# Differential and difference equations for recurrence coefficients of orthogonal polynomials with hypergeometric weights and Bäcklund transformations of the sixth Painlevé equation

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**Abstract**

It is known [11] that the recurrence coefficients of discrete orthogonal polynomials on the non-negative integers with hypergeometric weights satisfy a system of non-linear difference equations. There is also a connection to the solutions of the $\sigma$-form of the sixth Painlevé equation (one of the parameters of the weights being an independent variable in the differential equation) [11]. In this paper, we derive a second-order nonlinear difference equation from the system and present explicit formulas showing how it arises from the Bäcklund transformations of the sixth Painlevé equation. We also present an alternative way to derive the connection between the recurrence coefficients and the solutions of the sixth Painlevé equation.

Keywords: Discrete orthogonal polynomials; hypergeometric weights; Painlevé VI; Bäcklund transformations.

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## 1 Introduction

Recently there has been considerable interest in understanding the connection between recurrence coefficients of semi-classical orthogonal polynomials and solutions of discrete or differential Painlevé equations (see, for instance, [18] and numerous references therein). The Painlevé equations, the famous second-order nonlinear differential equations appearing in many applications, possess the so-called Bäcklund transformations which link solutions with various values of the parameters [6]. Discrete Painlevé equations also have many nice properties and they are classified in [16] (see also [14]). Some of the discrete Painlevé equations are obtained from the Bäcklund transformations of the differential Painlevé equations.

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In the study of semi-classical orthogonal polynomials one can obtain non-linear difference equations or systems of such equations for the recurrence coefficients by various methods (e.g., by ladder operators technique [1, 4]). Sometimes, mostly for simple weights, these equations or systems are easily identified with one of the discrete Painlevé equations. However, in the majority of cases, such identification and the necessary change of variables is not easily seen. We show that if one combines such difference equations or systems with the differential Toda-type systems for the recurrence coefficients one can get a (cumbersome) nonlinear differential equations with independent variable being one of the parameters of the weights, which can be identified with the Painlevé equations. Then the non-linear difference equations can be considered as compositions of standard (seed) Bäcklund transformations and explicit formulas can be obtained. Such an approach is rather computational but it helps to understand the connection between the recurrence coefficients of orthogonal polynomials and the solutions of the Painlevé equations much better. Explicit formulas can further be used in applications, for instance, to study asymptotics. This method was successfully applied to orthogonal polynomials with respect to certain weights, both discrete and continuous (see [2, 8, 9, 10]).

The main objective of this paper is to derive a second-order nonlinear difference equation from the non-linear difference system, which was obtained in [11] for the recurrence coefficients of discrete orthogonal polynomials on the non-negative integers with certain hypergeometric weights, and to present explicit formulas showing how it arises from the Bäcklund transformations of the the sixth Painlevé equation. The equation is quite cumbersome and we only present the way to derive it, which can be repeated in any computer algebra system. In addition, we present an alternative way to derive the connection between the recurrence coefficients for the weights in [11] and the solutions of the sixth Painlevé equation.

# 2 Preliminaries

## 2.1 The sixth Painlevé equation and its Bäcklund transformations

The sixth Painlevé equation is given by

$$\begin{aligned} y'' =& \frac{1}{2}\left(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-t}\right)y'^2-\left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{y-t}\right)y' \\ &+\frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha_6+\beta_6\frac{t}{y^2}+\gamma_6\frac{(t-1)}{(y-1)^2}+\delta_6\frac{t(t-1)}{(y-t)^2}\right), \end{aligned} \tag{1}$$

where $' = d/dt$ and $\alpha_6$, $\beta_6$, $\gamma_6$, $\delta_6$ are arbitrary complex parameters. There also exists the so-called $\sigma$-form of the sixth Painlevé equation, see [15]. The sixth Painlevé equation has the following transformations, see [6].

- Symmetries.

Solutions of equation (1) are invariant with respect to symmetry transformations which form a group generated by transformations

$$S_j : y(t) = y(t, \alpha_6, \beta_6, \gamma_6, \delta_6) \to y_j(t, \alpha_6^j, \beta_6^j, \gamma_6^j, \delta_6^j), \quad j \in \{1, 2, 3\},$$

$$S_1: \quad y_1(t, -\beta_6, -\alpha_6, \gamma_6, \delta_6) = y^{-1}(1/t, \alpha_6, \beta_6, \gamma_6, \delta_6);$$

$$S_2: \quad y_2(t, -\beta_6, -\gamma_6, \alpha_6, \delta_6) = 1 - y^{-1}(\frac{1}{1-t}, \alpha_6, \beta_6, \gamma_6, \delta_6);$$

$$S_3: \quad y_3(t, -\beta_6, -\alpha_6, -\delta_6 + \frac{1}{2}, -\gamma_6 + \frac{1}{2}) = ty^{-1}(t, \alpha_6, \beta_6, \gamma_6, \delta_6).$$

In the following, we shall also use the symmetry

$$S_4: \quad y_4(t, 1/2 - \delta_6, -\gamma_6, -\beta_6, 1/2 - \alpha_6) = \frac{ty(t, \alpha_6, \beta_6, \gamma_6, \delta_6) - t}{y(t, \alpha_6, \beta_6, \gamma_6, \delta_6) - t}.$$

It can be obtained by various compositions of generators in the group, for instance, $S_4 = S_2 \circ S_2 \circ S_3 \circ S_2 = S_1 \circ S_2 \circ S_1 \circ S_3 \circ S_2$.

- Bäcklund transformations.

The Bäcklund transformations are given as follows. Let $y(t) = y(t, \alpha_6, \beta_6, \gamma_6, \delta_6)$ be a solution of (1), such that $R(t, y) := t(t-1)y' + (\eta_2 + \eta_3 + \eta_4 - 1)y^2 - (t\eta_2 + t\eta_3 + \eta_4 - 1)y + t\eta_2 \neq 0$. Then the transformation

$$T: y(t, \alpha_6, \beta_6, \gamma_6, \delta_6) \to \tilde{y}(t, \tilde{\alpha}_6, \tilde{\beta}_6, \tilde{\gamma}_6, \tilde{\delta}_6) = y - \frac{(\eta_1 + \eta_2 + \eta_3 + \eta_4 - 1)y(y-1)(y-t)}{R(t, y)}, \tag{2}$$

where $\eta_1^2 = 2\alpha_6$, $\eta_2^2 = -2\beta_6$, $\eta_3^2 = 2\gamma_6$, $\eta_4^2 = 1 - 2\delta_6$, determines solution $\tilde{y}(t, \tilde{\alpha}_6, \tilde{\beta}_6, \tilde{\gamma}_6, \tilde{\delta}_6)$ of equation (2) with new values of parameters $2\tilde{\alpha}_6 = \tilde{\eta}_1^2$, $2\tilde{\beta}_6 = -\tilde{\eta}_2^2$, $2\tilde{\gamma}_6 = \tilde{\eta}_3^2$, $1 - 2\tilde{\delta}_6 = \tilde{\eta}_4^2$, where $\tilde{\eta}_j = \eta_j - (\eta_1 + \eta_2 + \eta_3 + \eta_4)/2 + 1/2$, $j \in \{1, 2, 3, 4\}$.

Let the signs of $\eta_j$ be $\epsilon_j$ ($\epsilon_j = \pm 1$). Clearly the corresponding transformations depend on $\epsilon_j$, $j \in \{1, 2, 3, 4\}$. Thus, it is convenient to denote these transformations by $T_{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4}$. For instance, $\tilde{\eta}_j$ can then be written by

$$\tilde{\eta}_j = \epsilon_j \eta_j - (\epsilon_1 \eta_1 + \epsilon_2 \eta_2 + \epsilon_3 \eta_3 + \epsilon_4 \eta_4)/2 + 1/2, \qquad j \in \{1, 2, 3, 4\}.$$

Similarly one should put $\epsilon_j$ in (2). We also refer to paper [7] where the compositions of Bäcklund trasnformations were studied.

## 2.2 Discrete orthogonal polynomials with hypergeometric weights

Discrete orthonormal polynomials on $\mathbb{N} = \{0, 1, 2, \ldots\}$

$$\sum_{k=0}^{\infty} p_n(k) p_m(k) w_k = \delta_{m,n}, \qquad p_n(x) = \gamma_n x^n + \ldots$$

with

$$w_k = \frac{(\alpha)_k (\beta)_k}{(\gamma)_k k!} c^k, \qquad \alpha, \beta, \gamma > 0, \ 0 < c < 1,$$

were studied in [11]. They are referred to as discrete orthogonal polynomials with hypergeometric weights since the moments of this weight are given in terms of the Gauss hypergeometric function ${}_2F_1(\alpha, \beta; \gamma; c)$ and its derivatives. The symbol $(\cdot)_k$ is the Pochhammer symbol and $\delta_{m,n}$ is Kronecker's delta. The polynomials satisfy the three term recurrence relation

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x),$$

where $a_0 = 0$. The coefficients $a_n$ and $b_n$ are called the recurrence coefficients [5, 12, 17]. Note that the monic orthogonal polynomials $P_n = p_n/\gamma_n$ satisfy a similar three term recurrence relation

$$xP_n(x) = P_{n+1}(x) + b_n P_n(x) + a_n^2 P_{n-1}(x).$$

In [11, Theorem 3.1] a system of two first-order difference equations was obtained using the theory of ladder operators for discrete orthogonal polynomials [13]. In particular, new quantities $x_n$ and $y_n$ related to the recurrence coefficients $a_n^2$ and $b_n$ were introduced by

$$a_n^2 \frac{1-c}{c} = y_n + \sum_{k=0}^{n-1} x_k + \frac{n(n+\alpha+\beta-\gamma-1)}{1-c}, \tag{3}$$

$$b_n = x_n + \frac{n + (n+\alpha+\beta)c - \gamma}{1-c}. \tag{4}$$

In addition,

$$\frac{(1-c)^2}{c^2} a_n^2 x_n x_{n-1} = y_n(y_n - \alpha\beta + \frac{\gamma}{c}) + (\alpha\beta - y_n)\frac{1-c}{c}\sum_{k=0}^{n-1} x_k. \tag{5}$$

From (3) and (5) one can obtain an alternative expression for $a_n^2$:

$$a_n^2 = \frac{n\alpha\beta c(n+\alpha+\beta-\gamma-1) - c[n^2 + n(\alpha+\beta-\gamma-1) - \alpha\beta + \gamma]y_n - cy_n^2}{(c-1)^2(\alpha\beta - x_{n-1}x_n - y_n)}. \tag{6}$$

The sequences $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ satisfy the following discrete system (see [11, Theorem 3.1]):

$$\begin{aligned}(y_n - \alpha\beta + (\alpha+\beta+n)x_n - x_n^2)&(y_{n+1} - \alpha\beta + (\alpha+\beta+n+1)x_n - x_n^2) \\ &= \frac{1}{c}(x_n - 1)(x_n - \alpha)(x_n - \beta)(x_n - \gamma)\end{aligned} \tag{7}$$

and

$$\begin{aligned}&(x_n + Y_n)(x_{n-1} + Y_n) \\ &= \frac{(y_n + n\alpha)(y_n + n\beta)(y_n + n\gamma - (\gamma-\alpha)(\gamma-\beta))(y_n + n - (1-\alpha)(1-\beta))}{(y_n(2n+\alpha+\beta-\gamma-1) + n((n+\alpha+\beta)(n+\alpha+\beta-\gamma-1) - \alpha\beta + \gamma))^2},\end{aligned} \tag{8}$$

where

$$Y_n = \frac{y_n^2 + y_n(n(n+\alpha+\beta-\gamma-1) - \alpha\beta + \gamma) - \alpha\beta n(n+\alpha+\beta-\gamma-1)}{y_n(2n+\alpha+\beta-\gamma-1) + n((n+\alpha+\beta)(n+\alpha+\beta-\gamma-1) - \alpha\beta + \gamma)}.$$

The initial values $x_0$ and $y_0$ are given by

$$x_0 = \frac{\alpha\beta c}{\gamma}\frac{{}_2F_1(\alpha+1,\beta+1;\gamma+1;c)}{{}_2F_1(\alpha,\beta;\gamma;c)} + \frac{(\alpha+\beta)c - \gamma}{c-1}, \quad y_0 = 0. \tag{9}$$

For the hypergeometric weights the connection with the $\sigma$-form of the sixth Painlevé equation (with independent variable $c$) is known (see [11, Theorem 5.1]). The essential role is played by the Toda system for the recurrence coefficients (see, e.g., [12, §2.8] or [18, §3.2.2]). For the hypergeometric weights, it is given by

$$c\frac{d}{dc}a_n^2 = a_n^2(b_n - b_{n-1}), \qquad n \geq 1, \tag{10}$$

$$c\frac{d}{dc}b_n = a_{n+1}^2 - a_n^2, \qquad n \geq 0. \tag{11}$$

It is proved in [11, Theorem 5.1] that a simple linear change of variable transforms $S_n = \sum_{k=0}^{n-1} x_k$ into the solutions of the $\sigma$-form of the sixth Painlevé equation. Knowing $S_n$ one can find $x_n$, $y_n$ and, hence, the recurrence coefficients $a_n^2$, $b_n$ in terms of $S_n$ and its derivatives. Using formulas in [15] it is not difficult (see Theorem 5 in Subsection 3.3) to transform $S_n$ to the solutions of the sixth Painlevé equation with parameters $\alpha_6$, $\beta_6$, $\gamma_6$, $\delta_6$ given by

$$\alpha_6 = \frac{1}{2}(\gamma-1)^2, \;\; \beta_6 = -\frac{1}{2}(\alpha-\beta)^2, \;\; \gamma_6 = \frac{n^2}{2}, \;\; \delta_6 = \frac{1}{2}(1-(n+\alpha+\beta-\gamma)^2). \tag{12}$$

In this paper we shall present an alternative (computational) approach to derive a nonlinear differential equation for $x_n$ (hence, $b_n$) and show how it is related into the sixth Painlevé equation. The possibility of using such a computational approach was mentioned in [11] at the beginning of Section 5, but the details were not given and the exact formulas were not presented.

# 3 Main results

In this section first we obtain a nonlinear second order difference equation for $x_n$ (with respect to $n$) from system (7) and (8), next we obtain a nonlinear second order second degree differential equation for $x_n$ (with respect to the variable $c$), transform it to the sixth Painlevé equation and then show how the second-order difference equation is related to the composition of Bäcklund transformations of the sixth Painlevé equation.

## 3.1 Nonlinear difference equation for $x_n$

The first objective is to derive a nonlinear difference equation for $x_n$ (hence $b_n$) explicitly. To obtain a second-order difference equation

$$F(x_{n+1}, x_n, x_{n-1}, c) = 0 \tag{13}$$

we need to eliminate $y_n$ and $y_{n+1}$ from equations (7) and (8). The calculations are very cumbersome and should be done using any computer algebra software. We only present an algorithm to derive the result.

First we solve equation (7) with respect to $y_{n+1}$. This gives an expression of $y_{n+1}$ in terms of $y_n$ and $x_n$. Next, we replace $n$ by $n+1$ in equation (8) and substitute $y_{n+1}$ from the previous step. We obtain a long expression involving $x_n$, $x_{n+1}$ and $y_n$. Notice that equation (8) with $n$ unchanged contains $x_{n-1}$, $x_n$ and $y_n$. So we can eliminate $y_n$ from these two equations (for instance by computing the resultant of the equivalent polynomial expressions) and hence we get an expression involving only $x_{n-1}$, $x_n$ and $x_{n+1}$. Getting rid of trivial factors the final expression is given in the following theorem.

**Theorem 1.** *The sequence $(x_n)_{n\in\mathbb{N}}$ satisfies a second-order difference equation*

$$F(x_{n+1}, x_n, x_{n-1}, c) = \sum_{i=0}^{2}\sum_{j=0}^{8}\sum_{k=0}^{2} c_{i,j,k}^{(n)} x_{n+1}^i x_n^j x_{n-1}^k = 0. \tag{14}$$

Equation (14) has many non-zero cumbersome coefficients and we do not present them in paper. Below are two shorter ones.

$$c_{0,7,0}^{(n)} = (c-1)^3(4n - 3 - \alpha - \beta - 5\gamma + c(3 + 4n + 5\alpha + 5\beta + \gamma)),$$

$$c_{0,6,1}^{(n)} = 2(c-1)^3(3n - 2 - \alpha - \beta - 4\gamma + c(3 + 3n + 4\alpha + 4\beta + \gamma)).$$

## 3.2 Differential equation for $x_n(c)$

In this subsection we find a second-order second-degree differential equation for $x_n(c)$ and show how it can be transformed to the sixth Painlevé equation for $q_n(c)$ using a method similar to the one in [9].

Let us describe the procedure how to obtain a differential equation for $x_n(c)$. The Toda system (10) and (11) is essential to derive the second order second degree differential equation for $x_n(c)$. First we replace $x_n$ and $y_n$ in (7) and (8) by $x_n(c)$ and $y_n(c)$ respectively. We can solve equation (7) for $y_{n+1}(c)$ in terms of $x_n(c)$ and $y_n(c)$. Similarly we can solve equation (8) for $x_{n-1}(c)$ in terms of $x_n(c)$ and $y_n(c)$. Next we replace $n$ in (7) by $n+1$ and substitute $y_{n+1}(c)$ found previously. This gives us an opportunity to find an expression for $x_{n+1}(c)$ in terms of $x_n(c)$ and $y_n(c)$. Next we need to modify the Toda system. Equations (4) and (6) are expressions of the recurrence coefficients $a_n^2(c)$ and $b_n(c)$ in terms of $x_n(c)$, $x_{n-1}(c)$ and $y_n(c)$. We substitute (4) and (6) into the Toda system. Next we substitute $y_{n+1}(c)$, $x_{n-1}(c)$ and their derivatives along with the expression for $x_{n+1}(c)$ found previously into the modified Toda system. This gives us a system of two first order differential equations of the form $x_n'(c) = R_1(x_n(c), y_n(c), c)$ and $y_n'(c) = R_2(x_n(c), y_n(c), c)$, where $R_1$ and $R_2$ are rational functions. Once such system is obtained, there is a standard method to derive a second order differential equation for one its variables. By examining the degrees of rational functions, we observe that it is more convenient to derive a differential equation for $x_n(c)$. We differentiate the first equation of this system with respect to $c$ and replace the first order derivatives $x_n'(c)$ and $y_n'(c)$ by using this system. We get $x_n''(c)$ in terms of $x_n$ and $y_n$ as a rational expression. So we have this expression and the first equation of the system, expressing rationally $x_n'(c)$ in terms of $x_n$ and $y_n$ as well. Writing down equivalent polynomial expressions $P_1(x_n'(c), x_n(c), y_n(c), c) = 0$ and $P_2(x_n''(c), x_n(c), y_n(c), c) = 0$ we can compute the resultant to eliminate $y_n(c)$. In the result we obtain a second order second degree differential equation for $x_n(c)$.

**Theorem 2.** *The sequence $(x_n(c))_{n\in\mathbb{N}}$ satisfies the following second order second degree differential equation:*

$$H(x_n'', x_n', x_n, c) = \sum_{i=0}^{2}\sum_{j=0}^{2}\sum_{k=0}^{6} l_{i,j,k}^{(n)} (x_n'')^i (x_n')^j x_n^k = 0. \tag{15}$$

*The list of non-zero coefficients $l_{i,j,k}^{(n)}$ is given in the Appendix.*

Note that equation similar to (15) was found in an amplify and forward problem [3]. Equation (15) can be transformed to the sixth Painlevé equation for $q_n = q_n(c)$:

$$
\begin{aligned}
q_n'' =& \frac{1}{2}\left(\frac{1}{q_n} + \frac{1}{q_n - 1} + \frac{1}{q_n - c}\right) q_n'^2 - \left(\frac{1}{c} + \frac{1}{c-1} + \frac{1}{q_n - c}\right) q_n' \\
&+ \frac{q_n(q_n - 1)(q_n - c)}{c^2(c-1)^2}\left(\alpha_6 + \frac{\beta_6 c}{q_n^2} + \frac{\gamma_6(c-1)}{(q_n-1)^2} + \frac{\delta_6 c(c-1)}{(q_n - c)^2}\right).
\end{aligned}
$$

Indeed, by taking

$$
x_n(c) = \frac{c(c-1)q_n' - (n+\beta)q_n^2 + (n - \gamma + (n + \alpha + \beta)c)q_n - (n + \alpha - \gamma)c}{2(c-1)q_n} \tag{16}
$$

we find parameters

$$
\alpha_6 = \frac{1}{2}(n+\beta)^2, \;\; \beta_6 = -\frac{1}{2}(n + \alpha - \gamma)^2, \;\; \gamma_6 = \frac{1}{2}(\beta - \gamma)^2, \;\; \delta_6 = -\frac{1}{2}(\alpha - 2)\alpha. \tag{17}
$$

This is not a unique transformation. One can use the method of undetermined coefficients and a suitable Ansatz for the transformation, assuming that $x_n(c)$ is a rational expression in $q_n(c)$ and its derivative.

**Theorem 3.** *Equation (15) is transformed by (16) to the sixth Painlevé equation with parameters (17).*

Note that the sixth Painlevé equation with parameters (17) and $n = 0$ admits the following one-parameter family of solutions which solve simultaneously the Riccati equation

$$
c(c-1)q_0'(c) = \beta q_0(c)^2 + ((\alpha - \beta)c - \gamma)q_0(c) + (\gamma - \alpha)c.
$$

This equation is consistent with the initial condition $x_0(c)$ in (9) by using (16) with $n = 0$.

### 3.3 Bäcklund transformations for $q_n(c)$

In this subsection we show how to find formulas for $q_{n+1}(c)$ and $q_{n-1}(c)$ in terms of $q_n(c)$ and its derivative using the Bäcklund transformations of the sixth Painlevé equation. These expressions along with (16) can be used to verify that equation (13) becomes zero.

Let

$$
F_1 : q_n(c) \to q_{n-1}(c) \tag{18}
$$

and

$$
F_2 : q_{n-1}(c) \to q_n(c). \tag{19}
$$

Then we have the following theorem.

**Theorem 4.** *Let $q_n(c)$ solves the sixth Painlevé equation with parameters*

$$
\alpha_6 = \frac{1}{2}(n+\beta)^2, \quad \beta_6 = -\frac{1}{2}(n + \alpha - \gamma)^2, \quad \gamma_6 = \frac{1}{2}(\beta - \gamma)^2, \quad \delta_6 = \frac{1}{2}(2 - \alpha)\alpha.
$$

*Then*

$$
F_1 = S_3 \circ T_{1,1,-1,-1} \circ T_{1,1,1,1}, \quad F_2 = T_{1,1,-1,-1} \circ T_{-1,-1,1,1} \circ S_3.
$$

This theorem can be used to find formulas for $q_{n+1}(c)$ and $q_{n-1}(c)$ in terms of the $q_n(c)$ and its derivative and one can verify using (16) that equation (13) becomes zero. The formulas are as follows.

$$q_{n-1} = \frac{f_{1,1}(q_n, q_n', n, c)}{f_{1,2}(q_n, q_n', n, c)}, \tag{20}$$

where

$$f_{1,1}(q_n, q_n', n, c) = \sum_{r=0}^{4}\sum_{s=0}^{2} c_{r,s}^{(n)} q_n^r (q_n')^s, \tag{21}$$

$$f_{1,2}(q_n, q_n', n, c) = \sum_{r=0}^{5}\sum_{s=0}^{2} d_{r,s}^{(n)} q_n^r (q_n')^s. \tag{22}$$

$$q_{n+1} = \frac{f_{2,1}(q_n, q_n', n, c)}{f_{2,2}(q_n, q_n', n, c)}, \tag{23}$$

where

$$f_{2,1}(q_n, q_n', n, c) = \sum_{r=0}^{4}\sum_{s=0}^{2} k_{r,s}^{(n)} q_n^r (q_n')^s, \tag{24}$$

$$f_{2,2}(q_n, q_n', n, c) = \sum_{r=0}^{5}\sum_{s=0}^{2} l_{r,s}^{(n)} q_n^r (q_n')^s. \tag{25}$$

The lists of non-zero coefficients of the formulas $(21)-(22)$ and $(24)-(25)$ are given in the Appendix.

Therefore, using the connection of $x_n(c)$ to $q_n(c)$ and its derivative we can easily obtain formulas for $x_{n+1}(c)$ and $x_{n-1}(c)$ in terms of $q_n(c)$ and its derivative. Substituting $x_{n+1}, x_n$ and $x_{n-1}$ into the nonlinear equation $F(x_{n+1}, x_n, x_{n-1}, c)$, we get identically zero.

In this paper we deal with $x_n(c)$ (hence, $b_n(c)$) and express it in terms of solution $q_n(c)$ of the sixth Painlevé equation. However, we also have $x_n = S_{n+1} - S_n$, where the function $S_n(c)$ was studied in [11] and it was expressed in terms of solutions of the $\sigma$-form of the sixth Painlevé equation. Let us find connection between $S_n(c)$ and solution $\tilde{q}_n(c)$ of the sixth Painlevé equation and show how $q_n(c)$ and $\tilde{q}_n(c)$ are related.

**Theorem 5.** *The function $S_n(c) = \sum_{k=0}^{n-1} x_k(c)$ can be expressed as*

$$S_n(c) = \frac{f(\tilde{q}_n(c), \tilde{q}_n{}'(c), n, c)}{4(c-1)(c-\tilde{q}_n(c))(\tilde{q}_n(c)-1)\tilde{q}_n(c)}, \tag{26}$$

*where*

$$f(\tilde{q}_n(c), \tilde{q}_n{}'(c), n, c) = \sum_{r=0}^{4}\sum_{s=0}^{2} e_{r,s}^{(n)} (\tilde{q}_n(c))^r (\tilde{q}_n{}'(c))^s$$

*and the function $\tilde{q}_n(c)$ satisfies the sixth Painlevé equation with parameters*

$$\tilde{\alpha}_6 = \frac{1}{2}(\gamma-1)^2, \quad \tilde{\beta}_6 = -\frac{1}{2}(\alpha-\beta)^2, \quad \tilde{\gamma}_6 = \frac{n^2}{2}, \quad \tilde{\delta}_6 = \frac{1-(n+\alpha+\beta-\gamma)^2}{2}.$$

*The list of non-zero coefficients $e_{r,s}^{(n)}$ is given in the Appendix.*

To prove this theorem one can use formulas in [15] and the corresponding Hamiltonian system.

Let

$$F_3 : \tilde{q}_n(c) \to \tilde{q}_{n-1}(c) \tag{27}$$

and

$$F_4 : \tilde{q}_{n-1}(c) \to \tilde{q}_n(c). \tag{28}$$

Then we have the following theorem.

**Theorem 6.** *Let $\tilde{q}_n(c)$ solves the sixth Painlevé equation with parameters*

$$\tilde{\alpha}_6 = \frac{1}{2}(\gamma-1)^2, \quad \tilde{\beta}_6 = -\frac{1}{2}(\alpha-\beta)^2, \quad \tilde{\gamma}_6 = \frac{n^2}{2}, \qquad \tilde{\delta}_6 = \frac{1-(n+\alpha+\beta-\gamma)^2}{2}.$$

*Then*

$$F_3 = T_{-1,-1,1,1} \circ T_{-1,1,1,1} \circ S_3, \quad F_4 = T_{-1,-1,1,1} \circ T_{1,-1,-1,-1} \circ S_3.$$

Using this theorem one can calculate $\tilde{q}_{n\pm 1}$ in terms of $\tilde{q}_n$.

Next, for completeness, we present relations between the functions $\tilde{q}_n(c)$ (which is related to $S_n(c)$) and $q_n(c)$ (which is related to $x_n(c)$):

$$F_5 : q_n(c, \alpha_6, \beta_6, \gamma_6, \delta_6) \to \tilde{q}_n(c, \tilde{\alpha}_6, \tilde{\beta}_6, \tilde{\gamma}_6, \tilde{\delta}_6) \tag{29}$$

and

$$F_6 : \tilde{q}_n(c, \tilde{\alpha}_6, \tilde{\beta}_6, \tilde{\gamma}_6, \tilde{\delta}_6) \to q_n(c, \alpha_6, \beta_6, \gamma_6, \delta_6). \tag{30}$$

**Theorem 7.** *Let $q_n(c)$ and $\tilde{q}_n(c)$ be, respectively, solutions of the sixth Painlevé equation with parameters*

$$\alpha_6 = \frac{1}{2}(n+\beta)^2, \quad \beta_6 = -\frac{1}{2}(n+\alpha-\gamma)^2, \quad \gamma_6 = \frac{1}{2}(\beta-\gamma)^2, \quad \delta_6 = \frac{1}{2}(2-\alpha)\alpha$$

*and*

$$\tilde{\alpha}_6 = \frac{1}{2}(\gamma-1)^2, \quad \tilde{\beta}_6 = -\frac{1}{2}(\alpha-\beta)^2, \quad \tilde{\gamma}_6 = \frac{n^2}{2}, \qquad \tilde{\delta}_6 = \frac{1-(n+\alpha+\beta-\gamma)^2}{2}.$$

*Then we have the following relations:*

$$F_5 = T_{-1,-1,-1,1} \circ S_4 \circ T_{1,1,-1,-1}, \quad F_6 = S_4 \circ T_{-1,1,-1,1}.$$

*Explicitly,*

$$\begin{aligned} \tilde{q}_n = & \frac{c(q_n-1)}{(c-q_n)\big[c(\gamma-n-\alpha) + q_n((c+1)n + c(\alpha+\beta-2\gamma)+\gamma) - (n+\beta)q_n^2 + (c-1)cq_n'\big]} \\ & \times \Big\{ c(n+\alpha-\gamma) + q_n[(c-2)\alpha - (1+c)n - c\beta + \gamma] + (n+\beta)q_n^2 - c(c-1)q_n' \Big\}, \end{aligned} \tag{31}$$

$$\begin{aligned} q_n = & -\frac{c(\tilde{q}_n-1)}{(c-\tilde{q}_n)\big[c(\alpha-\beta) + \tilde{q}_n(1-n-\gamma+c(n-\alpha+\beta)) + (\gamma-1)\tilde{q}_n{}^2 + c(c-1)\tilde{q}_n'\big]} \\ & \times \Big\{ c(\alpha-\beta) + \tilde{q}_n(1+(c-1)n - 2\alpha + c(\alpha+\beta-2\gamma)+\gamma) + (\gamma-1)\tilde{q}_n^2 + c(c-1)\tilde{q}_n' \Big\}. \end{aligned} \tag{32}$$

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## Appendix

- The list of non-zero coefficients $l^{(n)}_{i,j,k}(c)$ in $H(x''_n, x'_n, x_n, c)$ (see (15)):

$$\begin{aligned} l^{(n)}_{0,0,0}(c) =& - c^2\alpha\beta(1+2n+2\alpha+2\beta)^2\gamma - \alpha\beta\gamma(1-2n+2\gamma)^2 + c\big[4\beta^2\gamma^2+ \\ &+ 2\alpha\beta\gamma\big(4n^2+2\beta-1+4n(\beta-\gamma)+2\gamma\big) + 4\alpha^2\big(\gamma^2+\beta^2(1+\gamma)^2+\beta\gamma(2n+1)\big)\big], \end{aligned}$$

$$\begin{aligned} l^{(n)}_{0,0,1} =& \beta\gamma\big[c^2(1+2n+2\beta)^2+(1-2n+2\gamma)^2-2c\big(4n^2+6\beta+4n(\beta-\gamma)+6\gamma+ \\ &+ 4\beta\gamma-1\big)\big] + 4c\alpha^2\big[(c-2n+2cn-2\gamma-3)\gamma+2\beta^2(c-2)(1+\gamma)- \\ &- \beta(1+\gamma)(3+2n+2\gamma)+c\beta\big(1+8\gamma+2n(1+\gamma)\big)\big] + \alpha\Big[4c^2\beta^3(1+\gamma)+ \\ &+ \gamma\big((c+2cn)^2+(1-2n+2\gamma)^2+c(2-8n^2-12\gamma+8n\gamma)\big) - 4c\beta^2\big[(1+ \\ &+ \gamma)(3+2n+2\gamma)-c\big(1+8\gamma+2n(1+\gamma)\big)\big] + \beta(1+13\gamma+24\gamma^2+4\gamma^3)+ \\ &+ 4n^2\beta(1+\gamma)-4n\beta(1+7\gamma+2\gamma^2)+c^2\beta(1+2n)(1+17\gamma+2n(1+\gamma))- \\ &- 2c\beta\big(9\gamma-4n(\gamma-1)\gamma+6\gamma^2+4n^2(1+\gamma)-1\big)\Big] + 4c^2\alpha^3(\beta+\gamma+\beta\gamma), \end{aligned}$$

$$\begin{aligned} l^{(n)}_{0,0,2} =& - 4c^2\beta^3(1+\gamma) - 4c^2\alpha^3(1+\beta+\gamma) - \gamma\big((c+2cn)^2+(1-2n+2\gamma)^2+c(2- \\ &- 8n^2-12\gamma+8n\gamma)\big) + 4c\beta^2[3+7\gamma+2\gamma^2+2n(1+\gamma)-c(1+6\gamma+2n(1+\gamma))]- \\ &- \beta[1+13\gamma+24\gamma^2+4\gamma^3+4n^2(1+\gamma)-4n(1+7\gamma+2\gamma^2)+c^2(1+2n)(1+ \\ &+ 13\gamma+2n(1+\gamma))-2c\big(15\gamma+14\gamma^2-4n\gamma^2+4n^2(1+\gamma)-1\big)] + 4c\alpha^2[3+9\beta+ \\ &+ 4\beta^2+7\gamma+8\beta\gamma+2\gamma^2+2n(1+\beta+\gamma)-c\big(1+8\beta+2\beta^2+6\gamma+7\beta\gamma+2n(1+ \\ &+ \beta+\gamma)\big)] - \alpha\Big[1+9\beta+13\gamma+44\beta\gamma+24\gamma^2+20\beta\gamma^2+4\gamma^3+4n^2(1+\beta+\gamma)- \\ &- 4n\big(1+7\gamma+2\gamma^2+\beta(5+6\gamma)\big) - 2c\big[15\gamma+14\gamma^2+4n^2(1+\beta+\gamma)-1+2\beta^2(9+ \\ &+ 8\gamma)+\beta(15+34\gamma+12\gamma^2)+4n\big(\beta^2-\gamma^2+\beta(2+\gamma)\big)\big] + c^2\big[1+4\beta^3+13\gamma+ \\ &+ 4n^2(1+\beta+\gamma)+4\beta^2(8+7\gamma)+\beta(17+72\gamma)+4n\big(1+2\beta^2+7\gamma+\beta(9+8\gamma)\big)\big]\Big], \end{aligned}$$

$$\begin{aligned} l_{0,0,3}^{(n)} =& 1 + 9\alpha + 9\beta + 24\alpha\beta + 13\gamma + 44\alpha\gamma + 44\beta\gamma + 32\alpha\beta\gamma + 24\gamma^2 + 20\alpha\gamma^2 + \\ &+ 20\beta\gamma^2 + 4\gamma^3 + 4n^2(1+\alpha+\beta+\gamma) - 4n\big[1 + 7\gamma + 2\gamma^2 + \beta(5+6\gamma) + \\ &+ \alpha(5+4\beta+6\gamma)\big] - 2c(1+\alpha+\beta+\gamma)\big(4n^2 + 14\beta - 1 + 4n(\alpha+\beta - \\ &- \gamma) + 14\gamma + 12\beta\gamma + 2\alpha(7+8\beta+6\gamma)\big) + c^2\Big[1 + 4\alpha^3 + 13\beta + 24\beta^2 + \\ &+ 4\beta^3 + 9\gamma + 44\beta\gamma + 20\beta^2\gamma + 4n^2(1+\alpha+\beta+\gamma) + 4\alpha^2(6+7\beta+5\gamma) + \\ &+ 4n\big(1 + 2\alpha^2 + 7\beta + 2\beta^2 + 5\gamma + 6\beta\gamma + \alpha(7+8\beta+6\gamma)\big) + \alpha\big(13 + \\ &+ 28\beta^2 + 44\gamma + 8\beta(9+7\gamma)\big)\Big], \end{aligned}$$

$$\begin{aligned} l_{0,0,4}^{(n)} =& (c-1)\big[9 + 4n^2 + 24\alpha + 24\beta + 16\alpha\beta + 44\gamma + 32\alpha\gamma + 32\beta\gamma + 20\gamma^2 + \\ &- c\big(9 + 4n^2 + 20\alpha^2 + 44\beta + 20\beta^2 + 24\gamma + 32\beta\gamma + 4n(5 + 6\alpha + 6\beta + \\ &+ 4\gamma) + 4\alpha(11 + 14\beta + 8\gamma)\big) - 4n(5 + 4\alpha + 4\beta + 6\gamma)\big], \end{aligned}$$

$$l_{0,0,5}^{(n)} = 8(c-1)(2n - 2\alpha - 2\beta - 4\gamma - 3 + c(3 + 2n + 4\alpha + 4\beta + 2\gamma)),$$

$$l_{0,0,6}^{(n)} = -16(c-1)^2,$$

$$l_{0,1,0}^{(n)} = -2c\big(1 - 4c + 3c^2\big)\big(\beta\gamma + \alpha(\beta + \gamma + \beta\gamma)\big),$$

$$l_{0,1,1}^{(n)} = 4c(3c^2 + 1 - 4c)(\alpha + \beta + \gamma + \alpha\beta + \alpha\gamma + \beta\gamma),$$

$$l_{0,1,2}^{(n)} = -6c(1 - 4c + 3c^2)(1 + \alpha + \beta + \gamma),$$

$$l_{0,1,3}^{(n)} = 8(3c^2 - 4c + 1)c,$$

$$\begin{aligned} l_{0,2,0}^{(n)} =& c(c-1)^2\Big(n - n^2 - c^2(n + n^2 + \alpha + 2n\alpha + \alpha^2 + \beta + 2n\beta + 2\alpha\beta + \beta^2 - 2) - \\ &- \gamma + 2n\gamma - \gamma^2 - c\big(2 - 2n^2 + \alpha + \beta - 2n(\alpha + \beta - \gamma) + \gamma + 2\alpha\gamma + 2\beta\gamma\big)\Big), \end{aligned}$$

$$l_{0,2,1}^{(n)} = 2(1+c)(c-1)^2\big(1 - 2n + c(1 + 2n + 2\alpha + 2\beta) + 2\gamma\big)c,$$

$$l_{0,2,2}^{(n)} = -4(c^2 - 1)^2 c,$$

$$l_{1,0,0}^{(n)}(c) = -4(c-1)^2 c^2\big(\beta\gamma + \alpha(\beta + \gamma + \beta\gamma)\big),$$

$$l_{1,0,1}^{(n)} = 8c^2(c-1)^2(\alpha + \beta + \gamma + \alpha\beta + \alpha\gamma + \beta\gamma),$$

$$l_{1,0,2}^{(n)} = -12(c-1)^2 c^2(1 + \alpha + \beta + \gamma),$$

$$l_{1,0,3}^{(n)} = 16(c-1)^2 c^2,$$

$$l_{1,1,0}^{(n)} = (c-1)^3(3c-1)c^2,$$

$$l_{2,0,0}^{(n)} = (c-1)^4 c^3.$$

- The list of non-zero coefficients of $c_{r,s}^{(n)}$ (see (21)):

$$c_{0,0}^{(n)} = c^3(n+\alpha-\gamma)^2,$$
$$c_{0,1}^{(n)} = 2(c-1)c^3(n+\alpha-\gamma), \qquad c_{0,2}^{(n)} = (c-1)^2c^3,$$
$$c_{1,0}^{(n)} = -2c^2(n+\alpha-\gamma)(n+cn+\alpha+c\alpha-\gamma-c\gamma-1),$$
$$c_{1,1}^{(n)} = -2(c-1)c^2(n+cn+\alpha+c\alpha-\gamma-c\gamma-1),$$

$$\begin{aligned}c_{2,0}^{(n)} =& c\Big((1+c^2)n^2+c^2(\alpha-\beta)(\alpha+\beta-2\gamma)+\gamma(2-2\alpha+\gamma)+2n\big((1+c)^2\alpha-1-\\ &-\gamma-c^2\gamma-2c(\beta+\gamma)\big)+2c\big(2\alpha^2+\beta+\gamma+2\gamma^2-\alpha(2+\beta+3\gamma)\big)\Big),\end{aligned}$$

$$c_{2,1}^{(n)} = 2(c-1)c^2(n+\alpha-\gamma-1),$$
$$c_{3,0}^{(n)} = 2c\Big(n^2-\beta+2n\beta+\alpha\beta-\gamma+\alpha\gamma-\gamma^2+c\big(n^2-\alpha^2+n(2\beta-1)+\beta(\beta-2\gamma-1)+\alpha(1+\beta+\gamma)\big)\Big),$$
$$c_{4,0}^{(n)} = -c(n+\beta)(3n+2\alpha+\beta-2\gamma-2).$$

- The list of non-zero coefficients of $d_{r,s}^{(n)}$ (see (22)):

$$d_{1,0}^{(n)} = -c^2(n+\alpha-\gamma)(3n+\alpha+2\beta-\gamma-2),$$
$$d_{1,1}^{(n)} = -2(c-1)c^2(n+\beta-1),$$
$$d_{1,2}^{(n)} = (c-1)^2c^2,$$

$$\begin{aligned}d_{2,0}^{(n)} =& 2c\Big((c+1)n^2+\alpha\beta+\gamma-\alpha\gamma-2\beta\gamma+\gamma^2+c\big(\alpha^2+\beta-\beta^2+\alpha(\beta-\gamma-2)+\\ &+\gamma\big)+n\big(2(c+1)\alpha-2(c+1)\gamma-1\big)\Big),\end{aligned}$$

$$d_{2,1}^{(n)} = 2(c-1)c\big(n+\beta+c(n+\beta-1)\big),$$

$$\begin{aligned}d_{3,0}^{(n)} =& c^2\big(n^2+2\alpha-\alpha^2+2n(\beta-1)+(\beta-2)\beta\big)+(n+2\beta-\gamma)(n+\gamma)-\\ &-2c\big(\beta+\alpha\beta-2\beta^2+2n(\alpha-\beta-\gamma)+\gamma-\alpha\gamma\big),\end{aligned}$$

$$d_{3,1}^{(n)} = -2(c-1)(n+\beta)c,$$
$$d_{4,0}^{(n)} = -2(n+\beta)\big(n+\beta+c(n+\beta-1)\big),$$
$$d_{5,0}^{(n)} = (n+\beta)^2.$$

- The list of non-zero coefficients of $k_{r,s}^{(n)}$ (see (24)):

$$k_{0,0}^{(n)} = c^3(n+\alpha-\gamma)^2,$$

$$k_{0,1}^{(n)} = -2(c-1)c^3(n+\alpha-\gamma), \qquad k_{0,2}^{(n)} = (c-1)^2c^3,$$

$$k_{1,0}^{(n)} = -2c^2(n+\alpha-\gamma)(1+n+cn+\alpha+c\alpha-\gamma-c\gamma),$$

$$k_{1,1}^{(n)} = 2(c-1)c^2(1+n+cn+\alpha+c\alpha-\gamma-c\gamma),$$

$$\begin{aligned} k_{2,0}^{(n)} =& c\Big((1+c^2)n^2 + c(4+c)\alpha^2 - 2c\beta - c^2\beta^2 - 2\gamma - 2c\gamma + 2c^2\beta\gamma + \gamma^2 + 4c\gamma^2 + \\ &+ 2n\big(1+(1+c)^2\alpha - \gamma - c^2\gamma - 2c(\beta+\gamma)\big) - 2\alpha\big(\gamma + c^2\gamma + c(\beta+3\gamma) - 2\big)\Big), \end{aligned}$$

$$k_{2,1}^{(n)} = -2(c-1)c^2(1+n+\alpha-\gamma),$$

$$\begin{aligned} k_{3,0}^{(n)} =& 2c\Big(n^2 - 2\alpha + \beta + 2n\beta + \alpha\beta + \gamma + \alpha\gamma - \gamma^2 + c\big(n + n^2 - \alpha^2 + \\ &+ 2n\beta + \beta(1+\beta-2\gamma) + \alpha(1+\beta+\gamma)\big)\Big), \end{aligned}$$

$$k_{4,0}^{(n)} = -c(n+\beta)(2+3n+2\alpha+\beta-2\gamma).$$

- The list of non-zero coefficients of $l_{r,s}^{(n)}$ (see (25)):

$$l_{1,0}^{(n)} = -c^2(n+\alpha-\gamma)(2+3n+\alpha+2\beta-\gamma),$$

$$l_{1,1}^{(n)} = 2(c-1)c^2(1+n+\beta), \qquad l_{1,2}^{(n)} = (c-1)^2c^2,$$

$$\begin{aligned} l_{2,0}^{(n)} =& 2c\Big((1+c)n^2 + c\alpha^2 - c\beta - c\beta^2 - \gamma - c\gamma - 2\beta\gamma + \gamma^2 + \\ &+ n\big(1 + 2(1+c)\alpha - 2(1+c)\gamma\big) + \alpha\big(2 + (1+c)\beta - (1+c)\gamma\big)\Big), \end{aligned}$$

$$l_{2,1}^{(n)} = -2(c-1)(n+\beta+c(1+n+\beta))c,$$

$$\begin{aligned} l_{3,0}^{(n)} =& c^2\big(n^2 + 2\alpha - \alpha^2 + 2n(1+\beta) + \beta(2+\beta)\big) + (n+2\beta-\gamma)(n+\gamma) + \\ &+ 2c(\beta + 2n\beta + 2\beta^2 - \alpha(2+2n+\beta-\gamma) + \gamma + 2n\gamma), \end{aligned}$$

$$l_{3,1}^{(n)} = 2(c-1)(n+\beta)c,$$

$$l_{4,0}^{(n)} = -2(n+\beta)(n+\beta+c(1+n+\beta)),$$

$$l_{5,0}^{(n)} = (n+\beta)^2.$$

- The list of non-zero coefficients of $e_{r,s}^{(n)}$ in Theorem 5 (see (26)):

$$e_{0,0}^{(n)} = c^2(\alpha-\beta)^2,$$

$$e_{0,2}^{(n)} = -(c-1)^2c^2,$$

$$e_{1,0}^{(n)} = -2c[c\alpha^2 + c\beta^2 - 2\gamma + \beta\big((c-1)n + \gamma + 1\big) + \alpha\big((c-1)n - 2(c+1)\beta + \gamma + 1\big)],$$

$$\begin{aligned} e_{2,0}^{(n)} =& (c-1)^2n^2 + c^2(\alpha-\beta)^2 + (\gamma-1)^2 + 2n(c-1)\big((\alpha+\beta)c + \gamma + 1\big) + \\ &+ 4c(\alpha + \beta - 2\gamma - 2\alpha\beta + \alpha\gamma + \beta\gamma), \end{aligned}$$

$$e_{3,0}^{(n)} = -2\big((\gamma-1)^2 + n(c-1)(\gamma+1) + c(\alpha+\beta-2\gamma-2\alpha\beta+\alpha\gamma+\beta\gamma)\big),$$

$$e_{4,0}^{(n)} = (\gamma-1)^2.$$

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