Asymptotics of the Largest Eigenvalue Distribution of the Laguerre Unitary Ensemble

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Abstract

We study the probability that all the eigenvalues of $n \times n$ Hermitian matrices, from the Laguerre unitary ensemble with the weight $x^{\gamma}\mathrm{e}^{-4nx},\ x \in [0,\infty),\ \gamma > -1$, lie in the interval $[0,\alpha]$. By using previous results for finite n obtained by the ladder operator approach of orthogonal polynomials, we derive the large n asymptotics of the largest eigenvalue distribution function with α ranging from 0 to the soft edge. In addition, at the soft edge, we compute the constant conjectured by Tracy and Widom [Commun. Math. Phys. **159** (1994), 151–174], later proved by Deift, Its and Krasovsky [Commun. Math. Phys. **278** (2008), 643–678]. Our results are reduced to those of Deift et al. when $\gamma = 0$.

Keywords: Laguerre unitary ensemble; Largest eigenvalue distribution; Asymptotic behavior; Ladder operators; Fredholm determinant.

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1 Introduction

We consider the Laguerre unitary ensemble (LUE for short) of $n \times n$ Hermitian matrices whose eigenvalues have the following joint probability density function [24]

$$p(x_1, x_2, \ldots, x_n) = \frac{1}{Z_n} \prod_{1 \le i < j \le n} (x_i - x_j)^2 \prod_{k=1}^{n} w(x_k; \gamma, n),$$

where $w(x; \gamma, n)$ is the scaled Laguerre weight

$$w(x; \gamma, n) = x^{\gamma} \mathrm{e}^{-4nx}, \qquad x \in [0, \infty), \quad \gamma > -1,$$

and Z_n is the partition function which reads

$$Z_n := \int_{[0,\infty)^n} \prod_{1 \le i < j \le n} (x_i - x_j)^2 \prod_{k=1}^{n} w(x_k; \gamma, n) dx_k.$$

The probability that all the eigenvalues in this LUE lie in the interval $[0, \alpha]$, or the largest eigenvalue is not greater than α, is given by

$$\mathbb{P}(n, \gamma, \alpha) = \frac{D_n(\alpha)}{D_n(\infty)}, \tag{1.1}$$

where $D_n(\alpha)$ is defined by

$$D_n(\alpha) := \frac{1}{n!} \int_{[0,\alpha]^n} \prod_{1 \le i < j \le n} (x_i - x_j)^2 \prod_{k=1}^{n} w(x_k; \gamma, n) dx_k.$$

It is apparent that $D_n(\infty) = Z_n / n!$.

In this paper, we are interested in the asymptotic behavior of $\mathbb{P}(n, \gamma, \alpha)$ at the soft edge. Deift, Its and Krasovsky [13] studied the special case $\mathbb{P}(n, 0, \alpha)$, namely the largest eigenvalue distribution on $[0, \alpha]$ of LUE with the weight e^{-4nx}. By using the Riemann-Hilbert approach, they obtained the constant conjectured by Tracy and Widom [32], which appears in the asymptotic formula for $\mathbb{P}(n, 0, \alpha)$ at the soft edge. We would like to generalize their results to generic γ.

By changing variables $4nx_\ell = y_\ell$, $\ell = 1, 2, \ldots, n$, in $D_n(\alpha)$, we get

$$\begin{aligned} D_n(\alpha) &= (4n)^{-n(n+\gamma)} \cdot \frac{1}{n!} \int_{[0,4n\alpha]^n} \prod_{1 \le i < j \le n} (y_i - y_j)^2 \prod_{k=1}^{n} y_k^{\gamma} \mathrm{e}^{-y_k} dy_k \\ &=: (4n)^{-n(n+\gamma)} \widehat{D}_n(4n\alpha), \end{aligned}$$

where $\widehat{D}_n(\cdot)$ is defined by

$$\widehat{D}_n(t) := \frac{1}{n!}\int_{[0,t]^n} \prod_{1\le i<j\le n}(x_i - x_j)^2 \prod_{k=1}^{n} x_k^{\gamma} \mathrm{e}^{-x_k} dx_k. \tag{1.2}$$

It follows from (1.1) that

$$\mathbb{P}(n,\gamma,\alpha) = \frac{\widehat{D}_n(4n\alpha)}{\widehat{D}_n(\infty)}. \tag{1.3}$$

Denoting by $\widehat{\mathbb{P}}(n,\gamma,t)$ the probability that all the eigenvalues lying in $[0,t]$ of LUE with the normal Laguerre weight $x^{\gamma}\mathrm{e}^{-x}$, we have (see [20])

$$\widehat{\mathbb{P}}(n,\gamma,t) = \frac{\widehat{D}_n(t)}{\widehat{D}_n(\infty)}. \tag{1.4}$$

Note that $\widehat{D}_n(\infty)$ has the following closed-form expression [24]:

$$\begin{aligned}\widehat{D}_n(\infty) =& \frac{1}{n!}\prod_{j=1}^{n}\Gamma(j+1)\Gamma(j+\gamma)\\ =& \frac{G(n+1)G(n+\gamma+1)}{G(\gamma+1)},\end{aligned} \tag{1.5}$$

where $G(\cdot)$ is the Barnes G-function which satisfies the relation

$$G(z+1) = \Gamma(z)G(z), \qquad\qquad G(1) := 1.$$

See [3, 35] for more properties of this function.

A combination of (1.3) and (1.4) gives us a connection between the largest eigenvalue distribution of LUE with the weight $x^{\gamma}\mathrm{e}^{-4nx}$ and the weight $x^{\gamma}\mathrm{e}^{-x}$:

$$\mathbb{P}(n,\gamma,\alpha) = \widehat{\mathbb{P}}(n,\gamma,4n\alpha).$$

Therefore, to study $\mathbb{P}(n,\gamma,\alpha)$, we first turn our attention to $\widehat{\mathbb{P}}(n,\gamma,t)$.

It is well known that $\widehat{\mathbb{P}}(n,\gamma,t)$, i.e. the probability that the interval (t,∞) is free of eigenvalues, can be expressed as a Fredholm determinant [24], namely,

$$\widehat{\mathbb{P}}(n,\gamma,t) = \det\left(I - K_n\chi_{(t,\infty)}\right),$$

where $\chi_{(t,\infty)}(\cdot)$ is the characteristic function of the interval (t,∞) and the integral operator $K_n\chi_{(t,\infty)}$ has kernel $K_n(x,y)\chi_{(t,\infty)}(y)$, with $K_n(x,y)$ given by the Christoffel-Darboux formula [31]:

$$\begin{aligned}K_n(x,y) =& \sum_{j=0}^{n-1}\varphi_j(x)\varphi_j(y)\\ =& \sqrt{n(n+\gamma)}\,\frac{\varphi_{n-1}(x)\varphi_n(y) - \varphi_{n-1}(y)\varphi_n(x)}{x-y}.\end{aligned}$$

Here $\{\varphi_j(x)\}_{j=0}^{\infty}$ are obtained by orthonormalizing the sequence $\{x^j x^{\gamma/2}\mathrm{e}^{-x/2}\}_{j=0}^{\infty}$ over $[0,\infty)$, and

$$\varphi_j(x)=\sqrt{\frac{\Gamma(j+1)}{\Gamma(j+\gamma+1)}}x^{\frac{\gamma}{2}}\mathrm{e}^{-\frac{x}{2}}L_j^{(\gamma)}(x),$$

with $L_j^{(\gamma)}(x)$ denoting the Laguerre polynomial of degree j.

The kernel $K_n(x,y)$ tends to the Airy kernel at the soft edge [15], i.e.,

$$\lim_{n\to\infty}2^{\frac{4}{3}}n^{\frac{1}{3}}K_n\left(4n+2\gamma+2+2^{\frac{4}{3}}n^{\frac{1}{3}}u,4n+2\gamma+2+2^{\frac{4}{3}}n^{\frac{1}{3}}v\right)=K_{\mathrm{Airy}}(u,v),$$

where $K_{\mathrm{Airy}}(u,v)$ is the Airy kernel defined by

$$K_{\mathrm{Airy}}(u,v):=\frac{\mathrm{Ai}(u)\mathrm{Ai}'(v)-\mathrm{Ai}(v)\mathrm{Ai}'(u)}{u-v}.$$

Here $\mathrm{Ai}(\cdot)$ is the Airy function of the first kind [19]. See also [28, 32] on the study of the Airy kernel. Tracy and Widom [32] showed that $\widehat{\mathbb{P}}(n,\gamma,t)$ can be expressed in terms of a Painleveé II transcendent and it satisfies the celebrated Tracy-Widom distribution at the soft edge.

At the hard edge, $K_n(x,y)$ tends to the Bessel kernel [15], that is,

$$\lim_{n\to\infty}\frac{1}{4n}K_n\left(\frac{u}{4n},\frac{v}{4n}\right)=K_{\mathrm{Bessel}}(u,v),$$

where

$$K_{\mathrm{Bessel}}(u,v):=\frac{J_\gamma(\sqrt{u})\sqrt{v}J_\gamma'(\sqrt{v})-\sqrt{u}J_\gamma'(\sqrt{u})J_\gamma(\sqrt{v})}{2(u-v)}.$$

Tracy and Widom [33] proved that the log-derivative of $\widehat{\mathbb{P}}(n,\gamma,t)$ satisfies a particular Painlevé III equation when t approaches the hard edge.

The level density of the LUE with the weight $x^\gamma\mathrm{e}^{-x}$ is given by [24, p.356]

$$\rho(x)=\frac{1}{2\pi}\sqrt{\frac{4n-x}{x}},\qquad 0<x<4n,$$

which is an example of the Marčenko-Pastur law [23]. Hence, in [13] and also in this paper, the scaled Laguerre weight with n appears in the exponent instead of the standard one is considered, in order to make the equilibrium density of the eigenvalues supported on $(0,1)$ as opposed to $(0,4n)$.

For finite n, Tracy and Widom [34] established a particular Painlevé V equation satisfied by the log-derivative of $\widehat{\mathbb{P}}(n,\gamma,t)$. Adler and van Moerbeke [1] derived the same results via differential operators. By using the ladder operator approach of orthogonal polynomials, Basor and Chen [4] investigated the Hankel determinant generated by the Laguerre weight with a jump, which including $\widehat{D}_n(t)$ as a special case, and a Painlevé V equation shows up as is expected. Based on their results, Lyu and Chen [20] considered the asymptotic behavior of $x^{\gamma/2}\mathrm{e}^{-x/2}P_j(x)$ at the soft edge, with $P_j(x),\ j=0,1,\ldots$ denoting the monic polynomials orthogonal with respect to $x^\gamma\mathrm{e}^{-x}$ on $[0,t]$. We mention here that the ladder operator method is effective and straightforward in the finite dimension analysis of problems in unitary ensembles, for example, the gap probability [5, 9, 21, 22, 25], the partition function for weights with singularities [26, 27, 29].

In the present paper, in order to derive the asymptotic formula for $\mathbb{P}(n,\gamma,\alpha)$ at the soft edge, we proceed from two aspects. On one hand, we first derive a large n asymptotic expansion for $\frac{d}{d\alpha}\ln\mathbb{P}(n,\gamma,\alpha)$, by using differential equations for finite n from [4]. Then we integrate the expansion from α_0 to α with arbitrary $\alpha_0<\alpha$ to obtain an asymptotic formula for $\ln\mathbb{P}(n,\gamma,\alpha)-\ln\mathbb{P}(n,\gamma,\alpha_0)$. On the other hand, we make use of the definition of $D_n(\alpha)$, i.e. the multiple integral, to get an approximate expression for $\ln\mathbb{P}(n,\gamma,\alpha_0)$ when α_0 is close to 0. Taking the sum of these two asymptotic expansions together, and by sending α_0 to 0, we come to an asymptotics for $\ln\mathbb{P}(n,\gamma,\alpha)$ in large n with α ranging from 0 to the soft edge, where a term which is independent of α and tends to 0 as $n\to\infty$ is included. Finally, by setting $\alpha=1-\frac{s}{(2n)^{2/3}}$ and sending n to ∞, we obtain the asymptotic formula of $\ln\mathbb{P}(n,\gamma,\alpha)$ at the soft edge for large s:

$$\lim_{n\to\infty}\ln\mathbb{P}\left(n,\gamma,1-\frac{s}{(2n)^{2/3}}\right)=-\frac{s^3}{12}-\frac{1}{8}\ln s+\frac{1}{24}\ln 2+\zeta'(-1)+O(s^{-3}).$$

Here the celebrated Tracy-Widom constant [32] shows up.

The above method is motivated by Deift, Its and Krasovsky [13] where they studied the special case $\mathbb{P}(n,0,\alpha)$, namely the largest eigenvalue distribution on $[0,\alpha]$ of LUE with the weight e^{-4nx}. They used the Riemann-Hilbert approach to get the asymptotic expansion for $\frac{d}{d\alpha}\ln\mathbb{P}(n,0,\alpha)$, while in this paper, as is mentioned above, we relate the weight $x^\gamma\mathrm{e}^{-4nx}$ to $x^\gamma\mathrm{e}^{-x}$ and make use of the established results for the latter weight. The Riemann-Hilbert method [14] is a very powerful tool to investigate the asymptotic behavior of many unitary ensembles. See, for instance, the gap probability problem [11, 38], correlation kernel [7, 36], partition functions [2, 6, 12] and orthogonal polynomials [8].

This paper is organized as follows. In Sec. 2, we present some important results from [4] which are related to the largest eigenvalue distribution of LUE with the weight $x^\gamma\mathrm{e}^{-x}$. Sec. 3 is devoted to the derivation of the asymptotic formula for $\frac{d}{d\alpha}\ln\mathbb{P}(n,\gamma,\alpha)$ when n is large. Our main results are developed in Sec. 4.

2 Preliminaries

In this section, we present some important results of Basor and Chen [4], which are crucial for the analysis of the asymptotic behavior of the largest eigenvalue distribution in the LUE with the scaled Laguerre weight.

It is well known that the multiple integral $\widehat{D}_n(t)$ defined by (1.2) can be written as the determinant of a Hankel matrix and also as the product of the square of the L^2-norms of the corresponding monic orthogonal polynomials [17], namely

$$\begin{aligned}\widehat{D}_n(t) :=&\frac{1}{n!}\int_{[0,t]^n}\prod_{1\le i<j\le n}(x_i-x_j)^2\prod_{k=1}^{n}x_k^{\gamma}\mathrm{e}^{-x_k}dx_k\\ =&\det\left(\int_0^t x^{i+j}x^{\gamma}\mathrm{e}^{-x}dx\right)_{i,j=0}^{n-1}\\ =&\prod_{j=0}^{n-1}h_j(t),\end{aligned}$$

where

$$h_j(t)\delta_{jk} := \int_0^t P_j(x,t)P_k(x,t)x^{\gamma}\mathrm{e}^{-x}dx, \tag{2.1}$$

and δ_{jk} is the Dirac delta function. Here $P_n(x,t),\ n=0,1,2,\ldots,$ are monic polynomials of degree n defined by

$$P_n(x,t) = x^n + \mathrm{p}(n,t)x^{n-1} + \cdots + P_n(0,t). \tag{2.2}$$

Note that, in the following discussions, n stands for any nonnegative integer instead of the dimension of the Hermitian matrices.

The orthogonality (2.1) implies the following three-term recurrence relation [10, 31]:

$$xP_n(x,t) = P_{n+1}(x,t) + \alpha_n(t)P_n(x,t) + \beta_n(t)P_{n-1}(x,t), \tag{2.3}$$

subject to the initial conditions

$$P_0(x,t) := 1, \qquad\qquad \beta_0(t)P_{-1}(x,t) := 0.$$

As an easy consequence of (2.1)–(2.3), we have

$$\alpha_n(t) = \mathrm{p}(n,t) - \mathrm{p}(n+1,t),$$

$$\beta_n(t) = \frac{h_n(t)}{h_{n-1}(t)}.$$

In addition, $\alpha_n(t)$ and $\beta_n(t)$ admit the following integral representations,

$$\alpha_n(t) = \frac{1}{h_n(t)} \int_0^t x P_n^2(x,t) x^\gamma \mathrm{e}^{-x} dx,$$

$$\beta_n(t) = \frac{1}{h_{n-1}(t)} \int_0^t x P_n(x,t) P_{n-1}(x,t) x^\gamma \mathrm{e}^{-x} dx.$$

From the recurrence relation (2.3), one can derive the famous Christoffel-Darboux formula [31],

$$\sum_{k=0}^{n-1} \frac{P_k(x,t)P_k(y,t)}{h_k(t)} = \frac{P_n(x,t)P_{n-1}(y,t) - P_n(y,t)P_{n-1}(x,t)}{h_{n-1}(t)(x-y)}.$$

For convenience, we will not display the t dependence of relevant quantities unless it is required in the following discussions.

Basor and Chen [4] studied the Hankel determinant generated by the discontinuous Laguerre weight $x^\gamma \mathrm{e}^{-x}(A + B\theta(x-t))$, where $\theta(\cdot)$ is the Heaviside step function. We observe that the special case where $A = 1$ and $B = -1$ corresponds to $\widehat{D}_n(t)$.

It is proved in [4] that the monic orthogonal polynomials defined by (2.1) satisfy the lowering operator equation

$$\left(\frac{d}{dz} + B_n(z)\right) P_n(z) = \beta_n A_n(z) P_{n-1}(z),$$

and the raising operator equation

$$\left(\frac{d}{dz} - B_n(z) - \mathrm{v}'(z)\right) P_{n-1}(z) = -A_{n-1}(z) P_n(z),$$

where $\mathrm{v}(z) := -\ln(z^\gamma \mathrm{e}^{-z}) = z - \gamma \ln z$, and

$$A_n(z) := \frac{R_n(t)}{z-t} + \frac{1}{h_n(t)} \int_0^t \frac{\mathrm{v}'(z) - \mathrm{v}'(y)}{z-y} P_n^2(y) y^\gamma \mathrm{e}^{-y} dy,$$

$$B_n(z) := \frac{r_n(t)}{z-t} + \frac{1}{h_{n-1}(t)} \int_0^t \frac{\mathrm{v}'(z) - \mathrm{v}'(y)}{z-y} P_n(y) P_{n-1}(y) y^\gamma \mathrm{e}^{-y} dy.$$

Here the auxiliary quantities $R_n(t)$ and $r_n(t)$ are defined by

$$R_n(t) := -\frac{t^\gamma \mathrm{e}^{-t}}{h_n(t)} P_n^2(t,t),$$

$$r_n(t) := -\frac{t^\gamma \mathrm{e}^{-t}}{h_{n-1}(t)} P_n(t,t) P_{n-1}(t,t),$$

and $P_n(t,t) := P_n(z,t)|_{z=t}$.

From the definitions of $A_n(z)$ and $B_n(z)$, Basor and Chen [4] derived two identities valid for $z \in \mathbb{C} \cup \{\infty\}$:

$$B_{n+1}(z) + B_n(z) = (z - \alpha_n) A_n(z) - \mathrm{v}'(z), \tag{S_1}$$

$$1 + (z - \alpha_n)(B_{n+1}(z) - B_n(z)) = \beta_{n+1} A_{n+1}(z) - \beta_n A_{n-1}(z). \tag{S_2}$$

A combination of (S_1) and (S_2) produces

$$B_n^2(z) + \mathrm{v}'(z) B_n(z) + \sum_{j=0}^{n-1} A_j(z) = \beta_n A_n(z) A_{n-1}(z), \tag{S_2'}$$

which may give us a better insight into the recurrence coefficients.

Computing $A_n(z)$ and $B_n(z)$ by using their definitions and substituting the resulting expressions into the compatibility conditions (S_1), (S_2) and (S_2'), Basor and Chen [4] obtained the following results.

Proposition 2.1. *$A_n(z)$ and $B_n(z)$ are given by*

$$A_n(z) = \frac{R_n(t)}{z - t} + \frac{1 - R_n(t)}{z},$$

$$B_n(z) = \frac{r_n(t)}{z - t} - \frac{n + r_n(t)}{z}.$$

Proposition 2.2. *The quantity*

$$S_n(t) := 1 - \frac{1}{R_n(t)},$$

satisfies the second-order differential equation

$$S_n'' = \frac{(3S_n - 1)(S_n')^2}{2S_n(S_n - 1)} - \frac{S_n'}{t} - \frac{\gamma^2}{2} \frac{(S_n - 1)^2}{t^2 S_n} + \frac{(2n + 1 + \gamma) S_n}{t} - \frac{S_n(S_n + 1)}{2(S_n - 1)}, \tag{2.4}$$

which is a particular Painlevé V equation, $P_V\left(0, -\frac{\gamma^2}{2}, 2n + 1 + \gamma, -\frac{1}{2}\right)$, following the convention of [16].

Let

$$\sigma_n(t) := t \frac{d}{dt} \ln \widehat{D}_n(t).$$

Then, in view of (1.4), we have

$$\sigma_n(t) = t\frac{d}{dt}\ln\widehat{\mathbb{P}}(n,\gamma,t). \tag{2.5}$$

Recall that $\widehat{\mathbb{P}}(n,\gamma,t)$ represents the largest eigenvalue distribution function on $[0,t]$ of LUE with the weight $x^{\gamma}\mathrm{e}^{-x}$. The following results come from [4] and [20].

Proposition 2.3. *The quantity $\sigma_n(t)$ satisfies the Jimbo-Miwa-Okamoto σ-form of Painlevé V [18],*

$$(t\sigma_n'')^2 = 4(\sigma_n')^2\left(\sigma_n - n(n+\gamma) - t\sigma_n'\right) + \left((2n+\gamma-t)\sigma_n' + \sigma_n\right)^2,$$

and $\sigma_n(t)$ is expressed in terms of $S_n(t)$ by

$$\sigma_n(t) \;=\; -\frac{\gamma^2}{4S_n} + \frac{t(4n+2\gamma-t)}{4(S_n-1)} - \frac{t^2}{4(S_n-1)^2} + \frac{t^2(S_n')^2}{4S_n(S_n-1)^2}. \tag{2.6}$$

In the next section, we will make use of the above results to study $\mathbb{P}(n,\gamma,\alpha)$, that is the largest eigenvalue distribution on $[0,\alpha]$ of the LUE with the weight $x^{\gamma}\mathrm{e}^{-4nx}$.

3 Logarithmic Derivative of the Largest Eigenvalue Distribution Function

We consider the LUE defined by the scaled Laguerre weight,

$$w(x;\gamma,n) = x^{\gamma}\mathrm{e}^{-4nx}, \qquad x\in[0,\infty), \quad \gamma > -1.$$

As is shown in the introduction, the probability that the largest eigenvalue in this LUE is not greater than α is equal to the probability that the largest eigenvalue is not greater than $4n\alpha$ in the LUE with the weight $x^{\gamma}\mathrm{e}^{-x}$, i.e.,

$$\mathbb{P}(n,\gamma,\alpha) = \widehat{\mathbb{P}}(n,\gamma,4n\alpha). \tag{3.1}$$

According to the results in the previous section with $t = 4n\alpha$, we come to the following result.

Lemma 3.1. *As $n\to\infty$, $\frac{d}{d\alpha}\ln\mathbb{P}(n,\gamma,\alpha)$ has the following asymptotic expansion:*

$$\begin{aligned}\frac{d}{d\alpha}\ln\mathbb{P}(n,\gamma,\alpha) \;=\; & \frac{(1-\alpha)^2}{\alpha}n^2 + \frac{\gamma(1-\alpha)}{\alpha}n + \frac{\alpha+2\gamma^2(1-\alpha)}{4(1-\alpha^2)} - \frac{\gamma\left(\alpha+\gamma^2(1-\alpha)^2\right)}{4n(1-\alpha^2)^2} \\ & + O\left(\frac{1}{(1-\alpha)^4 n^2}\right).\end{aligned} \tag{3.2}$$

Proof. Let

$$t = 4n\alpha,$$

and denote

$$F_n(\alpha) := S_n(t) = S_n(4n\alpha). \tag{3.3}$$

Then equation (2.4) becomes

$$F_n'' = \frac{(3F_n - 1)(F_n')^2}{2F_n(F_n - 1)} - \frac{F_n'}{\alpha} - \frac{\gamma^2(F_n - 1)^2}{2\alpha^2 F_n} + \frac{4n(2n + 1 + \gamma)F_n}{\alpha} - \frac{8n^2 F_n(F_n + 1)}{F_n - 1}. \tag{3.4}$$

By disregarding the derivative terms in this equation, we get a cubic equation for $\tilde{F}_n(\alpha)$,

$$\left(16n^2\alpha(1-\alpha) + 8n\alpha(1+\gamma) - \gamma^2\right)\tilde{F}_n^3 - \left(16n^2\alpha(1+\alpha) + 8n\alpha(1+\gamma) - 3\gamma^2\right)\tilde{F}_n^2 - 3\gamma^2\tilde{F}_n + \gamma^2 = 0.$$

It has only one real solution which has the following large n expansion,

$$\tilde{F}_n(\alpha) = \frac{1+\alpha}{1-\alpha} - \frac{\alpha(1+\gamma)}{n(1-\alpha)^2} + \frac{\alpha(1+\alpha)^2(1+2\gamma) + \alpha(1+3\alpha)\gamma^2}{2n^2(1-\alpha)^3(1+\alpha)^2} + O(n^{-3}).$$

Hence we suppose that $F_n(\alpha)$ has the following series expansion,

$$F_n(\alpha) = \sum_{i=0}^{\infty} a_i(\alpha) n^{-i}, \qquad n \to \infty.$$

Substituting the above expression into equation (3.4), we obtain

$$\begin{aligned} F_n(\alpha) \quad = \quad & \frac{1+\alpha}{1-\alpha} - \frac{\alpha(1+\gamma)}{n(1-\alpha)^2} + \frac{\alpha + \alpha^2 - \alpha^4 + 2\alpha(1-\alpha)(1+\alpha)^2\gamma + \alpha(1-\alpha)(1+3\alpha)\gamma^2}{2n^2(1-\alpha)^4(1+\alpha)^2} \\ & + O\left(\frac{1}{(1-\alpha)^5 n^3}\right). \end{aligned} \tag{3.5}$$

Furthermore, from (2.5) and (3.1), it follows that

$$\begin{aligned} \sigma_n(t) =& \sigma_n(4n\alpha) = \alpha \frac{d}{d\alpha} \ln \widehat{\mathbb{P}}(n, \gamma, 4n\alpha) \\ =& \alpha \frac{d}{d\alpha} \ln \mathbb{P}(n, \gamma, \alpha). \end{aligned}$$

Hence, by using (2.6) and in view of (3.3), we are able to express $\frac{d}{d\alpha} \ln \mathbb{P}(n, \gamma, \alpha)$ in terms of $F_n(\alpha)$,

$$\frac{d}{d\alpha} \ln \mathbb{P}(n, \gamma, \alpha) = -\frac{\gamma^2}{4\alpha F_n} + \frac{2n(2n(1-\alpha) + \gamma)}{F_n - 1} - \frac{4n^2\alpha}{(F_n - 1)^2} + \frac{\alpha(F_n')^2}{4F_n(F_n - 1)^2}. \tag{3.6}$$

Substituting (3.5) into (3.6), we arrive at (3.2). $\square$

Remark 1. *When $\gamma = 0$, our formula (3.2) is consistent with the expression (152) of Deift et al. [13].*

In order to obtain the asymptotic formula of the largest eigenvalue distribution function $\mathbb{P}(n, \gamma, \alpha)$ as $n \to \infty$, we can integrate identity (3.2) from α_0 to any α, where α_0 is close to zero from above and $\alpha_0 < \alpha \leq \frac{1}{4n}(4n - 2^{4/3}n^{1/3}s_0) = 1 - \frac{s_0}{(2n)^{2/3}}$ with finite $s_0 > 0$. See [20, 30] on the soft edge scaling in LUE. So we need to know the asymptotics of $\mathbb{P}(n, \gamma, \alpha)$ when α is close to zero. We will analyze it in the next section following the method in [13].

4 Asymptotic Behavior of the Largest Eigenvalue Distribution Function

Returning to our problem, we recall that the probability that all the eigenvalues lie in $[0, \alpha]$ is given by

$$\mathbb{P}(n, \gamma, \alpha) = \frac{D_n(\alpha)}{D_n(\infty)},$$

where

$$D_n(\alpha) = \frac{1}{n!} \int_{[0,\alpha]^n} \prod_{1 \leq i < j \leq n} (x_i - x_j)^2 \prod_{k=1}^{n} x_k^{\gamma} \mathrm{e}^{-4nx_k} dx_k,$$

and

$$D_n(\infty) = \frac{1}{n!} \int_{[0,\infty)^n} \prod_{1 \leq i < j \leq n} (x_i - x_j)^2 \prod_{k=1}^{n} x_k^{\gamma} \mathrm{e}^{-4nx_k} dx_k.$$

By changing variables $x_\ell = \alpha t_\ell, \ell = 1, 2, \ldots, n$, we find

$$D_n(\alpha) = \alpha^{n(n+\gamma)} \frac{1}{n!} \int_{[0,1]^n} \prod_{1 \leq i < j \leq n} (t_i - t_j)^2 \prod_{k=1}^{n} t_k^{\gamma} \mathrm{e}^{-4n\alpha t_k} dt_k.$$

For fixed n and as $\alpha \to 0$, we have

$$\mathrm{e}^{-4n\alpha t_k} = 1 - 4n\alpha t_k + O(\alpha^2),$$

so that

$$\begin{aligned} D_n(\alpha) &= \alpha^{n(n+\gamma)} \frac{1}{n!} \int_{[0,1]^n} \prod_{1 \leq i < j \leq n} (t_i - t_j)^2 \prod_{k=1}^{n} t_k^{\gamma} (1 - 4n\alpha t_k + O(\alpha^2)) dt_k \\ &= \alpha^{n(n+\gamma)} A_n(\gamma)(1 + o_n(\alpha)), \end{aligned}$$

where $o_n(\alpha) \to 0$ as $\alpha \to 0$ for fixed n, and

$$A_n(\gamma) := \frac{1}{n!} \int_{[0,1]^n} \prod_{1 \le i < j \le n} (t_i - t_j)^2 \prod_{k=1}^{n} t_k^{\gamma} dt_k.$$

Hence, we find as $\alpha \to 0$,

$$\begin{aligned} \ln \mathbb{P}(n,\gamma,\alpha) &= \ln D_n(\alpha) - \ln D_n(\infty) \\ &= n(n+\gamma)\ln\alpha + \ln A_n(\gamma) - \ln D_n(\infty) + o_n(\alpha). \end{aligned} \tag{4.1}$$

According to identity (17.1.3) in [24], we have

$$\begin{aligned} A_n(\gamma) &= \frac{1}{n!} \prod_{j=0}^{n-1} \frac{\Gamma(j+1)\Gamma(j+2)\Gamma(j+\gamma+1)}{\Gamma(j+n+\gamma+1)} \\ &= \frac{G^2(n+1)G^2(n+\gamma+1)}{G(\gamma+1)G(2n+\gamma+1)}. \end{aligned}$$

Now we look at $D_n(\infty)$, i.e.

$$D_n(\infty) = \frac{1}{n!} \int_{[0,\infty)^n} \prod_{1 \le i < j \le n} (x_i - x_j)^2 \prod_{k=1}^{n} x_k^{\gamma} \mathrm{e}^{-4nx_k} dx_k.$$

By changing variables $4nx_\ell = y_\ell, \ell = 1, 2, \ldots, n$, we get

$$\begin{aligned} D_n(\infty) &= (4n)^{-n(n+\gamma)} \cdot \frac{1}{n!} \int_{[0,\infty]^n} \prod_{1 \le i < j \le n} (y_i - y_j)^2 \prod_{k=1}^{n} y_k^{\gamma} \mathrm{e}^{-y_k} dy_k \\ &= (4n)^{-n(n+\gamma)} \widehat{D}_n(\infty) \\ &= (4n)^{-n(n+\gamma)} \frac{G(n+1)G(n+\gamma+1)}{G(\gamma+1)}, \end{aligned}$$

where we have used (1.5). Substituting the above expressions for $A_n(\gamma)$ and $D_n(\infty)$ into (4.1), we arrive at, as $\alpha \to 0$,

$$\ln \mathbb{P}(n,\gamma,\alpha) = n(n+\gamma)\ln(4n\alpha) + \ln G(n+1) + \ln G(n+\gamma+1) - \ln G(2n+\gamma+1) + o_n(\alpha).$$

By using the asymptotic formula of Barnes G-function (see, for example, formula (A.6) in [35]), i.e.,

$$\ln G(z+1) = z^2 \left(\frac{\ln z}{2} - \frac{3}{4} \right) + \frac{z}{2} \ln(2\pi) - \frac{\ln z}{12} + \zeta'(-1) + O(z^{-1}), \qquad z \to \infty, \tag{4.2}$$

where $\zeta(\cdot)$ is the Riemann zeta function, we obtain

$$\begin{aligned}
\ln \mathbb{P}(n,\gamma,\alpha) &= \left(\frac{3}{2}n^2 + n\gamma - \frac{1}{12}\right)\ln n + \left(\frac{(n+\gamma)^2}{2} - \frac{1}{12}\right)\ln(n+\gamma) \\
&\quad - \left(\frac{(2n+\gamma)^2}{2} - \frac{1}{12}\right)\ln(2n+\gamma) + n(n+\gamma)\left(\frac{3}{2} + \ln(4\alpha)\right) \\
&\quad + \zeta'(-1) + \tilde{\delta}_n(\gamma) + o_n(\alpha), \qquad (4.3)
\end{aligned}$$

where $\tilde{\delta}_n(\gamma)$ depends only on n and γ, and $\tilde{\delta}_n(\gamma) \to 0$ as $n \to \infty$ for any given γ.

Remark 2. *When $\gamma = 0$, formula (4.3) is coincident with formula (27) of Deift et al. [13].*

To continue, we integrate identity (3.2) from α_0 to any α with $0 < \alpha_0 < \alpha \le 1 - s_0/(2n)^{2/3},\ s_0 > 0$, and find

$$\begin{aligned}
&\ln \mathbb{P}(n,\gamma,\alpha) - \ln \mathbb{P}(n,\gamma,\alpha_0) \\
&= n^2\left(\ln\frac{\alpha}{\alpha_0} + \frac{\alpha^2 - \alpha_0^2}{2} - 2(\alpha - \alpha_0)\right) + n\gamma\left(\ln\frac{\alpha}{\alpha_0} - (\alpha - \alpha_0)\right) \\
&\quad + \frac{1}{8}\left((4\gamma^2 - 1)\ln\frac{1+\alpha}{1+\alpha_0} - \ln\frac{1-\alpha}{1-\alpha_0}\right) + \frac{\gamma\,(2(1-\alpha)\gamma^2 - 1)}{8n(1-\alpha^2)} \\
&\quad - \frac{\gamma\,(2(1-\alpha_0)\gamma^2 - 1)}{8n(1-\alpha_0^2)} + O\left(\frac{1}{n^2(1-\alpha)^3}\right) - O\left(\frac{1}{n^2(1-\alpha_0)^3}\right). \qquad (4.4)
\end{aligned}$$

Substituting formula (4.3) for $\ln \mathbb{P}(n,\gamma,\alpha_0)$ into (4.4) and taking the limit $\alpha_0 \to 0$, we establish the following theorem.

Theorem 4.1. *For any $0 < \alpha \le 1 - \frac{s_0}{(2n)^{2/3}},\ s_0 > 0$, we have as $n \to \infty$,*

$$\begin{aligned}
\ln \mathbb{P}(n,\gamma,\alpha) &= n^2\left(\frac{3}{2} - 2\alpha + \frac{\alpha^2}{2} + \ln(4\alpha)\right) + n\gamma\left(\frac{3}{2} - \alpha + \ln(4\alpha)\right) + \left(\frac{3}{2}n^2 + n\gamma - \frac{1}{12}\right)\ln n \\
&\quad + \left(\frac{(n+\gamma)^2}{2} - \frac{1}{12}\right)\ln(n+\gamma) - \left(\frac{(2n+\gamma)^2}{2} - \frac{1}{12}\right)\ln(2n+\gamma) \\
&\quad + \frac{1}{8}\left((4\gamma^2 - 1)\ln(1+\alpha) - \ln(1-\alpha)\right) + \zeta'(-1) + \frac{\gamma\,(2(1-\alpha)\gamma^2 - 1)}{8n(1-\alpha^2)} \\
&\quad + O\left(\frac{1}{n^2(1-\alpha)^3}\right) + \delta_n(\gamma), \qquad (4.5)
\end{aligned}$$

where $\delta_n(\gamma)$ depends only on n and γ, and $\delta_n(\gamma) \to 0$ as $n \to \infty$ for any given γ.

Remark 3. *From the asymptotic formula (4.2) of the Barnes G-function, we can show that* $\delta_n(\gamma) = O(\frac{1}{n})$ $(n \to \infty)$ *for any given* γ*. In addition, if* $\gamma = 0$*, then (4.5) becomes*

$$\begin{aligned} \ln \mathbb{P}(n, 0, \alpha) &= n^2 \left(\frac{3}{2} + \ln \alpha - 2\alpha + \frac{\alpha^2}{2} \right) - \frac{1}{12} \ln n - \frac{1}{8} \ln(1 - \alpha^2) \\ &\quad + \frac{1}{12} \ln 2 + \zeta'(-1) + O\left(\frac{1}{n^2(1-\alpha)^3} \right) + \delta_n, \end{aligned}$$

where δ_n *depends only on* n*, and* $\delta_n \to 0$ *as* $n \to \infty$*. This agrees with (162) of Deift et al. [13].*

In the end, we give the asymptotic formula of the Fredholm determinant

$$\det(I - K_{\text{Airy}}),$$

where K_{Airy} is the integral operator with the Airy kernel

$$K_{\text{Airy}}(x, y) = \frac{\text{Ai}(x)\text{Ai}'(y) - \text{Ai}(y)\text{Ai}'(x)}{x - y}$$

acting on $L^2(-s, \infty)$.

For any $s > s_0$ and sufficiently large n, we set

$$\alpha = 1 - \frac{s}{(2n)^{2/3}}.$$

Substituting it into (4.5) and taking the limit $n \to \infty$, the r.h.s. of (4.5) becomes

$$-\frac{s^3}{12} - \frac{1}{8} \ln s + \frac{1}{24} \ln 2 + \zeta'(-1) + O(s^{-3}),$$

and the l.h.s. of (4.5) approaches $\ln \det(I - K_{\text{Airy}})$. Therefore, we establish the following asymptotic formula of the Airy determinant as $s \to +\infty$:

$$\ln \det(I - K_{\text{Airy}}) = -\frac{s^3}{12} - \frac{1}{8} \ln s + \frac{1}{24} \ln 2 + \zeta'(-1) + O(s^{-3}). \tag{4.6}$$

Remark 4. *The constant term in (4.6), i.e.* $\frac{1}{24} \ln 2 + \zeta'(-1)$*, was conjectured by Tracy and Widom [32], and proved by Deift et al. [13] where (4.6) was also derived but with the order term* $O(s^{-3/2})$*. Our order term* $O(s^{-3})$ *coincides with formula (1.19) in [32].*

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