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CONFORMAL TRANSFORMATIONS AND HARDY SPACES ARISING IN CLIFFORD ANALYSIS

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ABSTRACT. Vahlen matrices are used to study the conformal covariance of various types of Hardy spaces over surfaces in euclidean space. These are Hilbert spaces associated to monogenic function spaces. Particular emphesis is made for the cases where the surfaces bound unbounded domains, particularly Lipschitz domains. An anlaogous study is made of Hardy type spaces over special types of surfaces lying in the conformal closure of $\mathbb{C}^n$.

KEYWORDS: *Hardy spaces over surfaces, monogenic function spaces, Lipschitz domains, Vahlen matrices, conformal covariance.*

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INTRODUCTION

Clifford analysis has its roots in various attempts to generalize one variable complex analysis. This can be seen in the early work of Dixon ([10]), the later independant works of Fueter ([11]), Moisil and Teodorescu ([22]), and others. Many of the basic results can be found in the text of Brackx, Delanghe and Sommen ([5]).

More recently strong, and unexpected links have been discovered between this subject, classical harmonic analysis and several complex variables; see for instance [3], [9], [12], [17], [18], [20], [21], 22], [30], [35]. Also, quite considerable work has been done recently on the links between Clifford analysis and the conformal group, see for instance [4], [8], [16], [23], [28], [29]. This arose from the rediscovery and development by Ahlfors ([1]) in the 1980's of long forgotten work from the turn of the century by Vahlen ([36]). In that work it is shown that arbitrary Moebius transformations over Euclidean space can be expressed in much the same form as Moebius transformations in the complex plane, now using matrices of 2×2 matrices with Clifford algebra valued coefficents, satisfying some constraints.

This paper will build on the general theme developed in [8], [23], [28], [29]. The main thrust will be to show that some of the hard worked for results obtained over Lipschitz graphs and domains in [17], [18] may be easily and elagantly transformed via Vahlen matrices to conformally equivalent surfaces and domains. A particular transformation to bear in mind here is the case where the Moebius transformation is the Cayley transformation. In this case Lipschitz graphs are transformed to perturbations of the sphere. It should be noted that the type of monogenic functions defined on sector domains, and used in [17], [18] satisfy a basic inequality which makes them particularly suitable for study under conformal transformations.

To set the stage we begin by looking at the interplay between Vahlen matrices, and kernels acting over the L^2 spaces of sufficently smooth surfaces. We briefly illustrate how most basic properties of these Hilbert spaces carry over isometrically from one surface to any other surface which is conformally equivalent to that surface. In particular we are able to construct an orthonormal basis of functions on $\mathbf{R}^{n-1}$ which extend to monogenic functions on upper and lower half space in $\mathbf{R}^n$. This gives rise to an alternative proof of the L^2 boundedness of the singular Cauchy transform over $\mathbf{R}^{n-1}$ which bypasses the use of the Fourier transform. It also allows us to give an alternative description to that given in [20] of the orthogonal decomposition of $L^2(\mathbf{R}^{n-1})$ into appropriate Hardy spaces. In the last section of the paper we turn to a slightly different topic, and show the conformal invariance of special of manifolds lying in $\mathbf{C}^n$. These real n-dimensional manifolds are described in [29], and references therein. In [29] we show that these manifolds are invariant under certain types of conformal transformations. There we were forced to introduce certain artifical constraints. We present here a simple argument to overcome this problem.

1. PRELIMINARIES

In this section we shall introduce some neccessary background material on Clifford algebras and the conformal group. We shall consider the real 2^n dimensional Clifford algebra A_n generated from $\mathbf{R}^n$ equipped with a negative definite inner product. We shall assume that $\mathbf{R}^n \subset A_n$. So that for each vector $x \in \mathbf{R}^n$ we have that $x^2 = -\|x\|^2$. Consequently, if $e_1, \ldots, e_n$ is an orthonormal basis for $\mathbf{R}^n$, then we have the anti-commutation relationship $e_ie_j + e_je_i = -2\delta_{ij}$,where δ_{ij} is the Kroneker delta function. An extremely important point here for Clifford analysis is that each non-zero vector $x \in \mathbf{R}^n$ is invertible in the algebra, with inverse x^{-1} given by $-\frac{x}{\|x\|^2}$. Up to the minus sign this corresponds to the Kelvin inverse of a vector in euclidean space.

The reason for choosing the Clifford algebra with a negative definite inner product rather than the Clifford algebra with a positive definite inner product is so that we can facilitate the introduction of the covering group, $\mathrm{Pin}(n)$, of the orthogonal group $O(n)$. It may be observed that the algebra A_n has as basis elements the vectors $1, e_1, \ldots, e_n, \ldots, e_{j_1} \ldots e_{j_r}, \ldots, e_1 \ldots e_n$, where $j_1 < \cdots < j_r$ and $1 \leqslant r \leqslant n$. Consequently, $A_1 = \mathbb{C}$, the complex field and $A_2 = H$, the quaternionic division algebra. When $n = 3$ we have that $E_+ = \frac{1}{2}(1 + e_1e_2e_3)$ and $E_- = \frac{1}{2}(1 - e_1e_2e_3) \in A_3$. Moreover, $E_+E_- = E_-E_+ = 0$. So for $n \geqslant 3$ the algebra A_n is no longer a division algebra. Many basic properties of Clifford algebras may be found in the paper of Atiyah, Bott and Shapiro ([2]), the books of Harvey and Porteous ([13], [25]), and other basic references.

Suppose now that $y \in \mathbb{S}^{n-1} \subset \mathbf{R}^n$, then the action $y\mathbf{R}^n y$ describes a reflection on euclidean n-space in the direction of y. Inductively, for $y_1, \ldots, y_p \in \mathbb{S}_{n-1}$ and each $p \in \mathbf{N}$ we have that $a\mathbf{R}^n\widetilde{a}$ describes an orthogonal transformation over $\mathbf{R}^n$, where $a = y_1 \cdots y_p$ and $\widetilde{a} = y_p \cdots y_1$. In fact this gives rise to the Lie group $\mathrm{Pin}(n) = \{a \in A_n : a = y_1 \cdots y_p, p \in \mathbf{N} \text{ and } y_1, \ldots, y_p \in \mathbb{S}^{n-1}\}$. It can be fairly easily deduced that $\mathrm{Pin}(n)$ is a double covering of the orthogonal group $O(n)$, see for instance ([2]). When $\mathrm{Pin}(n)$ is restricted to the even subalgebra of A_n we obtain the spin group $\mathrm{Spin}(n)$, which in turn is a double covering of the special orthogonal group $SO(n)$. It should be noted that in describing the reflection of euclidean space in the direction of the vector y, that it was here that we specifically used the negative definite inner product on euclidean space.

The transformation $\sim$ on $\mathrm{Pin}(n)$ extends to an anti-automorphism $\sim: A_n \to A_n : e_{j_1} \cdots e_{j_r} \to e_{j_r} \cdots e_{j_1}$. For an element $A \in A_n$ we shall write $\widetilde{A}$ instead of $\sim (A)$.

Also, we have the anti-automorphism

$$- : A_n \to A_n : e_{j_1} \cdots e_{j_r} \to (-1)^n e_{j_r} \cdots e_{j_1}.$$

Again we write $\overline{A}$ for $-(A)$. This antiautomorphism is a generalization of complex conjugation. In fact the real part of $\overline{A}A$ gives $\|A\|^2 = a_0^2 + \cdots + a_{1\ldots n}^2$, where $A = a_0 + \cdots + a_{1\ldots n}e_1 \cdots e_n$, with $a_0, \ldots, a_{1\ldots n} \in \mathbf{R}$.

We now turn to introduce Vahlen matrices. These matrices can be used to facilitate the study of Moebius transformations over the one point compactification of euclidean space. Moebius transformations are the transformations belonging to the group of diffeomorphisms over $\mathbf{R}^n \cup \{\infty\}$ generated by orthogonal transformations, dilations, and translations over $\mathbf{R}^n$ and Kelvin inversion. Any such transformation can be expressed as $(ax+b)(cx+d)^{-1}$, where $a, b, c, d \in A_n$ and satisfy the following constraints:

(i) a, b, c, d are all products of vectors;

(ii) $a\tilde{c}, c\tilde{d}, d\tilde{b}$ and $d\tilde{a} \in \mathbf{R}^n$;

(iii) $a\tilde{d} - b\tilde{c} = \pm 1$.

Each 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with coefficents a, b, c, and d belonging to A_n and satisfying (i)–(iii) is refered to as a Vahlen matrix. They were introduced by Vahlen in [36], reintroduced by Maass in [19] and consequently rediscovered by Ahlofors ([1]) in the 1980's. Since that time they have been used by many authors, see for instance references given in the introduction.

The set of Vahlen matrices, $V(n)$, forms a group under matrix multiplication. This group is a covering group of the group of Moebius transformations over the one point compactification of euclidean space. Examples of Vahlen matrices include $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} a & 0 \\ 0 & \tilde{a}^{-1} \end{pmatrix}$, where $a \in \mathrm{Pin}(n)$, $\begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}$, where $v \in \mathbf{R}^n$, and $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, where $\lambda \in \mathbf{R}^+$. These matrices are generators of the group $V(n)$. They correspond respectively to Kelvin inversion, orthogonal transformations, translation and dilation.

2. HILBERT SPACES ON SURFACES

DEFINITION 2.1. A topological manifold is said to be *locally Lipschitz* if it has an atlas consisting of Lipschitz continuous charts.

It should be noted that each $\mathbf{C}^1$, $\mathbf{C}^r$, $\mathbf{C}^\infty$ or $\mathbf{C}^\omega$ manifold is an example of such a manifold,where here $1 < r < \infty$.

DEFINITION 2.2. We shall assume here that a *surface* is a real $(n-1)$ dimensional, locally Lipschitz, orientable manifold locally embedded, but possibly globally immersed, in $\mathbf{S}^n = \mathbf{R}^n \cup \{\infty\}$.

This definition is slightly more general than the usal definition of a surface. For instance, it allows self intersections. We now state what is meant for surfaces to be conformally equivalent.

DEFINITION 2.3. Two surfaces S_1 and S_2 are said to be *conformally equivalent* if there is a Moebius transformation φ over $\mathbb{S}^n$ such that $\varphi(S_1) = S_2$.

Although we have defined surfaces and the conformal equivalence of surfaces over $\mathbb{S}^n$, we shall be working with these structures primarily within $\mathbf{R}^n$.

Given a surface S lying in $\mathbf{R}^n$ we shall denote the space of square integrable functions $f : S \to A_n$ by $L^2(S, A_n)$. For each $f, g \in L^2(S, A_n)$ we have that the integral $\int_S f(x)\overline{g(x)}\,\mathrm{d}\sigma(x)$ is well defined, where σ denotes the usual Lebesgue measure on S. The real part of this integral gives the inner product of f and g over S. By decomposing the Moebius transformation $y = \varphi(x) = (ax+b)(cx+d)^{-1}$ into its generators we may, following [8], obtain:

LEMMA 2.4. *Suppose that $\varphi(S_1) = S_2$ then for each $f, g \in L^2(S_2, A_n)$ we have that*

$$\int_{S_2} \overline{f}(y)g(y)\,\mathrm{d}\sigma(y) = \pm \int_{S_1} \overline{f}(\varphi(x))g(\varphi(x)) \parallel cx + d \parallel^{-2n+2} \mathrm{d}\sigma(x).$$

The minus sign in the previous formula arises when the diffeomorphism φ is orientation reversing, for instance, when φ is a reflection.

It is straightforward to see that $\|cx+d\|^{-2n+2} = \overline{J}(\varphi, x)J(\varphi, x)$, where $J(\varphi, x) = \frac{\widetilde{cx+d}}{\|cx+d\|^n}$. It follows that if $f(y) \in L^2(S_2, A_n)$, then $J(\varphi, x)f(\varphi(x)) \in L^2(S_1, A_n)$. It is now a simple matter to deduce

PROPOSITION 2.5. *The map*

$$\varphi : L^2(S_2, A_n) \to L^2(S_1, A_n);\ \varphi(f) = J(\varphi, x)f(\varphi(x))$$

is an isommetry.

From this isomorphism we also obtain:

PROPOSITION 2.6. *If $H(S_2)$ is a Hilbert subspace of $L^2(S_2, A_n)$, then $\varphi(H(S_2))$ is a Hilbert subspace of $L^2(S_1, A_n)$.*

We shall use $H(S_1)$ to denote the space $\varphi(H(S_2))$. Suppose now that T_1 and T_2 are surfaces. Moreover, $\varphi(T_1) = T_2$. Then it may be observed that for each linear operator $P : H(S_2) \to H(T_2)$ there corresponds a linear operator $Q : H(S_1) \to H(T_1)$ such that $Q(J(\varphi, x)f(\varphi(x))) = J(\varphi, x)g(\varphi(x))$ whenever $P(f(y)) = g(y)$. Here $H(T_1)$ and $H(T_2)$ are Hilbert subspaces of $L^2(T_1, A_n)$ and $L^2(T_2, A_n)$ respectively. It may be observed that if P is bounded then so is Q. Also, if P is compact then so is Q, and if P is a Fredholm operator then again so is Q. In this last case it can be seen that the dimension of the kernel of P is the same as the dimension of the kernel of Q. In fact for each linear operator P we have that $f(y) \in \ker(P)$ if and only if $J(\varphi, x)f(\varphi(x)) \in \ker(Q)$. All of these observations follow straightforwardly from Proposition 2.5. Moreover, it easily follows from our constructions that if P can be represented by a kernel $K(y, u)$ then Q is represented by the kernel $J(\varphi, x)K(\varphi(x), \varphi(v))\bar{J}(\varphi, v)$, where $\varphi(v) = u$ with $u \in T_2$ and $v \in T_1$.

So far we have shown how Clifford algebras can be used to easily describe relationships between Hilbert spaces over conformally equivalent surfaces. It can be argued that instead of using the function $J(\varphi, x)$ one could avoid the use of Clifford algebras and simply use the function $\|cx + d\|^{-n+1}$. However, this does not permit one to access the important link with spaces of monogenic functions, and particular classes of H^p spaces.

DEFINITION 2.7. Given a domain U lying in $\mathbf{R}^n$, then a differentiable function $f : U \to A_n$ is called *left monogenic* if it satisfies the equation $Df = 0$, where D is the Dirac operator $\sum_{i=1}^{n} e_i \frac{\partial}{\partial x_i}$.

A similar definition can be given for right monogenic functions. Collectively such functions are refered to as monogenic functions, though also the terms left or right regular functions, or Clifford holomorphic functions are used to refer to the same class of functions. The study of the properties and applications of these functions and related functions is refered to as Clifford analysis. This analysis has been pursued by a number of authors, see for instance some of the references referred to in the introduction, and in other parts of the text. In particular, we have the following version of Cauchy theorem and Cauchy integral formula:

THEOREM 2.8. *Suppose that $f, g : U \to A_n$ are respectively left and right monogenic functions, and $S \subseteq U$ is a compact surface bounding a domain in U. Then $\int_S g(x)n(x)f(x)\, d\sigma(x) = 0$, where $n(x)$ is the outward pointing normal vector to S at x.*

THEOREM 2.9. *Suppose now that f is as in Theorem 2.8 and u belongs to the domain bounded by S. Then $f(u) = \int_S G(x-u)n(x)f(x)\, d\sigma(x)$, where $G(x) = \frac{1}{\omega_n}\frac{x}{\|x\|^{n-1}}$, and ω_n is the surface area of the unit sphere in $\mathbf{R}^n$.*

The previous two theorems are basic to Clifford analysis, and can be found in many places in the literature. The function $G(x)$ is both left and right monogenic, and is often refered to as the Cauchy-Clifford kernel.

We now begin to explore the relationship between function spaces defined on surfaces, and special classes of monogenic functions. The results presented here follow similar lines to analogous arguments in the complex plane. We begin with:

DEFINITION 2.10. Suppose S is a surface which bounds a domain U. A homotopy $H : S \times [0,1] \to U \cup S$ is called a *homotopy of Hardy type* if:

(i) $H(S,t)$ is a surface for each $t \in [0,1]$;

(ii) $H(S,0) = S$;

(iii) for each $t \in (0,1]$ the set $H(S,t) \cap S$, regarded as a subset of S, is a set of measure zero;

(iv) the function H is Lipschitz continuous.

We shall denote the space of A_n valued L^p functions on S by $L^p(S, A_n)$, where $1 \leqslant p \leqslant \infty$. Using Theorems 2.8 and 2.9 and the previous definition we may deduce:

THEOREM 2.11. *Suppose S is a compact surface bounding a bounded domain U, and $f : U \cup S \to A_n$ is such that:*

(i) *$f|U$ is left mongenic;*

(ii) *$f|S \in L^p(S, A_n)$;*

(iii) *the function $F : [0,1] \to L^p(S, A_n) : F(t) = f(H(,t))$ is continuous. Then for each $u \in U$ we have $\int_S G(x-u)n(x)f(x)\,\mathrm{d}\sigma(x) = f(u)$.*

Proof (Outline). For each $u \in U$ there exists $T(u) \in (0,1]$ such that for each $t \in [0, T(u)]$ the vector u does not lie in $H(S, [0, T(u)])$. Consequently, for each $t \in (0, T(u)]$ we have that $f(u) = \int_{H(S,t)} G(x-u)n(x)f(x)\,\mathrm{d}\sigma(x)$. The result now follows from the completeness of the normed space $L^2(S, A_n)$, and the continuity of F. ∎

From the completeness of the normed space $L^2(S, A_n)$ we get:

PROPOSITION 2.12. *Suppose the compact surface S bounds the bounded domain U, and we have a homotopy, H, of Hardy type. Then space of all functions, f, satisfying conditions* (i)–(iii) *in the previous theorem is a complete right A_n module.*

A good example to consider is the case where $S = \mathbb{S}^{n-1}$ and $H(S,t) = \mathbb{S}^{n-1}(1-t)$, where $\mathbb{S}^{n-1}(1-t)$ is the sphere in $\mathbf{R}^n$, centred at 0 and of radius $1-t$. In this case we obtain the Hardy spaces considered in [7].

Given a bounded surface S, bounding a domain U, and any pair of homotopies H_1 and H_2 from $S \times [0,1]$ to $U \cup S$ it can be deduced that there is a Lipschitz continuous homotopy $H_{1,2} : S \times [0,1]^2 \to U \cup S$ such that

(i) $H_{1,2}(S,t,0) = H_1(S,t)$

(ii) $H_{1,2}(S,t,1) = H_2(S,t)$; and

(iii) $H_{1,2} : S \times [0,1] \times \{s\} \to U \cup S$ is a homotopy of Hardy type for each $s \in [0,1]$.

It follows that if the surface S bounds a bounded domain U, and $f : U \cup S \to A_n$ satisfies conditions (i)–(iii) from Theorem 2.11 for a homotopy H_1 of Hardy type, then the function $F : [0,1] \to L^p(S, A_n)$ introduced in Theorem 2.11 extends to a continuous function, $F_{1,2}$ from $(0,1] \times [0,1]$ to $L^p(S, A_n)$, for each homotopy $H_{1,2}$ described in the previous paragraph. Moreover, as F is continuous on $[0,1]$ it follows that $F_{1,2}$ extends continuously to $[0,1] \times [0,1]$. Furthermore, $F_{1,2}(0,s) = f|S$ for each $s \in [0,1]$. Consequently, the integral given in Theorem 2.11 is independent of the particular choice of homotopy of Hardy type, and also the completeness result given in the previous proposition is also independent of the choice of homotopy of Hardy type. This prompts us to make the following definition.

DEFINITION 2.13. Suppose S is a surface bounding a bounded domain U, then the right A_n module of functions $f : U \times S \to A_n$ satisfying conditions (i)–(iii) from Theorem 2.11 for some homotopy of Hardy type, and for some $p \in [0, \infty]$, is called a *Hardy space for the domain* U, and it is denoted by $H^p(U)$.

The space $H^p(U)$ corresponds to the complete space described in Proposition 2.12.

We shall now turn to look more closely at the special case $p = 2$. First we need the following two results from [29].

PROPOSITION 2.14. *Suppose* $f, g \in L^2(S_2, A_n)$, *then*

$$\int_{S_2} f(y)n(y)g(y)\, d\sigma(y) = \int_{S_1} f(\varphi(x))\widetilde{J}(\varphi, x)n(x)J(\varphi, x)g(\varphi(x))\, d\sigma(x).$$

THEOREM 2.15. *Suppose that* U *and* V *are domains and* $\varphi(U) = V$. *Then the function* $f(y)$ *is left monogenic on the domain* V *if and only if the function* $J(\varphi, x)f(\varphi(x))$ *is left monogenic on the domain* U.

The previous theorem was first proved in the quaternionic case by Sudbery, [34]. The theorem is proved via Cauchy theorem and breaking the Moebius transformation $\varphi(x) = (ax+b)(cx+d)^{-1}$ into its generators, and establishing the result for each generator.

Using a combination of translation and Kelvin inversion, or the Cayley transform, it may be seen that each unbounded domain whose complement contains a non empty open set can be conformally transformed to a bounded domain. Also, following [23] we have that

$$G(\varphi(x) - \varphi(u)) = J(\varphi, x)^{-1}G(u - x)\widetilde{J}(\varphi, u)^{-1}. \tag{2.1}$$

This formula is again obtained by splitting φ up into a composition of generators, and noting by that $x^{-1} - u^{-1} = u^{-1}(u - x)x^{-1}$.

Combining equation (2.1), Proposition 2.14 and Theorem 2.15 it now follows that we may obtain, for the case $p = 2$, the following generalizations of Theorem 2.11 and Proposition 2.12.

THEOREM 2.16. *Suppose that S is a surface which bounds a domain U, suppose also that $f : U \cup S \to A_n$, and for the special case $p = 2$, f satisfies* (i)–(iii) *from Theorem* 2.11. *Then for each $u \in U$ we have that $f(u) = \int_S G(x - u)n(x)f(x)\,\mathrm{d}\sigma(x)$.*

Using a conformal transformation of S to a bounded surface bounding a bounded domain, and the arguments following Theorem 2.11, it may be deduced that the integral formula given in Theorem 2.15 is independent of the choice of homotopy of Hardy type. Consequently, we may introduce the following space.

DEFINITION 2.17. For S a surface which bounds a domain, the class of functions $f :\to U \cup S$ satisfying the conditions layed out in Theorem 2.15 is called the *Hardy* 2 *space of* U, and it is denoted by $H^2(U)$.

By the same arguments to those used to deduce Proposition 2.12 we can automatically deduce:

PROPOSITION 2.18. *The right A_n module $H^2(U)$ is complete.*

For the moment we shall just work with the space $H^2(U)$. Of particular interest here is the case where U is a Lipschitz domain.

DEFINITION 2.19. Suppose that $\mathbf{R}^{n-1} = \{x_2e_2 + \cdots + x_ne_n : x_2, \ldots, x_n \in \mathbf{R}\}$ and $\eta : \mathbf{R}^{n-1} \to \mathbf{R}$ is a Lipschitz continuous function. Then the graph of $\eta = \{\eta(x)e_1 + x : x \in \mathbf{R}^{n-1}\}$ is called a *Lipschitz graph.* We denote it by Σ. Moreover, the domains $\Sigma^+ = \{x_1e_1 + x : x_1 > \eta(x)$ and $x \in \mathbf{R}^{n-1}\}$ and $\Sigma^- = \{x_1e_1 + x : x_1 < \eta(x)$ and $x \in \mathbf{R}^{n-1}\}$ are called *Lipschitz domains.*

The study of Hardy spaces over Lipschitz domains has been the focus of considerable attention in recent years, see for instance [3], [17], [18], [20], [21], [12]. Theorem 2.15 gives a Cauchy integral formula for each $f \in H^2(\Sigma^\pm)$ and with the integral taken over the surface Σ.

From the Cauchy integral formula given in Theorem 2.15 we immediately obtain the following decomposition theorem.

THEOREM 2.20. *Suppose that S is a surface bounding a domain U and $f \in H^2(U)$. Suppose also that $S = S_1 \cup \cdots \cup S_K$, where each S_j is a surface, and $S_i \cap S_j$, seen as a subset of either S_i or S_j is a set of measure zero. Then $f = f_1 + \cdots + f_K$, where $f_j \in H^2(U_j)$ for $1 \leqslant j \leqslant K$ and U_j is the maximal domain in $\mathbf{R}^n$ bounded by S_j and containing U.*

A special case of this theorem is used in [?] to set up three line theorems on the infinite strip $\{x_1e_1 + x_2e_2 + \cdots + x_ne_n : 0 \leqslant x_1 \leqslant 1, \text{ and } x_2, \ldots, x_n \in \mathbf{R}\}$. A different proof of this theorem is given in [?].

It follows from Propositions 2.5 and 2.6 and arguments presented after the statement of Proposition 2.6 that if S_1 and S_2 are conformally equivalent surfaces bounding domains U_1 and U_2 respectively, then the Hardy spaces $H^2(U_1)$ and $H^2(U_2)$ are isometric Hilbert spaces. It follows from Propositions 2.5 and 2.6 and the arguments following Proposition 2.6 that one can relate canonically operators acting over $H^2(U_1)$ to operators acting over $H^2(U_2)$. In particular in [8] it is shown that the reproducing kernels, or Szego kernels, for these spaces are canonically related. This argument is used to explicitly produce a Szego kernel, $S_{\mathbf{R}^{n,+}}(x,u)$, for the upper half space, $\mathbf{R}^{n,+} = \{x_1e_1 + x_2e_2 + \cdots + x_ne_n : x_1 > 0, \text{ and } x_2, \ldots, x_n \in \mathbf{R}\}$ and then taking $\frac{\partial}{\partial x_1} S_{\mathbf{R}^{n,+}}(x,u)$ to obtain the Bergman kernel for the space of square integrable left mongenic functions, with suitable decay at infinity, on upper half space.

Similarly $2B_{\Sigma^\pm}(x,u) = \frac{\partial}{\partial x_1} S_{\Sigma^\pm}(x,u)$, where $B_{\Sigma^\pm}(x,u)$ is the Bergman kernel for square integrable left monogenic functions with suitable decay at infinity, on the Lipschitz domain $\Sigma^\pm$, and $S_{\Sigma^\pm}(x,u)$ is the Szego kernel for $H^2(\Sigma^\pm)$.

So far we have only dealt with L^2 and H^2 spaces associated to unbounded domains via conformal transformations from bounded domains. The story for other choices of p is less successful. In these cases the L^p spaces conformally transform to proper subspaces of the corresponding L^p spaces. This is easy to see for the surfaces $\mathbf{S}^{n-1}$ and $\mathbf{R}^{n-1}$, which are conformally equivalent via the Cayley transform.

3. THE CAYLEY TRANSFORM

The Cayley transform is given by $(e_1x+1)(e_1x+1)^{-1}$. It transforms upper half space, $\mathbf{R}^{n,+}$, to the unit disc, $\mathbf{D}(0,1)=\{x\in\mathbf{R}^n : \|x\|<1\}$. It is generally well known that each $f\in L^2(\mathbf{S}^{n-1},A_n)$ can be expressed as a sequence $\sum_{i=0}^{\infty}h_i$ where each h_i is the restriction to h_i of a harmonic polynomial homogeneous of degree i. Now it is known that for each such h_i we have $h_i(x)=p_i(x)+xp_{i-1}(x)$ for $i>0$, where p_i and p_{i-1} are left monogenic polynomials homogeneous of degree i and $i-1$ respectively. The calculation of this fact is quite simple. Restricting to the sphere $\mathbf{S}^{n-1}$ we have, using Kelvin inversion, that $xp_{i-1}(x)=G(-x)p_{i-1}(-x^{-1})$. The function on the right side of this expression extends to a left monogenic function, homogeneous of degree $-n-i+2$ on $\mathbf{R}^n\setminus\{0\}$. This decomposition was first introduced in the quaternionic case by Sudbery ([34]), and later independently introduced in euclidean space by Sommen ([32]). Using this decompossition the following result automatically follows.

PROPOSITION 3.1. $L^2(\mathbf{S}^{n-1},A_n)=H^2(\mathbf{D})\oplus H^2(\mathbf{R}^n\setminus(\mathbf{D}\cup\mathbf{S}^{n-1}))$.

It follows from Cauchy theorem and the construction of this decomposition that the decomposition is orthogonal. This proposition is essentially contained in [32]. The construction of orthonormal bases for $H^2(\mathbf{D})$ in terms of left monogenic, homogeneous polynomials is given in [15], see also [26]. From the construction of the decomposition given in Proposition 3.7 we also may deduce:

LEMMA 3.2. *Suppose that* $\{\varphi_{i,m(i)}(y) : 0\leqslant i\leqslant\infty$, *and* $0\leqslant m(i)\leqslant M(i)$, *where* $M(i)$ *is the order of the collection of monomials in* $n-1$ *variables of degree* $i\}$ *is an orthonormal basis for* $H^2(\mathbf{D})$, *comprising of left monogenic polynomials homogeneous of degree i, with* $0\leqslant i\leqslant\infty$. *Then* $\{G(-y)\varphi_{i,m(i)}(-y^{-1}) : 0\leqslant i\leqslant\infty$ *and* $0\leqslant m(i)\leqslant M(i)\}$ *is an orthonormal basis for* $H^2(\mathbf{R}^n\setminus(\mathbf{D}\cup\mathbf{S}^{n-1}))$.

We automatically obtain from the previous proposition, the previous lemma, and the inverse of the Cayley transformation, φ:

THEOREM 3.3. $L^2(\mathbf{R}^{n-1},A_n)=H^2(\mathbf{R}^{n,+})\oplus H^2(\mathbf{R}^{n,-})$, *where* $\mathbf{R}^{n,-}=\{x_1e_1+x_2e_2+\cdots+x_ne_n : x_1<0$ *and* $x_2,\ldots,x_n\in\mathbf{R}\}$, *is lower half space. This decomposition is an orthogonal decomposition. Moreover,* $\{J(\varphi,x)\varphi_{i,m(i)}(\varphi(x)) : 0\leqslant i\leqslant\infty$, *and* $0\leqslant m(i)\leqslant M(i)\}$ *is an orthonormal basis for* $H^2(\mathbf{R}^{n,+})$, *while* $\{J(\varphi,x)G(-\varphi(x))\varphi_{i,m(i)}(\varphi(x)^{-1}) : 0\leqslant i\leqslant\infty$ *and* $1\leqslant m(i)\leqslant M(i)\}$ *is an orthonormal basis for* $H^2(\mathbf{R}^{n,-})$.

This orthogonal decomposition for $L^2(\mathbf{R}^{n-1},A_n)$ was first obtained using a mixture of Clifford analysis and Fourier analysis by McIntosh in [20].

Suppose that $\lambda : [0,1] \to \mathbf{D} \cup \mathbf{S}^{n-1}$ is a piecewise C^1 function satisfying:
(i) $\lambda((0,1]) \subset D$;
(ii) $\lambda(1) = u \in \mathbf{S}^{n-1}$;
(iii) λ has a right derivative at 1 which is not tangental to $\mathbf{S}^{n-1}$.
Then from simple continuity and Cauchy integral formula we have

$$\lim_{t\to 0} \int_{\mathbf{S}^{n-1}} G(y-\lambda(t))n(y)\varphi_{i,m(i)}(y)\,d\sigma(y) = \varphi_{i,m(i)}(u)$$

for each basis function $\varphi_{i,m(i)}(y)$ appearing in the basis given in Lemma 3.2. As $\mathbf{S}^{n-1}$ is sufficently smooth, and as $\varphi_{i,m(i)}(y)$ is a real analytic function it follows from the Plemelj formulae given in [14] that

$$\lim_{t\to 0} \int_{\mathbf{S}^{n-1}} G(y-\lambda(t))n(y)\varphi_{i,m(i)}(y)\,d\sigma(y)$$
$$= \frac{1}{2}\varphi_{i,m(i)}(u) + \text{P.V.} \int_{\mathbf{S}^{n-1}} G(y-u)n(y)\varphi_{i,m(i)}(y)\,d\sigma(y).$$

Consequently, P.V. $\int_{\mathbf{S}^{n-1}} G(y-u)n(y)\varphi_{i,m(i)}(y)\,d\sigma(y) = \frac{1}{2}\varphi_{i,m(i)}(u)$.

By similar arguments, and also by being slightly more careful with the orientation of $\mathbf{S}^{n-1}$ we may deduce that

$$\text{P.V.} \int_{\mathbf{S}^{n-1}} G(y-u)n(y)G(-y)\varphi_{i,m(i)}(-y^{-1})\,d\sigma(y) = -\frac{1}{2}G(-u)\varphi_{i,m(i)}(-u^{-1}).$$

Consequently, we have:

THEOREM 3.4. *The singular integral operator*

$$T_G : L^2(\mathbf{S}^{n-1}, A_n) \to L^2(\mathbf{S}^{n-1}, A_n),\ T_G(f) = \text{P.V.} \int_{\mathbf{S}^{n-1}} G(y-u)n(y)f(y)\,d\sigma(y)$$

is well defined and has the spectral decompositions

$$T_G : H^2(\mathbf{D}) \to H^2(\mathbf{D}),\ T_G(f) = \frac{1}{2}f$$

and

$$T_G : H^2(\mathbf{R}^n \setminus (\mathbf{D} \cup \mathbf{S}^{n-1})) \to H^2(\mathbf{R}^n \setminus (\mathbf{D} \cup \mathbf{S}^{n-1}));\ T_G(f) = -\frac{1}{2}f.$$

The L^2 boundedness of the operator T_G over the sphere has previously been described in [31]. As shown in [29] the singular integral T_G conformally transforms to itself, so we immediately obtain via the Cayley transform and our previous results:

THEOREM 3.5. *The singular integral operator*

$$T_G : L^2(\mathbf{R}^{n-1}, A_n) \to L^2(\mathbf{R}^{n-1}, A_n)$$

$$T_G(f) = \text{P.V.} \int_{\mathbf{R}^{n-1}} G(x-v)e_1 f(x)\, dx^{n-1}$$

is well defined and has the spectral decompositions

$$T_G : H^2(\mathbf{R}^{n,+}) \to H^2(\mathbf{R}^{n,+}),\ T_G(f) = \frac{1}{2}f$$

and

$$T_G : H^2(\mathbf{R}^{n,-}) \to H^2(\mathbf{R}^{n,-}),\ T_G(f) = -\frac{1}{2}f.$$

The fact that the operator T_G is L^2 bounded on $\mathbf{R}^{n-1}$ is well known; see for instance [20]. What is different here is the use of the Cayley transform and the Hardy space decomposition of $L^2(\mathbb{S}_{n-1}, A_n)$ to obtain this result. Usually one notes that $T_G|\mathbf{R}^{n-1} = \sum_{i=1}^{n-1} R_i e_i$, where R_i is the i th Riesz potential on $\mathbf{R}^{n-1}$. Then one uses Fourier transform techniques as layed out in [33].

One also readily has from Theorems 3.4 and 3.5 that $4T_G^2 = \text{Id}$, the identity map. Here, of course, T_G is seen as acting over $L^2(\mathbb{S}^{n-1}, A_n)$ or $L^2(\mathbf{R}^{n-1}, A_n)$.

In [17] it is established that for each Lipschitz surface Σ we have that $L^2(\Sigma, A_n) = H^2(\Sigma^+) \oplus H^2(\Sigma^-)$. Using the Cayley transformation φ^{-1} the Lipschitz surface Σ is transformed to a surface $\varphi^{-1}(\Sigma) = \Pi$, while the domain Σ^+ is transformed to a bounded domain Π^+ and the domain Σ^- is transformed to an unbounded domain Π^-. Consequently:

THEOREM 3.6. $L^2(\Pi, A_n) = H^2(\Pi^+) \oplus H^2(\Pi^-)$.

In [18] and [17] the following sector domains are introduced:

$$S_\mu = \left\{ y_1e_1 + y_2e_2 + \cdots + y_ne_n : |y_1| < \tan\mu \|y_2e_2 + \cdots + y_ne_n\|, \text{ with } 0 < \mu < \frac{\pi}{2} \right\}.$$

Also, right monogenic functions $f : S_\mu \to A_n$ are considered. These functions satisfy the inequality

$$\|f(y)\| \leqslant C_\nu \|y\|^{-n+1} \tag{3.1}$$

and

$$\sup_{r\in\mathbf{R}^+} \theta(r) \leqslant C \tag{3.2}$$

for some C_ν and $C \in \mathbf{R}^+$, for $0 < \nu < \mu$, and where

$$\theta(R) - \theta(r) = \int_{A(r,R)\cap \mathbf{R}^{n-1}} f(x)e_1 \, dx^{n-1},$$

with $A(r, R)$ the annulus, or spherical shell, in $\mathbf{R}^n$ of radii r and R, with $0 < r < R$. As observed in [18] and [17] the function θ is uniquely determined up to a constant by the right monogenic function f. One reason for introducing the function θ is in order to introduce generalizations of the Plemelj formulae on Lipschitz surfaces. This is pointed out in [17]. The existence of a reasonably wide class of such functions, f, is established in [17] using a combination of Fourier analysis and several complex variables techniques.

For a general Moebius transformation $y = \varphi(x) = (ax+b)(cx+d)^{-1}$ we have for such a function

$$\|f(\varphi(x))J(\varphi, x)\| \leqslant C_{1,\nu}\|ax + b\|^{-n+1}.$$

When $\varphi(x) = x^{-1}$ then $\varphi^{-1}(S_\mu) = S_\mu$, and the left A_n module of right monogenic functions introduced in the previous paragraph is transformed to a left A_n module of right monogenic functions which are defined on the sector domain S_μ and bounded on each sector domain S_ν, where $0 < \nu < \mu$.

When $y = \varphi(x) = (e_1x + 1)(e_1x + 1)^{-1}$ is the Cayley transformation, then the sector domain is transformed to a bounded subdomain $\varphi(S_\mu)$ of the disc $\mathbf{D}$. This domain is bounded by the surfaces Υ_1 and Υ_2, where

$$\Upsilon_1 = \{e_1\} \cup \left\{y \in \mathbf{R}^n : \|y\|^2 = \frac{(1 - \tan\mu\|x\|)^2 + \|x\|^2}{(-1 - \tan\mu\|x\|)^2 + \|x\|^2} : x \in \mathbf{R}^{n-1}\right\}$$

and

$$\Upsilon_2 = \{e_1\} \cup \left\{y \in \mathbf{R}^n : \|y\|^2 = \frac{(1 + \tan\mu\|x\|)^2 + \|x\|^2}{(-1 - \tan\mu\|x\|)^2 + \|x\|^2} : x \in \mathbf{R}^{n-1}\right\}.$$

It may be noted that $\Upsilon_1 \cap \Upsilon_2 = \{e_1, -e_1\}$.

It follows that each right monogenic function $f(y)$ defined on a sector domain S_μ and satisfying (3.1) and (3.2) on each sub-sector domain S_ν, is transformed via the Cayley map to a right monogenic function $f(\varphi(x))J(\varphi, x)$ defined on the domain $\varphi(S_\mu)$ and satisfying

$$\|f(\varphi(x))J(\varphi, x)\| \leqslant C_{2,\nu}\|x - e_1\|^{-n+1}$$

on each subdomain $\varphi(S_\nu)$.

Also, each left monogenic function, f, on S_μ which is bounded on each sector subdomain S_ν is transformed via the Cayley transform to a right monogenic function $f(\varphi(x))J(\varphi, x)$ on $\varphi(S_\mu)$ which satisfies (3.1) and (3.2) on each domain $\varphi(S_\nu)$.

In [18] it is shown that each right monogenic function, f, defined on the sector domain S_μ and satisfying the inequalities (3.1) and (3.2) on each sector subdomain S_ν can be expressed as $f = f_1 + f_2$ where $f_1 : S_\mu^+ \to A_n$ and $f_2 : S_\mu^- \to A_n$ are right monogenic functions and $S_\mu^+ = S_\mu \cup \mathbf{R}^{n-1,+}$ and $S_\mu^- = S_\mu \cup \mathbf{R}^{n-1,-}$. Moreover, [18], on each subdomain S_ν^+ the function $f_1(y)$ satisfies the inequality $\|f_1(y)\| \leqslant C_\nu \|y\|^{-n+1}$, and on each subdomain S_ν^- the function $f_2(y)$ satisfies the inequality $\|f(y)\| \leqslant C_\nu \|y\|^{-n+1}$, see also [17].

This extension conformally transforms to give us the following three propositions.

PROPOSITION 3.7. *Suppose that $f : S_\mu \to A_n$ is a right monogenic function which is bounded on each sector subdomain S_ν, and*

$$\sup_{R\in\mathbf{R}^+} \left\| \int_{\mathbf{R}^{n-1}\setminus B(0,R)} f(y)G(y)\,\mathrm{d}y^{n-1} \right\| \leqslant C.$$

Then $f = f_1 + f_2$, where f_1 is right monogenic on S_μ^+ and f_2 is right monogenic on S_μ^-. Moreover, f_1 is bounded on each subdomain S_ν^+ and f_2 is bounded on each subdomain S_ν^-.

PROPOSITION 3.8. *Suppose that ψ is the Cayley transformation, and that $f : \psi(S_\mu) \to A_n$ is a right monogenic function satisfying $\|f(x)\| \leqslant C_\nu \|x + e_1\|^{-n+1}$ for each $x \in \psi(S_\nu)$ with $0 < \nu < \mu$. Suppose also that*

$$\sup_{R\in\mathbf{R}^+} \left\| \int_{\mathbf{S}^{n-1}\cap B(e_1,R)} f(x)n(x)G(x+e_1)\,\mathrm{d}\sigma(x) \right\| \leqslant C,$$

for some $C \in \mathbf{R}^+$. Then $f = f_1 + f_2$, where $f_i : \Delta_i \to A_n$ is a right monogenic function for $i = 1, 2$. Moreover, Δ_1 is the bounded domain bounded by the surface Υ_1, and Δ_2 is the unbounded domain bounded by the surface Υ_2. Also, $\|f_i(x)\| \leqslant C_\nu \|x + e_1\|^{-n+1}$ for $i = 1$ and $x \in \psi(S_\nu^+)$ and for $i = 2$ and $x \in \psi(S_\nu^-)$, where $0 < \nu < \mu$.

PROPOSITION 3.9. *Suppose that $f : \psi(S_\mu) \to A_n$ is a right monogenic function satisfying $\|f(x)\| \leqslant C_\nu \|x - e_1\|^{-n+1}$ for each $x \in \psi(S_\nu)$ with $0 < \nu < \mu$. Suppose also that*

$$\sup_{R\in\mathbf{R}^+} \left\| \int_{\mathbf{S}^{n-1}\cap B(-e_1,R)} f(x)n(x)G(x+e_1)\,\mathrm{d}\sigma(x) \right\| \leqslant C,$$

for some $C \in \mathbf{R}^+$. Then $f = f_1 + f_2$, where $f_i : \Delta_i \to A_n$ is a right monogenic function for $i = 1, 2$. Moreover, $\|f_i(x)\| \leqslant C_\nu \|x - e_1\|^{-n+1}$ for $i = 1$ and $x \in \psi(S_\nu^+)$ and for $i = 2$ and $x \in \psi(S_\nu^-)$, where $0 < \nu < \mu$.

The open cone $T_\mu = \{y \in \mathbf{R}^n : y_1 > \|y_2 e_2 + \cdots + y_n e_n\| \cot \mu\}$ is introduced in [18] and [17]. This domain conformally transforms itself via the Cayley transformation, φ, to the domain bounded by the surface

$$\Big\{x \in \mathbf{R}^n : \|x\|^2 = \frac{(1 + \cot \mu \|w\|^2) + \|w\|^2}{(-1 - \cot \mu \|w\|^2) + \|w\|^2},$$

$$\text{where } w = y_2 e_2 + \cdots + y_n e_n, \text{ and } \varphi(x) = y\Big\}.$$

We shall denote this domain by Λ_μ. This domain contains the line interval $\{x_1 e_1 : -1 < x_1 < 1\}$, as do the bounded domains bounded by Υ_1 and Υ_2. All three of these domains are symmetric with respect to this line interval and any plane in $\mathbf{R}^n$ which contains this line interval. Also, all three domains are invariant with respect to any orthogonal transformation over the space spanned by $e_2, \ldots, e_n$.

Suppose now that $\Sigma = \{(\varphi(x)e_1 + x)\}$ is a Lipschitz surface, then the Lipschitz constant of this surface is defined to be $\inf\{c \in \mathbf{R}^+ : \|\varphi(y) - \varphi(u)\| \leqslant c\|y - u\|\}$. Suppose now that C is the Lipschitz constant for the Lipschitz surface Σ, and $C < \tan \mu$. Then in [18], [17] and [20] it is shown that for almost all $y \in \Sigma$ the outward pointing normal vector $n(y)$, to Σ at y, lies in the open cone T_μ. Moreover, for each $\varepsilon \in \mathbf{R}^+$ the vector $\varepsilon n(y) \in T_\mu$. As Moebius transformations preserve angles it follows that for almost all $x \in \varphi(\Sigma)$, and for ε sufficently small, the vector $\varepsilon n(x)$ lies in the domain Λ_μ.

Following [18], [17], [20] and [35] we introduce for each monogenic function $f : S_\mu \to A_n$, satisfying the inequalities (3.1) and (3.2), the function $\theta : T_\mu \to A_n$. This function satisfies

$$\theta(p) - \theta(q) = \int_{S(p,q)} n(y) f(y)\, d\sigma(y) \tag{3.3}$$

where $S(p, q)$ is a smooth, orientable surface lying in T_μ, and with boundary. The boundary of $S(p, q)$ consists of two spheres of dimension $(n - 2)$. Both spheres are centred at the origin. The first sphere has radius $\|p\|$, and is orthogonal to p. The second sphere has radius $\|q\|$ and it is orthogonal to the vector q. From Cauchy theorem it follows that the integral appearing in equation (3.3) is independant of the choice of surface $S(p, q)$. It also follows from (3.3) that the function θ is well defined up to a constant. All of these points are observed in [18], [17] and [20]. This function is a continuation of the function θ defined on $\mathbf{R}^+$ introduced earlier.

Suppose now that φ is a Moebius transformation then the function θ is conformally transformed to a function $\theta_\varphi : \varphi^{-1}(T_\mu) \to A_n$ such that

$$\theta_\varphi(s) - \theta_\varphi(t) = \int_{\varphi^{-1}(S(p,q))} \widetilde{J}(\varphi, x)n(x)J(\varphi, x)f(\varphi(x))\,\mathrm{d}\sigma(x)$$

where $\varphi(s) = p$ and $\varphi(t) = q$. It follows from (3.3) that the function θ_φ is well defined up to a constant.

The following theorem is proved in [18], and also in [35]:

THEOREM 3.10. *Suppose that $f : S_\mu \to A_n$ is a right monogenic function which satisfies the inequalities* (3.1) *and* (3.2)*; then for each Lipschitz surface Σ with Lipschitz constant $C < \tan\mu$, the operator*

$$T_{f,\theta} : L^p(\Sigma, A_n) \to L^p(\Sigma, A_n)$$

defined by

$$\lim_{\varepsilon\to 0}\Big(\int_{\Sigma\setminus B(y,\varepsilon)} f(y-u)n(y)h(y)\,\mathrm{d}\sigma(y) + \theta(\varepsilon n(u))h(y)\Big),$$

is well defined and L^p bounded, where $B(y,\varepsilon)$ is the open ball centred at y and of radius ε, and $h \in L^p(\Sigma, A_n)$ for $1 < p < \infty$.

The Lipschitz constant C of the surface Σ is chosen so that for each $y \in \Sigma$ the surface $\Sigma \setminus \{y\}$ lies in the sector $S_\mu + y$. For the special case $p = 2$ the previous theorem conformally transforms as follows:

THEOREM 3.11. *Suppose that S_μ, Σ and f are as in the previous theorem. Suppose also that φ is a Moebius transformation. Then the operator*

$$T_{J(\varphi,)f,\theta_\varphi} : L^2(\varphi^{-1}(\Sigma), A_n) \to L^2(\varphi^{-1}(\Sigma), A_n)$$

defined by

$$\lim_{\varepsilon\to 0}\int_{\varphi^{-1}(\Sigma)\setminus B(x,\varepsilon)} J(\varphi, v)f(\varphi(x) - \varphi(v))\widetilde{J}(\varphi, x)n(x)g(x)\,\mathrm{d}\sigma(x) + \theta_\varphi(\varepsilon n(v))g(v),$$

for $g \in L^2(\varphi^{-1}(\Sigma), A_n)$, is a well defined L^2 bounded operator.

In [18] and [17] it is shown via the Plemelj formulae that when f is both left and right monogenic then the decomposition of the function f into f_1 and f_2 gives rise to a decomposition of the operator $T_{f,\theta}$ into

$$T_1(f,\theta) : H^p(\Sigma_+) \to H^p(\Sigma_+)$$

and

$$T_2(f,\theta) : H^p(f,\Sigma_-) \to H^p(\Sigma_-).$$

Our arguments show that the operator $T_{J(\varphi,)f,\theta_\varphi}$ similarly decomposes into operators $T_1(J(\varphi,)f,\theta_\varphi$ and $T_2(J(\varphi,)f,\theta_\varphi$ acting over the respective H^2 spaces. It would be interesting to see if the H_∞ functional calculus described in [17] for the operators acting over the Lipschitz surfaces carries over to a suitable analogue via Moebius transformations to the analogous type of operators introduced here.

When the left monogenic function f is also right monogenic it should be noted that the function $J(\varphi,v)f(\varphi(x)-\varphi(v))\widetilde{J}(\varphi,x)$ is left monogenic with respect to the variable v and right monogenic with respect to the variable x. Basic properties of functions with this property are studied in some papers of Brackx and Pincket, see for instance [6].

4. THE SEVERAL COMPLEX VARIABLE CASE

We begin this section by complexifying the algebra A_n to obtain the complex Clifford algebra $A_n(\mathbf{C})$. It should be noted that the anti-automorphisms $\sim$ and $-$ automatically extend from the real algebra A_n to be anti-automorphisms on the complex algebra $A_n(\mathbf{C})$. We shall use the same notation to denote these anti-automorphisms. Lying in this algebra is the complexification of $\mathbf{R}^n$, namely $\mathbf{C}^n$. Under Clifford multiplication each vector $z = z_1e_1 + \cdots + z_ne_n$ gives $z^2 = -z_1^2 - \cdots - z_n^2$. Consequently, not all non-zero vectors in $\mathbf{C}^n$ are invertible in $A_n(\mathbf{C})$. However, it is possible to introduce Vahlen matrices which act over the conformal closure of $\mathbf{C}^n$.

DEFINITION 4.1. A 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ which satisfies:

(i) a, b, c and $d \in A_n(\mathbf{C})$;

(ii) a, b, c and d are all products of vectors in $\mathbf{C}^n$;

(iii) $a\widetilde{c}, c\widetilde{d}, d\widetilde{b}$ and $b\widetilde{a} \in \mathbf{C}^n$;

(iv) $a\widetilde{d} - b\widetilde{c} = \pm 1$, is called a *complex Vahlen matrix*.

We shall denote the set of all such matrices by $V_{\mathbf{C}}(n)$. The basic properties of these matrices are described in [8] and [28]. In particular it is known that $V_{\mathbf{C}}(n)$ is a group under matrix multiplication. The group is generated by the matrices $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} a & 0 \\ 0 & \widetilde{a}^{-1} \end{pmatrix}$ where a is a product of invertible vectors in $\mathbf{C}^n$, and $\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$, where $u \in \mathbf{C}^n$.

This group has the lower triangular subgroup

$$V_{\mathbf{C},\Delta}(n) = \left\{ \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \in V_{\mathbf{C}}(n) \right\}.$$

In [?] we show that the quotient space $V_{\mathbf{C}}(n) \setminus V_{\mathbf{C},\Delta}(n)$ is a complex homogeneous manifold of complex dimension n. Furthermore, this manifold contains $\mathbf{C}^n$ as an open, dense subset. Also the group $V_{\mathbf{C}}(n)$ acts transitively over this homogeneous manifold. We can consequently regard this homogeneous manifold to be the conformal closure of $\mathbf{C}^n$. We shall use the symbol $\mathbf{C}^{n,\sharp}$ to denote the conformal closure of $\mathbf{C}^n$. We shall regard the action of an element of the Lie group $V_{\mathbf{C}}(n)$ on $\mathbf{C}^{n,\sharp}$ as a Moebius transformation over this space. We shall call such a transformation a complex Moebius transformation. On an open, dense subset of $\mathbf{C}^n$ such a transformation can be expressed as $w = \varphi(z) = (az+b)(cz+d)^{-1}$, where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in V_{\mathbf{C}}(n)$.

A real, $(n-1)$ dimensional, locally Lipschitz continuous, orientable manifold, S, locally embedded but possibly globally immersed in $\mathbf{C}^{n,\sharp}$ is called a *surface in* $\mathbf{C}^n$. We shall be interested primarily in the case where the surface bounds a particular type of manifold. In [27] and elsewhere we introduce the following type of manifolds:

DEFINITION 4.2. A connected smooth, real, n-dimensional manifold, M, lying in $\mathbf{C}^n$ is called a *domain manifold* if for each $z \in M$ we have that $TM_z \cup N(z) = \{z\}$ and $M \cup N(z) = \{z\}$, where TM_z is the tangent space of M at z, and $N(z)$ is the null cone $\{w \in \mathbf{C}^n : (z-w)^2 = 0\}$.

Such a manifold may or may not have a boundary. If it does have a boundary we shall impose the further restriction that the boundary is locally Lipschitz continuous. When M is a subset of $\mathbf{R}^n, \subset \mathbf{C}^n$, then M is simply a domain in $\mathbf{R}^n$.

In [29] it is shown that given a domain manifold M, then $\varphi^{-1}(M)$ is also a domain manifold for φ belonging to a wide class of Moebius transformations. A technical problem prevented us from showing that domain manifolds remain invariant under general complex Moebius transformations. A case to consider is when $M = B(0,1)$, and the Moebius transformation is Kelvin inversion. In this case the domain manifold is the image of the domain manifold $\mathbf{R}^n \setminus (\mathbf{S}^{n-1} \cup B(0,1))$. However, if we homotopically deform $B(0,1)$ within C^n keeping the boundary of $B(0,1)$ fixed, and such that at each point in time the homotopy deformation of the domain manifold is again a domain manifold, then in general it is not clear that the inverse images under Kelvin inversion is again a domain manifold.

It is clear, looking at the generators of $V_{\mathbf{C}}(n)$, that the only generator that can cause a problem is Kelvin inversion, or rather the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Now $\mathbf{C}^n$ can be re-expressed as $\left\{ \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} : z \in \mathbf{C}^n \right\}$. Following [28] this set can be seen as an open dense subset of $\mathbf{C}^{n,1}$. Now $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & z \end{pmatrix}$. It follows that under Kelvin inversion the set $\left\{ \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} : z \in \mathbf{C}^n \right\} \subset \mathbf{C}^{n,1}$ is transformed to the set $\left\{ \begin{pmatrix} 0 & 1 \\ 1 & z \end{pmatrix} : z \in \mathbf{C}^n \right\} \subset \mathbf{C}^{n,1}$. It follows that Kelvin inversion transforms any domain manifold lying in $\mathbf{C}^n$ to a domain manifold lying in $\mathbf{C}^{n,1}$.

The previous argument is a special case of the argument where one takes a general matrix from $V_C(n)$ and let this matrix act on a quotient space representative for $\mathbf{C}^n$ within $\mathbf{C}^{n,1}$. It is a straightforward calculation to see that in each case we obtain another quotient space representative of $\mathbf{C}^n$ within $\mathbf{C}^{n,1}$. Consequently, in this way it may be seen that domain manifolds are preserved under complex conformal transformations.

Suppose that S is a surface, then for any measurable function $f : S \to A_n(\mathbf{C})$ the following inequality is satisfied:

$$\left\| \int_S f(z)\overline{f}(z)\, \mathrm{d}\sigma(z) \right\| \leqslant C(n) \int_S \sum_A \|f_A(z)\|^2\, \mathrm{d}|\sigma(z)|,$$

where $\mathrm{d}\sigma(z)$ is the complex Borel measure on S, $C(n)$ is a dimensional constant, $\mathrm{d}|\sigma(z)|$ is the real measure resulting from taking the modulus of the measure $\mathrm{d}\sigma(z)$, and A is an index for the basis of A_n, so $f = \sum_A f_A e_A$. We shall be interested in the case where the right hand side of the previous inequality is finite. We shall denote the space of such functions by $L^2(S, A_n(\mathbf{C}))$. For each $f, g \in L^2(S, A_n(\mathbf{C}))$ it can be seen that the integral $\int_S f(z)\overline{g}(z)\, \mathrm{d}\sigma(z)$ is bounded. The space $L^2(S, A_n(\mathbf{C}))$ is a Banach space. We may also consider the Banach space $L^p(S; A_n(\mathbf{C}))$, consisting of $A_n(\mathbf{C})$ valued functions defined on S and L^p integrable with respect to the measure $\mathrm{d}|\sigma(z)|$. Here $1 \leqslant p \leqslant \infty$.

Suppose now that S_1 and S_2 are surfaces in $\mathbf{C}^n$, and φ is a complex Moebius transformation. Moreover, $\varphi(S_1) = S_2$. Suppose also that $f, g \in L^2(S_2, A_n(\mathbf{C}))$. Then by similar arguments to those given in the real case earlier in this paper we can deduce that

$$\int_{S_2} f(w)\overline{g}(w)\, \mathrm{d}\sigma(w) = \int_{S_1} f(\varphi(z))\overline{g}(\varphi(z))((cz+d)(\widetilde{cz+d}))^{-n+1}\, \mathrm{d}\sigma(z).$$

When n is even $((cz+d)\widetilde{(cz+d)})^{-n+1} = J(\varphi,z)\widetilde{J}(\varphi,z)$, where

$$J(\varphi,z) = \widetilde{(cz+d)}((cz+d)\widetilde{(cz+d)})^{-\frac{n}{2}}.$$

It follows that we have the following result:

PROPOSITION 4.3. *Suppose that S_1 and S_2 are surfaces lying in $\mathbb{C}^n$, with n even, and that φ is a complex Moebius transformation, with $\varphi(S_1) = S_2$. Then there is a canonical isometry*

$$\varphi : L^2(S_2, A_n(\mathbb{C})) \to L^2(S_1, A_n(\mathbb{C})) : f(w) \to J(\varphi,z)f(\varphi(z)).$$

Similar results to those described for the real case earlier in the paper now automatically extend to complex even dimensions. Again the most interesting results occur when combined with the theory of monogenic functions.

DEFINITION 4.4. Suppose Ω is a domain lying in $\mathbb{C}^n$, and $f : \Omega \to A_n(\mathbb{C})$ is a holomorphic function. Then f is called *a complex left monogenic function* if f satifies the equation $\sum_{j=1}^{n} e_j \frac{\partial}{\partial z_j} f(z) = D_{\mathbb{C}} f(z) = 0$.

A similar definition can be given for complex right monogenic functions.

The following holomorphic continuation result is given in [27] and elsewhere.

THEOREM 4.5. *Suppose that n is even, and that $f : \Omega \to A_n$ is a complex left monogenic function. Suppose also that $M \subset \Omega$ is a bounded domain manifold with boundary a surface $S \subset \Omega$. The f holomorphically extends to the component $H(S)$ of $\mathbb{C}^n \setminus \bigcup_{z\in S} N(z)$ which contains the interior of M. Moreover, for each w in $H(S)$ the holomorphic continuation of f is given by $\int_S G(z-w)\mathrm{D}z f(z)$, where G is the holomorphic continuation to $\mathbb{C}^n \setminus N(0)$ of the Clifford-Cauchy kernel used in the euclidean setting, and $\mathrm{D}z$ is the differential form $\sum_{j=1}^{n}(-1)^j e_j \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_{j-1} \wedge \mathrm{d}z_{j+1} \wedge \cdots \wedge \mathrm{d}z_n$.*

It can be seen from the integral appearing in the previous theorem that the holomorphic continuation of f to $H(M)$ is also complex left monogenic.

DEFINITION 4.6. Suppose that the surface S is the boundary of a domain manifold M. Then a homotopy $H : S \times [0,1] \to M \cup S$ is called a *homotopy of Hardy type* if:

(i) $H(z,0) = z$ for each $z \in S$;

(ii) H is Lipschitz continuous;

(iii) $H(S,t)$ is a surface for each $t \in [0,1]$;

(iv) for each $t \in (0,1]$ the set $S \cap H(S,t)$ seen as a subset of S is a set of measure zero with respect to the Borel measure $d\sigma(z)$.

Using the previous definition and theorem we have:

THEOREM 4.7. *Suppose that M is a bounded domain manifold, with boundary S. Suppose also that n is even, and $f : M \cup S \to A_n(\mathbf{C})$ is such that:*

(i) $f|M$ is the restriction to M of a complex left monogenic function on $H(M)$;

(ii) $f|S \in L^p(S, A_n(\mathbf{C}))$ for some $p \in [0, \infty]$;

(iii) there is a homotopy of Hardy type, $H : S \times [0,1] \to M \cup S$ and a continuous function $F : [0,1] \to L^p(S, A_n(\mathbf{C})) : F(t) = f|H(S,t)$.

Then for each $\zeta \in H(S)$ we have that

$$f(\zeta) = \int_S G(z-\zeta)\mathrm{D}z f(z).$$

By the same arguments given earlier in the paper for the euclidean case it may be observed that the integral formula given in the previous theorem is independent of the choice of homotopy of Hardy type within $M \cup S$. Similarly, if $H : M \times [0,1] \to H(S)$ is such that:

(i) $H(z,0) = z$ for each $z \in M$,

(ii) $H(M,t)$ is a domain manifold for each $t \in [0,1]$; and

(iii) each domain manifold $H(M,t)$ has the surface S as its boundary,

then it can be deduced by simple continuity arguments that the integral formula given in Theorem 4.7 is independent of the choice of M, but only depends on the domain $H(M)$. For this reason we shall denote the right $A_n(\mathbf{C})$ module of functions satisfying the conditions described in Theorem 4.7 by $H^p(H(S), A_n(\mathbf{C}))$. We call this space *the Hardy p-space of the domain $H(S)$*. Using Theorem 4.7 it may be observed that this space is a complete subspace of $L^p(S, A_n(\mathbf{C}))$, where S is the boundary of M.

In conclusion we now point out that it is a relatively simple exercise to to adapt the arguments developed earlier in this paper and set up the module $H^2(H(S), A_n(\mathbf{C}))$ for the cases where n is even, and S now bounds an unbounded domain manifold. In fact we obtain:

THEOREM 4.8. *Suppose that S_1 and S_2 are surfaces in $\mathbf{C}^n$, with n even. Suppose also that S_1 bounds a domain manifold M_1 and S_2 bounds a domain manifold M_2, and φ is a conformal transformation such that $\varphi(M_1) = M_2$. Then*

$$\varphi : H^2(H(S_2, A_n(\mathbf{C})) \to H^2(S_1, A_n(\mathbf{C})) : f(w) \to J(\varphi, z)f(\varphi(z))$$

is an isomorphism.

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