $b \in H^{\infty}(S^{o}_{\omega,-})$. By noticing that $H^{\infty}(S^{o}_{\omega,\pm})$ are subclasses of $H^{\infty}(S^{o}_{\omega})$ and every function in $H^{\infty}(S^{o}_{\omega})$ has a decomposition into a sum of a function in $H^{\infty}(S^{o}_{\omega,+})$ and of a function in $H^{\infty}(S^{o}_{\omega,-})$ in an obvious manner, we obtain a definition of $b(D_{\gamma}), b\in H^{\infty}(S^{o}_{\omega})$ which is a bounded operator in $L^{2}(\gamma)$. We note that in this approach we don't assume $\gamma$ bounded.




picture (A)

**Theorem 1.** *Mapping* $\mathscr{G}:\bigcup_{0<\mu<\omega}H^{\infty}(S^{o}_{\mu,+})\to\bigcup_{0<\mu<\omega}K(C^{\infty}_{\mu,+}), 0<\mu<\omega<\frac{\pi}{2}$, *is one to one, onto, and for every* $b\in H^{\infty}(S^{o}_{\omega,+})$, *by denoting* $\varphi=\mathscr{G}(b),\varphi_{1}=\mathscr{G}_{1}(b)$ *we have*

$$b(D_{\gamma})f(z)=\lim_{t\to 0+}\int_{\gamma}\varphi(z-\zeta+it)f(\zeta)d\zeta$$

$$=\lim_{\varepsilon\to 0+}\left(\int_{|z-\zeta|>\varepsilon}\varphi(z-\zeta)f(\zeta)d(\zeta)+\varphi_{1}(\varepsilon t(z))f(z)\right),\quad \text{a.e., } f\in L^{2}(\gamma), \tag{5}$$

*where* $t(z)$ *is the unit tangent vector to* $\gamma$ *which is defined at almost all* $z\in\gamma$.

The same conclusion holds for $\mathscr{G}:\bigcup_{0<\mu<\omega}H^{\infty}(S^{o}_{\mu,-})\to\bigcup_{0<\mu<\omega}K(C^{\infty}_{\mu,-}),\ 0<\mu<\omega<\frac{\pi}{2}$, except unsing limit $t\to 0-$.

For $b\in H^{\infty}(S^{o}_{\omega})$ we define $\mathscr{G}b=\mathscr{G}b_{+}+\mathscr{G}b_{-}$ and

$$\mathscr{G}_{1}b=\mathscr{G}_{1}b_{+}+\mathscr{G}_{1}b,\quad \text{where } b_{\pm}=b\chi_{\{\pm \mathrm{Re}z>0\}}\in H^{\infty}(S^{o}_{\omega,\pm}).$$

If we look the mapping $\mathcal{G}:\bigcup_{0<\mu<\omega} H^{\infty}(S_{\mu}^{o})\to\bigcup_{0<\mu<\omega} K(C_{\mu}^{\infty})$, $0<\mu<\omega<\frac{\pi}{2}$, it is still one to one, but no longer "onto". It does map $\bigcup_{0<\mu<\omega} H^{\infty}(S_{\mu}^{o})$ onto a nice subclass of $\bigcup_{0<\mu<\omega} K(S_{\mu}^{o})$, however.

**Theorem 2.** *Mapping* $\mathcal{G}:\bigcup_{0<\mu<\omega} H^{\infty}(S_{\mu}^{o})\to\bigcup_{0<\mu<\omega} \Phi(S_{\mu}^{o})$ *is one to one and onto, where* $0<\mu<\omega<\frac{\pi}{2}$ *and*

$$\Phi(S_{\mu}^{o})=\left\{\varphi:S_{\mu}^{o}\to\mathbb{C}\mid \varphi=\varphi_{0}+\varphi'_{1},\ \varphi_{0},\ \varphi_{1}\ \text{are odd and}\ \varphi_{0}\in K(S_{\mu}^{o}),\varphi_{1}\in H^{\infty}(S_{\mu}^{o})\right\}.$$

*For every* $b\in H^{\infty}(S_{\mu}^{o})$ *we have* $\varphi=\mathcal{G}b=\varphi_{0}+\varphi'_{1}=\varphi_{0}+\left(\frac{1}{2}\mathcal{G}_{1}b\right)',\varphi_{1}=\frac{1}{2}\mathcal{G}_{1}b$, *and*

$$b(D_{\gamma})f(z)=\lim_{\varepsilon\to 0+}\left(\int_{|z-\zeta|>\varepsilon}\varphi(z-\zeta)f(\zeta)+2\varphi_{1}(\varepsilon t(z)f(z)\right),\ \text{a.e.},\ f\in L^{2}(\gamma). \tag{6}$$

**Theorem 3.** *Let* $\varphi$ *be a function holomorphic and satisfy* $|\varphi(z)|\leqslant\frac{C}{|z|}$ *on* $S_{\omega}^{o}$. $\gamma=x+iA(x)$ *is a Lipschitz curve,* $\|A'\|_{\infty}<\tan\omega$. *Then there exists a* $L^{2}(\gamma)$ *bounded operator* $T$ *such that* $Tf(z)=\int\varphi(z-\zeta)f(\zeta)d\zeta$, $\forall f\in C_{c}(\gamma),\forall z\bar{\in}\mathrm{supp}f$, *where* $C_{c}(\gamma)$ *denotes the class of continuous functions with compact support on* $\gamma$, *if and only if there exists a function* $\varphi_{1}\in H^{\infty}(S_{\omega}^{o})$ *such that* $\varphi'_{1}(z)=\frac{1}{2}(\varphi(z)+\varphi(-z))$, $z\in S_{\omega}^{o}$.

This work was supported by the Commonwealth of Australia through the Australian Research Council. It is a plearsure to thank the School of Mathematical Sciences in the Flinders University of South Australia for its hospitality where the final write–up was prepared. The authors also with to thank Miss Jolie Hutchinson for the time she spent in typing an early draft of this work.

## § 2 Proffs of the Theorems

*Proof of Theorem* 1. The ray $p\theta$ has been chosen so that the integrand in the defintion of $\mathcal{G}$ is expoenentially decreasing at $\infty$. This permits to verify directly that

$$|\varphi(z)|\leqslant\left\{2\pi\ \mathrm{dist}(z,\mathbb{C}\setminus C_{\omega,+}^{o})\right\}^{-1}\|b\|_{\infty}.$$

which shows that $\varphi\in K(C_{\mu,+}^{o})$ with any $\mu$ such that $0<\mu<\omega$.

To prove that $\mathcal{G}$ is one to one, let us show that $\mathcal{G}_{b}\equiv 0$ for $z\in C_{\mu,+}^{o}$ implies $b\equiv 0$ in $\mathbf{R}$, and so that $b\equiv 0$ in $S_{\omega,+}^{o}$. In fact, for $\varepsilon_{0}>0$

$$0=\mathcal{G}(b)(x+i\varepsilon_0)=\frac{1}{2\pi}\int_0^\infty e^{ix\xi}e^{-\varepsilon_0\xi}b(\xi)d\xi=(b_{\varepsilon_0})^{\vee}(x),$$

where $b_{\varepsilon_0}=e^{-\varepsilon_0\xi}b(\xi)\chi_+(\xi)\in L^2(\mathbf{R})$. From the inverse formula of fourier transform we have $b_{\varepsilon_0}=0$ a.e. in $\mathbf{R}$, and, so, $b_{\varepsilon_0}=0$ and $b$ has to be zero on $\mathbf{R}$.

To prove that $\mathcal{G}$ is onto we use the following mapping

$$\mathcal{F}_1 K(C^o_{\mu,+})\to H^\infty\left(S^o_{\mu_1,+}\right),\quad 0<\mu_1<\mu:$$

$$\mathcal{F}_1\varphi(\zeta)=\int_{\sigma^A_\theta}\varphi^{-iz\zeta}\varphi(z)dz,\quad \zeta\in S^o_{\mu,+},$$

where

$$\sigma^A_\theta=\begin{cases} te^{i\theta}, & t \text{ is from } A \text{ to } \infty,\\ te^{i(\pi-\theta)}, & t \text{ is from } \infty \text{ to } A,\\ Ae^{it}, & t \text{ is from } \pi \text{ to } 0\end{cases}$$

with $A>0,\theta<0,|\theta|>|\arg\zeta|$. The defintion is independent of $A,\theta$. Since the integrand in the defintion is expoenentially decreasing, it is drect to verify that

$$\mathcal{F}_1\varphi\in H^\infty\left(S^o_{\mu_1,+}\right)\qquad \forall\,\varphi\in K\left(S^o_{\mu,+}\right).$$

We'll prove $\mathcal{G}\mathcal{F}_1\varphi=\varphi,\forall\,\varphi\in K\left(S^o_{\mu,+}\right)$ and so $\mathcal{G}$ is "onto". In fact, for a fixed $\xi>0$, chose integral contours $\rho_\psi$ and $\sigma^A_\theta$ such that $\theta<\psi<|\theta|$, $\theta<0$ and $A>0$ is small enough so that $\mathrm{Im}(\zeta z)<0,\mathrm{Im}(\xi\zeta)>0$ and $\mathrm{Im}(\zeta(\xi-z))>0$ for all $\zeta\in\rho_\psi,z\in\sigma^A_\theta$, then from a partial inverse of Fubini's theorem we have

$$\begin{aligned}\mathcal{G}\mathcal{F}_1\varphi(\xi)&=\frac{1}{2\pi}\int_{\rho_\psi}e^{i\xi\zeta}\left(\int_{\sigma^A_\theta}e^{-i\zeta z}\varphi(z)dzd\zeta\right.\\ &=\frac{1}{2\pi}\int_{\sigma^A_\theta}\varphi(z)\left(\int_{\rho_\psi}e^{i\zeta(\xi-z)}d\zeta\right)dz=\frac{1}{2\pi i}\int_{\sigma^A_\theta}\frac{\varphi(z)}{z-\xi}dz=\varphi(\xi).\end{aligned}$$

This concludes that $\mathcal{G}\mathcal{F}_1\varphi=\varphi$ on the positive part of the real line. Since $\mathcal{G}\mathcal{F}_1\varphi$ and $\varphi$ are both holomorphic we proved $\mathcal{G}\mathcal{F}_1\varphi=\varphi$ on $S^o_{\mu,+}$ which shows that $\mathcal{G}$ is onto.

To prove the equalities (5) we need the following lemma:

**Lemma 1.** $1^\circ$ *If* $b\in H^\infty\left(S^o_{\omega,+}\right)$ *satisfies* $|b(\zeta)|\leqslant C_s|\zeta|^s$ *for some* $s\in(-1,\infty)$, *then*

$$|\mathcal{G}(b)(z)| \leqslant \frac{C_s}{2\pi\left[\operatorname{dist}(z,\partial C^o_{\omega,+})\right]^{1+s}}, \quad z \in C^o_{\omega,+}.$$

2° *If* $b \in \Psi\left(S^o_{\omega,+}\right)$, *then* $\varphi = \mathcal{G}b \in L^1(\gamma)$ *and* $b(D_\gamma)f = \varphi * f$ *for* $f \in L^p(\gamma), 1 \leqslant p < \infty$.

*Proof.* 1° Let $z = |z|e^{i\theta_0} \in C^o_{\omega,+}$. By choosing an appropriate contour $\rho_\theta$ we have

$$|\mathcal{G}(b)(z)| \leqslant \frac{C_s}{2\pi}\int_0^\infty e^{-|z|(\sin(\theta+\theta_0))t} t^s ds$$
$$= \frac{C_s}{2\pi}\frac{1}{(|z|\sin(\theta+\theta_0))^{1+s}} \leqslant \frac{C_s}{2\pi}\frac{1}{(\operatorname{dist}(z,\partial C^o_{\omega,+}))^{1+s}}.$$

2° $b \in \Psi\left(S^o_{\omega,+}\right)$ implies that $\exists\, s \in (0,1)$ and $C_s$ such that $|b(\zeta)| \leqslant C_s \min\{|\zeta|^s, |\zeta|^{-s}\}, \forall\, \zeta \in S^o_{\omega,+}$. By using the result in 1° we have $|\varphi(z)| \leqslant C\min\left\{\frac{1}{|z|^{1+s}}, \frac{1}{|z|^{1-s}}\right\} \in L^1(\gamma)$.

Denote $\rho_{\pm\theta} = re^{i(\pm\theta)}$, $r$ is from 0 to $+\infty$, $0 < \tan^{-1}M < \theta < \omega$. Then from formula (4) we have

$$b(D_\gamma)f(z) = \frac{1}{2\pi i}\int_{(\rho_{+\theta})\cup(\rho_{-\theta})} b(\zeta)(\zeta I - D_\gamma)^{-1}f(z)d\zeta$$
$$= \frac{-1}{2\pi}\int_{-\rho_\theta} b(\zeta)d\zeta\int_{\gamma_z^-} e^{i\zeta(z-\xi)}f(\xi)d\xi$$
$$+ \frac{1}{2\pi}\int_{\rho_{-\theta}} b(\zeta)d\zeta\int_{\gamma_z^+} e^{i\zeta(z-\xi)}f(\xi)d(\xi) = -I_1 + I_2,$$

where we used the convolution expression of resolvent $(\zeta I - D_\gamma)^{-1}$ (See [3] and $\gamma_z^{\pm} = \{\xi \in \gamma \mid \pm Re(z-\xi) > 0\}$. By the proof of the fact $\varphi \in L^1(\gamma)$, we actually have

$$\int_{\rho_{\pm\theta}} \left|e^{i\zeta(z-\xi)}b(\zeta)\right||d\zeta| \in L^1(\gamma), \quad \xi \in \gamma_z^{\mp}.$$

Since $f \in L^p(\gamma), 1 \leqslant p < \infty, \varphi \in L^1(\gamma)$, by the Hansdoff–Young inequality and a partial converse of Fubini's theorem, we have

$$-I_1 = \int_{\gamma_z^-} f(\xi)d\xi\frac{1}{2\pi}\int_{\rho_\theta} e^{i\zeta(z-\xi)}b(\zeta)d\zeta = \int_{\gamma_z^-} f(\xi)\varphi(z-\xi)d\xi$$

$$I_2 = \int_{\gamma_s^+} f(\xi)d(\xi)\frac{1}{2\pi}\int_{\rho_{-\theta}} e^{i\zeta(z-\xi)} b(\zeta)d\zeta = \int_{\gamma_s^+} f(\xi)\varphi(z-\xi)d\xi.$$

By adding the last two formulas we conclude $b(D_\gamma)f = \varphi * f$.

Now we prove the equalities (5). First for a fixed $t > 0$, $b_{t,s}(\zeta) = \frac{\zeta^s}{1+\zeta^{2s}} b(\zeta)e^{-it\zeta}$ $\in \Psi(S^o_{\omega,+})$ are uniformly bounded in $H^\infty(S^o_{\omega,+})$ and $\lim_{s\to 0} b^{(\zeta)}_{t,s} = b_t(\zeta) = b(\zeta)e^{-it\zeta}$ uniformly in any compact set contained in $S^o_{\omega,+}$. Therefore by [2], § 4

$$\lim_{s\to 0} b_{t,s}(D_\gamma)f = b_t(D_\gamma)f,$$

in $L^2(\gamma)$ norm for every $f\in L^2(\gamma)$. Denote $\varphi_{t,s} = \mathcal{G} b_{t,s}$, by 2° of Lemma 1

$$b_{t,s}(D_\gamma)f(z) = \int \varphi_{t,s}(z-\xi)f(\xi)d\xi.$$

Since $\exists\, \alpha, \beta > 0$ such that

$$|\varphi_{t,s}(z-\xi)| \leqslant \int_{\rho_{\pm\theta}} |e^{i\zeta(z-\xi)} b_t(\zeta)||d\zeta| = \frac{C}{\beta t + \alpha|z-\zeta|} \in L^2(\gamma)$$

and

$$\lim_{s\to 0} \varphi_{t,s}(z-\xi) = \varphi_t(z-\xi),$$

by the Lebesgue dominated convergence theorem, we have

$$\begin{aligned} b_t(D_\gamma)f(z) &= \int \varphi_t(z-\xi)f(\xi)d\xi \\ &= \int \varphi(z-\xi+it)f(\xi)d\xi, \qquad f\in L^2(\gamma). \end{aligned}$$

Using the convergence theorem in [2] § 4 once again to the left hand side of the above equality, we have

$$b(D_\gamma)f(z) = \lim_{t\to 0+} \int \varphi(z-\zeta+it)f(\xi)d\xi, \quad \forall\, f\in L^2(\gamma), \tag{7}$$

where the convergence is in $L^2(\gamma)$ norm.

To prove the second part of the equalities (5) we first assume

$f\in L_{\mathrm{lip},c}(\gamma) = \{f{:}\gamma\to\mathbb{C} \mid f$ is a Lipschitz function with compact support$\}$.

Then we can prove

$$\left| b(D_\gamma)f(z) - \int_{|z-\zeta|>\varepsilon} \varphi(z-\zeta)f(\zeta)d\zeta - f(z)\int_{\zeta\in C^+_{z,\varepsilon}} \varphi(z-\zeta)d\zeta \right| \leqslant C\varepsilon \cdot \|f'\|_{L^\infty(\gamma)}, \tag{8}$$

where

$$C^+_{z,\varepsilon} = \{\zeta \in \mathbb{C} \mid |\zeta - z| = \varepsilon,\quad \mathrm{Im}\zeta > A(\mathrm{Re}\zeta)\}.$$

In fact,

$$\begin{aligned} b(D_\gamma)f(z) &= \lim_{t\to 0+} \int_{|z-\zeta|>\varepsilon} \varphi(z-\zeta+it)f(\zeta)d\zeta \\ &+ \lim_{t\to 0+} \int_{|z-\zeta|\leqslant\varepsilon} \varphi(z-\zeta+it)(f(\zeta)-f(z))d\zeta \\ &+ \lim_{t\to 0+} f(z)\int_{|z-\zeta|\leqslant\varepsilon} \varphi(z-\zeta+it)d\zeta = \sum_{i=1}^{3} J_i, \end{aligned}$$

where

$$J_1 = \int_{|z-\zeta|>\varepsilon} \varphi(z-\zeta)f(\zeta)d\zeta,$$

$$|J_2| \leqslant \int_{|z-\zeta|\leqslant\varepsilon} |\varphi(z-\zeta)||f(\zeta)-f(z)||d\zeta| \leqslant C\varepsilon\|f'\|_{L^\infty(\gamma)},$$

$$J_3 = \lim_{t\to 0+} f(z)\int_{\zeta\in C^+_{z,\varepsilon}} \varphi(z-\zeta+it)d\zeta = f(z)\int_{\zeta\in C^+_{z,\varepsilon}} \varphi(z-\zeta)d\zeta.$$

These prove (8).

By noticing that

$$\lim_{\varepsilon\to 0} \int_{\zeta\in C^+_{z,\varepsilon}} \varphi(z-\zeta)d\zeta = \lim_{\varepsilon\to 0} \varphi_1(\varepsilon t(z))$$

and taking limit $\varepsilon\to 0$ in (8), we conclude

$$b(D_\gamma)f(z) = \lim_{\varepsilon\to 0}\left( \int_{|z-\zeta|>\varepsilon} \varphi(z-\zeta)f(\zeta)d\zeta + f(z)\varphi_1(\varepsilon t(z)) \right), \quad a.e.\, z\in\gamma \quad \forall f\in L_{\mathrm{lip},c}(\gamma). \tag{9}$$

To extend $f\in L_{\mathrm{lip},c}(\gamma)$ to $f\in L^2(\gamma)$ in (9) we need the following lemma. We omit the standard proof of it.

**Lemma 2.** *Denote*

$$T_\varepsilon f(z) = \int_{|z-\zeta|>\varepsilon} \varphi(z-\zeta) f(\zeta) d\zeta,$$

$$T^* f(z) = \sup_{\varepsilon>0} |T_\varepsilon f(z)|$$

*Then*

$$\|T^* f\|_p \leqslant C_p \|f\|_p, \qquad 1 < p < \infty.$$

Now let $f \in L^2(\gamma)$. For any $\delta > 0$, $\exists f_\delta \in L_{\text{lip},c}(\gamma)$ such that $\|f - f_\delta\|_{L^2(\gamma)} \leqslant \delta$. Then for any $\alpha > 0$

$$|\{Z \in \gamma| \ |b(D_\gamma)f(z) - \lim_{\varepsilon\to 0}\left(\int_{|z-\zeta|>\varepsilon} \varphi(z-\zeta)f(\zeta)d\zeta + f(z)\varphi_1(\varepsilon t(z))\right)| > \alpha\}|$$

$$\leqslant |\{z \in \gamma| \ |b(D_\gamma)(f-f_\delta)(z) + \lim_{\varepsilon\to 0}\left(\int_{|z-\zeta|>\varepsilon} \varphi(z-\zeta)(f-f_\delta)(\zeta)d\zeta + (f-f_\delta)(z)\varphi_1(z-z_\varepsilon^-)\right)| > \frac{\alpha}{2}\}|$$

$$+ |\{z \in \gamma| \ |b(D_\gamma)f_\delta(z) - \lim_{\varepsilon\to 0}\left(\int_{|z-\zeta|>\varepsilon} \varphi(z-\zeta)f_\delta(\zeta)d\zeta + f_\delta(z)\varphi_1(z-z_\varepsilon^-)\right)| > \frac{\alpha}{2}\}|$$

$$\leqslant |\{z \in \gamma| \ |b(D_\gamma)(f-f_\delta(z)| > \frac{\alpha}{6}\}| + |\{z \in \gamma| \ T^*(f-f_\delta)(z) > \frac{\alpha}{6}\}|$$

$$+ |\{z \in \gamma| \ \sup_{\varepsilon>0}|(f-f_\delta)(z)\varphi_1(z-z_\varepsilon^-)| > \frac{\alpha}{6}\}|$$

$$\leqslant C\left(\frac{\delta}{\alpha}\right)^2,$$

where $z_\varepsilon^-$ is the intersection point of the circle with centre $z$ and radius $\varepsilon$ and the left part of $\gamma$:$\mathrm{Re}\gamma(x) < \mathrm{Re}z$, where we have used Lemma2 and $\varphi_1 \in H^\infty\left(S^o_{\mu,+}\right)$. Since $\delta$ can be chosen as small as we want, we conclude (9) for $F \in L^2(\gamma)$. This completes the proof of the theorem.

*Proof of theorem 2.* It is easy to see that

$$H^\infty\left(S^o_\omega\right) = \{b: S^o_\omega \to \mathbb{C} \mid B(\xi) = b_0(\xi) + 2\pi i\xi b_1(\xi), \text{ where } b_0, b_1 \text{ are odd and }$$
$$b_0 \in H^\infty\left(S^o_\omega\right),\ b_1 \in K\left(S^o_\omega\right)\}. \qquad (10)$$

And, for every $b \in H^\infty\left(S^o_\omega\right)$ there is a unique such decomposition as shown in the right hand side of (10). Denote

$$\Phi_0\left(S^o_\omega\right) = \left\{\varphi \in \Phi\left(S^o_\mu\right) | \varphi \text{ is odd}\right\},$$
$$H^\infty_0\left(S^o_\omega\right) = \left\{b \in H^\infty\left(S^o_\omega\right) | b \text{ is odd}\right\},$$

and define a new mapping $\mathscr{F}_0 : \Phi_0\left(S^o_\omega\right) \to H^\infty_0\left(S^o_\omega\right)$, by

$$\mathscr{F}_0 \varphi_0 = b^+ + b^-, \qquad \forall\, \varphi_0 \in \Phi_0\left(S^o_\omega\right),$$

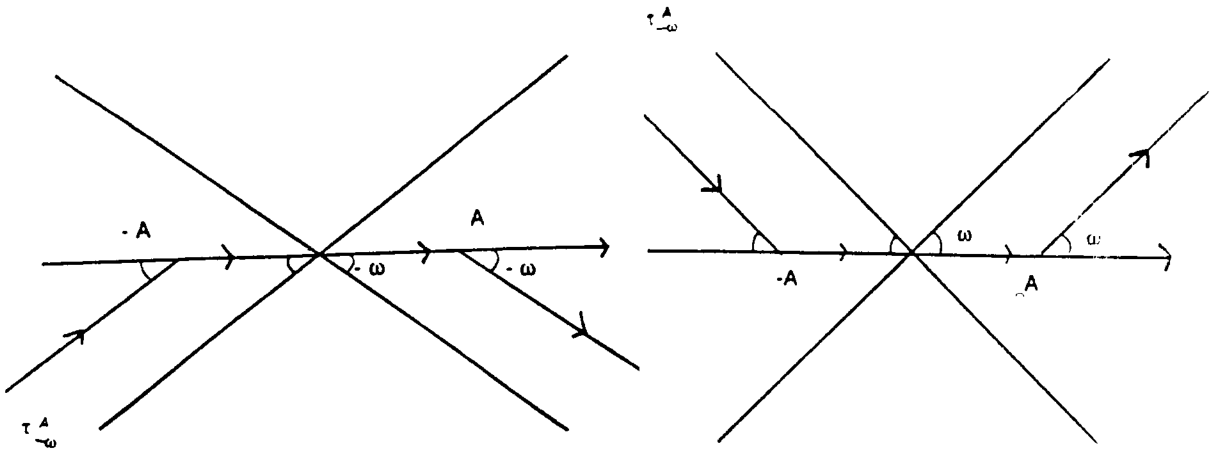
where

$$b^+(\zeta) = \begin{cases} \text{p.v.}\int_{\tau^A_{-\omega}} e^{-i\zeta z}\varphi_0(z)dz, & \text{Re}\zeta > 0, \\ 0, & \text{Re}\zeta < 0, \end{cases}$$

and

$$b^-(\zeta) = \begin{cases} 0, & \text{Re}\zeta > 0, \\ \text{p.v.}\int_{\gamma^A_\omega} e^{-\zeta z}\varphi_0(z)dz, & \text{Re}\zeta < 0, \end{cases}$$

where $\tau^A_{-\omega}$ and $\tau^A_\omega$ are shown in the pictures below.




Define, for $\varphi \in \Phi\left(S^o_\omega\right)$, $b \in H^\infty\left(S^o_\omega\right)$, that

$$\mathscr{F}\varphi(\zeta) = (\mathscr{F}_0\varphi_0)(\zeta) + 2\pi i\zeta(\mathscr{G}\varphi_1)(\zeta), \quad \zeta \in S^o_\mu, \quad 0 < \mu < \omega,$$
$$\mathscr{G}b(z) = (\mathscr{G}b_0)(z) + (\mathscr{F}_0 b_1)'(z), \quad z \in S^o_\mu, \quad 0 < \mu < \omega.$$

we will show that

$$\hat{\mathscr{F}}\mathscr{G} = id_{H^\infty(S^o_\omega)}, \tag{11}$$

$$\mathscr{G}\hat{\mathscr{F}} = id_{\Phi(S^o_\omega)} \tag{12}$$

$$\hat{\mathscr{G}} = \mathscr{G}. \tag{13}$$

And these result that $\mathscr{G}:\bigcup_{0<\mu<\omega} H^\infty\left(S^o_\mu\right) \to \bigcup_{0<\mu<\omega} \Phi\left(S^o_\mu\right)$ is one to one and onto for every $\mu \in (0,\omega)$.

It is easy to check that $\hat{\mathscr{F}}$ and $\hat{\mathscr{G}}$ are "into" and

$$\mathscr{G}:\Phi_0\left(S^o_\omega\right) \to H^\infty_0\left(S^o_\mu\right),$$

(11) and (12) reduces to

$$\hat{\mathscr{F}}\mathscr{G}b = \mathscr{F}_0\mathscr{G}b_0 + 2\pi i\zeta(\mathscr{G}\mathscr{F}_0 b_1) = b_0 + 2\pi i\zeta b_1$$
$$\mathscr{G}\hat{\mathscr{F}}\varphi = \mathscr{G}\mathscr{F}_0\varphi_0 + (\mathscr{F}_0\mathscr{G}\varphi_1)' = \varphi_0 + \varphi'_1,$$

therefore, it is sufficient to show that

$$1^\circ \quad \mathscr{G}\mathscr{F}_0\varphi_0 = \varphi_0, \quad \forall\, \varphi_0 \in \Phi_0\left(S^o_\omega\right), \quad 2^\circ \quad \mathscr{F}_0\mathscr{G}b_0 = b_0, \quad \forall\, b_0 \in H^\infty_0\left(S^o_\omega\right).$$

To prove $1^\circ$, for a fixed $x>0$, chose $A > x$, $0 < \psi < \omega$, we have

$$b_0^\pm(\zeta) = \begin{cases} \int_{\tau^A_{\pm\omega}} e^{-i\zeta z}\varphi_0(z)d\zeta, & \pm \mathrm{Re}\zeta > 0, \\ 0, & \pm \mathrm{Re}\zeta < 0, \end{cases}$$

and, by the definition of $\mathscr{G}$

$$\mathscr{G}b_+(x) = \frac{1}{2\pi}\int_{\rho_\psi} e^{ix\zeta} b_0^+(\zeta)d\zeta,$$

$$\mathscr{G}b_-(x) = \frac{1}{2\pi}\int_{-\rho_{\psi+\pi}} e^{ix\zeta} b_0^-(\zeta)d\zeta.$$

Since there exists

$$\int_{\tau^{A,\delta}_{\pm\omega}} |e^{-i\zeta z}\varphi_0(z)||dz| \leqslant CA|\zeta|, \qquad 0 < \zeta < A,$$

where $\tau^{A,\delta}_{\pm\omega} = \tau^A_{\pm\omega}\backslash(-\delta,\delta)$, we can change the order of taking integrals and take limit $\delta \to 0$ in the below:

$$\mathscr{G}\mathscr{F}_0\varphi_0(x) = \frac{1}{2\pi}\int_{\rho_\psi} e^{ix\zeta}d\zeta \,\mathrm{p.v}\int_{\tau^A_{-\omega}} e^{-i\zeta z}\varphi_0(z)dz + \frac{1}{2\pi}\int_{-\rho_{\psi+\pi}} e^{ix\zeta}d\zeta \,\mathrm{p.v.}\int_{\tau^A_\omega} e^{-i\zeta z}\varphi_0(z)dz$$

$$= \frac{1}{2\pi}\int_{\rho_\psi} e^{ix\zeta}d\zeta \ \lim_{\delta\to 0}\int_{\tau_{-\omega}^{A,\delta}} e^{-i\zeta z}\gamma_0(z)dz + \frac{1}{2\pi}\int_{\rho_{\psi+\pi}} e^{ix\zeta}d\zeta\lim_{\delta\to 0}\int_{\tau_{\omega}^{A,\delta}} e^{-i\zeta z}\varphi_0(z)dz$$

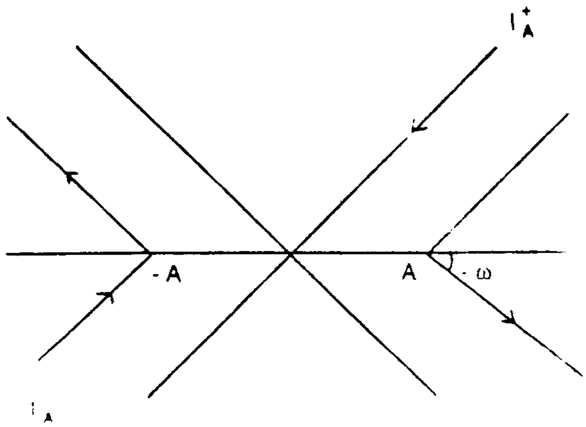
$$= \lim_{\delta\to 0}\frac{1}{2\pi}\left(\int_{\rho_\psi} e^{ix\zeta}d\zeta\int_{\tau_{-\omega}^{A,\delta}} e^{-i\zeta z}\varphi_0(z)dz + \int_{-\rho_{\pi+\psi}} e^{ix\zeta}d\zeta\int_{\tau_{\omega}^{A,\delta}} e^{-i\zeta z}\varphi_0(z)dz\right).$$

$$= \lim_{\delta\to 0}\frac{1}{2\pi i}\int_{\tau_{-\omega}^{A,\delta}\cup(-\tau_{\omega}^{A,\delta})} \varphi_0(z)\frac{1}{z-x}dz$$

$$= \lim_{\delta\to 0}\left(\frac{1}{2\pi i}\int_{l_A^+}\frac{\varphi_0(z)}{z-x}dz + \frac{1}{2\pi i}\int_{l_A^-}\frac{\varphi_0(z)}{z-x}dz\right)$$

$$= \varphi_0(x),$$

where contours $l_A^+, l_A^-$ are as shown in the picture.




Since $\varphi_0$ is holomorphic in $S^o_{\omega,+}$ and $x>0$ is arbitrary, we conclude $\mathscr{G}\mathscr{F}_0 = id$.

Now we prove 2°. Let $b_0\in H_0^\infty\left(S^o_\omega\right)$ and $b_0 = b_0^+ + b_0^-$ where $b_0^\pm = \chi_{\{\pm \mathrm{Re} z>0\}}\cdot b_0$. By the definition of $\mathscr{G}$

$$\varphi_0 = \mathscr{G}b_0 = \mathscr{G}(b_0^+) + \mathscr{G}(b_0^-) = \varphi_0^+ + \varphi_0^-.$$

On the other hand, by the difinition of $\mathscr{F}_0$,

$$\mathscr{F}_0\mathscr{G}b_0 = \mathscr{F}_0\varphi_0 = b^+ + b^-. \tag{14}$$

For $\zeta:\mathrm{Re}\zeta>0$, we will prove:

$$b^{+}(\zeta)=\text{p.v.}\int_{\tau_{-\infty}^{A}} e^{-i\zeta z}\varphi_0(z)dz$$

$$\int_{\sigma_{\omega,+}^{A}} e^{-i\zeta z}\varphi_0^{+}(z)dz+\int_{\sigma_{\omega,-}^{A}} e^{-i\zeta z}\varphi_0^{-}(z)dz, \tag{15}$$

where

$$\sigma_{\omega,\pm}^{A}=\begin{cases} te^{-i\omega}, & t \text{ is from } A \text{ to } +\infty, \\ te^{i(\pi+\omega)}, & t \text{ is from } +\infty \text{ to } A, \\ Ae^{\pm it}, & t \text{ is from } \pi \text{ to } 0. \end{cases}$$

In fact

$$\text{p.v.}\int_{-A}^{A} e^{-i\zeta z}\varphi_0(z)dz$$

$$=\lim_{\varepsilon\to 0}\left(\int_{-A}^{-\varepsilon}+\int_{\varepsilon}^{A}\right)e^{-i\zeta z}(\varphi_0^{+}(z)+\varphi_0^{-}(z))dz$$

$$=\lim_{\varepsilon\to 0}\left[\left(\int_{-A}^{-\varepsilon}+\int_{\varepsilon}^{A}\right)e^{-i\zeta z}\varphi_0^{+}(z)dz+\left(\int_{-A}^{-\varepsilon}+\int_{\varepsilon}^{A}\right)e^{i\zeta z}\varphi_0^{-}(z)dz\right]$$

$$=\lim_{\varepsilon\to 0}\left[\left(\int_{S_\varepsilon^{+}}+\int_{-S_A^{+}}\right)e^{-i\zeta z}\varphi_0^{+}(z)dz+\left(\int_{-S_\varepsilon^{-}}+\int_{S_A^{-}}\right)e^{-i\zeta z}\varphi_0^{-}(z)dz\right],$$

where $S_r^{\pm}=\{z\in\mathbb{C}\mid z=re^{i\left(\theta+\frac{\pi}{2}\right)},\ \theta \text{ is from } \mp\frac{\pi}{2} \text{ to} \pm\frac{\pi}{2}\}$ and $-S_r^{\pm}=\{z\in\mathbb{C}\mid z=re^{i\left(\theta-\frac{\pi}{2}\right)},\ \theta \text{ is from } \mp\frac{\pi}{2} \text{ to} \pm\frac{\pi}{2}\}$.

Since $\varphi_0^{+}(z)=-\varphi_0^{-}(-z)$ for $z\in C_{\omega,+}^{0}$, we have

$$\int_{S_\varepsilon^{+}}\varphi_0^{+}(z)dz+\int_{-S_\varepsilon^{-}}\varphi_0^{-}(z)dz=0.$$

Therefore,

$$\left|\int_{S_\varepsilon^{+}} e^{-i\zeta z}\varphi_0^{+}(z)dz+\int_{-S_\varepsilon^{-}} e^{-i\zeta z}\varphi_0^{-}(z)dz\right|$$

$$\leqslant\left|\int_{S_\varepsilon^{+}}(e^{i\zeta z}-1)\varphi_0^{+}(z)dz\right|+\left|\int_{-S_\varepsilon^{-}}(e^{-i\zeta z}-1)\varphi_0^{-}(z)dz\right|$$

$$\leqslant C\varepsilon\to 0,\ \text{as } \varepsilon\to 0.$$

This concludes (15). Since we can transform $\sigma^{A}_{\omega,-}$ into below without changing the value of the integral

$$\int_{\sigma^{A}_{\omega,-}} e^{-i\zeta z}\varphi_0^-(z)dz$$

by the Cauchy theorem, it has to be zero and (15) gives

$$b^+(\zeta)=\int_{\sigma^{A}_{\omega,+}} e^{-i\zeta z}\varphi_0^+(z)dz=\mathscr{F}_1\mathscr{G}b_0^+(\zeta)=b_0^+(\zeta)$$

where we used $\mathscr{F}_1\mathscr{G}=id$ that is a consequence of $\mathscr{G}\mathscr{F}_1=id$ and $\mathscr{G}$ is one to one which are from the proof of Theorem 1.

Similarly we can prove $b^-=b_0^-$ and these conclude $\mathscr{F}_0\mathscr{G}=id$ from (14).

Now we prove (13). Let $b\in H^{\infty}(S^0_{\omega})$. If $b$ is odd, then $\tilde{\mathscr{G}}b=\mathscr{G}b$ from the definition of $\tilde{\mathscr{G}}$. Suppose $b(\zeta)=2\pi i\zeta b_1(\zeta)$ where $b_1\in\Phi_0(S^0_{\omega})$, then we have, for $z\in S^0_{\omega,+}$

$$\begin{aligned}\tilde{\mathscr{G}}b(z)&=(\mathscr{F}_0 b_1)'(z)=\left(\text{p.v.}\int_{\tau^{A}_{-\omega}} e^{-i\zeta z}b_1(\zeta)d\zeta\right)'\\&=\left(\text{p.v.}\int_{\tau^{A}_{\omega}} e^{i\zeta z}b_1(\zeta)d\zeta\right)'\\&=\int_{\tau^{A}_{\omega}} i\zeta e^{i\zeta z}b_1(\zeta)d\zeta.\end{aligned}$$

On the other hand, by definition, for suitable $\theta$, $\theta'$

$$\begin{aligned}\mathscr{G}b(z)&=\frac{1}{2\pi}\int_{\rho_\theta} e^{i\zeta z}2\pi i\zeta b_1(\zeta)d\zeta+\frac{1}{2\pi}\int_{-\rho_{\theta'}} e^{i\zeta z}2\pi i\zeta b_1(\zeta)d\zeta\\&=\int_{\rho_\theta\cup(-\rho_{\theta'})} e^{i\zeta z}i\zeta b_1(\zeta)d\zeta\\&=\int_{\tau^{A}_{\omega}} i\zeta e^{i\zeta z}b_1(\zeta)d\zeta\\&=\tilde{\mathscr{G}}b(z),\end{aligned}$$

by the Cauchy theorem.

It is similar in case of $z\in S^0_{\omega,-}$ and therefroe $\mathscr{G}b=\tilde{\mathscr{G}}$. Since $\mathscr{G}$and $\tilde{\mathscr{G}}$ are both linear, we conclude (13).

So far to complete the proof we need to show $\mathscr{G}_1 b = \varphi_1$. Now for every $b \in H^\infty(S^0_\omega)$ we have $\varphi = \mathscr{G}b = \varphi_0 + \varphi_1'$ where $\varphi_0$, $\varphi_1$ are odd, $\varphi_0 \in K(S^0_\mu)$, $\varphi_1 \in H^\infty(S^0_\mu)$ for any $\mu \in (0,\omega)$. So,

$$\begin{aligned}(\mathscr{G}_1 b)'(z) &= (\mathscr{G}_1 b^+)(z) + (\mathscr{G}_1 b^-)'(z)\\ &= \left(\int_{l^+(-z,z)} \mathscr{G}b^+(\zeta)d\zeta + \int_{l^-(-z,z)} \mathscr{G}b^-(\zeta)d\zeta\right)'\\ &= \varphi^+(z) + \varphi^+(-z) + \varphi^-(z) + \varphi^-(-z)\\ &= \varphi(z) + \varphi(-z),\\ &= 2\varphi'_1(z),\end{aligned}$$

where $l^\pm(-z,z)$ are two integral contours lying in $C^0_{\omega,\pm}$ respectively. Since both $\mathscr{G}_1 b$ and $\varphi_1$ ard odd, we conclude $\mathscr{G}_1 b = \varphi_1$ and finish the proof.

*Proof of Theorem* 3. "if" part: In this case $\varphi \in \Phi(S^0_\omega)$, then by Theorem 2 there exists $b \in H^\infty(S^0_\mu)$ so that (6) holds. We simply let $T = b(D_\gamma)$ which meets all the requirements of the theorem.

"only part": Since $T$ is $L^2(\gamma)$ bounded, the formula for $T$ can be extended to

$$Tf(z) = \int_\gamma \varphi(z-\zeta)f(\zeta)d\zeta$$

where $f = \chi_Q$, $Q$ is any finite interval on $\gamma$ and $z \in \bar{Q}$. Denote $\varphi_\varepsilon = \varphi\chi_{\{z\in\mathbf{C}\,||z|>\varepsilon\}}$, $T_\varepsilon f(z) = \int \varphi_\varepsilon(z-\zeta)f(\zeta)d\zeta$. A standard argument gives that $\|T_\varepsilon\|_{L^2(\gamma)\to L^2(\gamma)}$ is uniformly bounded, and Hölder's inequality implies that for any open interval $Q$ of $\gamma$

$$\int_Q |T_\varepsilon X_Q||d_\zeta| < C|Q| \tag{16}$$

uniformly in $\varepsilon$

Denote $\gamma_z = \gamma - z$, $z \in \gamma$, which is a Lipschitz curve passing the origin, $Q_{z,\eta} = \{\zeta \in \gamma_z \,||\zeta| < \eta\}$. Let $z_0 \in \gamma$ be fixed. For $z_1 \in Q_{z_0,\frac{\eta}{2}}$, we'll prove that

$$\left|T_\varepsilon \chi_{Q_{z_1,\eta}}(z_1) - \int_{\zeta\in\gamma_{z_1}\ \varepsilon<|\zeta|<\eta} \varphi(\zeta)d\zeta\right| \leqslant C < \infty \tag{17}$$

where $C$ is a constant independent of $\varepsilon$, $\eta$ and $z_1 \in Q_{z_0,\frac{\eta}{2}}$.

In fact, denote $\gamma^\pm(z_0,\eta)$ the right and the left end points of $Q_{z_0,\eta}$, we have

$$T_\varepsilon \chi_{Q_{z_0,\eta}}(z_1) - \int_{\zeta\in\gamma_{z_0},\ \varepsilon<|\zeta|<\eta} \varphi(\zeta)d\zeta$$

$$= \left\{ \int_{J_1} + \int_{J_2} \right\} \varphi(\zeta)d\zeta$$

$$J_1 = \left\{ \zeta\in\gamma_{z_1},\ \text{from } z_1+\gamma^+(z_0,\eta) \text{ to } z_1+\gamma^-(z_0,\eta),\ |\zeta|>\varepsilon \right\}$$

$$J_2 = \left\{ \zeta\in\gamma_{z_0},\ \text{from } \gamma^-(z_0,\eta) \text{ to } \gamma^+(z_0,\eta),\ |\zeta|>\varepsilon \right\}.$$

By using the Cauchy Theorem we can reduce the above integral to an integral along circles of radius $\eta$ and $\varepsilon$ and along the directions of radius within $\{ z\in\mathbb{C} \mid \frac{\eta}{2} \leqslant |z| \leqslant \frac{3}{2}\eta\}$. Then from condition $|\varphi(z)| \leqslant \frac{C}{|z|}$ we can conclude (17).

From (17) we have

$$\left| \int_{\zeta\in\gamma_{z_0},\ \varepsilon<|\zeta|<\eta} \varphi(\zeta)d\zeta \right| \leqslant C + \left| + T_\varepsilon \chi_{Q_{z_0,\eta}}(z_1) \right|.$$

Taking average to both sides of this inequality w. r. t. $z_1\in Q_{z_0,\frac{\eta}{2}}$ and using (16) we conclude

$$\left| \int_{\zeta\in\gamma_{z_0},\ \varepsilon<|\zeta|<\eta} \varphi(\zeta)d\zeta \right| \leqslant C \tag{18}$$

for any $0<\varepsilon<\eta<\infty$. From the Cauchy theorem, condition $|\varphi(z)| \leqslant \frac{c}{|z|}$ and inequality (18) we have

$$\left| \int_{l^-(z_1^-,z_2^-)} + \int_{l^+(z_1^+,z_2^+)} \varphi(\zeta)d\zeta \right| \leqslant C,$$

where $z_1^\pm, z_2^\pm \in S^0_{\omega,\pm}$, $l(z_1^-,z_2^-)$ is a contour lying in $S^0_{\omega,-}$ from $z_1^-$ to $z_2^-$, $l^+(z_1^+,z_2^+)$ is a contour lying in $S^0_{\omega,+}$ from $z_2^+$ to $z_1^+$, and $|z_1^-|=|z_1^-|,|z_2^-|=|z_2^+|$. We let

$$\varphi_1(z) = \frac{1}{2}\left( \int_{l^-(-1,\mp z)} \varphi(\zeta)d\zeta + \int_{l^+(+1,\pm z)} \varphi(\zeta)d\zeta \right)$$

for $z\in S^0_{\omega,\pm}$. Now it is easy to check that $\varphi_1 \in H^\infty(S^0_\omega)$ and $\varphi'_1 = \frac{1}{2}(\varphi(z)+\varphi(-z))$ for $z\in S^0_\omega$.

## References

[1] Coifman, R. et Meyer, Y., Fourier Analysis of Multilinear Convolution, Calderón's Theorem and Analysis on Lipschitz Curves, Springer–Verlag, Lecture Notes in Maths. 779, 104–122.

[2] McIntosh, A., Operators which have an $H_\infty$–Functional Calculus, Miniconference on Operator Theory and Partial Differential Equations, 1986. Proceedings of the Centre for Mathematical Analysis, ANU, Canberra, 14(1986).

[3] McIntosh, A. and Qian, T., Fourier Multipiers on Lipschitz Curves, to appear.

[4] McIntosh, A. and Qian, T., Convolution Singular Integrals on Lipschitz Curves, to appear in Springer–Verlag Lecture Notes in Math.

A. McIntosh
Macquarie University
Australia

Qian
Acad mia Sinica,
China