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Gaussian unitary ensembles with two jump discontinuities, PDEs, and the coupled Painlevé II and IV systems

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Abstract

We consider the Hankel determinant generated by the Gaussian weight with two jump discontinuities. Utilizing the results of Min and Chen [*Math. Methods Appl Sci*. 2019;42:301-321] where a second-order partial differential equation (PDE) was deduced for the log derivative of the Hankel determinant by using the ladder operators adapted to orthogonal polynomials, we derive the coupled Painlevé IV system which was established in Wu and Xu [arXiv: 2002.11240v2] by a study of the Riemann-Hilbert problem for orthogonal polynomials. Under double scaling, we show that, as $n \to \infty$, the log derivative of the Hankel determinant in the scaled variables tends to the Hamiltonian of a coupled Painlevé II system and it satisfies a second-order PDE. In addition, we obtain the asymptotics for the recurrence coefficients of orthogonal polynomials, which are connected with the solutions of the coupled Painlevé II system.

KEYWORDS

Gaussian unitary ensembles, Hankel determinant, orthogonal polynomials, Painlevé equations

1 INTRODUCTION

The n-dimensional Gaussian unitary ensemble (GUE for short) is a set of $n \times n$ Hermitian random matrices whose eigenvalues have the following joint probability density function

$$p(x_1, x_2, \ldots, x_n) = \frac{1}{C_n} \cdot \frac{1}{n!} \prod_{1 \le i < j \le n} (x_i - x_j)^2 \prod_{k=1}^{n} \mathrm{e}^{-x_k^2}, \tag{1}$$

where $x_k \in (-\infty, \infty), k = 1, 2, \ldots, n$ (See Ref. 1, sections 2.5, 2.6, and 3.3). The normalization constant $n!C_n$, also known as the partition function, has the following explicit representation (Ref. 1, eq. 17.6.7)

$$\begin{aligned} n!C_n :&= \int_{(-\infty,\infty)^n} \prod_{1 \le i < j \le n} (x_i - x_j)^2 \prod_{k=1}^{n} \mathrm{e}^{-x_k^2} dx_k \\ &= (2\pi)^{n/2} 2^{-n^2/2} \prod_{k=1}^{n} k!, \end{aligned}$$

namely,

$$C_n = (2\pi)^{n/2} 2^{-n^2/2} \prod_{k=1}^{n-1} k!.$$

We consider the Hankel determinant generated by the moments of the Gaussian weight multiplied by a factor that has two jumps, ie,

$$D_n(s_1, s_2) := \det \left(\int_{-\infty}^{\infty} x^{i+j} w(x; s_1, s_2) dx \right)_{i,j=0}^{n-1},$$

where the weight function reads

$$w(x; s_1, s_2) := \mathrm{e}^{-x^2} (A + B_1 \theta(x - s_1) + B_2 \theta(x - s_2)), \quad x \in (-\infty, \infty),$$

with $s_1 < s_2$ and $A \ge 0, A + B_1 \ge 0, A + B_1 + B_2 \ge 0, B_1 B_2 \ne 0$. Here, $\theta(x)$ is 1 for $x > 0$ and 0 otherwise. For any interval $I \subset (-\infty, \infty)$, it is well known that (see Ref. 2, sections 2.1 and 2.2)

$$\frac{1}{n!} \int_{I^n} \prod_{1 \le i < j \le n} (x_i - x_j)^2 \prod_{k=1}^{n} \mathrm{e}^{-x_k^2} dx_k = \det \left(\int_I x^{i+j} \mathrm{e}^{-x^2} dx \right)_{i,j=0}^{n-1}.$$

Therefore, the probability that the interval (s_1, s_2) has all or no eigenvalues of GUE is given by $D_n(s_1, s_2)/C_n$ with $A = 0, B_1 = 1, B_2 = -1$ and $A = 1, B_1 = -1, B_2 = 1$, respectively. The former was studied in Ref. 3 via the ladder operator approach,[4] a formalism adapted to monic orthogonal polynomials, and its log derivative was shown to satisfy a two-variable generalization of the Painlevé IV system.

By using the ladder operator formalism and with the aid of four auxiliary quantities, Min and one of the authors[5] derived a second-order partial differential equation (PDE for short) satisfied by

$$\sigma_n(s_1, s_2) := \left(\frac{\partial}{\partial s_1} + \frac{\partial}{\partial s_2}\right) \ln D_n(s_1, s_2).$$

In a recent paper,[6] Wu and Xu studied the special case of $D_n(s_1, s_2)$ where $A = 1, B_1 = \omega_1 - 1, B_2 = \omega_2 - \omega_1$ with $\omega_1, \omega_2 \geq 0$. In addition, through the Riemann-Hilbert (RH for short) formalism of orthogonal polynomials,[7] they showed that $\sigma_n(s_1, s_2) + n(s_1 + s_2)$ is the Hamiltonian of a coupled Painlevé IV system. When s_1 and s_2 tend to the soft edge of the spectrum of GUE, by applying Deift-Zhou nonlinear steepest descent analysis[8] to the RH problem (we call it RH method below), the asymptotic formulas for $D_n(s_1, s_2)$ and the associated orthogonal polynomials were deduced, which are expressed in terms of the solution of a coupled Painlevé II system.

Comparing the finite n results of the above two papers concerning $D_n(s_1, s_2)$, we ask naturally whether they are compatible with each other. To the best knowledge of the authors, it is not easy to obtain the second-order PDE of Ref. 5 from the coupled Painlevé IV system of Ref. 6. What about the other side? It transpires that the Hamiltonian of the coupled Painlevé IV system of Ref. 6 can be derived by using the results of Ref. 5. This is the main purpose of this paper, which may provide new insights into the connection between the ladder operator approach and RH problems.

As we know, the ladder operator approach and the RH method are both very effective tools and have their own advantages in the study of unitary ensembles. The former is elementary in the sense that it uses the very basic theory of orthogonal polynomials and it provides a quite straightforward way to derive classical Painlevé equations (Painlevé I-VI, XXXIV) for finite dimensional problems particularly those involving one variable. This variable could be the deformation variable in a deformed weight function,[9] the interval variable in gap probability problems,[10,11] the jump variable in the weight function which has jump discontinuities,[12,13] the perturbation variable in a singularly perturbed weight function,[14,15] and so on. For problems involving two such variables, a second-order PDE can be deduced. The two variables could be, for example, two deformation variables,[16] two interval variables,[3] two jump variables,[5] one interval variable together with one perturbation variable.[17] As the dimension of the unitary ensemble tends to ∞ and under suitable double scaling, based on the aforementioned Painlevé equations or PDEs, we can have a rough understanding of the asymptotic expansion of the interested quantities. However, the analysis is not rigorous since the existence and convergence of the asymptotic series are not proved.

Comparatively speaking, the RH method is powerful and rigorous in asymptotic analysis of unitary ensembles under double scaling. For problems involving a single variable, the asymptotic expansion of the quantities of interest is characterized by classical Painlevé equations.[18–20] For problems involving $k(k \geq 2)$ variables, one can establish a system of coupled ordinary differential equations (ODEs for short) which can be viewed as a generalization of Painlevé equations since the system is reduced to the classical Painlevé equations when k is assumed to be 1 or the variables involved take certain values which convert the k-variable problem to one-variable problem (see, for instance, Refs. 21–23).

The four-dimensional Painlevé-type equations, ie, the system of four coupled ODEs, were classified by Kawakami, Nakamura, and Sakai.[24,25] In the classification, the coupled Painlevé systems showed up. They have recently attracted a great deal of attention in random matrix theory, appearing in the study of unitary ensembles to characterize two-variable problems. In Ref. 26, the symmetric gap probability of a circular unitary ensemble with Fisher-Hartwig singularities was represented by the integral of the Hamiltonian for a coupled Painlevé V system. In Ref. 27, the gap probability of the unitary ensemble with the weight function $|x|^{2\alpha}e^{-nV(x)}$ was studied via the Fredholm determinant of Painlevé II and XXXIV kernels, and an integral representation in terms of the coupled Painlevé II system was derived. The coupled Painlevé II system was applied by Claeys and Doeraene[28] to characterize the Airy point process. It also appeared in the work of Amir, Corwin, and Quastel[29] which investigated the free energy of directed polymer or the Kardar-Parisi-Zhang equation (see proposition 2.1 and section 5.2 therein).

This paper is built up as follows. In Section 2, we present some notations and results of Ref. 5. We make use of them in Section 3 to show that the four auxiliary quantities allied with the orthogonal polynomials satisfy a coupled Painlevé IV system and $\sigma_n(s_1, s_2) + n(s_1 + s_2)$ is the Hamiltonian of that system. Section 4 is devoted to the discussion of the double scaling limit of the Hankel determinant. By using the finite n results given in Section 2, we deduce that, as $n \to \infty$, the log derivative of the Hankel determinant in the scaled variables tends to the Hamiltonian of a coupled Painlevé II system and it satisfies a second-order PDE. In addition, for the recurrence coefficients of the monic orthogonal polynomials associated with $w(x; s_1, s_2)$, we obtain their asymptotic expansions in large n with the coefficients of the leading order term expressed in terms of the solutions of the coupled Painlevé II system.

2 | NOTATIONS AND SOME RESULTS OF REF. 5

In this section, we present some results of Ref. 5 which will be used for our later derivation in subsequent sections.

Denote the Gaussian weight by $w_0(x)$, ie,

$$w_0(x) := e^{-\mathrm{v}_0(x)}, \qquad \mathrm{v}_0(x) = x^2.$$

Then the weight function of our interest reads

$$w(x; s_1, s_2) = w_0(x)(A + B_1\theta(x - s_1) + B_2\theta(x - s_2)).$$

It is well known that the associated Hankel determinant admits the following representation (see Ref. 4, pp. 16-19)

$$\begin{aligned} D_n(s_1, s_2) &= \det\left(\int_{-\infty}^{\infty} x^{i+j} w(x; s_1, s_2)dx\right)_{i,j=0}^{n-1} \\ &= \prod_{j=0}^{n-1} h_j(s_1, s_2). \end{aligned} \tag{2}$$

Here, $h_j(s_1, s_2)$ is the square of the L^2-norm of the jth-degree monic polynomial orthogonal with respect to $w(x; s_1, s_2)$, namely,

$$h_j(s_1, s_2)\delta_{jk} := \int_{-\infty}^{\infty} P_j(x; s_1, s_2)P_k(x; s_1, s_2)w(x; s_1, s_2)dx, \tag{3}$$

for $j, k = 0, 1, 2, \ldots$, and

$$P_j(x; s_1, s_2) := x^j + p(j, s_1, s_2)x^{j-1} + \cdots + P_j(0; s_1, s_2).$$

From the orthogonality, there follows the three term recurrence relation

$$xP_n(x; s_1, s_2) = P_{n+1}(x; s_1, s_2) + \alpha_n(s_1, s_2)P_n(x; s_1, s_2) + \beta_n(s_1, s_2)P_{n-1}(x; s_1, s_2) \tag{4}$$

with $n \geq 0$, subject to the initial conditions

$$P_0(x; s_1, s_2) := 1, \qquad \beta_0(s_1, s_2)P_{-1}(x; s_1, s_2) := 0.$$

The recurrence coefficients are given by

$$\alpha_n(s_1, s_2) = p(n, s_1, s_2) - p(n+1, s_1, s_2), \tag{5}$$

$$\beta_n(s_1, s_2) = \frac{h_n(s_1, s_2)}{h_{n-1}(s_1, s_2)}, \tag{6}$$

and it follows from (5) that

$$\sum_{j=0}^{n-1} \alpha_j(s_1, s_2) = -p(n, s_1, s_2). \tag{7}$$

For ease of notations, in the following discussion, we do not display the s_1 and s_2 dependence unless necessary.

The recurrence relation implies the Christoffel-Darboux formula

$$\sum_{j=0}^{n-1} \frac{P_j(x)P_j(y)}{h_j} = \frac{P_n(x)P_{n-1}(y) - P_{n-1}(x)P_n(y)}{h_{n-1}(x-y)}.$$

Here we point out that this identity and the recurrence relation hold for general monic polynomials orthogonal with respect to any given positive function which has moments of all orders. See, for example, Ref. 2 (section 3.2) for more details.

With all the above identities, one can derive a pair of ladder operators adapted to $P_n(z) = P_n(z; s_1, s_2)$:

$$P_n'(z) = \beta_n A_n(z)P_{n-1}(z) - B_n(z)P_n(z),$$

$$P_{n-1}'(z) = \big(B_n(z) + \mathrm{v}_0'(z)\big)P_{n-1}(z) - A_{n-1}(z)P_n(z),$$

where $\mathrm{v}_0(z) = z^2$, $A_n(z)$ and $B_n(z)$ have simple poles at s_1 and s_2, reading

$$A_n(z) = \frac{R_{n,1}(s_1, s_2)}{z - s_1} + \frac{R_{n,2}(s_1, s_2)}{z - s_2} + 2,$$

$$B_n(z) = \frac{r_{n,1}(s_1, s_2)}{z - s_1} + \frac{r_{n,2}(s_1, s_2)}{z - s_2},$$

with the residues defined by

$$R_{n,i}(s_1, s_2) := \frac{B_i P_n^2(s_i) \mathrm{e}^{-s_i^2}}{h_n}, \tag{8}$$

$$r_{n,i}(s_1, s_2) := \frac{B_i P_n(s_i) P_{n-1}(s_i) \mathrm{e}^{-s_i^2}}{h_{n-1}}. \tag{9}$$

Here, $P_j(s_i) = P_j(x; s_1, s_2)|_{x=s_i}$ for $j = n-1, n$. Moreover, one can show that $A_n(z)$ and $B_n(z)$ satisfy three compatibility conditions

$$(B_{n+1}(z) + B_n(z)) = (z - \alpha_n)A_n(z) - \mathrm{v}_0'(z), \tag{S$_1$}$$

$$1 + (z - \alpha_n)(B_{n+1}(z) - B_n(z)) = \beta_{n+1}A_{n+1}(z) - \beta_n A_{n-1}(z), \tag{S$_2$}$$

$$B_n^2(z) + \mathrm{v}_0'(z)B_n(z) + \sum_{j=0}^{n-1} A_j(z) = \beta_n A_n(z)A_{n-1}(z), \tag{S$'_2$}$$

where (S_2') results from (S_1) and (S_2) (see Ref. 5, theorem 3.1). Concerning the discussion of ladder operators and their compatibility conditions for general weight functions with jumps, we refer to lemma 1, remark 1, and remark 2 of Ref. 30.

Substituting $A_n(z)$ and $B_n(z)$ into (S_1) and (S_2'), by equating the residues on their both sides, it was found that the recurrence coefficients can be expressed in terms of the auxiliary quantities which satisfy a system of difference equations (see Ref. 5, eqs. 3.7-3.14). The results are presented below.

Proposition 1.

(a) $R_{n,i}$ and $r_{n,i}$, $i = 1, 2$, satisfy the following system of difference equations:

$$\beta_n R_{n,i} R_{n-1,i} = r_{n,i}^2, \tag{10}$$

$$r_{n+1,i} + r_{n,i} = (s_i - \alpha_n)R_{n,i}. \tag{11}$$

(b) The recurrence coefficients are expressed in terms of $R_{n,i}$ and $r_{n,i}$ $(i = 1, 2)$ by

$$\alpha_n = \frac{1}{2}\left(R_{n,1} + R_{n,2}\right), \tag{12}$$

$$\beta_n = \frac{1}{2}\left(r_{n,1} + r_{n,2} + n\right). \tag{13}$$

(c) The quantity $\sum_{j=0}^{n-1}(R_{j,1} + R_{j,2})$ has the following representation:

$$\sum_{j=0}^{n-1}\left(R_{j,1} + R_{j,2}\right) = -2s_1 r_{n,1} - 2s_2 r_{n,2} + 2\beta_n\left(R_{n,1} + R_{n,2} + R_{n-1,1} + R_{n-1,2}\right). \tag{14}$$

By taking the derivatives of (3) with $j = k = n$ and $j = k + 1 = n$, the auxiliary quantities turn out to be the partial derivatives of $-\ln h_n(s_1, s_2)$ and $\ln p(n, s_1, s_2)$ with respect to s_1 and s_2. Refer to eqs. (3.15), (3.16), (3.19), and (3.20) of Ref. 5. For ease of notations, in what follows, we denote $\frac{\partial}{\partial s_i}$ and $\frac{\partial^2}{\partial s_i \partial s_j}(i, j = 1, 2)$ by ∂_{s_i} and $\partial^2_{s_i s_j}$, respectively.

Proposition 2. *The following differential relations hold:*

$$\partial_{s_i} \ln h_n(s_1, s_2) = -R_{n,i}, \tag{15}$$

$$\partial_{s_i} p(n, s_1, s_2) = r_{n,i}, \tag{16}$$

with $i = 1, 2$. In view of $\alpha_n = p(n, s_1, s_2) - p(n + 1, s_1, s_2)$ and $\beta_n = h_n/h_{n-1}$, it follows that

$$\partial_{s_i}\alpha_n(s_1, s_2) = r_{n,i} - r_{n+1,i}, \tag{17}$$

$$\partial_{s_i}\beta_n(s_1, s_2) = \beta_n\left(R_{n-1,i} - R_{n,i}\right). \tag{18}$$

Define

$$\sigma_n(s_1, s_2) := \left(\partial_{s_1} + \partial_{s_2}\right)\ln D_n(s_1, s_2).$$

With the fact that $D_n(s_1, s_2) = \prod_{j=0}^{n-1} h_j(s_1, s_2)$ and by using (12), one finds

$$\sigma_n(s_1, s_2) = -\sum_{j=0}^{n-1}\left(R_{j,1} + R_{j,2}\right). \tag{19}$$

According to (7) and (15), there follows

$$\sigma_n(s_1, s_2) = 2p(n, s_1, s_2), \tag{20}$$

so that, in light of (16),

$$\partial_{s_i}\sigma_n = 2r_{n,i}, \qquad i = 1, 2.$$

Hence, the compatibility condition $\partial^2_{s_1 s_2}\sigma_n = \partial^2_{s_2 s_1}\sigma_n$ gives us

$$\partial_{s_2} r_{n,1} = \partial_{s_1} r_{n,2}. \tag{21}$$

Combining (14) with (19), and taking account of (10) and (13), we obtain the expression of $\sigma_n(s_1, s_2)$ in terms of the auxiliary quantities

$$\sigma_n = 2\left(s_1 r_{n,1} + s_2 r_{n,2} - \frac{r^2_{n,1}}{R_{n,1}} - \frac{r^2_{n,2}}{R_{n,2}}\right) - \left(r_{n,1} + r_{n,2} + n\right)\left(R_{n,1} + R_{n,2}\right). \tag{22}$$

By using the above identities, a second-order PDE was established for $\sigma_n(s_1, s_2)$ (see Theorem 3.3, Ref. 5).

Proposition 3. *$\sigma_n(s_1, s_2)$ satisfies the following equation:*

$$\left(\left(2s_1 \cdot \partial_{s_1}\sigma_n + 2s_2 \cdot \partial_{s_2}\sigma_n - 2\sigma_n\right)^2 - \Delta_1 - \Delta_2\right)^2 = 4\Delta_1\Delta_2,$$

where Δ_1 and Δ_2 are defined by

$$\Delta_1 := \left(\partial^2_{s_1 s_1}\sigma_n + \partial^2_{s_2 s_1}\sigma_n\right)^2 + 4\left(\partial_{s_1}\sigma_n\right)^2\left(\partial_{s_1}\sigma_n + \partial_{s_2}\sigma_n + 2n\right),$$
$$\Delta_2 := \left(\partial^2_{s_2 s_2}\sigma_n + \partial^2_{s_1 s_2}\sigma_n\right)^2 + 4\left(\partial_{s_2}\sigma_n\right)^2\left(\partial_{s_1}\sigma_n + \partial_{s_2}\sigma_n + 2n\right).$$

3 | PDEs SATISFIED BY $R_{n,i}$ AND COUPLED PAINLEVÉ IV SYSTEM

Based on the results presented in the previous section, we will derive coupled PDEs satisfied by $R_{n,1}$ and $R_{n,2}$ in this section, which we will see in the next section are crucial for the derivation of the coupled Painlevé II system under double scaling. We will also deduce the coupled Painlevé IV system satisfied by quantities allied with $R_{n,i}$ and $r_{n,i}$.

3.1 | Analogs of Riccati equations for $R_{n,i}$ and $r_{n,i}$, and coupled PDEs satisfied by $R_{n,i}$

Combining the expressions involving the recurrence coefficients together, namely, (12), (13), (17), and (18), with the aid of the difference equations (10) and (11), we arrive at the following four first-order PDEs for $R_{n,i}$ and $r_{n,i}$.

Lemma 1. *The quantities $R_{n,i}$ and $r_{n,i}$, $i = 1, 2$, satisfy the analogs of Riccati equations*

$$\partial_{s_i}\left(R_{n,1} + R_{n,2}\right) = 4r_{n,i} + (R_{n,1} + R_{n,2} - 2s_i)R_{n,i}, \tag{23}$$

$$\partial_{s_i}\left(r_{n,1} + r_{n,2}\right) = \frac{2r_{n,i}^2}{R_{n,i}} - \left(n + r_{n,1} + r_{n,2}\right)R_{n,i}. \tag{24}$$

Proof. Removing $r_{n+1,i}$ from (17) by using (11), we get

$$\partial_{s_i}\alpha_n(s_1, s_2) = 2r_{n,i} + (\alpha_n - s_i)R_{n,i}.$$

Inserting (12) into the above equation, we obtain (23).
Getting rid of $R_{n-1,i}$ in (18) by using (10), we find

$$\partial_{s_i}\beta_n = \frac{r_{n,i}^2}{R_{n,i}} - \beta_n R_{n,i}.$$

Plugging (13) into this identity, we come to (24). □

From (23), we readily get the expressions of $r_{n,i}$ in terms of $R_{n,i}$ and their first-order partial derivatives. Substituting them into (24), we arrive at a coupled PDEs satisfied by $R_{n,i}$.

Theorem 1. *The quantities $R_{n,i}$, $i = 1, 2$, satisfy the following coupled PDEs:*

$$\begin{aligned}&\left(\partial^2_{s_1 s_1} + \partial^2_{s_1 s_2}\right)\left(R_{n,1} + R_{n,2}\right) - \partial_{s_1}\left(R_{n,1} + R_{n,2}\right) \cdot \left(\frac{\partial_{s_1}(R_{n,1} + R_{n,2})}{2R_{n,1}} + R_{n,2}\right) + 2(s_2 - s_1)\left(\partial_{s_1} R_{n,2}\right)\\ &\quad + R_{n,1}\left(\partial_{s_2}(R_{n,1} + R_{n,2}) - \frac{3}{2}(R_{n,1} + R_{n,2})^2 + 2\left(2s_1 R_{n,1} + (s_1 + s_2)R_{n,2} - s_1^2 + 2n + 1\right)\right) = 0,\end{aligned} \tag{25a}$$

and

$$\begin{aligned}&\left(\partial^2_{s_2 s_2} + \partial^2_{s_2 s_1}\right)\left(R_{n,1} + R_{n,2}\right) - \partial_{s_2}\left(R_{n,1} + R_{n,2}\right) \cdot \left(\frac{\partial_{s_2}(R_{n,1} + R_{n,2})}{2R_{n,2}} + R_{n,1}\right) + 2(s_1 - s_2)\left(\partial_{s_2} R_{n,1}\right)\\ &\quad + R_{n,2}\left(\partial_{s_1}(R_{n,1} + R_{n,2}) - \frac{3}{2}(R_{n,1} + R_{n,2})^2 + 2\left((s_1 + s_2)R_{n,1} + 2s_2 R_{n,2} - s_2^2 + 2n + 1\right)\right) = 0.\end{aligned} \tag{25b}$$

Remark 1. Interchanging s_1 with s_2, $R_{n,1}$ with $R_{n,2}$ in (25a), we get (25b). This observation agrees with the symmetry in position of s_1 and s_2 in the weight function $w(x; s_1, s_2)$ and the definitions of $R_{n,i}$.

Remark 2. If $B_2 = 0$, then $R_{n,2} = 0$ and $R_{n,1}$ depends only on s_1. Equation (25a) is reduced to an ODE satisfied by $R_n(s_1) := R_{n,1}(s_1, 0)$

$$R_n'' = \frac{(R_n')^2}{2R_n} + \frac{3}{2}R_n^3 - 4s_1 R_n^2 + 2\left(s_1^2 - 2n - 1\right) R_n, \tag{26}$$

which is identical with (2.37) of Ref. 5 where t_1 is used instead of s_1. As was pointed out there, (26) can be transformed into a Painlevé IV equation satisfied by $y(s_1) := R_n(-s_1)$.

In case $B_1 = 0$, via a similar argument, we find that $R_{n,2}(0, s_2)$ satisfies (26) with s_1 replaced by s_2.

3.2 | Coupled Painlevé IV system

Define

$$x := \frac{s_1 + s_2}{2}, \qquad s := \frac{s_2 - s_1}{2},$$

and introduce four quantities allied with $R_{n,i}(s_1, s_2)$ and $r_{n,i}(s_1, s_2)$:

$$a_i(x, s) := \frac{r_{n,i}^2}{R_{n,i}(r_{n,1} + r_{n,2} + n)},$$

$$b_i(x, s) := \frac{R_{n,i}}{r_{n,i}}(r_{n,1} + r_{n,2} + n),$$

with $i = 1, 2$. We have

$$s_1 = x - s, \qquad s_2 = x + s,$$

and

$$\begin{aligned} R_{n,i}(s_1, s_2) &= \frac{a_i b_i^2}{a_1 b_1 + a_2 b_2 + n}, \\ r_{n,i}(s_1, s_2) &= a_i b_i. \end{aligned} \tag{27}$$

By making use of the results from Section 2, we show that a_i and b_i satisfy a coupled Painlevé IV system with $(\partial_{s_1} + \partial_{s_2}) \ln D_n(s_1, s_2) + n(s_1 + s_2)$ being the Hamiltonian.

Theorem 2. *The quantity*

$$H_{IV}(a_1, a_2, b_1, b_2; x, s) := \sigma_n(s_1, s_2) + n(s_1 + s_2)$$

with $\sigma_n(s_1,s_2) = (\partial_{s_1} + \partial_{s_2}) \ln D_n(s_1,s_2)$ satisfying the second-order PDE given by Proposition 3, is expressed in terms of $a_i(x,s)$ and $b_i(x,s)$ by

$$\begin{aligned} H_{IV}(a_1,a_2,b_1,b_2;x,s) = &-2(a_1b_1+a_2b_2+n)(a_1+a_2) - (a_1b_1^2+a_2b_2^2) \\ &+ 2((x-s)a_1b_1 + (x+s)a_2b_2 + nx), \end{aligned} \tag{28}$$

and it is the Hamiltonian of the following coupled Painlevé IV system

$$\begin{aligned} \partial_x a_1 &= \partial_{b_1} H_{IV} = -2a_1(a_1+a_2+b_1-x+s), \\ \partial_x a_2 &= \partial_{b_2} H_{IV} = -2a_2(a_1+a_2+b_2-x-s), \\ \partial_x b_1 &= \partial_{a_1} H_{IV} = b_1^2 + 2b_1(2a_1+a_2-x+s) + 2(a_2b_2+n), \\ \partial_x b_2 &= \partial_{a_2} H_{IV} = b_2^2 + 2b_2(a_1+2a_2-x-s) + 2(a_1b_1+n). \end{aligned} \tag{29}$$

Expression (28) follows directly from (22) and (27). To derive the coupled Painlevé IV system, we establish four linear equations in the variables $\partial_x a_i$ and $\partial_x b_i$, $i = 1, 2$. Before proceeding further, we first present some results which will be used later for the derivation.

Since $\partial_x = \partial_{s_1} + \partial_{s_2}$, we readily get from (15) and (16) that

$$\partial_x \ln h_n(s_1,s_2) = -\left(R_{n,1} + R_{n,2}\right), \tag{30}$$

$$\partial_x\, p(n,s_1,s_2) = r_{n,1} + r_{n,2}. \tag{31}$$

Noting that $\alpha_n = p(n,s_1,s_2) - p(n+1,s_1,s_2)$ and $\beta_n = h_n/h_{n-1}$, we find

$$\partial_x \alpha_n(s_1,s_2) = \sum_{i=1,2} \left(r_{n,i} - r_{n+1,i}\right), \tag{32}$$

$$\partial_x \beta_n(s_1,s_2) = \beta_n \sum_{i=1,2} \left(R_{n-1,i} - R_{n,i}\right). \tag{33}$$

As an immediate consequence of (12) and (13), we have

Lemma 2. *The recurrence coefficients are expressed in terms of a_i and b_i by*

$$\alpha_n = \frac{a_1b_1^2 + a_2b_2^2}{2(a_1b_1+a_2b_2+n)}, \tag{34}$$

$$\beta_n = \frac{1}{2}(a_1b_1+a_2b_2+n). \tag{35}$$

Replacing $r_{n,1}+r_{n,2}+n$ by $2\beta_n$ in the definitions of $a_i(x,s)$, which is due to (13), with the aid of (10), we build the direct relationships between a_i and the quantities with index $n-1$, ie, $R_{n-1,i}, \alpha_{n-1}$ and $\partial_x \ln h_{n-1}$.

Lemma 3. *We have*

$$a_i(x,s) = \frac{R_{n-1,i}}{2}, \qquad i = 1, 2, \tag{36}$$

so that, in view of (12) and (30),

$$\alpha_{n-1}(s_1,s_2) = a_1(x,s) + a_2(x,s) = -\frac{1}{2}\,\partial_x h_{n-1}(s_1,s_2). \tag{37}$$

Now we are ready to deduce the four linear equations in $\partial_x a_i$ and $\partial_x b_i$, each of which will be stated as a lemma. We start from the combination of (34) and (35) which gives us

$$a_1 b_1^2 + a_2 b_2^2 = 4\alpha_n \beta_n. \tag{38}$$

Lemma 4. *We have*

$$\begin{aligned} &b_1^2(\partial_x a_1) + b_2^2(\partial_x a_2) + 2a_1 b_1(\partial_x b_1) + 2a_2 b_2(\partial_x b_2) \\ &\qquad = 4(a_1 a_2 + b_1 b_2 + n)(a_1 a_2 + b_1 b_2) + 2a_1 b_1^2(a_1 + a_2 - s_1) + 2a_2 b_2^2(a_1 + a_2 - s_2). \end{aligned}$$

Proof. Taking the derivative on both sides of (38) with respect to x, we have

$$\partial_x\left(a_1 b_1^2 + a_2 b_2^2\right) = 4\beta_n(\partial_x \alpha_n) + 4\alpha_n(\partial_x \beta_n). \tag{39}$$

Now we make use of (32) and (33) to derive the expressions of $\partial_x \alpha_n$ and $\partial_x \beta_n$ in terms of a_i, b_i or $R_{n,i}, r_{n,i}$. Using (11) to get rid of $r_{n+1,i}, i = 1, 2$, in (32), we find

$$\partial_x \alpha_n(s_1,s_2) = \sum_{i=1,2}\left(2r_{n,i} + (\alpha_n - s_i)R_{n,i}\right).$$

On account of (36), we replace $R_{n-1,i}$ by $2a_i$ in (33) and get

$$\partial_x \beta_n(s_1,s_2) = -\beta_n\left(R_{n,1} + R_{n,2}\right) + 2\beta_n(a_1 + a_2).$$

Plugging the above two identities into (39), we obtain

$$\begin{aligned} &b_1^2(\partial_x a_1) + 2a_1 b_1(\partial_x b_1) + b_2^2(\partial_x a_2) + 2a_2 b_2(\partial_x b_2) \\ &\qquad = 4\beta_n\left(2(r_{n,1} + r_{n,2}) - s_1 R_{n,1} - s_2 R_{n,2}\right) + 8\alpha_n \beta_n(a_1 + a_2). \end{aligned}$$

On substituting (27), (34), and (35) into this equation, we come to the desired result. $\square$

Replacing n by $n-1$ in (32) and (11), we have

$$\partial_x \alpha_{n-1}(s_1, s_2) = \sum_{i=1,2} \left(r_{n-1,i} - r_{n,i}\right),$$

$$r_{n,i} + r_{n-1,i} = (s_i - \alpha_{n-1})R_{n-1,i}.$$

Using the second equality to remove $r_{n-1,i}$ in the first one, we are led to

$$\partial_x \alpha_{n-1}(s_1, s_2) = \sum_{i=1,2} \left((s_i - \alpha_{n-1})R_{n-1,i} - 2r_{n,i}\right).$$

According to (37) and (36), we replace α_{n-1} by $a_1 + a_2$ and $R_{n-1,i}$ by $2a_i$ in the above identity. By taking note that $r_{n,i} = a_i b_i, i = 1, 2$, we come to the following equation.

Lemma 5. *We have*

$$\partial_x a_1 + \partial_x a_2 = -2a_1(a_1 + a_2 + b_1 - s_1) - 2a_2(a_1 + a_2 + b_2 - s_2).$$

The next equation is obtained by combining the two expressions involving β_n and $\partial_x \beta_n$.

Lemma 6. *We have*

$$b_i(\partial_x a_i) + a_i(\partial_x b_i) = 2a_i(a_1 b_1 + a_2 b_2 + n) - a_i b_i^2, \qquad i = 1, 2.$$

Proof. Plugging (13) into (18), we get

$$\partial_{s_i}\left(r_{n,1} + r_{n,2}\right) = \left(r_{n,1} + r_{n,2} + n\right)\left(R_{n-1,i} - R_{n,i}\right), \qquad i = 1, 2.$$

In view of (21), ie, $\partial_{s_2} r_{n,1} = \partial_{s_1} r_{n,2}$, we find

$$\left(\partial_{s_1} + \partial_{s_2}\right) r_{n,1} = \left(r_{n,1} + r_{n,2} + n\right)\left(R_{n-1,1} - R_{n,1}\right),$$
$$\left(\partial_{s_1} + \partial_{s_2}\right) r_{n,2} = \left(r_{n,1} + r_{n,2} + n\right)\left(R_{n-1,2} - R_{n,2}\right).$$

Since $\partial_x = \partial_{s_1} + \partial_{s_2}$, it follows that

$$\partial_x r_{n,i} = \left(r_{n,1} + r_{n,2} + n\right)\left(R_{n-1,i} - R_{n,i}\right), \qquad i = 1, 2.$$

Using (27) to replace $r_{n,i}$ and $R_{n,i}$ in this expression, and substituting $2a_i$ for $R_{n-1,i}$, which is due to (36), we complete the proof. $\square$

Proof of Theorem 2 Now we have four linear equations in $\partial_x a_1, \partial_x a_2, \partial_x b_1$, and $\partial_x b_2$, namely,

$$\begin{aligned} &b_1^2(\partial_x a_1) + b_2^2(\partial_x a_2) + 2a_1 b_1(\partial_x b_1) + 2a_2 b_2(\partial_x b_2) \\ &\quad = 4(a_1 a_2 + b_1 b_2 + n)(a_1 a_2 + b_1 b_2) + 2a_1 b_1^2(a_1 + a_2 - s_1) + 2a_2 b_2^2(a_1 + a_2 - s_2), \end{aligned} \tag{40}$$

$$\partial_x a_1 + \partial_x a_2 = -2a_1(a_1 + a_2 + b_1 - s_1) - 2a_2(a_1 + a_2 + b_2 - s_2), \tag{41}$$

$$b_1(\partial_x a_1) + a_1(\partial_x b_1) = 2a_1(a_1 b_1 + a_2 b_2 + n) - a_1 b_1^2, \tag{42}$$

$$b_2(\partial_x a_2) + a_2(\partial_x b_2) = 2a_2(a_1 b_1 + a_2 b_2 + n) - a_2 b_2^2. \tag{43}$$

Subtracting (40) from the sum of (42) multiplied by $2b_1$ and (43) multiplied by $2b_2$, we get

$$b_1^2(\partial_x a_1) + b_2^2(\partial_x a_2) = -2a_1 b_1^2(a_1 + a_2 + b_1 - s_1) - 2a_2 b_2^2(a_1 + a_2 + b_2 - s_2). \tag{44}$$

Combining (41) with (44) to solve for $\partial_x a_1$ and $\partial_x a_2$, and substituting the resulting expressions into (42) and (43), we arrive at the desired coupled Painlevé IV system (29). □

Remark 3. The Hamiltonian of the coupled Painlevé IV system presented in Theorem 2 is the same as the one given by (1.15) and (1.16) of Ref. 6 which was derived by considering the RH problem for $P_j(x; s_1, s_2)$, ie, the monic polynomial orthogonal with respect to $w(x; s_1, s_2)$. Taking note that our symbols β_n and h_n correspond to β_n^2 and γ_n^{-2} of Ref. 6, we find that our Equations (34), (35), and (37) are consistent with (1.23), (1.24), and (1.26) of Ref. 6, respectively.

Compared with the finite-dimensional analysis in Ref. 6, our derivation presented in this section seems to be more elementary and straightforward.

4 | COUPLED PAINLEVÉ II SYSTEM AT THE SOFT EDGE

We remind the reader that our weight function is obtained by multiplying the Gaussian weight by a factor with two jumps, ie,

$$w(x; s_1, s_2) = \mathrm{e}^{-x^2}(A + B_1\theta(x - s_1) + B_2\theta(x - s_2)),$$

where $B_1 B_2 \neq 0$. In this section, we discuss the asymptotic behavior of the associated Hankel determinant when s_1 and s_2 tend to the soft edge of the spectrum of GUE, namely,

$$s_i := \sqrt{2n} + \frac{t_i}{\sqrt{2}n^{1/6}}, \qquad i = 1, 2.$$

This double scaling may be explained in the following way. As we know, the classical Hermite polynomials $H_n(x)$ are orthogonal with respect to the Gaussian weight e^{-x^2}, $x \in (-\infty, \infty)$. Under the double scaling $x = \sqrt{2n} + \frac{t}{\sqrt{2}n^{1/6}}$ and as $n \to \infty$, the Hermite function $\mathrm{e}^{-x^2/2}H_n(x)$ is approximated by the Airy function $A(x)$ multiplied by a factor involving n (Ref. 2, formula 8.22.14). See also Ref. (31, formula 3.6) and Ref. (32, theorem 2.1) for more explanation about this double scaling.

When $B_1 = 0$ or $B_2 = 0$, our weight function has only one jump. This case was studied in Ref. 5 and the expansion formula for $R_n(s_1) := R_{n,1}(s_1, 0)$ in large n was given by

$$R_n(s_1) = n^{-1/6}v_1(t_1) + n^{-1/2}v_2(t_1) + n^{-5/6}v_3(t_1) + O\left(n^{-7/6}\right),$$

where $v_k(t_1), k = 1, 2, 3$, are certain functions of t_1. It was obtained by using the second-order ODE satisfied by $R_n(s_1)$ (see theorem 2.10, Ref. 5). Hence, for our two jump case where $B_1B_2 \neq 0$, we assume

$$R_{n,1}(s_1, s_2) = \sum_{i=1}^{\infty} \mu_i(t_1, t_2) \cdot n^{(1-2i)/6}, \tag{45a}$$

$$R_{n,2}(s_1, s_2) = \sum_{i=1}^{\infty} \nu_i(t_1, t_2) \cdot n^{(1-2i)/6}. \tag{45b}$$

From the compatibility condition $\partial^2_{s_1 s_2} R_{n,i} = \partial^2_{s_2 s_1} R_{n,i}, i = 1, 2$, it follows that

$$\begin{aligned} \partial^2_{t_1 t_2}\mu_1(t_1, t_2) &= \partial^2_{t_2 t_1}\mu_1(t_1, t_2), \\ \partial^2_{t_1 t_2}\nu_1(t_1, t_2) &= \partial^2_{t_2 t_1}\nu_1(t_1, t_2). \end{aligned} \tag{46}$$

We keep these two relations in mind in the subsequent discussions.

Substituting (45) into the left-hand side of (25a) and (25b), by taking their series expansions in large n and setting the leading coefficients to be zero, we get a coupled PDEs satisfied by μ_1 and ν_1.

Theorem 3. *The leading coefficients in the expansions of $R_{n,i}$ in large n, ie,*

$$\mu_1(t_1, t_2) = \lim_{n\to\infty} n^{1/6} R_{n,1}(s_1, s_2),$$

$$\nu_1(t_1, t_2) = \lim_{n\to\infty} n^{1/6} R_{n,2}(s_1, s_2),$$

satisfy the following coupled PDEs

$$\left(\partial^2_{t_1 t_1} + \partial^2_{t_1 t_2}\right)(\mu_1 + \nu_1) - \frac{\left(\partial_{t_1}(\mu_1 + \nu_1)\right)^2}{2\mu_1} + 2\mu_1(\sqrt{2}(\mu_1 + \nu_1) - t_1) = 0, \tag{47a}$$

$$\left(\partial^2_{t_2 t_2} + \partial^2_{t_1 t_2}\right)(\mu_1 + \nu_1) - \frac{\left(\partial_{t_2}(\mu_1 + \nu_1)\right)^2}{2\nu_1} + 2\nu_1(\sqrt{2}(\mu_1 + \nu_1) - t_2) = 0. \tag{47b}$$

Plugging (45) into (23), we get

$$r_{n,1}(s_1, s_2) = \frac{\mu_1}{\sqrt{2}} n^{1/3} + \frac{\mu_2}{\sqrt{2}} + \frac{\sqrt{2}}{4}\partial_{t_1}(\mu_1 + \nu_1) + O(n^{-1/3}), \tag{48a}$$

$$r_{n,2}(s_1, s_2) = \frac{\nu_1}{\sqrt{2}} n^{1/3} + \frac{\nu_2}{\sqrt{2}} + \frac{\sqrt{2}}{4} \partial_{t_2}(\mu_1 + \nu_1) + O(n^{-1/3}). \tag{48b}$$

Hence, according to (21), ie, $\partial_{s_2} r_{n,1} = \partial_{s_1} r_{n,2}$, we find

$$\partial_{t_2} \mu_1(t_1, t_2) = \partial_{t_1} \nu_1(t_1, t_2). \tag{49}$$

To continue, we define

$$\begin{aligned} v_1(t_1, t_2 - t_1) &:= -\frac{\mu_1(t_1, t_2)}{\sqrt{2}}, \\ v_2(t_1, t_2 - t_1) &:= -\frac{\nu_1(t_1, t_2)}{\sqrt{2}}. \end{aligned}$$

With the aid of (49), we establish the following differential relations.

Lemma 7. *We have*

$$\partial_{t_1}(\mu_1(t_1, t_2) + \nu_1(t_1, t_2)) = -\sqrt{2} v_{1\xi}(t_1, t_2 - t_1), \tag{50a}$$

$$\partial_{t_2}(\mu_1(t_1, t_2) + \nu_1(t_1, t_2)) = -\sqrt{2} v_{2\xi}(t_1, t_2 - t_1), \tag{50b}$$

where $v_{i\xi}(i = 1, 2)$ denotes the first-order derivative of $v_i(\xi, \eta)$ with respect to ξ.

Proof. By the definition of $v_1(t_1, t_2 - t_1)$, we find

$$\begin{aligned} \partial_{t_1} \mu_1(t_1, t_2) &= -\sqrt{2} \cdot \partial_{t_1} v_1(t_1, t_2 - t_1) \\ &= -\sqrt{2} \big(v_{1\xi}(t_1, t_2 - t_1) - v_{1\eta}(t_1, t_2 - t_1) \big), \\ \partial_{t_2} \mu_1(t_1, t_2) &= -\sqrt{2} \cdot \partial_{t_2} v_1(t_1, t_2 - t_1) \\ &= -\sqrt{2} \cdot v_{1\eta}(t_1, t_2 - t_1), \end{aligned}$$

so that

$$\big(\partial_{t_1} + \partial_{t_2}\big) \mu_1(t_1, t_2) = -\sqrt{2} v_{1\xi}(t_1, t_2 - t_1).$$

In view of (49), we obtain (50a). Via a similar argument, we can prove (50b). ∎

With the aid of (49) and (50), we establish the following equations for v_1 and v_2 by using the coupled PDEs (47).

Theorem 4. *The quantities $v_1(t_1, t_2 - t_1)$ and $v_2(t_1, t_2 - t_1)$ satisfy a coupled nonlinear equations*

$$v_{i\xi\xi} - \frac{v_{i\xi}^2}{2v_i} - 2v_i(2(v_1 + v_2) + t_i) = 0, \tag{51}$$

where $v_{i\xi}$ and $v_{i\xi\xi}$ denote the first- and second-order derivative of $v_i(\xi, \eta)$ with respect to ξ, respectively.

Proof. Differentiation of both sides of (50a) over t_1 and t_2 gives us

$$\partial^2_{t_1 t_1}(\mu_1(t_1, t_2) + \nu_1(t_1, t_2)) = -\sqrt{2}\big(v_{1\xi\xi}(t_1, t_2 - t_1) - v_{1\xi\eta}(t_1, t_2 - t_1)\big),$$
$$\partial^2_{t_1 t_2}(\mu_1(t_1, t_2) + \nu_1(t_1, t_2)) = -\sqrt{2} v_{1\xi\eta}(t_1, t_2 - t_1),$$

where in the second equality we make use of (46). It follows that

$$\left(\partial^2_{t_1 t_1} + \partial^2_{t_1 t_2}\right)(\mu_1(t_1, t_2) + \nu_1(t_1, t_2)) = -\sqrt{2} v_{1\xi\xi}(t_1, t_2 - t_1). \tag{52a}$$

Similarly, by differentiating both sides of (50b) over t_1 and t_2, we get

$$\left(\partial^2_{t_2 t_2} + \partial^2_{t_1 t_2}\right)(\mu_1(t_1, t_2) + \nu_1(t_1, t_2)) = -\sqrt{2} v_{2\xi\xi}(t_1, t_2 - t_1). \tag{52b}$$

Plugging (52) and (50) into (47), we arrive at the desired equations. □

Now we look at $\sigma_n(s_1, s_2)$ which is defined by

$$\sigma_n(s_1, s_2) := \left(\partial_{s_1} + \partial_{s_2}\right) \ln D_n(s_1, s_2).$$

Recall that it is expressed in terms of $R_{n,i}$ and $r_{n,i}$ by (22). Substituting the expansions of $R_{n,i}$ and $r_{n,i}$ into this expression, we establish the following results.

Theorem 5. *$\sigma_n(s_1, s_2)$ has the following asymptotic expansion in large n*

$$\sigma_n(s_1, s_2) = \sqrt{2} n^{1/6} H_{II}(t_1, t_2 - t_1) + O(n^{-1/6}), \tag{53}$$

where $H_{II}(t_1, t_2 - t_1)$ is the Hamiltonian of the following coupled Painlevé II system

$$\begin{cases} v_{i\xi} = \dfrac{\partial H_{II}}{\partial w_i} = 2 v_i w_i, & \text{(54a)} \\ w_{i\xi} = -\dfrac{\partial H_{II}}{\partial v_i} = 2\left(v_1 + v_2\right) + t_i - w_i^2, & \text{(54b)} \end{cases}$$

which is given by

$$H_{II}(t_1,t_2-t_1)=v_1w_1^2+v_2w_2^2-(v_1+v_2)^2-t_1v_1-t_2v_2. \tag{55}$$

Here, $v_i=v_i(t_1,t_2-t_1)$ *and* $w_i=w_i(t_1,t_2-t_1)$*. Moreover,* H_{II} *satisfies the following second-order second degree PDE*

$$\begin{aligned}\left(\partial_{t_1}H_{II}\right)\cdot\left(\partial^2_{t_2t_2}H_{II}+\partial^2_{t_2t_1}H_{II}\right)^2+\left(\partial_{t_2}H_{II}\right)\cdot\left(\partial^2_{t_1t_1}H_{II}+\partial^2_{t_1t_2}H_{II}\right)^2\\=4\left(\partial_{t_1}H_{II}\right)\left(\partial_{t_2}H_{II}\right)\left(t_1\cdot\partial_{t_1}H_{II}+t_2\cdot\partial_{t_2}H_{II}-H_{II}\right).\end{aligned} \tag{56}$$

Proof. Recall (22), ie,

$$\sigma_n(s_1,s_2)=2\left(s_1r_{n,1}+s_2r_{n,2}-\frac{r_{n,1}^2}{R_{n,1}}-\frac{r_{n,2}^2}{R_{n,2}}\right)-\left(r_{n,1}+r_{n,2}+n\right)\left(R_{n,1}+R_{n,2}\right).$$

Substituting (45) and (48) into the right-hand side of this expression, by taking its series expansion in large n, we obtain

$$\sigma_n(s_1,s_2)=\left(-\frac{\left(\partial_{t_1}(\mu_1+\nu_1)\right)^2}{4\mu_1}-\frac{\left(\partial_{t_2}(\mu_1+\nu_1)\right)^2}{4\nu_1}-\frac{(\mu_1+\nu_1)^2}{\sqrt{2}}+t_1\mu_1+t_2\nu_1\right)n^{1/6}+O\left(n^{-1/6}\right).$$

Replacing the derivative terms in the coefficient of $n^{1/6}$ by using (50), and substituting $-\sqrt{2}v_1$ and $-\sqrt{2}v_2$ for μ_1 and ν_1, respectively, we find

$$\sigma_n(s_1,s_2)=\sqrt{2}n^{1/6}\left(\frac{v_{1\xi}^2}{4v_1}+\frac{v_{2\xi}^2}{4v_2}-(v_1+v_2)^2-t_1v_1-t_2v_2\right)+O\left(n^{-1/6}\right).$$

On writing

$$w_i(t_1,t_2-t_1):=\frac{v_{i\xi}(t_1,t_2-t_1)}{2v_i(t_1,t_2-t_1)},$$

we get (53).

From the above definition of w_i, we readily see that (54a) holds. Taking the derivative on both sides of (54a), we are led to

$$v_{i\xi\xi}=4v_iw_i^2+2v_iw_{i\xi}.$$

Inserting it and (54a) into (51), after simplification, we produce (54b).

To derive (56), we plug (53) into the PDE satisfied by σ_n, ie, (3). By taking the series expansion of its left-hand side and setting the leading coefficient to be zero, we obtain (56). □

Recall (12) and (13) which express the recurrence coefficients in terms of $R_{n,i}$ and $r_{n,i}$, namely,

$$\alpha_n = \frac{1}{2}\left(R_{n,1} + R_{n,2}\right),$$

$$\beta_n = \frac{1}{2}\left(r_{n,1} + r_{n,2} + n\right).$$

Substituting (45) and (48) into the above expressions, after simplification, we get the asymptotic expansions of α_n and β_n in large n.

Theorem 6. *The recurrence coefficients of the monic polynomials orthogonal with respect to the Gaussian weight with two jump discontinuities have the following asymptotics for large n*

$$\alpha_n(s_1, s_2) = -\frac{v_1 + v_2}{\sqrt{2}n^{1/6}} + O(n^{-1/2}),$$

$$\beta_n(s_1, s_2) = \frac{n}{2} - \frac{v_1 + v_2}{2}n^{1/3} + O(1).$$

Here $v_i = v_i(t_1, t_2 - t_1), i = 1, 2$, satisfy the coupled Painlevé II system (54).

Remark 4. Theorems 4, 5, and 6 are consistent with eq. (1.33), Lemma 1, and Theorem 4 of Ref. 6, respectively. However, it is necessary to point out the lack of rigor of our derivation.

To prove Theorem (3) on which the derivation of the coupled Painlevé II system relies, we make the assumption that the asymptotic expansions of $R_{n,i}(s_1, s_2), i = 1, 2$, are (45), without proving the existence and convergence of the series. Hence, our derivation is not rigorous. This is the main disadvantage of our approach in the asymptotic analysis. Nevertheless, we would like to mention that, by substituting (45) into (25), we may get information of μ_j and ν_j for $j \geq 2$, which will provide possible insights into the higher-order terms of the asymptotics of the coupled Painlevé IV and II systems.

In comparison to our analysis, the one presented in Ref. 6 which is based on the RH method is rigorous. However, the properties of the higher-order terms in the asymptotics of the coupled Painlevé IV and II systems are not clear.

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