Gap probabilities in the Laguerre Unitary ensemble and discrete Painlevé equations

Gap Probabilities in the Laguerre Unitary Ensemble and Discrete Painlevé Equations

Jie Hu
Faculty of Science and Technology, Department of Mathematics, University of Macau, Av. Padre Tomás Pereiro, Taipa, Macau, P.R.China
E-mail: hujie_0610@163.com

Anton Dzhamay
School of Mathematical Sciences, The University of Northern Colorado, Greeley, CO 80526, USA
E-mail: adzham@unco.edu

Yang Chen
Faculty of Science and Technology, Department of Mathematics, University of Macau, Av. Padre Tomás Pereiro, Taipa, Macau, P.R.China
E-mail: yayangchen@umac.mo

Keywords: orthogonal polynomials, Askey-Wilson scheme, Painlevé equations, difference equations, birational transformations.

MSC2010: 333C47, 34M55, 39A99, 42C05, 3D45, 34M55, 34M56, 14E07, 39A13

To the memory of Jon Nimmo

Abstract

In this paper we study a certain recurrence relation, that can be used to generate ladder operators for the Laguerre Unitary ensemble, from the point of view of Sakai's geometric theory of Painlevé equations. On one hand, this gives us one more detailed example of the appearance of discrete Painlevé equations in the theory of orthogonal polynomials. On the other hand, it serves as a good illustration of the effectiveness of a recently proposed procedure on how to reduce such recurrences to some canonical discrete Painlevé equations.

1 Introduction

By now it is clear that there are many fundamental connections between the theory of Random Matrices, Orthogonal Polynomials, and Painlevé Equations, both differential and discrete. Some conceptual understanding of this fact has been given in a series of papers by Alexei Borodin and his collaborators [AB06, AB07, Bor03] and especially [BB03], see also a recent monograph of Walter Van Assche, [VA18]. In a way, the geometric setting of Hidetaka Sakai's theory of Painlevé equations [Sak01] seems to provide the natural framework for questions involving the study of various orthogonal polynomial ensembles, and so it is not surprising that various objects of interest, such as the *gap probabilities*, coefficients for *three-term recurrence relations*, or *ladder operators*, can be described in terms of solution of either differential or discrete Painlevé equations.

The purpose of the present paper is to study, from the geometric perspective of Sakai's theory, an example of a recurrence relation obtained by Shulin Lyu and Yang Chen in their study of the largest eigenvalue distribution [LC17] for the *Laguerre Unitary Ensemble*, focusing on the reduction of this recurrence to a canonical form following step-by-step procedure recently proposed in [DFS19].

Thus, we let the weight function be $w(x;\alpha) = x^\alpha \exp(-x)$, where $x > 0$ and $\alpha > -1$ is a parameter and consider a family of monic polynomials

$$P_j(x,t) = x^j + p_{j-1}(t)x^{j-1} + \cdots + p_0(t) \tag{1.1}$$

that are orthogonal with respect to the weight $w(x;\alpha)$ on the interval $[0,t]$, $0 < t \leq \infty$, i.e.,

$$\int_0^t P_n(x,t)P_m(x,t)x^\alpha \exp(-x)dx = \delta_{n,m}h_n(t), \tag{1.2}$$

where $h_n(t)$ is the square of the $L^2\left([0,t]; x^\alpha \exp(-x)dx\right)$ norm of $P_n(x,t)$. This unitary ensemble is called the *Laguerre Unitary Ensemble* (or *LUE* for short) since when $t=\infty$, the family $\{P_n(x) = P_n(x,\infty)\}$ is the well-known family of monic Laguerre polynomials orthogonal w.r.t. the weight $x^\alpha \exp(-x)$.

As usual, the orthogonality condition immediately implies the three term recurrence relations

$$xP_n(x,t) = P_{n+1}(x,t) + \alpha_n(t)P_n(x,t) + \beta_n(t)P_{n-1}(x,t), \qquad n \geq 0,$$

with initial conditions $P_{-1}(x,t) = 0$, $P_0(x,t) = 1$.

The paper [LC17] is concerned with the study of the probability $\mathbb{P}(n,t)$ that the largest eigenvalue in LUE on $[0,\infty)$ is not larger than t, where n is the size of the corresponding random matrix. This probability can be computed as

$$\mathbb{P}(n,t) = \frac{D_n(t)}{D_n(\infty)},$$

where

$$D_n(t) := \det\left[\int_0^t x^{j+k}x^\alpha \exp(-x)dx\right]_{0\leq j,k\leq n-1}$$

is the $n \times n$ *Hankel determinant*, a fundamental object in the theory of orthogonal polynomials [Sze67], that can be evaluated as $D_n(t) = \prod_{j=0}^{n-1} h_j(t)$, and

$$D_n(\infty) = \frac{G(n+1)G(n+\alpha+1)}{\Gamma(\alpha+1)}, \quad \text{where } G(.) \text{ is the Barnes } G\text{-function.}$$

One way to study (and generate) the family of orthogonal polynomials $\{P_n(x,t)\}$ is to use the lowering and raising ladder operators,

$$\left(\frac{\mathrm{d}}{\mathrm{d}z} + B_n(z,t)\right)P_n(z,t) = \beta_n(t)A_n(z,t)P_{n-1}(z,t),$$
$$\left(\frac{\mathrm{d}}{\mathrm{d}z} - B_n(z,t) - v'(z)\right)P_{n-1}(z,t) = -A_{n-1}(z,t)P_n(z,t)$$

where $A_n(z,t)$ and $B_n(z,t)$ can be parameterized by the functions $R_n(t)$ and $r_n(t)$,

$$A_n(z,t) = \frac{R_n(t)}{z-t} + \frac{1-R_n(t)}{z}, \qquad B_n(z,t) = \frac{r_n(t)}{z-t} - \frac{r_n(t)+n}{z},$$

and where

$$R_n(t) := -\frac{P_n^2(t,t)}{h_n(t)}t^\alpha e^{-t}, \qquad r_n(t) := -\frac{P_n(t,t)P_{n-1}(t,t)}{h_{n-1}(t)}t^\alpha e^{-t}, \qquad \text{and } P_j(t,t) := P_j(z,t)|_{z=t}.$$

Let us now introduce a different parameterization $x_n(t), y_n(t)$ via

$$x_n(t) = 1 - \frac{1}{R_{n-1}(t)} \quad y_n(t) = -r_n(t).$$

Then Lyu and Chen [LC17, Remark 2.3] showed that these variables satisfy the following recurrence relations in n:

$$\begin{cases} x_n x_{n+1} = \dfrac{(y_n - n)(y_n - (n+\alpha))}{y_n^2} \\ y_n + y_{n-1} = -\dfrac{(-t+2n-1+\alpha)x_n - (2n-1+\alpha)}{(x_n-1)^2}. \end{cases} \tag{1.3}$$

This is the recurrence that we are interested in studying. We show, following the step-by-step procedure of [DFS19], that this recurrence is a discrete Painlevé equation that is equivalent to one of the standard examples in the d-P $\left(\mathrm{A}_3^{(1)}/\mathrm{D}_5^{(1)}\right)$ family. Our main result is the following Theorem.

Theorem 1. *The recurrence (1.3) is equivalent to the standard discrete Painlevé equation (A.17) written in [KNY17]. This equivalence is achieved via the following change of variables:*

$$x(q,p) = \frac{q(p+t)+a_2}{qp+a_2}, \qquad y(q,p) = \frac{(p+t)(qp+a_2)}{t}. \tag{1.4}$$

The inverse change of variables is given by

$$q(x,y) = \frac{(x-1)(x-1)y+n}{tx}, \qquad p(x,y) = \frac{t(y-n)}{(x-1)y+n}. \tag{1.5}$$

The relationship between the Laguerre weight recurrence parameters and the root variables of discrete Painlevé equations is given by

$$a_0 = n+\alpha, \quad a_1 = -n, \quad a_2 = n, \quad a_3 = 1-n-\alpha. \tag{1.6}$$

Remark 2. Note that for our recurrence the root variables are constrained by the condition $a_1 + a_2 = 0$ (or, equivalently, $a_0 + a_3 = 1$).

These recurrences then are particular combinations of elementary mappings that can be thought of as Bäcklund transformations of a differential $P_{\rm V}$ equation that is associated with the same geometry. This is not surprising, since, if we put $\sigma_n(t) := t\frac{\mathrm{d}}{\mathrm{d}t}\ln\mathbb{P}(n,t)$, then it can be shown that it is the σ function of a particular Painlevé V equation. Estelle Basor and Yang Chen [BC09] gave an alternate derivation of this result without relying on the Christoffel-Darboux kernel (or the reproducing kernel). Note also that the quantity $S_n(t) = 1 - 1/R_n(t)$, satisfies

$$S_n''(t) = \frac{3S_n(t)-1}{2(S_n^2(t)-S_n(t))}(S_n'(t))^2 - \frac{S_n'(t)}{t} - \frac{\alpha^2(S_n(t)-1)^2}{2S_n(t)t^2} + (2n+1+\alpha)\frac{S_n(t)}{t} - \frac{1}{2}\frac{S_n(t)(S_n(t)+1)}{S_n(t)-1},$$

which is a $P_{\rm V}$ with parameters

$$\alpha_5 = 0, \qquad \beta_5 = -\frac{\alpha^2}{2}, \qquad c_5 = 2n+1+\alpha, \qquad d_5 = -1/2.$$

The function $\beta_n'(t) = tr_n'(t)$ satisfies a rather large second order non-linear ordinary differential equation in t, and we will not reproduce it here.

2 The Identification Procedure

2.1 The Singularity Structure

To determine whether a given second-order nonlinear (non-autonomous) recurrence relation is one of discrete Painlevé equations, see the recent survey [KNY17], the first step is to understand the singularity structure of the mapping defined by this recurrence relation. As is very common in this class of examples, our recurrence relation defines two natural mappings, the *forward mapping* $\psi_1^{(n)} : (x_n, y_n) \mapsto (x_{n+1}, y_n)$ defined by solving the first equation in (1.3) for x_{n+1} and the *backward mapping* $\psi_2^{(n)} : (x_n, y_n) \mapsto (x_n, y_{n-1})$ defined by solving the second equation in (1.3) for y_{n-1}. We are interested in studying the composed mapping $\psi^{(n)} = \left(\psi_2^{(n+1)}\right)^{-1} \circ \psi_1^{(n)} : (x_n, y_n) \mapsto (x_{n+1}, y_{n+1})$. We put $x := x_n$, $\overline{x} := x_{n+1}$, $y := y_n$, $\overline{y} := y_{n+1}$ and sometimes omit the index n in the mapping notation. The map $\psi : (x,y) \mapsto (\overline{x}, \overline{y})$ then becomes

$$\begin{cases} \overline{x} = \dfrac{(y-n)(y-(n+\alpha))}{xy^2}, \\ \overline{y} = \dfrac{-y(y-n)^2(y-(n+\alpha))^2 + xy^2(y-n)(y-(n+\alpha))(t+2y-2n-1-\alpha) + x^2y^4(2n+1+\alpha-y)}{((y-n)(y-(n+\alpha))-xy^2)^2}. \end{cases} \tag{2.1}$$

Compactifying the mapping from $\mathbb{C}^2$ to $\mathbb{P}^1 \times \mathbb{P}^1$ by introducing the coordinates $X = 1/x$ and $Y = 1/y$, it is easy to see that there are four affine base points of the mapping, and as we see below, it is convenient to label them as

$$q_1(0,n), \quad q_2(0,n+\alpha), \quad q_3(1,\infty), \quad q_7(\infty,0);$$

(for example, it is immediate that at q_1 and q_2 both the numerator and the denominator of the mapping vanish, other points are found in the same way in other charts). We resolve base point singularities using the blowup procedure, see, e.g., [Sha13]. That is, for each base point $q_i(x_i, y_i)$ we construct two new local charts (u_i, v_i) and (U_i, V_i) given by $x = x_i + u_i = x_i + U_i V_i$ and $y = y_i + u_i v_i = y_i + V_i$. The coordinates $v_i = 1/U_i$ represent all possible slopes of lines passing through the point q_i, and so this variable change "separates" all curves passing through q_i based on their slopes. This change of variables is a bijection away from q_i, but the point q_i is replaced by the $\mathbb{P}^1$-line of all possible slopes, called the *central fiber* or the *exceptional divisor* of the blowup. We denote this central fiber by F_i, it is given in the blowup charts by local equations $u_i = 0$ and $V_i = 0$. We then extend the mapping to these new charts via the above coordinate substitution, find and resolve new base points (those would only appear on the exceptional divisors $u_i = V_i = 0$) and continue this process until it terminates (it should, in the discrete Painlevé case). We summarize the result in the following Lemma.

Lemma 3. *The base points of the mapping* (2.1) *are*

$$\begin{aligned}
&q_1(x=0, y=n), \qquad q_2(x=0, y=n+\alpha), \qquad\qquad (2.2)\\
&q_3\left(x=1, Y=\frac{1}{y}=0\right) \leftarrow q_4\left(u_3 = x-1 = 0, v_3 = \frac{1}{y(x-1)} = 0\right)\\
&\qquad \leftarrow q_5\left(U_4 = y(1-x)^2 = t, V_4 = \frac{1}{y(x-1)} = 0\right)\\
&\qquad \leftarrow q_6\left(U_5 = (x-1)y((x-1)^2 y - t) = t(1-2n+t-\alpha), V_5 = \frac{1}{y(x-1)} = 0\right),\\
&q_7\left(X = \frac{1}{x} = 0, y = 0\right) \leftarrow q_8\left(U_7 = \frac{1}{xy} = 0, V_7 = y = 0\right).
\end{aligned}$$

Considering the inverse mapping does not add any new base points.

Resolving these base points lifts our birational mapping $\psi : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ to the isomorphism, also denoted by ψ, between the corresponding algebraic surfaces, $\psi : \mathcal{X}_{\mathbf{b}} \to \mathcal{X}_{\overline{\mathbf{b}}}$. The subscript $\mathbf{b}$ indicates that the coordinates of the base points (and hence the resulting surface) depend on the parameters of the mapping, $\mathbf{b} = \{\alpha, t, n\}$. These parameters can (and do) change under the mapping and so $\overline{\mathbf{b}}$ denotes the evolved set of parameters. Sometimes we drop the parameters subscript and use the notation $\overline{\mathcal{X}}$ for the range of the mapping.

2.2 The Induced Mapping on Pic($\mathcal{X}$)

The next step in the identification procedure is to compute the induced mapping on the Picard lattice. Recall that for a regular algebraic variety $\mathcal{X}$, its *Picard group* (or *Picard lattice*) is the quotient of the *divisor group* $\operatorname{Div}(\mathcal{X}) = \operatorname{Span}_{\mathbb{Z}}(D)$ that is a free Abelian group generated by closed irreducible subvarieties D of codimension 1, by the subgroup $\mathrm{P}(\mathcal{X})$ of *principal divisors* (i.e., by the relation of *linear equivalence*),

$$\operatorname{Pic}(\mathcal{X}) \simeq \operatorname{Cl}(\mathcal{X}) = \operatorname{Div}(\mathcal{X})/\,\mathrm{P}(\mathcal{X}) = \operatorname{Div}(\mathcal{X})/\sim,$$

see [SKKT00] or [Sha13]. In our case, it is enough to know that $\operatorname{Pic}(\mathbb{P}^1 \times \mathbb{P}^1) = \operatorname{Span}_{\mathbb{Z}}\{\mathcal{H}_x, \mathcal{H}_y\}$, where $\mathcal{H}_x = [H_{x=a}]$ is the class of a *vertical* and $\mathcal{H}_y = [H_{y=b}]$ is the class of a *horizontal* line on $\mathbb{P}^1 \times \mathbb{P}^1$. Each blowup procedure at a point q_i adds the class $\mathcal{F}_i = [F_i]$ of the *exceptional divisor* (i.e., the *central fiber*) of the blowup, so $\mathrm{Pic}(\mathcal{X}_n) = \mathrm{Span}_{\mathbb{Z}}\{\mathcal{H}_x, \mathcal{H}_y, \mathcal{F}_1, \ldots, \mathcal{F}_8\}$. Further, the Picard lattice is equipped with the symmetric bilinear *intersection form* given by

$$\mathcal{H}_x \bullet \mathcal{H}_x = \mathcal{H}_y \bullet \mathcal{H}_y = \mathcal{H}_x \bullet \mathcal{F}_i = \mathcal{H}_y \bullet \mathcal{F}_j = 0, \qquad \mathcal{H}_x \bullet \mathcal{H}_y = 1, \qquad \mathcal{F}_i \bullet \mathcal{F}_j = -\delta_{ij} \tag{2.3}$$

on the generators, and then extended by the linearity.

The mapping ψ induces a linear mapping $\psi_* : \mathrm{Pic}(\mathcal{X}) \to \mathrm{Pic}(\overline{\mathcal{X}})$. Note that $\mathrm{Pic}(\mathcal{X})$ and $\mathrm{Pic}(\overline{\mathcal{X}})$ are canonically isomorphic, so we sometimes just use the notation $\mathrm{Pic}(\mathcal{X})$. We also use $\overline{F}_i$ to denote the divisor of the central fiber of the blowup at the point $\overline{q}_i = \psi(q_i)$, and similarly for the backwards mapping and for the classes; notation $\mathcal{F}_{i\cdots j}$ stands for $\mathcal{F}_i + \cdots + \mathcal{F}_j$.

Lemma 4. *The action of the mapping* $\psi_* : \mathrm{Pic}(\mathcal{X}) \to \mathrm{Pic}(\overline{\mathcal{X}})$ *is given by*

$$\begin{aligned}
&\mathcal{H}_x \mapsto 5\overline{\mathcal{H}}_x + 2\overline{\mathcal{H}}_y - \overline{\mathcal{F}}_{12} - 2\overline{\mathcal{F}}_{3456} - \overline{\mathcal{F}}_{78}, && \mathcal{H}_y \mapsto 2\overline{\mathcal{H}}_x + \overline{\mathcal{H}}_y - \overline{\mathcal{F}}_{3456},\\
&\mathcal{F}_1 \mapsto 2\overline{\mathcal{H}}_x + \overline{\mathcal{H}}_y - \overline{\mathcal{F}}_{23456}, && \mathcal{F}_5 \mapsto \overline{\mathcal{H}}_x - \overline{\mathcal{F}}_4\\
&\mathcal{F}_2 \mapsto 2\overline{\mathcal{H}}_x + \overline{\mathcal{H}}_y - \overline{\mathcal{F}}_{13456}, && \mathcal{F}_6 \mapsto \overline{\mathcal{H}}_x - \overline{\mathcal{F}}_3\\
&\mathcal{F}_3 \mapsto \overline{\mathcal{H}}_x - \overline{\mathcal{F}}_6, && \mathcal{F}_7 \mapsto 2\overline{\mathcal{H}}_x + \overline{\mathcal{H}}_y - \overline{\mathcal{F}}_{34568},\\
&\mathcal{F}_4 \mapsto \overline{\mathcal{H}}_x - \overline{\mathcal{F}}_5, && \mathcal{F}_8 \mapsto 2\overline{\mathcal{H}}_x + \overline{\mathcal{H}}_y - \overline{\mathcal{F}}_{34567}.
\end{aligned}$$

The evolution of parameters (and hence, the base points) is given by $\mathbf{b} = \{\alpha, t, n\} \mapsto \overline{\mathbf{b}} = \{\alpha, t, n+1\}$.

Proof. The proof of this Lemma is a standard direct computation and is omitted, see [DFS19] or [DT18] for similar examples worked out in detail. □

2.3 The Surface Type

Given that our mapping is completely regularized by eight blowups, we know that it should fit into the discrete Painlevé equations framework. To determine the type of the resulting algebraic surface, we need to find the configuration of the irreducible components of (the proper transform of) a bi-degree $(2,2)$ (or *bi-quadratic*) curve Γ on which these points lie. Since the proper transform of Γ for a generic choice of parameters is the unique *anti-canonical divisor* (i.e., the polar divisor of a symplectic form ω), we denote it by $-K_{\mathcal{X}}$. We also denote by η the projection mapping back to $\mathbb{P}^1 \times \mathbb{P}^1$,

$$\eta : \mathcal{X}_{\mathbf{b}} = \mathrm{Bl}_{q_1 \cdots q_8}(\mathbb{P}^1 \times \mathbb{P}^1) \to \mathbb{P}^1 \times \mathbb{P}^1.$$

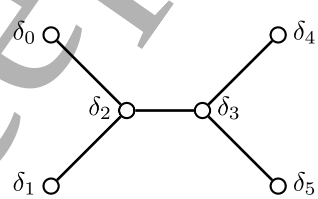
Lemma 5. *Base points* $q_1, \ldots, q_8$ *of the mapping* (2.1) *lie on the bi-quadratic curve* Γ *given in the affine chart by the equation* $x = 0$ *(the homogeneous equation of* Γ *is* $x^0 x^1 y^0 y^1 = 0$*, where* $x = x^0/x^1$ *and* $y = y^0/y^1$*, so* Γ *is indeed bi-quadratic); note that some points come in infinitely-close degeneration cascades. The irreducible components* d_i *of the* proper transform $-K_{\mathcal{X}}$ *of* Γ*,*

$$-K_{\mathcal{X}} = 2H_x + 2H_y - F_1 - \cdots - F_8 = d_0 + d_1 + 2d_2 + 2d_3 + d_4 + d_5,$$

are given by

$$d_0 = H_x - F_1 - F_2,\ d_1 = H_x - F_7 - F_8,\ d_2 = H_y - F_3 - f_4,\ d_3 = F_4 - F_5,\ d_4 = F_3 - F_4,\ d_5 = F_5 - F_6; \tag{2.4}$$

they define the surface root basis $\delta_1, \ldots, \delta_5$ *of* -2*-classes in* $\mathrm{Pic}(\mathcal{X})$ *whose configuration is described by the Dynkin diagram of type* $D_5^{(1)}$*:*



$$\begin{aligned}
\delta_0 &= \mathcal{H}_x - \mathcal{F}_1 - \mathcal{F}_2, & \delta_3 &= \mathcal{F}_4 - \mathcal{F}_5,\\
\delta_1 &= \mathcal{H}_x - \mathcal{F}_7 - \mathcal{F}_8, & \delta_4 &= \mathcal{F}_3 - \mathcal{F}_4,\\
\delta_2 &= \mathcal{H}_y - \mathcal{F}_3 - \mathcal{F}_4, & \delta_5 &= \mathcal{F}_5 - \mathcal{F}_6.
\end{aligned} \tag{2.5}$$

Figure 1: The Surface Root Basis for the Laguerre Weight Recurrence

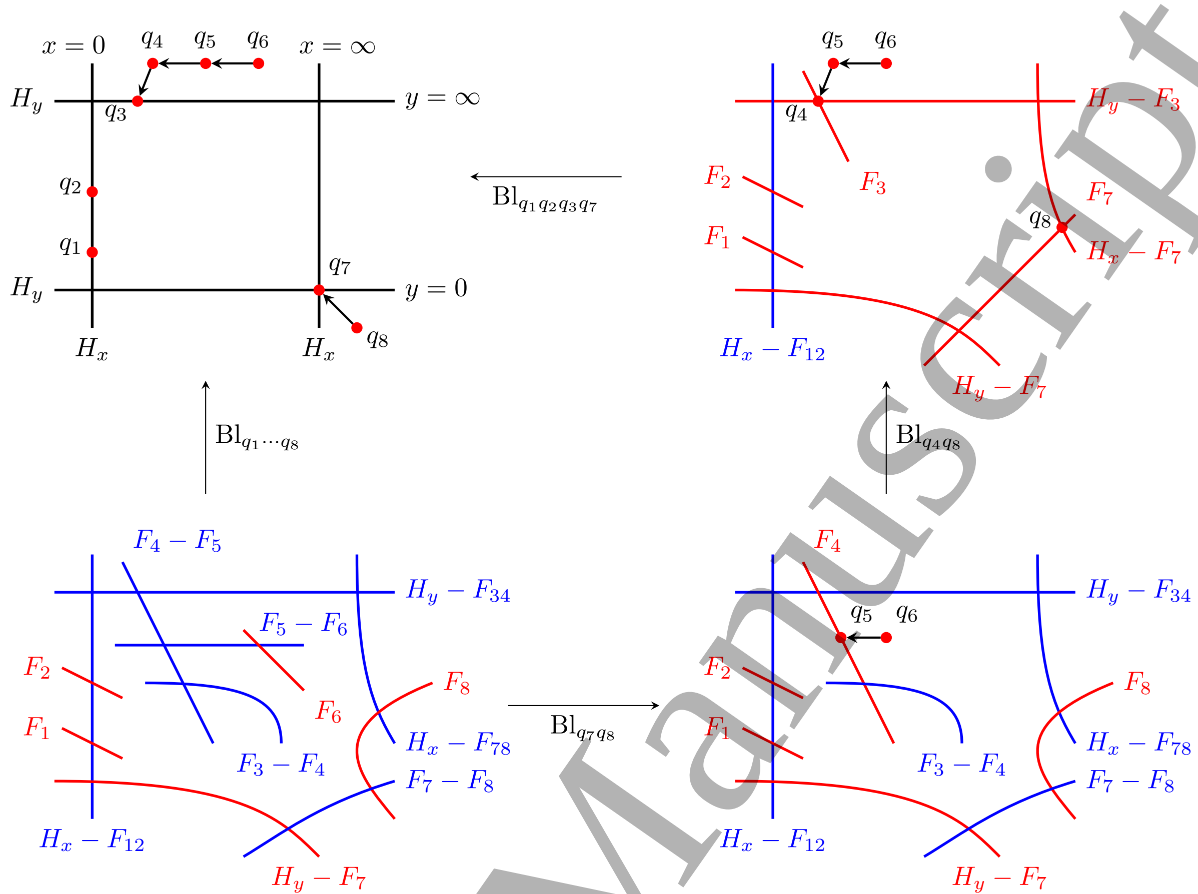


Figure 2: The Sakai Surface for the Laguerre Weight Recurrence ($F_{i \cdots j} = F_i + \cdots + F_j$).

We show some intermediate stages of the blowup process and the resulting $D_5^{(1)}$ surface on Figure 2. Thus our recurrence belongs to the d-P $\left(\mathrm{A}_3^{(1)}/\mathrm{D}_5^{(1)}\right)$ family with the symmetry group $\widetilde{W}\left(A_3^{(1)}\right)$. We describe the choice of the standard d-P $\left(\mathrm{A}_3^{(1)}/\mathrm{D}_5^{(1)}\right)$ point configuration, choices of the root bases for the surface and the symmetry sub-lattices, and other data, in the Appendix; we follow [KNY17] in our conventions.

Remark 6. Looking at Figure 2, we notice another -2-curve $F_7 - F_8$ that is disjoint from the irreducible components of the anti-canonical divisor. Such curves form the class Δ^{nod}, see [Sak01, Section 3.3]

2.4 Initial Geometry Identification

The next step in the identification process is to find some change of basis in Pix$(\mathcal{X})$ from the basis $\{\mathcal{H}_x, \mathcal{H}_y, \mathcal{F}_i\}$ to the basis $\{\mathcal{H}_q, \mathcal{H}_p, \mathcal{E}_j\}$ that correspond to the standard geometry configuration that identifies the surface root bases; we refer to this step as *matching the geometry*. At this point there are many possible choices of such basis change, we later may have to adjust it to *match the dynamics*.

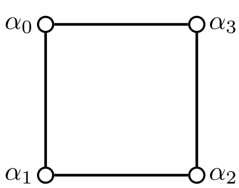
Lemma 7. *The following change of basis of* $\mathrm{Pic}(\mathcal{X})$ *identifies the root bases between the standard* $D_5^{(1)}$ *surface* *and the surface that we obtained for the Laguerre weight recurrence:*

$$
\begin{aligned}
\mathcal{H}_x &= \mathcal{H}_p, & \mathcal{H}_q &= 2\mathcal{H}_x + \mathcal{H}_y - \mathcal{F}_1 - \mathcal{F}_3 - \mathcal{F}_4 - \mathcal{F}_7,\\
\mathcal{H}_y &= \mathcal{H}_q + 2\mathcal{H}_p - \mathcal{E}_1 - \mathcal{E}_3 - \mathcal{E}_5 - \mathcal{E}_6, & \mathcal{H}_p &= \mathcal{H}_x,\\
\mathcal{F}_1 &= \mathcal{H}_p - \mathcal{E}_1, & \mathcal{E}_1 &= \mathcal{H}_x - \mathcal{F}_1,\\
\mathcal{F}_2 &= \mathcal{E}_2, & \mathcal{E}_2 &= \mathcal{F}_2,\\
\mathcal{F}_3 &= \mathcal{H}_p - \mathcal{E}_6, & \mathcal{E}_3 &= \mathcal{H}_x - \mathcal{F}_7,\\
\mathcal{F}_4 &= \mathcal{H}_p - \mathcal{E}_5, & \mathcal{E}_4 &= \mathcal{F}_8,\\
\mathcal{F}_5 &= \mathcal{E}_7, & \mathcal{E}_5 &= \mathcal{H}_x - \mathcal{F}_4,\\
\mathcal{F}_6 &= \mathcal{E}_8, & \mathcal{E}_6 &= \mathcal{H}_x - \mathcal{F}_3,\\
\mathcal{F}_7 &= \mathcal{H}_p - \mathcal{E}_3, & \mathcal{E}_7 &= \mathcal{F}_5,\\
\mathcal{F}_8 &= \mathcal{E}_4, & \mathcal{E}_8 &= \mathcal{F}_6.
\end{aligned}
$$

Proof. This is a direct computation based on comparing the surface root bases on Figure 1 and Figure 5. □

2.5 The Symmetry Roots and the Translations

We are now in a position to compare the dynamics. Note that there are two *non-equivalent* model examples of discrete Painlevé equations, that we label as $[\bar{\mathbf{1}}\mathbf{1}\bar{\mathbf{1}}\mathbf{1}]$ and $[\bar{\mathbf{1}}\mathbf{001}]$, on the $D_5^{(1)}$-surface that are described in Section A.4 in the Appendix. It is interesting that the mapping (1.3) has the *multiplicative-additive* from that looks very similar to the mapping (A.21), but instead it is equivalent to the mapping (A.17) that has the purely *additive* form. To show that, we start with the standard choice of the symmetry root basis (A.3) and use the change of basis in Lemma 7 to get the symmetry roots for the applied problem shown on Figure 3. From the action of ψ_* on $\mathrm{Pic}(\mathcal{X})$ given in Lemma 4 we can now obtain the corresponding translation on



$$
\begin{aligned}
\alpha_0 &= \mathcal{F}_1 - \mathcal{F}_2, & \alpha_2 &= \mathcal{F}_7 - \mathcal{F}_8,\\
\alpha_1 &= \mathcal{H}_y - \mathcal{F}_1 - \mathcal{F}_7, & \alpha_3 &= 2\mathcal{H}_x + \mathcal{H}_y - \mathcal{F}_1 - \mathcal{F}_3 - \mathcal{F}_4 - \mathcal{F}_5 - \mathcal{F}_6 - \mathcal{F}_7.
\end{aligned} \tag{2.6}
$$

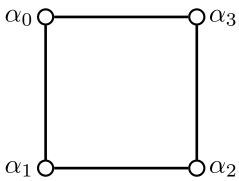
$$\delta = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3.$$

Figure 3: The Symmetry Root Basis for the Laguerre Weight Recurrence (preliminary choice)

the root lattice, decompose it in terms of the generators of the extended affine Weyl symmetry group, and compare the results with the standard mappings φ and ϕ given in Section A.4. We get

$$
\begin{aligned}
\psi_* &: \upalpha = \langle \alpha_0, \alpha_1, \alpha_2, \alpha_3 \rangle \mapsto \psi_*(\upalpha) = \upalpha + \langle 0, -1, 0, 1 \rangle \delta, & \psi &= \sigma_3\sigma_2 w_1 w_2 w_0 w_1,\\
\varphi_* &: \upalpha = \langle \alpha_0, \alpha_1, \alpha_2, \alpha_3 \rangle \mapsto \varphi_*(\upalpha) = \upalpha + \langle 1, -1, 1, -1 \rangle \delta, & \varphi &= \sigma_3\sigma_2 w_3 w_1 w_2 w_0,\\
\phi_* &: \upalpha = \langle \alpha_0, \alpha_1, \alpha_2, \alpha_3 \rangle \mapsto \varphi_*(\upalpha) = \upalpha + \langle -1, 0, 0, 1 \rangle \delta, & \phi &= \sigma_3\sigma_1 w_2 w_1 w_0.
\end{aligned}
$$

From here we immediately see that $\psi = w_1 \circ \varphi \circ w_1^{-1}$ (note that $w_1\sigma_3\sigma_2 = \sigma_3\sigma_2 w_3$ and that w_1 is an involution, $w_1^{-1} = w_1$). Thus, our dynamic is equivalent to the standard equation (A.17) written in [KNY17] but is different from equation (A.21) written in [Sak01] (i.e., $[\mathbf{0}\bar{\mathbf{1}}\mathbf{01}] = [\bar{\mathbf{1}}\mathbf{1}\bar{\mathbf{1}}\mathbf{1}]$). To find the change of variables matching the two equations we first need to adjust our change of basis in $\mathrm{Pic}(\mathcal{X})$ by acting on it by w_1, so that we match not only the geometry, but also the *dynamics*. We do it in the next section.



$$\begin{aligned} \alpha_0 &= \mathcal{H}_y - \mathcal{F}_2 - \mathcal{F}_7, & \alpha_2 &= \mathcal{H}_y - \mathcal{F}_1 - \mathcal{F}_8, \\ \alpha_1 &= \mathcal{F}_1 + \mathcal{F}_7 - \mathcal{H}_y, & \alpha_3 &= 2\mathcal{H}_x + \mathcal{H}_y - \mathcal{F}_1 - \mathcal{F}_3 - \mathcal{F}_4 - \mathcal{F}_5 - \mathcal{F}_6 - \mathcal{F}_7. \\ \delta &= \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3. \end{aligned} \tag{2.7}$$

Figure 4: The Symmetry Root Basis for the Laguerre Weight Recurrence (final choice)

2.6 Final Geometry Identification

Lemma 8. *After the change of basis of* $\mathrm{Pic}(\mathcal{X})$ *given by*

$$\begin{aligned} \mathcal{H}_x &= \mathcal{H}_q + \mathcal{H}_p - \mathcal{E}_5 - \mathcal{E}_6, & \mathcal{H}_q &= 2\mathcal{H}_x + \mathcal{H}_y - \mathcal{F}_1 - \mathcal{F}_3 - \mathcal{F}_4 - \mathcal{F}_7, \\ \mathcal{H}_y &= \mathcal{H}_q + 2\mathcal{H}_p - \mathcal{E}_1 - \mathcal{E}_3 - \mathcal{E}_5 - \mathcal{E}_6, & \mathcal{H}_p &= \mathcal{H}_x + \mathcal{H}_y - \mathcal{F}_1 - \mathcal{F}_7, \\ \mathcal{F}_1 &= \mathcal{H}_q + \mathcal{H}_p - \mathcal{E}_1 - \mathcal{E}_5 - \mathcal{E}_6, & \mathcal{E}_1 &= \mathcal{H}_x - \mathcal{F}_1, \\ \mathcal{F}_2 &= \mathcal{E}_2, & \mathcal{E}_2 &= \mathcal{F}_2, \\ \mathcal{F}_3 &= \mathcal{H}_p - \mathcal{E}_6, & \mathcal{E}_3 &= \mathcal{H}_x - \mathcal{F}_7, \\ \mathcal{F}_4 &= \mathcal{H}_p - \mathcal{E}_5, & \mathcal{E}_4 &= \mathcal{F}_8, \\ \mathcal{F}_5 &= \mathcal{E}_7, & \mathcal{E}_5 &= \mathcal{H}_x + \mathcal{H}_y - \mathcal{F}_1 - \mathcal{F}_4 - \mathcal{F}_7, \\ \mathcal{F}_6 &= \mathcal{E}_8, & \mathcal{E}_6 &= \mathcal{H}_x + \mathcal{H}_y - \mathcal{F}_1 - \mathcal{F}_3 - \mathcal{F}_7, \\ \mathcal{F}_7 &= \mathcal{H}_q + \mathcal{H}_p - \mathcal{E}_3 - \mathcal{E}_5 - \mathcal{E}_6, & \mathcal{E}_7 &= \mathcal{F}_5, \\ \mathcal{F}_8 &= \mathcal{E}_4, & \mathcal{E}_8 &= \mathcal{F}_6. \end{aligned}$$

the recurrence relations (1.3) *for variables* x_n *and* y_n *coincides with the discrete Painlevé equation given by* (A.17)*. The resulting identification of the symmetry root bases (the surface root bases do not change) is shown in Figure 4.*

Next we need to realize this change of basis on $\mathrm{Pic}(\mathcal{X})$ by an explicit change of coordinates. For that, it is convenient to first match the parameters between the applied problem and the reference example. This is done with the help of the *Period Map*.

2.7 The Period Map and the Identification of Parameters

For the root variable parameterization, let us consider a generic point configuration corresponding to the geometry of Figure 2. Using the action of the $\mathbf{PGL}_2(\mathbb{C}) \times \mathbf{PGL}_2(\mathbb{C})$ gauge group we can put $H_x - F_1 - F_2 = V(x)$, $H_x - F_7 - F_8 = V(X)$, $H_y - F_3 - F_4 = V(Y)$, and $q_7(\infty, 0)$. This leaves the scale freedom on the coordinates x and y; we use the scaling in the x-coordinate to put $q_3(1, \infty)$. Then our point configuration can be described in terms of generic parameters c_i as

$$q_1(0, c_1),\ q_2(0, c_2),\ q_3(1, \infty) \leftarrow q_4(u_3 = 0, v_3 = 0) \leftarrow q_5(c_5, 0) \leftarrow q_6(c_6, 0),\ q_7(\infty, 0) \leftarrow q_8(U_8 = 0, V_8 = 0)$$

with the remaining scaling gauge action in the y-coordinate given by

$$\begin{pmatrix} c_1 & c_2 \\ c_5 & c_6 \end{pmatrix}; \begin{matrix} x \\ y \end{pmatrix} \sim \begin{pmatrix} \lambda c_1 & \lambda c_2 \\ \lambda c_5 & \lambda^2 c_6 \end{matrix}; \begin{matrix} x \\ \lambda y \end{pmatrix}, \quad \lambda \neq 0.$$

It is immediate that the points q_i lie on the polar divisor of a symplectic form given in the affine (x, y) chart by $\omega = k\frac{dx \wedge dy}{x}$. We then have the following Lemma.

Lemma 9.

(i) The residues of the symplectic form $\omega = k\frac{dx\wedge dy}{x}$ along the irreducible components of the polar divisor are given by

$$\operatorname{res}_{d_0}\omega = k dy, \qquad \operatorname{res}_{d_2}\omega = 0, \qquad \operatorname{res}_{d_4}\omega = -k\frac{dv_3}{v_3^2},$$
$$\operatorname{res}_{d_1}\omega = -k dy, \qquad \operatorname{res}_{d_3}\omega = -3k\, dU_4, \qquad \operatorname{res}_{d_5}\omega = k\frac{3\, dU_5}{c_5}.$$

(ii) The root variables are given by

$$a_0 = -kc_2, \qquad a_1 = kc_1, \qquad a_2 = -kc_1, \qquad a_3 = k\left(-c_1 + c_5 - \frac{c_6}{c_5}\right), \tag{2.8}$$

and so the root variables are constrained by $a_1 + a_2 = 0$. Without loss of generality we can put $k = -1$ and then use the λ gauge scaling to ensure the standard normalization condition $a_0 + a_1 + a_2 + a_3 = 1$. Then we get

$$c_2 = a_0, \qquad c_1 = -a_1 = a_2, \qquad c_6 = c_5(1 - c_1 - c_2 + c_5), \tag{2.9}$$

which shows that the application parameters are in fact generic for this point configuration; putting $n = a_2$ and $\alpha = a_0 - a_2$, as well as denoting c_6 by t, establishes this equivalence. Note that the parameter evolution is now consistent between the root variables and the application parameters; $\overline{n} = n+1$, $\overline{\alpha} = \alpha$, and $\overline{t} = t$.

2.8 The Change of Coordinates

We are now ready to prove Theorem 1. Note that at this point we have not shown that the parameter t in (A.17) is the same as in (1.3), so we continue working with generic parameters c_i from the previous section.

Proof. (**Theorem 5**) The proof is standard, and so we only outline the key steps. From the linear change of basis on $\operatorname{Pic}(\mathcal{X})$ given in Lemma 8, we see that x is a projective coordinate on a pencil of $(1,1)$ curves in the (q,p)-plane passing through the points p_5 and p_6, and y is a projective coordinate on a pencil of $(1,2)$ curves in the (q,p)-plane passing through the points p_1, p_3, p_5, and p_6. The bases for these pencils are given by the curves with affine defining polynomials $\{q, qp - a_1\}$ and $\{1, p(q(p+t) - a_1)\}$, i.e.,

$$x = \frac{Aq + B(qp - a_1)}{Cq + D(qp - a_1)}, \qquad y = \frac{K + Lp(q(p+t) - a_1)}{M + Np(q(p+t) - a_1)}.$$

Using the correspondence between the exceptional divisor classes for F_i, $i = 1,2,3,4,7,8$ allows us to fix the values of the coefficients $A, \ldots, N$ to get $x = \frac{q(p+t)+a_2}{qp+a_2}$ and $y = \frac{(p+t)(qp+a_2)}{t}$. Moreover, the correspondence $F_7 - F_8 = H_q + H_p - E_3 - E_4 - E_5 - E_7$ imposes the $a_1 + a_2 = 0$ constraint, and the condition that $F_5 - F_6 = F_7 - F_8$ shows that $c_5 = t$, as expected. The inverse change of variables is obtained along the same lines. □

A Discrete Painlevé Equations in the d-P $\left(A_3^{(1)}/D_5^{(1)}\right)$ family

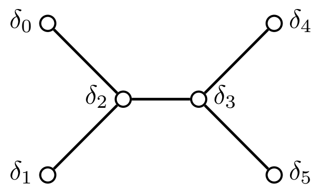
To make this paper self-contained, we collect in this Appendix some of the basic facts about the geometry of the $D_5^{(1)}$-family of Sakai surfaces and some standard discrete Painlevé equations associated with this surface family. The computations here are standard (see [KNY17], [DT18], [DFS19]) and are mostly omitted. We use (q,p)-coordinates for the standard example and follow the standard reference [KNY17] for the choice of the standard point configuration and the root bases.

A.1 The Point Configuration

We start with the root basis of the surface sub-lattice that is given by the classes δ_i of the irreducible components of the anti-canonical divisor

$$\delta = -\mathcal{K}_{\mathcal{X}} = 2\mathcal{H}_f + 2\mathcal{H}_g - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3 - \mathcal{E}_4 - \mathcal{E}_5 - \mathcal{E}_6 - \mathcal{E}_7 - \mathcal{E}_8 = \delta_0 + \delta_1 + 2\delta_2 + 2\delta_3 + \delta_4 + \delta_5.$$

The intersection configuration of those roots is given by the Dynkin diagram of type $D_5^{(1)}$, as shown on Figure 5.



$$\delta_0 = \mathcal{E}_1 - \mathcal{E}_2, \qquad \delta_3 = \mathcal{H}_p - \mathcal{E}_5 - \mathcal{E}_7,$$
$$\delta_1 = \mathcal{E}_3 - \mathcal{E}_4, \qquad \delta_4 = \mathcal{E}_5 - \mathcal{E}_6, \tag{A.1}$$
$$\delta_2 = \mathcal{H}_q - \mathcal{E}_1 - \mathcal{E}_3, \qquad \delta_5 = \mathcal{E}_7 - \mathcal{E}_8.$$

Figure 5: The Surface Root Basis for the standard d-P $\left(\mathrm{D}_5^{(1)}\right)$ point configuration

Using the action of the $\mathbf{PGL}_2(\mathbb{C}) \times \mathbf{PGL}_2(\mathbb{C})$ gauge group we can put divisors d_2 and d_3, with $\delta_i = [d_i]$, to be

$$d_2 = V(Q) = \{q = \infty\}, \qquad d_3 = V(P) = \{p = \infty\}.$$

This reduces the gauge group action to that of a four-parameter subgroup, $(q, p) \mapsto (\lambda q + \mu, \zeta p + \xi)$. The corresponding point configuration and the Sakai surface are shown on Figure 6.

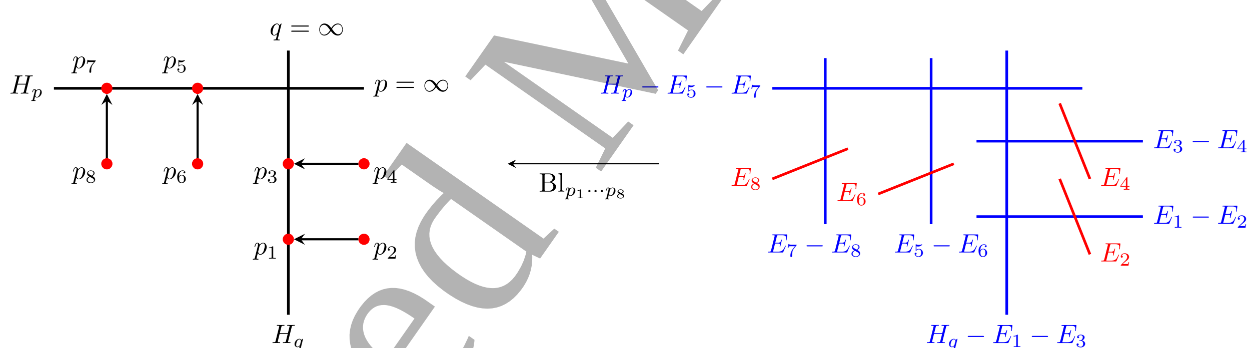


Figure 6: The model Sakai Surface for the d-$P\left(A_3^{(1)}/D_5^{(1)}\right)$ example

This point configuration can be parameterized by eight parameters $b_1, \ldots, b_8$ as follows:

$$p_1(\infty, b_1) \leftarrow p_2(\infty, b_1; q(p - b_1) = b_2), \quad p_5(b_5, \infty) \leftarrow p_6(b_5, \infty; (q - b_5)p = b_6),$$
$$p_3(\infty, b_3) \leftarrow p_4(\infty, b_3; q(p - b_3) = b_4), \quad p_7(b_7, \infty) \leftarrow p_8(b_7, \infty; (q - b_7)p = b_8).$$

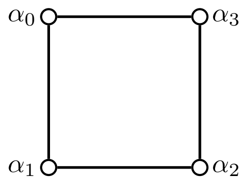
The four-parameter gauge group above acts on these configurations via

$$\begin{pmatrix} b_1 & b_2 & b_3 & b_4 ; & q \\ b_5 & b_6 & b_7 & b_8 ; & p \end{pmatrix} \sim \begin{pmatrix} \zeta b_1 + \xi & \lambda\zeta b_2 & \zeta b_3 + \xi & \lambda\zeta b_4 ; & \lambda q + \mu \\ \lambda b_5 + \mu & \lambda\zeta b_6 & \lambda b_7 + \mu & \lambda\zeta b_8 ; & \zeta g + \xi \end{pmatrix}, \ \lambda, \zeta \neq 0, \tag{A.2}$$

and so the true number of parameters is four. The correct gauge-invariant parameterization is given by the *root variables* that we now describe.

A.2 The Period Map and the Root Variables

To define the root variables we begin by choosing a root basis in the *symmetry sub-lattice* $Q = \Pi(R^{\perp}) \triangleleft \mathrm{Pic}(\mathcal{X})$ and defining the symplectic form ω whose polar divisor $-K_{\mathcal{X}}$ is the configuration of -2-curves shown on Figure 6. For the symmetry root basis we take the same basis as in [KNY17], see Figure 7.



$$\begin{aligned} \alpha_0 &= \mathcal{H}_p - \mathcal{E}_1 - \mathcal{E}_2, & \alpha_2 &= \mathcal{H}_p - \mathcal{E}_3 - \mathcal{E}_4, \\ \alpha_1 &= \mathcal{H}_q - \mathcal{E}_5 - \mathcal{E}_6, & \alpha_3 &= \mathcal{H}_q - \mathcal{E}_7 - \mathcal{E}_8. \end{aligned} \qquad \delta = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3. \tag{A.3}$$

Figure 7: The Standard Root Basis for the d-$P\left(A_3^{(1)}\right)$ Symmetry Sub-lattice

A symplectic form $\omega \in -\mathcal{K}_{\mathcal{X}}$ such that $[\omega] = \delta_0 + \delta_1 + 2\delta_2 + 2\delta_3 + \delta_4 + \delta_5$ can be given in local coordinate charts as

$$\omega = k dq \wedge dp = -k\frac{dQ \wedge dp}{Q^2} = -k\frac{dq \wedge dP}{P^2} = k\frac{dQ \wedge dP}{Q^2 P^2} = -k\frac{du_i \wedge dv_i}{u_i} = -k\frac{dU_j \wedge dV_j}{V_j}, \tag{A.4}$$

where, as usual, $Q = 1/q$, $P = 1/p$ are the coordinates centered at infinity, the blowup coordinates u_i, v_i at the points p_i, $i = 1, 3$, are given by $Q = u_i$, $p = b_i + u_i v_i$, and the blowup coordinates (U_j, V_j) at the points p_j, $j = 5, 7$, are given by $q = b_j + U_j V_j$ and $P = V_j$; k is some non-zero proportionality constant that we normalize later. Then we have the following Lemma.

Lemma 10.

(i) The residue of the symplectic form ω along the irreducible components of the polar divisor is given by

$$\mathrm{res}_{d_0}\,\omega = -k\,dv_1,\ \mathrm{res}_{d_1}\,\omega = -k\,dv_3,\ \mathrm{res}_{d_2}\,\omega = \mathrm{res}_{d_3}\,\omega = 0,\ \mathrm{res}_{d_4}\,\omega = k\,dU_5,\ \mathrm{res}_{d_4}\,\omega = k\,dU_7. \tag{A.5}$$

(ii) The root variables a_i are given by

$$a_0 = kb_2, \quad a_1 = -kb_6, \quad a_2 = kb_4, \quad a_3 = -kb_8. \tag{A.6}$$

It is convenient to take $k = -1$. We can then use the gauge action (A.2) to normalize $b_3 = b_5 = 0$, $b_7 = 1$, and $\chi(\delta) = a_0 + a_1 + a_2 + a_3 = 1$. In view of the relation of this example to differential Painlevé equations, it is also convenient to denote b_1 by $-t$. Then we get the following parameterization of this point configuration in terms of root variables:

$$b_1 = -t, \quad b_2 = -a_0, \quad b_3 = 0, \quad b_4 = -a_2, \quad b_5 = 0, \quad b_6 = a_1, \quad b_7 = 1, \quad b_8 = a_3. \tag{A.7}$$

Note that if we use the notation

$$p_{12}\left(\frac{1}{\varepsilon}, -t - \varepsilon a_0\right), \quad p_{34}\left(\frac{1}{\varepsilon}, -\varepsilon a_2\right), \quad p_{56}\left(a_1\varepsilon, \frac{1}{\varepsilon}\right), \quad p_{78}\left(1 + a_3\varepsilon, \frac{1}{\varepsilon}\right),$$

and impose the normalization $a_0 + a_1 + a_2 + a_3 = 1$, we get exactly the parameterization of the point configuration in section 8.2.18 of [KNY17].

A.3 The Extended Affine Weyl Symmetry Group

We now describe the birational representation of the extended affine Weyl symmetry group $\widetilde{W}\left(A_3^{(1)}\right) = \operatorname{Aut}\left(A_3^{(1)}\right) \ltimes W\left(A_3^{(1)}\right)$, which is a *semi-direct product* of the usual affine Weyl group $W\left(A_3^{(1)}\right)$ and the group of Dynkin diagram automorphisms $\operatorname{Aut}\left(A_3^{(1)}\right) \simeq \mathbb{D}_4$.

The abstract affine Weyl group $W\left(A_3^{(1)}\right)$ is defined in terms of generators $w_i = w_{\alpha_i}$ and relations that are encoded by the affine Dynkin diagram $A_3^{(1)}$,

$$W\left(A_3^{(1)}\right) = W\left(\begin{array}{ccc} \alpha_0 \circ & \text{—} & \circ\, \alpha_3 \\ | & & | \\ \alpha_1 \circ & \text{—} & \circ\, \alpha_2 \end{array}\right) = \left\langle w_0, \dots, w_4 \;\middle|\; \begin{array}{ll} w_i^2 = e, \quad w_i \circ w_j = w_j \circ w_i & \text{when } \overset{\circ}{\alpha_i}\ \overset{\circ}{\alpha_j} \\ w_i \circ w_j \circ w_i = w_j \circ w_i \circ w_j & \text{when } \underset{\alpha_i\ \alpha_j}{\circ\!\!-\!\!\circ} \end{array} \right\rangle.$$

The natural action of this group on $\operatorname{Pic}(\mathcal{X})$ is given by reflections in the roots α_i,

$$w_i(\mathcal{C}) = w_{\alpha_i}(\mathcal{C}) = \mathcal{C} - 2\frac{\mathcal{C} \bullet \alpha_i}{\alpha_i \bullet \alpha_i}\alpha_i = \mathcal{C} + (\mathcal{C} \bullet \alpha_i)\,\alpha_i, \qquad \mathcal{C} \in \operatorname{Pic}(\mathcal{X}), \tag{A.8}$$

which can be extended to an action on point configurations by elementary birational maps (which lifts to isomorphisms $w_i : \mathcal{X}_{\mathbf{b}} \to \mathcal{X}_{\overline{\mathbf{b}}}$ on the family of Sakai's surfaces), this is known as a birational representation of $W\left(A_3^{(1)}\right)$.

Theorem 11. *Reflections w_i on $\operatorname{Pic}(\mathcal{X})$ are induced by the elementary birational mappings given below, and also denoted by w_i, on the family $\mathcal{X}_{\mathbf{b}}$. To ensure the group structure, we require that each mapping preserves our normalization, and so it is enough to describe the mappings in terms of the root variables (note that the parameter t can also change when we consider the Dynkin diagram automorphisms, so it is convenient to include it among the root variables):*

$$w_0 : \begin{pmatrix} a_0 & a_1 \\ a_2 & a_3 \end{pmatrix}; t ; \begin{matrix} q \\ p \end{matrix} \mapsto \begin{pmatrix} -a_0 & a_0 + a_1 \\ a_2 & a_0 + a_3 \end{pmatrix}; t ; \begin{matrix} q + \dfrac{a_0}{p+t} \\ p \end{matrix}, \tag{A.9}$$

$$w_1 : \begin{pmatrix} a_0 & a_1 \\ a_2 & a_3 \end{pmatrix}; t ; \begin{matrix} q \\ p \end{matrix} \mapsto \begin{pmatrix} a_0 + a_1 & -a_1 \\ a_1 + a_2 & a_3 \end{pmatrix}; t ; \begin{matrix} q \\ p - \dfrac{a_1}{q} \end{matrix}, \tag{A.10}$$

$$w_2 : \begin{pmatrix} a_0 & a_1 \\ a_2 & a_3 \end{pmatrix}; t ; \begin{matrix} q \\ p \end{matrix} \mapsto \begin{pmatrix} a_0 & a_1 + a_2 \\ -a_2 & a_2 + a_3 \end{pmatrix}; t ; \begin{matrix} q + \dfrac{a_2}{p} \\ p \end{matrix}, \tag{A.11}$$

$$w_3 : \begin{pmatrix} a_0 & a_1 \\ a_2 & a_3 \end{pmatrix}; t ; \begin{matrix} q \\ p \end{matrix} \mapsto \begin{pmatrix} a_0 + a_3 & a_1 \\ a_2 + a_3 & -a_3 \end{pmatrix}; t ; \begin{matrix} q \\ p - \dfrac{a_3}{q-1} \end{matrix}. \tag{A.12}$$

It is clear that the group of Dynkin diagram automorphisms $\operatorname{Aut}\left(A_3^{(1)}\right) \simeq \mathbb{D}_4$, so we only describe two generators σ_1, σ_2, as well as one more automorphism σ_3 that we need.

Theorem 12. *Consider the automorphisms $\sigma_1, \dots, \sigma_3$ of $\operatorname{Aut}\left(A_3^{(1)}\right)$ that act on the symmetry and the surface root bases as follows (here we use the standard cycle notations for permutations):*

$$\sigma_1 = (\alpha_0\alpha_3)(\alpha_1\alpha_2) = (\delta_0\delta_5)(\delta_1\delta_4)(\delta_2\delta_3), \qquad \sigma_2 = (\alpha_0\alpha_2) = (\delta_0\delta_1), \qquad \sigma_3 = (\alpha_1\alpha_3) = (\delta_4\delta_5). \tag{A.13}$$

Then σ_i act on the Picard lattice as

$$\sigma_1 = (\mathcal{E}_1\mathcal{E}_7)(\mathcal{E}_2\mathcal{E}_8)(\mathcal{E}_3\mathcal{E}_5)(\mathcal{E}_4\mathcal{E}_6)w_\rho, \qquad \sigma_2 = (\mathcal{E}_1\mathcal{E}_3)(\mathcal{E}_2\mathcal{E}_4), \qquad \sigma_3 = (\mathcal{E}_5\mathcal{E}_7)(\mathcal{E}_6\mathcal{E}_8),$$

where w_ρ is a reflection (A.8) in the root $\rho = \mathcal{H}_q - \mathcal{H}_p$ (note also that a transposition $(\mathcal{E}_i\mathcal{E}_j)$ is induced by a reflection in the root $\mathcal{E}_i - \mathcal{E}_j$). The induced elementary birational mappings are then given by the following expressions:

$$\sigma_1 : \begin{pmatrix} a_0 & a_1 \\ a_2 & a_3 \end{pmatrix} ; t ; \begin{matrix} q \\ p \end{matrix} \mapsto \begin{pmatrix} a_3 & a_2 \\ a_1 & a_0 \end{pmatrix} ; -t ; \begin{matrix} -\frac{p}{t} \\ qt \end{matrix}, \tag{A.14}$$

$$\sigma_2 : \begin{pmatrix} a_0 & a_1 \\ a_2 & a_3 \end{pmatrix} ; t ; \begin{matrix} q \\ p \end{matrix} \mapsto \begin{pmatrix} a_2 & a_1 \\ a_0 & a_3 \end{pmatrix} ; -t ; \begin{matrix} q \\ p+t \end{matrix}, \tag{A.15}$$

$$\sigma_3 : \begin{pmatrix} a_0 & a_1 \\ a_2 & a_3 \end{pmatrix} ; t ; \begin{matrix} q \\ p \end{matrix} \mapsto \begin{pmatrix} a_0 & a_3 \\ a_2 & a_1 \end{pmatrix} ; -t ; \begin{matrix} 1-q \\ -p \end{matrix}. \tag{A.16}$$

Finally, the semi-direct product structure is defined by the action of $\sigma \in \mathrm{Aut}\left(A_3^{(1)}\right)$ on $W\left(A_3^{(1)}\right)$ via $w_{\sigma(\alpha_i)} = \sigma w_{\alpha_i} \sigma^{-1}$.

A.4 Some standard discrete d-P $\left(\mathrm{A}_3^{(1)}/\mathrm{D}_5^{(1)}\right)$ equations

There are infinitely many different discrete Painlevé equations of the same type corresponding to the non-conjugate translations in the affine symmetry sub-lattice Q. Of those, we are interested in two particular equations that correspond to short translation vectors. One is equation (8.23) in [KNY17, Section 8.1.17], the other is the so-called d-P_{IV} equation in [Sak01], which also appears in a slightly different form (2.33–2.34) in [Sak07]. We label these equations by $[\bar{\mathbf{1}}\mathbf{1}\bar{\mathbf{1}}\mathbf{1}]$ and $[\bar{\mathbf{1}}\mathbf{001}]$ respectively, based on the induced action of the dynamics on the symmetry roots (see below), which is unambiguous. In the above references these equations are presented in a geometric way as mappings, similar to our approach. However, both classes of equations were obtained earlier by Basil Grammaticos, Alfred Ramani, and their collaborators using the singularity confinement approach; in their papers these equations are presented as recurrences with particular coefficient evolution. Equation $[\bar{\mathbf{1}}\mathbf{1}\bar{\mathbf{1}}\mathbf{1}]$ first appeared in [GNP⁺94] (where it was shown that this equation actually has d-$\mathrm{P}_{\mathrm{III}}$ and not d-P_{IV} as a continuous limit) and equation $[\bar{\mathbf{1}}\mathbf{001}]$ first appeared in [GORS98]; see also [TGR02] where equations (3.1–3.2) is essentially the mapping (A.17) and equations (3.24ab) is essentially the mapping (A.21)*.

Note that equations $[\bar{\mathbf{1}}\mathbf{1}\bar{\mathbf{1}}\mathbf{1}]$ and $[\bar{\mathbf{1}}\mathbf{001}]$ *are not equivalent* — this can be seen, for example, from the length of the corresponding words in the extended affine Weyl group, or from the lengths of the corresponding translations, or, probably in the simplest possible way, by computing the Jordan form of the matrix description of the evolution on $\mathrm{Pic}(\mathcal{X})$.

A.4.1 The $[\bar{\mathbf{1}}\mathbf{1}\bar{\mathbf{1}}\mathbf{1}]$ discrete Painlevé equation on the $D_5^{(1)}$ surface

In [KNY17], the standard example of a discrete Painlevé equation on the $D_5^{(1)}$-surface is given in Section 8.1.17 equation (8.23), and it has the following *additive* form, when written in coordinates (q,p):

$$\bar{q} + q = 1 - \frac{a_2}{p} - \frac{a_0}{p+t}, \qquad p + \underline{p} = -t + \frac{a_1}{q} + \frac{a_3}{q-1} \tag{A.17}$$

with the root variable evolution and normalization given by

$$\bar{a}_0 = a_0 + 1, \quad \bar{a}_1 = a_1 - 1, \quad \bar{a}_2 = a_2 + 1, \quad \bar{a}_3 = a_3 - 1, \qquad a_0 + a_1 + a_2 + a_3 = 1. \tag{A.18}$$

For this equation, the geometry of the corresponding point configuration is shown on Figure 6, with the parameterization by the root variables is given by (A.7). From the root variable evolution (A.18) we immediately see that the corresponding translation on the root lattice is

$$\varphi_* : \alpha = \langle \alpha_0, \alpha_1, \alpha_2, \alpha_3 \rangle \mapsto \varphi_*(\alpha) = \alpha + \langle -1, 1, -1, 1 \rangle \delta, \tag{A.19}$$

*We thank A. Ramani for his help with historical references.

which explains our labeling for this equation (we use $\overline{1}$ instead of -1 for compactness). Using the standard techniques, see [DT18] for a detailed example, we get the following decomposition of φ in terms of the generators of $\widetilde{W}\left(A_3^{(1)}\right)$:

$$\varphi = \sigma_3\sigma_2 w_3 w_1 w_2 w_0. \tag{A.20}$$

Note that equations (A.17) naturally define two *half-maps*, $\varphi_1 : (q,p) \to (\overline{q}, -p)$ and $\varphi_2 : (q,p) \to (q, -\underline{p})$ (the additional negative sign here is related to the Möbius group gauge action as explained in [DFS19, Section 2.9]), and the mapping φ that we are interested in is $\varphi = (\overline{\varphi_2})^{-1} \circ \varphi_1$. These individual mappings decompose as $\varphi_1 = \sigma_3 w_2 w_0$ and $\varphi_3 = \sigma_2 w_3 w_1$.

A.4.2 The $[\overline{1}001]$ discrete Painlevé equation on the $D_5^{(1)}$ surface

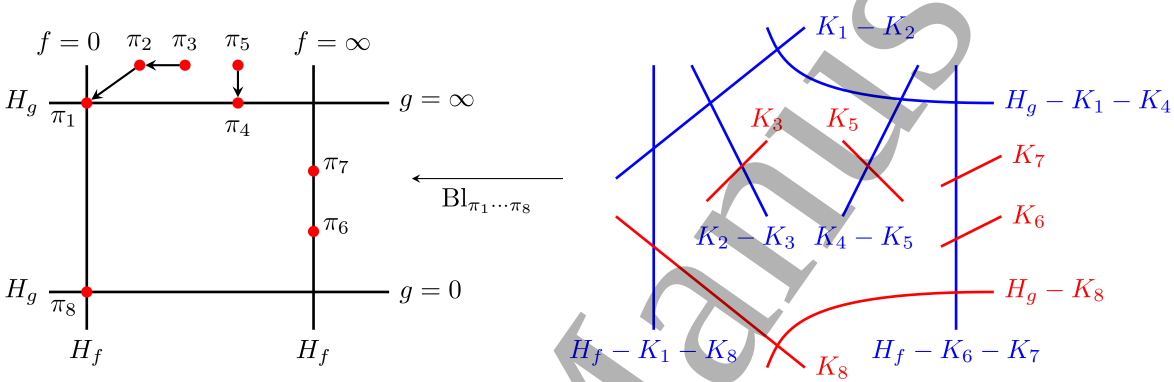


Figure 8: The Sakai Surface for the d-$\mathrm{P_{IV}}$ example

In [Sak01], the following mapping $\phi : (f,g) \to (\overline{f}, \overline{g})$, written in the *multiplicative-additive* form, is called a d-$\mathrm{P_{IV}}$ equation on the $D_5^{(1)}$ surface:

$$\overline{f}f = \frac{s\overline{g}}{(\overline{g} - a_3 + \lambda)(\overline{g} + a_0 + \lambda)}, \qquad \overline{g} + g = \frac{s}{f} + \frac{a_1 + a_0}{1 - f} - \lambda + a_3 - a_0, \tag{A.21}$$

where $\lambda = a_0 + a_1 + a_2 + a_3$ (without loss of generality it can be normalized to $\lambda = 1$), and the root variable evolution is given by $\overline{a}_0 = a_0 + \lambda$ and $\overline{a}_3 = a_3 - \lambda$. From the root variable evolution we see that the corresponding translation on the root lattice is

$$\phi_* : \upalpha = \langle \alpha_0, \alpha_1, \alpha_2, \alpha_3 \rangle \mapsto \varphi_*(\upalpha) = \upalpha + \langle -1, 0, 0, 1 \rangle \delta. \tag{A.22}$$

This map can be written in terms of generators as

$$\phi = \sigma_3\sigma_1 w_2 w_1 w_0 = w_3 w_2 w_1 \sigma_3 \sigma_1, \tag{A.23}$$

which is the same as given in Sakai's paper. However, the geometry of that example is slightly different from our reference model on Figure 6 and is given on Figure 8. This geometry can be matched to the standard one with the change of basis on $\operatorname{Pic}(\mathcal{X})$ given by

$$\begin{aligned}
\mathcal{H}_f &= \mathcal{H}_q + \mathcal{H}_p - \mathcal{E}_1 - \mathcal{E}_2, & \mathcal{H}_q &= \mathcal{H}_f + \mathcal{H}_g - \mathcal{K}_6 - \mathcal{K}_8,\\
\mathcal{H}_g &= \mathcal{H}_q + \mathcal{H}_p - \mathcal{E}_1 - \mathcal{E}_7, & \mathcal{H}_p &= \mathcal{H}_f + \mathcal{H}_g - \mathcal{K}_1 - \mathcal{K}_6,\\
\mathcal{K}_1 &= \mathcal{H}_q - \mathcal{E}_1, & \mathcal{E}_1 &= \mathcal{H}_f + \mathcal{H}_g - \mathcal{K}_1 - \mathcal{K}_6 - \mathcal{K}_8,\\
\mathcal{K}_2 &= \mathcal{E}_3, & \mathcal{E}_2 &= \mathcal{H}_g - \mathcal{K}_6,\\
\mathcal{K}_3 &= \mathcal{E}_4, & \mathcal{E}_3 &= \mathcal{K}_2,\\
\mathcal{K}_4 &= \mathcal{E}_5, & \mathcal{E}_4 &= \mathcal{K}_3,\\
\mathcal{K}_5 &= \mathcal{E}_6, & \mathcal{E}_5 &= \mathcal{K}_4,\\
\mathcal{K}_6 &= \mathcal{H}_q + \mathcal{H}_p - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_7, & \mathcal{E}_6 &= \mathcal{K}_5,\\
\mathcal{K}_7 &= \mathcal{E}_8, & \mathcal{E}_7 &= \mathcal{H}_f - \mathcal{K}_6,\\
\mathcal{K}_8 &= \mathcal{H}_p - \mathcal{E}_1, & \mathcal{E}_8 &= \mathcal{K}_7.
\end{aligned}$$

Note that this change of basis is chosen in such a way as to match the root variables between the two examples, however the parameters s and t differ by a sign, $s = -t$. The corresponding change of variables is given by

$$\begin{cases} f(q,p) = -\dfrac{p+t}{(q-1)(p+t)+a_0}, \\ g(q,p) = (q-1)(p+t), \end{cases} \qquad \begin{cases} q(f,g) = 1 - \dfrac{g}{f(g+a_0)}, \\ p(f,g) = s - f(g+a_0). \end{cases}$$

Acknowledgements

Yang Chen and Hu Jie were supported by the Macau Science and Technology Development Fund under grant numbers FDCT 130/2014/A3, FDCT 023/2017/A1 and by the University of Macau under grant numbers MYRG 2014-00011-FST, MYRG 2014-00004-FST. Part of this work was done when Anton Dzhamay visited Shanghai University and the University of Macau and he would like to thank both of these Universities for their support and hospitality. We also thank Alfred Ramani, Tomoyuki Takenawa, and Ralph Willox for helpful comments and discussions.

References

[AB06] D. Arinkin and A. Borodin, *Moduli spaces of d-connections and difference Painlevé equations*, Duke Math. J. **134** (2006), no. 3, 515–556.

[AB07] D Arinkin and A. Borodin, *Tau-function of discrete isomonodromy transformations and probability*, 2007.

[BB03] Alexei Borodin and Dmitriy Boyarchenko, *Distribution of the first particle in discrete orthogonal polynomial ensembles*, Comm. Math. Phys. **234** (2003), no. 2, 287–338.

[BC09] Estelle Basor and Yang Chen, *Painlevé V and the distribution function of a discontinuous linear statistic in the Laguerre unitary ensembles*, J. Phys. A **42** (2009), no. 3, 035203, 18.

[Bor03] Alexei Borodin, *Discrete gap probabilities and discrete Painlevé equations*, Duke Math. J. **117** (2003), no. 3, 489–542.

[DFS19] Anton Dzhamay, Galina Filipuk, and Alexander Stokes, *Recurrence coefficients for discrete orthogonal polynomials with hypergeometric weight and discrete Painlevé equations*, 2019.

[DT18] Anton Dzhamay and Tomoyuki Takenawa, *On some applications of Sakai's geometric theory of discrete Painlevé equations*, SIGMA Symmetry Integrability Geom. Methods Appl. **14** (2018), no. 075, 20.

[GNP+94] B. Grammaticos, F. W. Nijhoff, V. Papageorgiou, A. Ramani, and J. Satsuma, *Linearization and solutions of the discrete Painlevé III equation*, Phys. Lett. A **185** (1994), no. 5-6, 446–452.

[GORS98] B. Grammaticos, Y. Ohta, A. Ramani, and H. Sakai, *Degeneration through coalescence of the q-Painlevé VI equation*, J. Phys. A **31** (1998), no. 15, 3545–3558.

[KNY17] Kenji Kajiwara, Masatoshi Noumi, and Yasuhiko Yamada, *Geometric aspects of Painlevé equations*, J. Phys. A **50** (2017), no. 7, 073001, 164.

[LC17] Shulin Lyu and Yang Chen, *The largest eigenvalue distribution of the Laguerre unitary ensemble*, Acta Math. Sci. Ser. B (Engl. Ed.) **37** (2017), no. 2, 439–462.

[Sak01] Hidetaka Sakai, *Rational surfaces associated with affine root systems and geometry of the Painlevé equations*, Comm. Math. Phys. **220** (2001), no. 1, 165–229.

[Sak07] ______, *Problem: discrete Painlevé equations and their Lax forms*, Algebraic, analytic and geometric aspects of complex differential equations and their deformations. Painlevé hierarchies, RIMS Kôkyûroku Bessatsu, B2, Res. Inst. Math. Sci. (RIMS), Kyoto, 2007, pp. 195–208.

[Sha13] Igor R. Shafarevich, *Basic algebraic geometry 1*, Third ed., Springer, Heidelberg, 2013, Varieties in projective space.

[SKKT00] Karen E. Smith, Lauri Kahanpää, Pekka Kekäläinen, and William Traves, *An invitation to algebraic geometry*, Universitext, Springer-Verlag, New York, 2000.

[Sze67] Gábor Szegő, *Orthogonal polynomials*, third ed., American Mathematical Society, Providence, R.I., 1967, American Mathematical Society Colloquium Publications, Vol. 23.

[TGR02] T. Tokihiro, B. Grammaticos, and A. Ramani, *From the continuous* $\mathrm{P_V}$ *to discrete Painlevé equations*, J. Phys. A **35** (2002), no. 28, 5943–5950.

[VA18] Walter Van Assche, *Orthogonal polynomials and Painlevé equations*, Australian Mathematical Society Lecture Series, vol. 27, Cambridge University Press, Cambridge, 2018.