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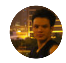
Yang Chen
University of Macau


Longjun Zhan
Fudan University

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Yang Chen,[1,a)] Galina Filipuk,[2,b)] and Longjun Zhan[3,c)]

AFFILIATIONS

[1] Department of Mathematics, University of Macau, Avenida da Universidade, Taipa, Macau, China
[2] Faculty of Mathematics, Informatics and Mechanics, University of Warsaw, Banacha 2, Warsaw, 02-097, Poland
[3] School of Mathematical Science, Fudan University, Shanghai 200433, China

[a)] yayangchen@umac.mo
[b)] filipuk@mimuw.edu.pl
[c)] **Author to whom correspondence should be addressed:** Zhan_Longjun@foxmail.com

ABSTRACT

The Painlevé equations arise from the study of Hankel determinants generated by moment matrices, whose weights are expressed as the product of "classical" weights multiplied by suitable "deformation factors," usually dependent on a "time variable" t. From ladder operators [see A. Magnus, J. Comput. Appl. Math. **57**(1-2), 215–237 (1995)], one finds second order linear ordinary differential equations for associated orthogonal polynomials with coefficients being rational functions. The Painlevé and related functions appear as the residues of these rational functions. We will be interested in the situation when n, the order of the Hankel matrix and also the degree of the polynomials $P_n(x)$ orthogonal with respect to the deformed weights, gets large. We show that the second order linear differential equations satisfied by $P_n(x)$ are particular cases of Heun equations when n is large. In some sense, monic orthogonal polynomials generated by deformed weights mentioned below are solutions of a variety of Heun equations. Heun equations are of considerable importance in mathematical physics, and in the special cases, they degenerate to the hypergeometric and confluent hypergeometric equations. In this paper, we look at three types of weights: the Jacobi type, the Laguerre type, and the weights deformed by the indicator function of $\chi_{(a,b)}(x)$ and the step function $\theta(x)$. In particular, we consider the following Jacobi type weights: (1.1) $x^{\alpha}(1-x)^{\beta}e^{-tx}, x\in[0,1], \alpha,\beta,t>0$; (1.2) $x^{\alpha}(1-x)^{\beta}e^{-t/x}, x\in(0,1], \alpha,\beta,t>0$; (1.3) $(1-x^2)^{\alpha}(1-k^2x^2)^{\beta}$, $x\in[-1,1]$, $\alpha,\beta>0$, $k^2\in(0,1)$; the Laguerre type weights: (2.1) $x^{\alpha}(x+t)^{\lambda}e^{-x}, x\in[0,\infty), t,\alpha,\lambda>0$; (2.2) $x^{\alpha}e^{-x-t/x}, x\in(0,\infty), \alpha,t>0$; and another type of deformation when the classical weights are multiplied by $\chi_{(a,b)}(x)$ or $\theta(x)$: (3.1) $e^{-x^2}(1-\chi_{(-a,a)}(x))$, $x\in\mathbb{R}$, $a>0$; (3.2) $(1-x^2)^{\alpha}(1-\chi_{(-a,a)}(x))$, $x\in[-1,1]$, $a\in(0,1)$, $\alpha>0$; (3.3) $x^{\alpha}e^{-x}(A+B\theta(x-t)), x\in[0,\infty)$, $\alpha,t>0, A\geq 0, A+B\geq 0$. The weights mentioned above were studied in a series of papers related to the deformation of "classical" weights.

I. INTRODUCTION

A. Heun equations

The general Heun equation is the second order linear Fuchsian ordinary differential equation (ODE) with four regular singular points in the complex plane.[26,47,49,50] It is a generalization of the well-studied Gauss hypergeometric equation with three regular singularities. However, it is much harder to study properties of the Heun functions. The additional singularity causes many complications in comparison with the hypergeometric case (for instance, solutions do not have integral representation). There also exist confluent Heun equations, see Refs. 1, 29, and 49, which are obtained by certain confluence of singularities of the general Heun equation.

The general Heun equation is given by

$$\frac{d^2y}{dz^2}+\left(\frac{\gamma}{z}+\frac{\delta}{z-1}+\frac{\epsilon}{z-a}\right)\frac{dy}{dz}+\frac{\alpha\beta z-q}{z(z-1)(z-a)}y=0, \tag{1.1}$$

where the parameters satisfy the Fuchsian relation,

$$1 + \alpha + \beta = \gamma + \delta + \epsilon. \tag{1.2}$$

This equation has four regular singular points at $z = 0, 1, a$, and ∞. Its solutions, the Heun functions, are usually denoted by $y = H(a, q; \alpha, \beta, \gamma, \delta; z)$, where ϵ is expressed in terms of $\alpha, \beta, \gamma, \delta$ via (1.2). The parameter q is called an accessory parameter.

It is well-known that the derivative of the hypergeometric function ${}_2F_1(a, b; c; x)$ is again a hypergeometric function with different values of the parameters. However, for the Heun function, it is generally not the case. The first order derivative of the general Heun function satisfies the second order Fuchsian differential equations with five regular singular points. It can be verified by direct computations that the function $v(z) = dy/dz$, where $y = y(z)$ is a solution of (1.1), satisfies the following equation:

$$\frac{d^2v}{dz^2} + \left(\frac{\gamma+1}{z} + \frac{\delta+1}{z-1} + \frac{\epsilon+1}{z-a} - \frac{\alpha\beta}{\alpha\beta z - q}\right)\frac{dv}{dz} + \frac{f(z)}{z(z-1)(z-a)(\alpha\beta z - q)}v = 0, \tag{1.3}$$

where $f(z) = z(\alpha\beta z - 2q)(\alpha\beta + \gamma + \delta + \epsilon) + q^2 + q(\gamma + a(\gamma + \delta) + \epsilon) - \alpha\beta\gamma a$. We see from the equation above that an additional singularity at $z = q/(\alpha\beta)$ appears.

There are four confluent limits of the general Heun equation: the confluent Heun, double confluent Heun, biconfluent Heun, and triconfluent Heun equations. When the singularity $z = a$ is merged with $z = \infty$, the confluent Heun equation is found. Translating $z = 1$ to $z = b$ followed by $a \to \infty, b \to 0$, one finds the double confluent Heun equation. The biconfluent Heun equation is obtained by $a \to \infty, b \to \infty$. The triconfluent Heun equation cannot be derived directly by confluence from the Heun equation in its standard form; we should be go back to the less specialized parameterization with singularities at a_1, a_2, a_3, and ∞, which is followed by $a_j \to \infty$, $j = 1, 2, 3$. These transformations are due to Heun (1889) (see Ref. 47), and they can be checked in Maple (https://www.maplesoft.com/). Other transformations can also be found in the literature; see for example, the work of Slavjanov and Lay.[49]

The list of Heun equations and equations for derivatives of the Heun functions is as follows.

The confluent Heun equation is given by

$$\frac{d^2y}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \epsilon\right)\frac{dy}{dz} + \frac{\alpha z - q}{z(z-1)}y = 0 \tag{1.4}$$

and the linear equation for the function $v = dy/dz$ is given by

$$\frac{d^2v}{dz^2} + \left(\frac{\gamma+1}{z} + \frac{\delta+1}{z-1} + \epsilon - \frac{\alpha}{\alpha z - q}\right)\frac{dv}{dz} + \frac{g(z)}{z(z-1)(\alpha z - q)}v = 0, \tag{1.5}$$

where $g(z) = (\alpha + \epsilon)(\alpha z - 2q)z + q^2 - (\gamma + \delta - \epsilon)q + \alpha\gamma$.

The double-confluent Heun equation is given by

$$\frac{d^2y}{dz^2} + \left(\frac{\gamma}{z^2} + \frac{\delta}{z} + \epsilon\right)\frac{dy}{dz} + \frac{\alpha z - q}{z^2}y = 0 \tag{1.6}$$

and the linear equation for the function $v = dy/dz$ is given by

$$\frac{d^2v}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta+2}{z-1} + \epsilon - \frac{\alpha}{\alpha z - q}\right)\frac{dv}{dz} + \frac{h(z)}{z^2(\alpha z - q)}v = 0, \tag{1.7}$$

where $h(z) = (\alpha + \epsilon)(\alpha z - 2q)z + q^2 - \gamma q - \alpha\gamma$.

The biconfluent Heun equation is given by

$$\frac{d^2y}{dz^2} + \left(\frac{\gamma}{z} + \delta + \epsilon z\right)\frac{dy}{dz} + \frac{\alpha z - q}{z}y = 0 \tag{1.8}$$

and the linear equation for the function $v = dy/dz$ is given by

$$\frac{d^2v}{dz^2} + \left(\frac{\gamma+1}{z} + \delta + \epsilon z - \frac{\alpha}{\alpha z - q}\right)\frac{dv}{dz} + \frac{k(z)}{z(\alpha z - q)}v = 0, \tag{1.9}$$

where $k(z) = (\alpha + \epsilon)(\alpha z - 2q)z + q^2 - \gamma q - \alpha\gamma$.

The triconfluent Heun equation is given by

$$\frac{d^2y}{dz^2} + \left(\gamma + \delta z + \epsilon z^2\right)\frac{dy}{dz} + (\alpha z - q)y = 0 \tag{1.10}$$

and the linear equation for the function $v = dy/dz$ is given by

$$\frac{d^2v}{dz^2} + \left(\gamma + \delta z + \epsilon z^2 - \frac{\alpha}{\alpha z - q}\right)\frac{dv}{dz} + \frac{p(z)}{\alpha z - q}v = 0, \tag{1.11}$$

where $p(z) = (\alpha + \epsilon)(\alpha z - 2q)z + q^2 - \gamma q - \alpha\gamma$.

Solutions of the Heun equation were given by Heun in 1889. In the 50 years following 1889,[26] solutions of Heun equations were obtained as power series in z. Svartholm[53] showed that solutions of the Heun equations may also be represented as series of hypergeometric functions. This was further developed by Erdélyi.[21–23] Schmidt[48] included the possibility of doubly infinite series of hypergeometric functions, similar to Laurent series. Kalnins and Miller[32] were concerned with the expansion of Heun polynomials based on group-theoretic methods and the technique of separation of variables on the n-sphere and deduced the expansion of a product of two Heun polynomials in terms of the product of Jacobi polynomials. There are certain cases where the confluent Heun functions could be expressed in terms of special functions of mathematical physics (see, for instance, recent results by Ishkhanyan, e.g., Ref. 35). Hence, it is of considerable interest to construct solutions of the Heun equations.

Looking at the Hermitian ensembles, namely, the "$\beta = 2$" problem, one very often finds a second order linear ODE (with rational coefficients) satisfied by the "deformed" orthogonal polynomials. Item 2.1 in the abstract describes the moment generating functions of the Shannon entropy; it is an example where the Heun equation makes its appearance. For a detailed account of the Heun equations, see the treatise by Ronveaux. Regarding the appearance of Heun equations in general relativity, see the work of Hortaçsu.[27,28] Suzuki *et al.* wrote about the perturbations of Kerr-de Sitter black hole[52] (See Kerr[33]), in astrophysics; see the work of Debosscher[19] and references therein. Malmendier[41] studied eigenvalue equation in a string theory (For solution of the Schrödinger equation of quantum mechanics see Bay,[7] and for dislocation theory see Lay[36]). The Heun equations also make their appearance on transmission problems in the theory of smooth electron wave guide[25] and in Bethe ansatz systems.[20] For quantum mechanical problems, see the work of Slavjanov.[49] See the work of Joyce[30,31] for lattice systems in statistical mechanics. Properties of Heun equations can be found in Ref. 45, Sec. 31.17.

In this paper, we will describe the orthogonal polynomials with respect to the deformed weights and show that for large n, the ordinary differential equations that they satisfy are Heun equations of various types.

B. Orthogonal polynomial and ladder operators

Let $\{P_j(x)\}_{j=1}^{\infty}$ be a sequence of monic polynomials of degree j orthogonal with respect to the weight $w(x)$ on the interval $[a, b]$, i.e.,

$$\int_a^b P_j(x)P_k(x)w(x)\mathrm{d}x = h_j\delta_{j,k}, \quad j, k = 0, 1, 2, \ldots,$$

where h_j denote the square of the weighted L^2 norm of $P_j(x)$ over $[a, b]$. We write

$$P_n(x) = x^n + p(n)x^{n-1} + \cdots.$$

It is known that

$$\prod_{1\le i<j\le N}(x_i - x_j) = \det\left(x_j^{i-1}\right)_{i,j=1}^{N} = \det\left(P_{i-1}(x_j)\right)_{i,j=1}^{N}.$$

The polynomials $P_n(x)$ can be constructed by the Gram-Schmidt orthogonalization process. Referring to the weights listed in the abstract, we see that the parameters $t, \alpha, \ldots$ will also appear in the polynomials and in the L^2 norm. However, to simplify notations, we will not usually display all the dependence.

If the moments of the deformed weight $\mu_j = \int_a^b x^j w(x)\mathrm{d}x$ exist and

$$\det\left(\mu_{i+j}\right)_{0\le i,j\le n-1} \neq 0,$$

then the theory of orthogonal polynomials states that monic orthogonal polynomials $P_n(z)$ for $n = 0, 1, 2, \ldots$ satisfy the three-term recurrence relation

$$zP_n(z) = P_{n+1}(z) + \alpha_n P_n(z) + \beta_n P_{n-1}(z)$$

with

$$P_0(z) = 1, \quad \beta_0 P_{-1}(z) = 0.$$

The recurrence coefficients α_n and β_n will depend on the parameters of the weight t, α, etc. Note that the monic polynomials $P_j(x)$ orthogonal with respect to weight $w(x)$ are defined on the real axis. However, they can be extended to the complex plane; hence, we will use the variable z in $P_j(z)$. For more details about orthogonal polynomials, see the work of Szegö.[51]

Let $w(a) = w(b) = 0$. In Ref. 13, it is shown that the following relations hold.

Lemma 1.1. Assume that $v(x) = -\log w(x)$ has derivative in some Lipschitz class with a positive exponent. The lowering and raising operators (ladder operators[40]) satisfy the following differential-difference formulas:

$$P_n'(z) = -B_n(z)P_n(z) + \beta_n A_n(z)P_{n-1}(z), \tag{1.12}$$

$$P_{n-1}'(z) = \left[B_n(z) + v'(z)\right]P_{n-1}(z) - A_{n-1}(z)P_n(z), \tag{1.13}$$

where

$$A_n(z) := \frac{1}{h_n}\int_a^b \frac{v'(z) - v'(y)}{z-y} P_n^2(y)w(y)\mathrm{d}y, \tag{1.14}$$

$$B_n(z) := \frac{1}{h_{n-1}}\int_a^b \frac{v'(z) - v'(y)}{z-y} P_n(y)P_{n-1}(y)w(y)\mathrm{d}y. \tag{1.15}$$

If $w(a) \neq 0, w(b) \neq 0$, then additional terms should be included in the definitions of $A_n(z)$ and $B_n(z)$ (see the work of Chen and Ismail[12,14]). The variable z shown in the equations above is complex; we assume that $v'(z)$ is an extension of $v'(x)$ off the real axis.

Lemma 1.2. The functions $A_n(z)$ and $B_n(z)$ defined by (1.14) and (1.15) satisfy the identities

$$B_{n+1}(z) + B_n(z) = (z - \alpha_n)A_n(z) - v'(z), \tag{S_1}$$
$$1 + (z-\alpha_n)\left[B_{n+1}(z) - B_n(z)\right] = \beta_{n+1}A_{n+1}(z) - \beta_n A_{n-1}(z). \tag{S_2}$$

It turns out that there is another supplementary condition involving $\sum_{j=0}^{n-1} A_j(z)$; we will call it (S_2'), which is widely used in the determination of recurrence coefficients α_n and β_n,

$$B_n(z)^2 + v'(z)B_n(z) + \sum_{j=0}^{n-1} A_j(z) = \beta_n A_n(z)A_{n-1}(z). \tag{S_2'}$$

Equation (S_2') should be thought of as an equation for $\sum_{j=0}^{n-1} A_j(z)$. See, for example, the work of Basor and Chen,[2] and Chen and Its.[15]

Eliminating $P_{n-1}(z)$ from ladder operators, we obtain the second order linear ordinary differential equation satisfied by $P_n(z)$,

$$P_n''(z) - \left(v'(z) + \frac{A_n'(z)}{A_n(z)}\right)P_n'(z) + \left(B_n'(z) - B_n(z)\frac{A_n'(z)}{A_n(z)} + \sum_{j=0}^{n-1} A_j(z)\right)P_n(z) = 0, \tag{1.16}$$

where $\sum_{j=0}^{n-1} A_j(z)$ is obtained from (S_2').

In Secs. II–IV, we will discuss the second order linear ordinary differential equations for large n, in the context of the deformed weights.

C. Coulomb fluid method

In this section, we treat the joint distribution function of the eigenvalues of the Hermitian ensembles as points of a fluid described by a continuous density $\rho(x)\mathrm{d}x$. We first present some basic description of the Coulomb fluid method, mainly from Refs. 16 and 12. The quantity

$$E(x_1, x_2, \ldots, x_n) = -2\sum_{1\le j<k\le n} \ln|x_j - x_k| + \sum_{j=1}^{n} v(x_j)$$

is the total energy of a system of n logarithmical repelling particles in one dimension subject to an external potential $v(x)$. The particles can be approximated as a continuous fluid with density ρ, for sufficiently large n. This density $\rho(x)$ assumed to be supported on $[a,b]$ will correspond to the equilibrium density of the fluid; this is obtained by the constrained minimization

$$\min_{\rho>0} F[\rho] \quad \text{subject to} \quad \int_a^b \rho(x)\mathrm{d}x = n,$$

where the free-energy function reads

$$F[\rho] := \int_a^b \rho(x)v(x)\mathrm{d}x - \int_a^b\int_a^b \rho(x)\ln|x-y|\rho(y)\mathrm{d}x\mathrm{d}y.$$

The equilibrium density satisfies the following integral equation (see the Frostman lemma[54]):

$$v(x) - 2\int_a^b \ln|x-y|\rho(y)\mathrm{d}y = A, \qquad x\in[a,b],$$

where A is the Lagrange multiplier that fixes the constraint $\int_a^b\rho(x)\mathrm{d}x = n$. For more details, see Ref. 12. After taking a derivative with respect to x, one obtains a singular integral equation,

$$2P\int_a^b \frac{\rho(y)}{x-y}\mathrm{d}y = v'(x), \tag{1.17}$$

where P denotes the Cauchy principal value. According to the standard theory of singular integral equations,[24,44] if $\rho(a)=\rho(b)=0$, then the density supported on $[a,b]$ reads

$$\rho(x) = \frac{\sqrt{(b-x)(x-a)}}{2\pi^2}P\int_a^b \mathrm{d}y\frac{v'(y)}{(y-x)\sqrt{(b-y)(y-a)}}. \tag{1.18}$$

The endpoints of the interval $[a,b]$ satisfy the condition $\int_a^b\rho(x)\mathrm{d}x = n$, as well as stability conditions

$$\begin{aligned}\int_a^b \frac{xv'(x)}{\sqrt{(b-x)(x-a)}}\mathrm{d}x &= 2n\pi,\\ \int_a^b \frac{v'(x)}{\sqrt{(b-x)(x-a)}}\mathrm{d}x &= 0.\end{aligned} \tag{1.19}$$

The end points of the support of the density are the solutions of (1.19) and are denoted by $a(n,t)$ and $b(n,t)$. They depend on the independent variables n, t, which play an important role in the asymptotics of the recurrence coefficients $\alpha_n(t)$ and $\beta_n(t)$, with

$$\alpha_n(t) \sim \frac{a(n,t)+b(n,t)}{2}, \tag{1.20}$$

$$\beta_n(t) \sim \left(\frac{b(n,t)-a(n,t)}{4}\right)^2; \tag{1.21}$$

see Refs. 12 and 42.

D. The structure of this paper

The second order linear differential equations satisfied by $P_n(z)$ related to several weight functions in the abstract (1.16) have coefficients that are rational functions of z, whose poles and residues depend on $r_n(t)$ and $R_n(t)$. Here, t is a "time parameter." It was found that $r_n(t)$ and $R_n(t)$ are evaluated as the "matrix elements" involving $P_n^2(x,t)$ and $P_n(x,t)P_{n-1}(x,t)$. In the ladder operators (1.12) and (1.13), with the weights given in the abstract, the functions $A_n(z)$ and $B_n(z)$ are the rational function of z. Conditions (S_1), (S_2), and (S_2') are used to obtain relations for recurrence coefficients α_n and β_n and auxiliary quantities R_n and r_n. In particular, one finds that the recurrence coefficients $\alpha_n(t)$ and $\beta_n(t)$ are expressed in terms of the auxiliary variables $r_n(t)$ and $R_n(t)$, which typically satisfy the coupled Riccati equations. Eliminating $r_n(t)$ gives a nonlinear second order ordinary differential equation for the function $R_n(t)$, which turns out to be equivalent (possibly after some change in variables or scaling) to one of the classical Painlevé equations.

TABLE I. Equations with respect to weight $w(x)$, when n goes to infinity.

Weight	Equation
Section II A: $x^{\alpha}(1-x)^{\beta}\mathrm{e}^{-tx}, x \in [0,1], \alpha, \beta, t > 0$	$n \to \infty$, $t \to 0^{+}$, $T = nt$, confluent Heun equation $P_n''(u) + \left(\frac{\alpha+1}{u} + \frac{\beta+1}{u-1} - \frac{T}{n}\right)P_n'(u) + \frac{Tu - T/2 - n(n+\alpha+\beta+1)}{u(u-1)}P_n(u) = 0$
Section II B: $x^{\alpha}(1-x)^{\beta}\mathrm{e}^{-t/x}, x \in (0,1], \alpha, \beta, t > 0$	$n \to \infty$, $t \to 0^{+}$, $T = 2n^2 t$ small, confluent Heun equation $P_n''(u) + \left(\frac{2\lambda-\alpha-\beta}{u} + \frac{\beta+1}{u-1} - s\right)P_n'(u) + \frac{-s\lambda u + \lambda(\lambda+s-\alpha)+a}{u(u-1)}P_n(u) = 0$
Section II C: $(1-x^2)^{\alpha}(1-k^2x^2)^{\beta}, x \in [-1,1],\ \alpha, \beta > 0,\ k^2 \in (0,1)$	$k^2 \to 0^{+}, \beta \to \infty, n \to \infty, k^2\beta = t$ is fixed, confluent Heun equation $P_n''(u) + \left(\frac{1}{2u} + \frac{\alpha+1}{u-1} - t\right)P_n'(u) + \frac{2ntu - n(n+2\alpha+1+t)}{4u(u-1)}P_n(u) = 0$
Section III A: $x^{\alpha}(x+t)^{\lambda}\mathrm{e}^{-x}, x \in [0,\infty), t, \alpha, \lambda > 0$	$n \to \infty$, confluent Heun equation $P_n''(u) + \left(\frac{\alpha+1}{u} + \frac{2\eta+\lambda}{u-1} + t\right)P_n'(u) + \left(\frac{4t(\eta-n)u + (2\eta+\lambda)(2(\alpha+\eta)+\lambda) - 4\lambda\sqrt{nt} + 4nt + 2\lambda t}{4u(u-1)}\right)P_n(u) = 0$
Section III B: $x^{\alpha}\mathrm{e}^{-x-t/x}, x \in (0,\infty), \alpha, t > 0$	$n \to \infty, t \to 0^{+}$, $s = (2n+\alpha+1)t$ fixed, double confluent Heun equation For large s, $P_n''(u) + \left(\frac{\gamma}{u^2} + \frac{\delta}{u} + \epsilon\right)P_n'(u) + \frac{au-q}{u^2}P_n(u) = 0$ For small s, $P_n''(u) + \left(\frac{s(1+\alpha)}{2\alpha n u^2} + \frac{\alpha+1}{u} - 1\right)P_n'(u) + \frac{nu + s/2\alpha}{u^2}P_n(u) = 0$
Section IV A: $\mathrm{e}^{-x^2}(1-\chi_{(-a,a)}(x)),\ x \in \mathbb{R},\ a > 0$	$n \to \infty$, confluent Heun equation $P_n''(u) + \left(\frac{1}{u-1} - \frac{1}{2u} - t\right)P_n'(u) + \frac{2ntu + \sqrt{2nt}}{4u(u-1)}P_n(u) = 0$
Section IV B: $(1-x^2)^{\alpha}(1-\chi_{(-a,a)}(x)),\ x \in [-1,1],\ a \in (0,1),\ \alpha > 0$	$n \to \infty$, general Heun equation $P_n''(u) + \left(-\frac{1}{2u} + \frac{\alpha+1}{u-1} + \frac{1}{u-t}\right)P_n'(u) - \frac{n\sqrt{t} + u(2\alpha+n+1)n}{4u(u-1)(u-t)}P_n(u) = 0$
Section IV C: $x^{\alpha}\mathrm{e}^{-x}(A + B\theta(x-t)), x \in [0,\infty), \alpha, t > 0, A \geq 0, A+B \geq 0$	For $A = 0, B = 1, n \to \infty, s = 4nt$, for large s, obtain double confluent Heun equation $P_n''(u) + \left(\frac{s - \alpha\sqrt{s}}{4nu^2} + \frac{\alpha+1}{u} - 1\right)P_n'(u) + \frac{4nu - 2\alpha\sqrt{s} + s + \alpha^2}{4u^2}P_n(u) = 0$ For $A = -1, B = 1, n \to \infty$, confluent Heun equation $P_n''(u) + \left(\frac{\alpha+1}{u} + \frac{1}{u-1} - t\right)P_n'(u) + \frac{ntu - n(n+\alpha+1+t/2)}{u(u-1)}P_n(u) = 0$

We show that in the situation where n tends to ∞, the linear second order ordinary differential equation (1.16) turns out to be the Heun equation. The large n behavior of $R_n(t)$ is found by using the nonderivative part of the equations satisfied by $R_n(t)$. From this approximation, we obtain the behavior of $r_n(t)$ and $R_n(t)$ under suitable double scaling and finally compute the recurrence coefficients, α_n and β_n. We see that the behavior of the recurrence coefficients obtained by this method is accurate and compare very well with the behavior of recurrence coefficients obtained from (1.19)–(1.21).

This paper is organized as follows. In Secs. II–IV, we study the deformed Jacobi type weights, deformed Laguerre type weights, and weights with gaps, respectively. We write the second order linear ordinary differential equations satisfied by orthogonal polynomials $P_n(z)$, which are usually known from the corresponding literature. Then, we deduce the Heun equations via some approximation procedure. The main results of the paper are summarized in Table I.

II. JACOBI TYPE WEIGHTS

In this section, we consider three deformed Jacobi type weights: $x^\alpha(1-x)^\beta e^{-tx}$ (see Ref. 55), $x^\alpha(1-x)^\beta e^{-t/x}$ (see Refs. 10 and 11), and $(1-x^2)^\alpha(1-k^2x^2)^\beta$ (see Ref. 5). The properties of polynomials orthogonal with respect to these weights and of their recurrence coefficients were studied in corresponding papers. Moreover, it was shown there that the auxiliary quantities $R_n(t), r_n(t)$, closely related to the recurrence coefficients $\alpha_n(t), \beta_n(t)$, satisfy certain Painlevé equations and Jimbo-Miwa-Okamoto σ-forms of the Painlevé equations. We show that monic orthogonal polynomial $P_n(z)$ for the weights above satisfy particular confluent Heun equations with parameters related to the parameters in the weight as n goes to infinity. We will study these Jacobi type weights in Subsections II A–II C.

A. $x^\alpha(1-x)^\beta e^{-tx}$, $x \in [0,1]$, $\alpha, \beta, t > 0$

In Ref. 55, the probability density function of the center of mass $\mathbb{P}(c,\alpha,\beta,n)$ was studied. The second order linear differential equation satisfied by monic polynomials orthogonal with respect to $x^\alpha(1-x)^\beta e^{-tx}$ is the Fuchsian equation with four singular points given by

$$P_n''(z) + Q_n(z,t)P_n'(z) + S_n(z,t)P_n(z) = 0, \tag{2.1}$$

where

$$\begin{aligned} Q_n(z,t) &= \frac{\alpha+1}{z} + \frac{\beta+1}{z-1} - t - \frac{1}{z - R_n(t)/t}, \\ S_n(z,t) &= \frac{ntz - n - \sum_{j=0}^{n-1} R_j(t)}{z(z-1)} + \frac{nz + r_n(t)}{z(z-1)(z-R_n(t)/t)}, \end{aligned}$$

with $R_n(t)$, $r_n(t)$, and $\beta_n(t)$ defined in Ref. 55. They satisfy the following relations:

$$r_n(t) = \frac{1}{2t}\left[tR_n'(t) + \alpha t - (2n+1+\alpha+\beta+t)R_n(t) + R_n^2(t)\right], \tag{2.2}$$

$$\sum_{j=0}^{n-1} R_j(t) = n(\alpha+\beta+n) - tr_n(t) - t^2\beta_n(t), \tag{2.3}$$

where

$$\beta_n(t) = \frac{n(\beta+n) + tr_n(t)[r_n(t)-\alpha]/R_n(t) + (\alpha+\beta+2n)r_n(t)}{t(t-R_n(t))}. \tag{2.4}$$

In order to further study the asymptotic expression of the second order differential equation (2.1), we will first find the asymptotic expression of $R_n(t)$; see Proposition 2.1. For convenience of the reader, we will use hollow symbols to define the new variable functions, such as $\mathbb{R}_n(T) := R_n(\frac{T}{n})$ and $\mathbb{X}(T) := X(\frac{T}{n})$.

Proposition 2.1. When $n \to \infty$, $t \to 0^+$, and $T = tn$ is fixed,

$$\begin{aligned} \mathbb{R}_n(T) = {} & 2n + \alpha + \beta + 1 + \frac{T}{2n} + \frac{T(\beta^2-\alpha^2)}{8n^3} + \frac{T(\alpha^2-\beta^2)(\alpha+\beta+1)}{8n^4} \\ & + \frac{(2\alpha^2+2\beta^2-1)T^2 - (\alpha^2-\beta^2)(3(\alpha+\beta)^2+6(\alpha+\beta)+4)T}{32n^5} \\ & + \mathcal{O}\left(\frac{1}{n^4}\right). \end{aligned} \tag{2.5}$$

Proof. The auxiliary quantity

$$R_n(t) := \frac{\alpha}{h_n}\int_0^1 \frac{P_n^2(x)}{x} x^\alpha (1-x)^\beta \mathrm{e}^{-tx}\mathrm{d}x$$

satisfies the second order nonlinear differential equation

$$\begin{aligned} R_n'' = {} & \frac{1}{2t^2(R_n-t)R_n}\big\{(2R_n-t)(tR_n')^2 - 2tR_n^2R_n' + 2R_n^5 - 2\alpha^2t^2R_n + \alpha^2t^3 \\ & - [2(2n+1+\alpha+\beta)+5t]R_n^4 + 4t(2n+1+\alpha+\beta+t)R_n^3 \\ & - [t^3+2t^2(2n+1+\alpha+\beta) - t(1+\alpha^2-\beta^2)]R_n^2\big\}, \end{aligned} \tag{2.6}$$

see Refs. 6 and 55, and it can further be reduced to the fifth Painlevé equation.

When n (the dimension of the Hankel determinant) tends to infinity, t tends to zero, and the product of n and t is fixed $nt = T$, the function $\mathbb{R}_n(T)$ satisfies the following equation:

$$\begin{aligned} & n^3\frac{\mathbb{R}_n{}^2}{T^2} + n^2\frac{\mathbb{R}_n\left(\alpha\mathbb{R}_n + \beta\mathbb{R}_n - \mathbb{R}_n{}^2 + \mathbb{R}_n - 2T\right)}{2T^2} \\ & \quad - n\frac{\alpha^2-\beta^2+1-3\mathbb{R}_n{}^2+2(\alpha+\beta+1)\mathbb{R}_n}{4T} \\ & \quad + n^2\frac{T\mathbb{R}_n\mathbb{R}_n'' - T\mathbb{R}_n'^2 + \mathbb{R}_n\mathbb{R}_n'}{2T\mathbb{R}_n} - n\frac{T\mathbb{R}_n'\left(T\mathbb{R}_n' - 2\mathbb{R}_n\right)}{4T\mathbb{R}_n{}^2} \\ & \quad - \frac{T^2\mathbb{R}_n'^2 - 2T\mathbb{R}_n\mathbb{R}_n' + \mathbb{R}_n{}^2(\mathbb{R}_n{}^2-\alpha^2-\beta^2+1)}{4\mathbb{R}_n{}^3} \\ & \quad + \frac{T(2T\mathbb{R}_n\mathbb{R}_n' - T^2\mathbb{R}_n'^2 + \beta^2\mathbb{R}_n{}^2 - \mathbb{R}_n{}^2)}{4\mathbb{R}_n{}^3(n\mathbb{R}_n - T)} = 0. \end{aligned} \tag{2.7}$$

Disregarding the derivative parts of (2.7) and considering the first two terms with n^3 and n^2 yield

$$\widehat{\mathbb{R}}_n(T) = \frac{1}{2}\left(2n+1+\alpha+\beta \pm \sqrt{(\alpha+\beta+2n+1)^2 - 8T}\right). \tag{2.8}$$

Expanding into the Taylor series as $n\to\infty$, we obtain two expressions of $\widehat{\mathbb{R}}_n(T)$,

$$\begin{aligned} \widehat{\mathbb{R}}_n(T)_1 &= 2n+\alpha+\beta+1 - \frac{T}{n} + \frac{(\alpha+\beta+1)T}{2n^2} + \mathcal{O}\left(\frac{1}{n^3}\right), \\ \widehat{\mathbb{R}}_n(T)_2 &= \frac{T}{n} - \frac{(\alpha+\beta+1)T}{2n^2} + \mathcal{O}\left(\frac{1}{n^3}\right). \end{aligned}$$

Next we assume that $\mathbb{R}_n(T)$ has the following expansion:

$$\mathbb{R}_n(T) = \sum_{j=0}^{\infty} a_j(T)n^{1-j}, \quad n\to\infty.$$

Substituting the expression above into (2.7) gives us (2.5) by comparing the corresponding coefficients on both sides. □

The proposition above is used to prove the following theorem.

Theorem 2.2. *When $n\to\infty$, $t\to 0^+$, and $T = tn$ is fixed, the monic polynomials $P_n(x)$ orthogonal with respect to the weight $w(x) = x^\alpha(1-x)^\beta e^{-tx}$ on $[0,1]$ satisfy the confluent Heun equation,*

$$P_n''(z) + \left(\frac{\widetilde{\gamma}}{z} + \frac{\widetilde{\delta}}{z-1} + \widetilde{\epsilon}\right)P_n'(z) + \frac{\widetilde{\alpha}z - \widetilde{q}}{z(z-1)}P_n(z) = 0, \tag{2.9}$$

with parameters

$$\widetilde{\gamma}=\alpha+1,\quad \widetilde{\delta}=\beta+1,\quad \widetilde{\epsilon}=-T/n,\quad \widetilde{\alpha}=T,\quad \widetilde{q}=n(n+\alpha+\beta+1)+T/2.$$

Proof. Substituting (2.2)–(2.4) into (2.1), the coefficients of (2.1), $Q_n(z,t)$ and $S_n(z,t)$, are given in terms of $R_n(t)$ and $R_n'(t)$. In particular,

$$\begin{aligned}
Q_n(z,t) &= \frac{\alpha+1}{z}+\frac{\beta+1}{z-1}-t-\frac{1}{z-R_n(t)/t},\\
S_n(z,t) &= \frac{nz+\left[tR_n'(t)+\alpha t-(2n+1+\alpha+\beta+t)R_n(t)+R_n^2(t)\right]/(2t)}{z(z-1)(z-R_n(t)/t)}\\
&\quad+\frac{tR_n'(t)+\alpha t-(2n+1+\alpha+\beta+t)R_n(t)+R_n^2(t)}{2z(z-1)}\\
&\quad+\frac{n(tz-1-\alpha-\beta-n)+tn(\beta+n)/(t-R_n(t))}{z(z-1)}\\
&\quad+\frac{(\alpha+\beta+2n)\left[tR_n'(t)+\alpha t-(2n+1+\alpha+\beta+t)R_n(t)+R_n^2(t)\right]}{2z(z-1)(t-R_n(t))}\\
&\quad+\frac{\left[tR_n'(t)+\alpha t-(2n+1+\alpha+\beta+t)R_n(t)+R_n^2(t)\right]^2}{4z(z-1)R_n(t)(t-R_n(t))}\\
&\quad-\frac{\alpha t\left[tR_n'(t)+\alpha t-(2n+1+\alpha+\beta+t)R_n(t)+R_n^2(t)\right]}{2z(z-1)R_n(t)(t-R_n(t))}.
\end{aligned}$$

Setting $T=nt$, the coefficients are further associated with $\mathbb{R}_n(T)$ and $\mathbb{R}_n'(T)$. From Proposition 2.1, we can substitute the asymptotic expression of $\mathbb{R}_n(T)$. Let n tends to infinity. We obtain

$$\begin{aligned}
\mathbb{Q}_n(z,T) &= \frac{\alpha+1}{z}+\frac{\beta+1}{z-1}-\frac{T}{n}+\mathcal{O}\left(n^{-2}\right),\\
\mathbb{S}_n(z,T) &= \frac{Tz-\left[n(n+\alpha+\beta+1)+T/2\right]}{z(z-1)}+\mathcal{O}\left(n^{-1}\right).
\end{aligned}$$

Substituting the above expressions into (2.1), we find the confluent Heun equation (2.9). Note that $P_n'(z)=n\sum_{j=0}^{n-1}b_jP_j(z)$; therefore, we take $\mathcal{O}\left(n^{-2}\right)$ for $\mathbb{Q}_n(z,T)$ and $\mathcal{O}\left(n^{-1}\right)$ for $\mathbb{S}_n(z,T)$. □

Corollary 2.3. When $t=0$, the weight $w(t,x)=x^\alpha(1-x)^\beta e^{-tx}$ reduces to the classical Jacobi weight $w(0,x)=x^\alpha(1-x)^\beta$ and the confluent Heun equation (2.9) reduces to the hypergeometric differential equation (Jacobi differential equation),

$$P_n''(z)+\left(\frac{\alpha+1}{z}+\frac{\beta+1}{z-1}\right)P_n'(z)-\frac{n(n+\alpha+\beta+1)}{z(z-1)}P_n(z)=0. \tag{2.10}$$

Proof. In the case when $t=0$, we have $T=0$. Then, $\widetilde{\epsilon}=\widetilde{\alpha}=0$ and $\widetilde{q}=n(n+\alpha+\beta+1)$, which directly gives (2.10).

Alternatively, we can use ladder operators to obtain the same result.

For $w(0,x)=x^\alpha(1-x)^\beta$, we have $v(x)=-\alpha\ln x-\beta\ln(1-x)$ and $v'(x)=-\alpha/x-\beta/(x-1)$. From (1.14) and (1.15), we find

$$\begin{aligned}
A_n(z) &= \frac{1}{h_n}\int_0^1\left(\frac{\alpha}{yz}+\frac{\beta}{(y-1)(z-1)}\right)P_n^2(y)y^\alpha(1-y)^\beta\,\mathrm{d}y,\\
B_n(z) &= \frac{1}{h_n}\int_0^1\left(\frac{\alpha}{yz}+\frac{\beta}{(y-1)(z-1)}\right)P_n(y)P_{n-1}(y)y^\alpha(1-y)^\beta\,\mathrm{d}y.
\end{aligned}$$

Integrating by parts, it follows that

$$A_n(z)=\frac{R_n}{z}-\frac{R_n}{z-1}. \tag{2.11}$$

Similarly, one has

$$B_n(z) = \frac{r_n}{z} - \frac{n + r_n}{z - 1}. \tag{2.12}$$

Here, R_n and r_n are defined by

$$R_n = R_n(\alpha, \beta) := \frac{\alpha}{h_n}\int_0^1 \frac{P_n^2(y)}{y} y^\alpha (1 - y)^\beta \, dy,$$
$$r_n = r_n(\alpha, \beta) := \frac{\alpha}{h_{n-1}}\int_0^1 \frac{P_n(y)P_{n-1}(y)}{y} y^\alpha (1 - y)^\beta \, dy.$$

Substituting (2.11) and (2.12) into (S_1) and equating residues of both sides of (S_1) at $z = 0$ and $z = 1$ give

$$r_n + r_{n+1} = -\alpha_n R_n + \alpha,$$
$$-(2n + 1 + r_n + r_{n+1}) = -(1 - \alpha_n)R_n + \beta.$$

Obviously, R_n can immediately be obtained by adding the two equalities above,

$$R_n = 2n + 1 + \alpha + \beta.$$

Then, we have

$$\sum_{j=0}^{n-1} R_j = n(n + \alpha + \beta).$$

Recall now (1.16),

$$-\left(v'(z) + \frac{A_n'(z)}{A_n(z)}\right) = \frac{\alpha + 1}{z} + \frac{\beta + 1}{z - 1},$$
$$B_n'(z) - B_n(z)\frac{A_n'(z)}{A_n(z)} + \sum_{j=0}^{n-1} A_j(z) = -\frac{n(n + \alpha + \beta + 1)}{z(z - 1)},$$

which produces (2.10).

Actually, we can obtain other relations for r_n, recurrence coefficients α_n and β_n by using (S_2) and (S_2'). See similar calculations for the classical Jacobi weight $(1 - x)^\alpha(1 + x)^\beta$ in the work of Chen and Ismail,[14] where explicit expressions for r_n, α_n, and β_n and also explicit expressions for the polynomials were obtained. □

Remark. Combining (2.2)–(2.4) with Proposition 2.1, and sending $n \to \infty$, $t \to 0^+$ and keeping $T = nt$ fixed, we find the following asymptotic expressions for $r_n(T/n)$, $\beta_n(T/n)$, and $\sum_{j=0}^{n-1}\mathbb{R}_j(T)$:

$$r_n(T/n) = -\frac{n}{2} + \frac{\alpha - \beta}{4} - \frac{T + \beta^2 - \alpha^2}{8n} + \frac{(\alpha - \beta)(\alpha + \beta)^2}{16n^2} + \frac{T\left(2\alpha^2 + 2\beta^2 - 1\right) - (\alpha - \beta)(\alpha + \beta)^3}{32n^3} + \mathcal{O}\left(\frac{1}{n^3}\right), \tag{2.13}$$

$$\beta_n(T/n) = \frac{1}{16} + \frac{1 - 2\alpha^2 - 2\beta^2}{64n^2} + \frac{\eta_n}{256n^3 T(\alpha + \beta + 1)} + \mathcal{O}\left(\frac{1}{n^4}\right), \tag{2.14}$$

$$\sum_{j=0}^{n-1}\mathbb{R}_j(T) = n(n + \alpha + \beta) + \frac{T}{2} + \frac{(\beta - \alpha)T}{4n} + \frac{(2(\alpha^2 - \beta^2) + T)T}{16n^2} - \frac{T(\alpha - \beta)(\alpha + \beta)^2}{16n^3} + \mathcal{O}\left(\frac{1}{n^4}\right), \tag{2.15}$$

where

$$\eta_n = \left(\alpha^2-\beta^2\right)\left[3T^2+16+5\alpha^4+20\alpha^3(\beta+1)+10\alpha^2(3\beta(\beta+2)+4)\right.$$
$$\left.+20\alpha(\beta+1)(\beta(\beta+2)+2)+5\beta(\beta+2)(\beta(\beta+2)+4)\right]$$
$$-2T\left[\alpha^4+4\alpha^3(\beta+2)+\alpha^2\left(6\beta^2+8\beta+9\right)+2\alpha(\beta+2)\left(2\beta^2-1\right)\right.$$
$$\left.+\beta(\beta(\beta(\beta+8)+9)-4)-5\right].$$

Next, we consider the second method (Dyson's Coulomb fluid method) to obtain asymptotic expression of the recurrence coefficient $\alpha_n(t)$. Using relation

$$R_n(t) = 2n+1+\alpha+\beta+t-t\alpha_n(t), \tag{2.16}$$

see Ref. 55, we then can deduce the asymptotic expression for $R_n(t)$. Using this method, we can verify the accuracy of $R_n(t)$ which was obtained in Proposition 2.1.

Proposition 2.4. Sending $n\to\infty$, we obtain the following asymptotic expressions of the recurrence coefficients:

$$\alpha_n(T/n) \sim \frac{1}{2}+\frac{\alpha^2-\beta^2}{8n^2}-\frac{(\alpha-\beta)(\alpha+\beta)^2}{8n^3}+\frac{3(\alpha-\beta)(\alpha+\beta)^3-2T\left(\alpha^2+\beta^2\right)}{32n^4}+\mathcal{O}\left(\frac{1}{n^3}\right), \tag{2.17}$$

$$\beta_n(T/n) \sim \frac{1}{16}-\frac{\alpha^2+\beta^2}{32n^2}+\frac{(\alpha+\beta)\left(\alpha^2+\beta^2\right)}{32n^3}-\frac{(\alpha+\beta)^2\left(5\alpha^2+2\alpha\beta+5\beta^2\right)-4T(\alpha^2-\beta^2)}{256n^4}+\mathcal{O}\left(\frac{1}{n^5}\right). \tag{2.18}$$

Proof. For $w(x) = x^\alpha(1-x)^\beta e^{-tx}$, we have

$$v(x) = -\alpha\ln x-\beta\ln(1-x)+tx,\quad v'(x) = -\frac{\alpha}{x}-\frac{\beta}{x-1}+t.$$

Substituting $v'(x)$ into (1.19), by using formulas (A1)–(A3) in the Appendix, we obtain the following two algebraic equations:

$$t+\frac{\beta}{\sqrt{(1-a)(1-b)}}-\frac{\alpha}{\sqrt{ab}} = 0, \tag{2.19}$$

$$2n+\alpha+\beta-\frac{\beta}{\sqrt{(1-a)(1-b)}}-\frac{(a+b)t}{2} = 0. \tag{2.20}$$

Adding two algebraic equations above, we obtain

$$2n+\alpha+\beta+t-\frac{\alpha}{\sqrt{ab}}-\frac{(a+b)t}{2} = 0. \tag{2.21}$$

Take $X(t) := \frac{1}{\sqrt{ab}}$ with $X(0) = (2n+\alpha+\beta)/\alpha$. Let $t=0$ in (2.21). Solving for $(a+b)/2$ from (2.21), we find

$$\frac{a+b}{2} = \frac{2n+\alpha+\beta+t-\alpha X(t)}{t}. \tag{2.22}$$

Substituting $(a+b)/2$ into the square root $\sqrt{(1-a)(1-b)}$ of (2.19) and using $X(t)$ to replace $1/\sqrt{ab}$, we get

$$t+\beta/\sqrt{1-\frac{2}{t}(2n+\alpha+\beta+t-\alpha X(t))+\frac{1}{X(t)^2}}-\alpha X(t) = 0.$$

After some simple calculations, we find that $X(t)$ satisfies the quintic equation,

$$2\alpha^3 X(t)^5 - \alpha^2(2\alpha + 2\beta + 5t + 4n)X(t)^4 + 4\alpha t(2n + \alpha + \beta + t)X(t)^3 \\ - t\big(4tn - \alpha^2 + 2\alpha t + (\beta + t)^2\big)X(t)^2 - 2\alpha t^2 X(t) + t^3 = 0, \tag{2.23}$$

with $X(0) = (2n + \alpha + \beta)/\alpha$.

Let $n \to \infty$, $t \to 0^+$, and $T = nt$ be fixed. We can find an equivalent quintic equation in terms of T. Consider the first two terms of n and n^0 of this new quintic equation,

$$\big(8\alpha T\widetilde{\mathbb{X}}(T)^3 + 2\alpha^3\widetilde{\mathbb{X}}(T)^5 - 2\alpha^3\widetilde{\mathbb{X}}(T)^4 - 2\alpha^2\beta\widetilde{\mathbb{X}}(T)^4\big) - 4n\big(\alpha^2\widetilde{\mathbb{X}}(T)^4\big) = 0.$$

Solving the equation above, we obtain two nonzero solutions,

$$\widetilde{\mathbb{X}}(T) = \frac{2n + \alpha + \beta \pm \sqrt{(2n + \alpha + \beta)^2 - 16T}}{2\alpha}.$$

Taking the Taylor series for large n, we obtain

$$\widetilde{\mathbb{X}}_1(T) = \frac{2T}{\alpha n} - \frac{(\alpha + \beta)T}{\alpha n^2} + \mathcal{O}\Big(\frac{1}{n^3}\Big),$$
$$\widetilde{\mathbb{X}}_2(T) = \frac{2n + \alpha + \beta}{\alpha} - \frac{2T}{\alpha n} + \frac{(\alpha + \beta)T}{\alpha n^2} + \mathcal{O}\Big(\frac{1}{n^3}\Big).$$

Assuming that $\mathbb{X}(T)$ has the form

$$\mathbb{X}(T) = \sum_{j=0}^{\infty} b_j(T) n^{1-j}, \quad n \to \infty,$$

and substituting the expression above into (2.23) with $X(0) = (2n + \alpha + \beta)/\alpha$, we obtain when $n \to \infty$

$$\mathbb{X}(T) = \frac{2n + \alpha + \beta}{\alpha} + \frac{T}{2\alpha n} - \frac{(\alpha^2 - \beta^2)T}{8\alpha n^3} + \frac{T(\alpha - \beta)(\alpha + \beta)^2}{8\alpha n^4} \\ + \frac{T\big(2T(\alpha^2 + \beta^2) - 3(\alpha - \beta)(\alpha + \beta)^3\big)}{32\alpha n^5} + \mathcal{O}\Big(\frac{1}{n^4}\Big). \tag{2.24}$$

Setting $T = nt$ in (2.22), we obtain

$$\alpha_n(T/n) \sim \frac{a + b}{2} = \frac{2n + \alpha + \beta + T/n - \alpha\mathbb{X}(T)}{T/n}$$

and

$$\beta_n(T/n) \sim \frac{[(b + a)/2]^2 - ab}{4} = \frac{1}{4}\Bigg[\Bigg(\frac{2n + \alpha + \beta + T/n - \alpha\mathbb{X}(T)}{T/n}\Bigg)^2 - \frac{1}{\mathbb{X}(T)^2}\Bigg].$$

Substituting $\mathbb{X}(T)$ into the expressions above and sending n to infinity, we obtain (2.17) and (2.18). □

Remark. From Dyson's Coulomb fluid approximation theory, we obtain the same order asymptotic expression of

$$\mathbb{R}_n(T) = 2n + 1 + \alpha + \beta + \frac{T}{n} - \frac{T\alpha_n(T/n)}{n}$$
$$\sim 2n + 1 + \alpha + \beta + \frac{T}{2n} - \frac{T(\alpha^2 - \beta^2)}{8n^3} + \mathcal{O}\Big(\frac{1}{n^4}\Big).$$

However, if we compare (2.18) with (2.14), they are not exactly the same. As we mentioned in Introduction, the Coulomb fluid method is suitable for sufficiently large n. We see that up to the order of $\mathcal{O}\big(\frac{1}{n}\big)$, they are equal.

Finally, if we consider Eq. (2.1) depending on functions $R_n(t)$ and $R_n'(t)$ satisfying Eq. (2.6) (without any reference to orthogonal polynomials), we can obtain that it is an equation for the derivative of the confluent Heun function in a special case.

Proposition 2.5. If $R_n(t)$ satisfies the Riccati equation

$$tR_n'(t) = R_n(t)^2 - (\alpha+\beta+t-1)R_n(t) + \alpha t$$

with solution

$$R_n(t) = \frac{t\Big(C_1\beta U(\beta+1,\alpha+\beta+1,t) + L_{-\beta-1}^{\alpha+\beta}(t)\Big)}{C_1 U(\beta,\alpha+\beta,t) + L_{-\beta}^{\alpha+\beta-1}(t)} + t, \tag{2.25}$$

then Eq. (2.1) reduces to Eq. (1.5),

$$\begin{aligned} &P_n''(z) + \left(\frac{\widetilde{\gamma}+1}{z} + \frac{\widetilde{\delta}+1}{z-1} + \widetilde{\epsilon} - \frac{\widetilde{\alpha}}{\widetilde{\alpha}z - \widetilde{q}}\right)P_n'(z) \\ &\quad + \frac{(\widetilde{\alpha}+\widetilde{\epsilon})\big(\widetilde{\alpha}z^2 - 2\widetilde{q}z\big) + \widetilde{\alpha}\widetilde{\gamma} + \widetilde{q}^2 - \widetilde{q}(\widetilde{\gamma}+\widetilde{\delta}-\widetilde{\epsilon})}{z(z-1)(\widetilde{\alpha}z-\widetilde{q})}P_n(z) = 0, \end{aligned} \tag{2.26}$$

with parameters

$$\widetilde{\gamma} = \alpha,\ \widetilde{\delta} = \beta,\ \widetilde{\epsilon} = -t,\ \widetilde{\alpha} = (n+1)t,\ \widetilde{q} = (n+1)R_n(t).$$

In the special case, when the constant $C_1 = 0$, we have

$$R_n(t) = \frac{\alpha t\, M(\beta;\alpha+\beta+1;t)}{(\alpha+\beta)\, M(\beta;\alpha+\beta;t)},$$

where $M(a;b;x)$, $U(a;b;x)$ is the Kummer function of the first and second kind (Ref. 45, Sec. 13.2) and $L_n^a(x)$ is the generalized Laguerre polynomial (Ref. 45, Sec. 18.1). The Riccati equation satisfies (2.6) when $n = -1$.

B. $x^\alpha(1-x)^\beta e^{-t/x}$, $x \in (0,1]$, $\alpha, \beta, t > 0$

The weight $x^\alpha(1-x)^\beta e^{-t/x}$ on $(0,1]$ was considered in the work of Chen,[10] Dai, and Chen.[11] The second order differential equation for $P_n(z)$ is as follows:

$$P_n''(z) + Q_n(z,t)P_n'(z) + S_n(z,t)P_n(z) = 0, \tag{2.27}$$

where

$$\begin{aligned} Q_n(z,t) &= \frac{t}{z^2} - \frac{1}{z - \tilde{R}_n(t)/[\tilde{R}_n(t) - R_n(t)]} + \frac{\alpha+2}{z} + \frac{\beta+1}{z-1}, \\ S_n(z,t) &= \frac{\tilde{R}_n(t)\big[n(z-1)^2 - r_n(t)\big] + R_n(t)\tilde{r}_n(t) - nR_n(t)z^2}{(z-1)z^2\big[z(R_n(t) - \tilde{R}_n(t)) + \tilde{R}_n(t)\big]} \\ &\quad + \frac{\sum_{j=0}^{n-1}\tilde{R}_j(t)}{z^2} - \frac{\sum_{j=0}^{n-1}R_j(t)}{z(z-1)}. \end{aligned}$$

See the work of Chen and Dai[11] for the definitions of $\tilde{r}_n(t)$, $r_n(t)$, $\tilde{R}_n(t)$, and $R_n(t)$. They satisfy the following relations (see Ref. 11 for details):

$$R_n(t) = R_n(\tilde{t}) - (\alpha+\beta+2n+1), \tag{2.28}$$

$$r_n(t) = \frac{R_n(t) - \beta - \alpha_n(t)R_n(t)}{2} + \frac{tR_n'(t) - \alpha_n(t)R_n(t)}{2(\alpha+\beta+2n+1)}, \tag{2.29}$$

$$\begin{aligned} \tilde{r}_n(t) &= \frac{R_n(t) - \beta - \alpha_n(t)R_n(t)}{2} + \frac{tR_n'(t) - \alpha_n(t)R_n(t)}{2(\alpha+\beta+2n+1)} \\ &\quad + \frac{\beta + t + (\alpha+\beta+2n+2)\alpha_n(t) - R_n(t)}{2}, \end{aligned} \tag{2.30}$$

$$\sum_{j=0}^{n-1} \tilde{R}_j(t) = n(t-\alpha-n) - (2n+\alpha+\beta)(\tilde{r}_n - r_n), \tag{2.31}$$

$$\sum_{j=0}^{n-1} R_j(t) = \sum_{j=0}^{n-1} \tilde{R}_j(t) + n(n+\alpha+\beta). \tag{2.32}$$

Equation (2.27) is neither Heun equation nor equation for the derivative of the Heun function. However, we will show that it is a confluent Heun equation using the asymptotic behavior of its coefficients. To obtain asymptotic behavior of $\alpha_n(t)$, we will use Dyson's Coulomb fluid method. For $R_n(t)$, we will use the formulas obtained by Chen and Dai[11] and Chen and Chen.[10] Here, we just show a brief statement; for further details and proof, see Ref. 10, Sec. 2. This will further be used to investigate the second order linear ordinary differential equation (2.27).

Proposition 2.6. (Ref. 10, Sec. 2). *Define*

$$f(t,\alpha,\beta) := n^2\left(\frac{R_n(t)}{2n+1+\alpha+\beta} - 1\right).$$

Let $t \to 0^+$, $n \to \infty$, and $T := 2n^2 t$ be fixed. If

$$F(T,\alpha,\beta) := \lim_{n\to\infty} f\left(\frac{T}{2n^2},\alpha,\beta\right),$$

then $F(T,\alpha,\beta)$ satisfies

$$F'' = \frac{F'^2}{F} - \frac{F'}{T} + \frac{2F^2}{T^2} + \frac{\alpha}{2T} - \frac{1}{4F} \tag{2.33}$$

with $F(0,\alpha,\beta) = 0$, $F'(0,\alpha,\beta) = 1/(2\alpha)$. Equations (2.33) is the third Painlevé equation $P_{III'}(8, 2\alpha, 0, -1)$.

Moreover, for $\alpha \neq \mathbb{Z}$, the following expansion holds:

$$F(T,\alpha,\beta) = \frac{T}{2\alpha} - \frac{T^2}{2\alpha^2(\alpha^2-1)} + \frac{3T^3}{2\alpha^3(\alpha^2-4)(\alpha^2-1)} + \mathcal{O}(T^4). \tag{2.34}$$

For convenience, we use hollow symbol to define a function of variable T by $\mathbb{R}_n(T) = R_n(T/(2n^2))$.

Proposition 2.7. For $T := 2n^2 t$ fixed, $t \to 0^+$, $n \to \infty$, and for small T, we have

$$\mathbb{R}_n(T) = (2n+\alpha+\beta+1)\left[1 + \frac{T}{2n^2\alpha} - \frac{T^2}{2n^2\alpha^2(\alpha^2-1)} + \mathcal{O}(T^3)\right],\ \alpha \neq \mathbb{Z}, \tag{2.35}$$

$$\alpha_n(T/(2n^2)) \sim \frac{1}{2} + \frac{\alpha^2-\beta^2}{8n^2} + \frac{T}{4\alpha n^2} - \frac{3T^2}{8\alpha^4 n^2} + \mathcal{O}(T^3), \tag{2.36}$$

$$\beta_n(T/(2n^2)) \sim \frac{1}{16} - \frac{\alpha^2+\beta^2}{32n^2} - \frac{T}{16\alpha n^2} + \frac{3T^2}{32\alpha^4 n^2} + \mathcal{O}(T^3). \tag{2.37}$$

Proof. From Proposition 2.6, it follows that

$$\mathbb{R}_n(T) = (2n+1+\alpha+\beta)\left[1 + \frac{F(T,\alpha,\beta)}{n^2}\right],$$

which gives (2.35).

Next we will use Dyson's Coulomb fluid method to obtain the asymptotic behavior of $\alpha_n(t)$. For the Pollaczek–Jacobi type weight $x^\alpha(1-x)^\beta e^{-t/x}$, we have

$$v(x) = -\ln w(x) = \frac{t}{x} - \alpha \ln x - \beta \ln(1-x) \text{ and } v'(x) = -\frac{t}{x^2} - \frac{\alpha}{x} - \frac{\beta}{x-1}.$$

Substituting $v'(x)$ into (1.19) and combining with formulas (A1), (A3), and (A4), we obtain two algebraic equations with respect to a and b,

$$\frac{a+b}{2(ab)^{3/2}}t+\frac{\alpha}{\sqrt{ab}}-\frac{\beta}{\sqrt{(1-a)(1-b)}}=0, \tag{2.38}$$

$$\frac{t}{\sqrt{ab}}-\frac{\beta}{\sqrt{(1-a)(1-b)}}+2n+\alpha+\beta=0. \tag{2.39}$$

Let $Y(t) := 1/(\sqrt{ab})$. Subtracting (2.38) from (2.39), we obtain

$$\frac{a+b}{2}=\frac{2n+\alpha+\beta-(\alpha-t)Y(t)}{tY(t)^3}. \tag{2.40}$$

Substituting the equality above into (2.39) yields

$$tY(t)-\beta/\sqrt{1-\frac{2}{tY(t)^3}[2n+\alpha+\beta-(\alpha-t)Y(t)]+\frac{1}{Y(t)^2}}+2n+\alpha+\beta=0.$$

Setting $T=2n^2t$ and $\mathbb{Y}(T)=Y(T/(2n^2))$, after some simple calculations, we obtain the quintic equation

$$\begin{aligned}\frac{T\mathbb{Y}(T)^3}{4n^2}-\frac{\beta^2\mathbb{Y}(T)^3T/(2n^2)}{2(\mathbb{Y}(T)T/(2n^2)+2n+\alpha+\beta)^2}+\alpha\mathbb{Y}(T)&\\ -\frac{\mathbb{Y}(T)T/(2n^2)}{2}-(2n+\alpha+\beta)=0&\end{aligned} \tag{2.41}$$

with $\mathbb{Y}(0)=(2n+\alpha+\beta)/\alpha$ (by setting $T=0$ in the equation above).

Consider the terms at T^0 and T,

$$T\left(\frac{\widetilde{\mathbb{Y}}(T)^3}{4n^2}-\frac{\beta^2\widetilde{\mathbb{Y}}(T)^3}{4n^2(\alpha+\beta+2n)^2}-\frac{\widetilde{\mathbb{Y}}(T)}{4n^2}\right)+(-\alpha-\beta-2n+\alpha\widetilde{\mathbb{Y}}(T))=0.$$

For large n, the term $-\beta^2\widetilde{\mathbb{Y}}(T)^3/(4n^2(\alpha+\beta+2n)^2)$ does not affect the form of the solution. So solving the equation above without this term, we obtain

$$\widetilde{\mathbb{Y}}(T)=\frac{\alpha+\beta+2n}{\alpha}-\frac{T\big((\beta+2n)\big(2\alpha^2+3\alpha\beta+(\beta+2n)^2+6\alpha n\big)\big)}{4(\alpha^4n^2)}+\mathcal{O}\big(T^2\big).$$

Then, we assume that $\mathbb{Y}(T)$ has the following form:

$$\mathbb{Y}(T)=\sum_{j=0}^{\infty}b_jT^j,\quad T\to0^+.$$

Substituting the expression above into (2.41) with $\mathbb{Y}(0)=(2n+\alpha+\beta)/\alpha$, we have

$$\begin{aligned}\mathbb{Y}(T)\sim&\frac{2n+\alpha+\beta}{\alpha}-\frac{(\alpha+\beta+2n)[\alpha\beta+2n(\alpha+\beta+n)]}{2\alpha^4n^2}T\\ &+\frac{(\alpha+\beta+2n)\eta_{n1}T^2}{4\alpha^7n^4}+\frac{(\alpha+\beta+2n)\eta_{n2}T^3}{8\alpha^{10}n^6}+\mathcal{O}\big(T^4\big),\quad T\to0^+,\end{aligned}$$

where

$$\begin{aligned}\eta_{n1}=&\,12n^4+24n^3(\alpha+\beta)+2n^2\big(7\alpha^2+18\alpha\beta+6\beta^2\big)+\alpha^2\beta(\alpha+2\beta)\\ &+2\alpha n(\alpha+\beta)(\alpha+6\beta),\\ \eta_{n2}=&\,\beta^3(\alpha+2n)\big(5\alpha^2+48n^2+48\alpha n\big)+\beta(\alpha+2n)\big(\alpha^2+6n^2+6\alpha n\big)\\ &\cdot\big(\alpha^2+24n^2+24\alpha n\big)+\beta^2\big(5\alpha^4+288n^4+576\alpha n^3+376\alpha^2n^2+88\alpha^3n\big)\\ &+2n(\alpha+n)\big(\alpha^4+48n^4+96\alpha n^3+63\alpha^2n^2+15\alpha^3n\big).\end{aligned}$$

Substituting $\mathbb{Y}(T)$ into (2.40) with $t = T/(2n^2)$, we get

$$\frac{a+b}{2} = \frac{2n+\alpha+\beta-(\alpha-T/2n^2)\mathbb{Y}(T)}{T/(2n^2)\mathbb{Y}(T)^3},$$

which gives (2.36) for small T and as $n \to \infty$. Since

$$\begin{aligned}\beta_n(T/(2n^2)) &\sim \frac{[(a+b)/2]^2 - ab}{4}\\ &= \frac{1}{4}\left[\left(\frac{2n+\alpha+\beta+(T/2n^2-\alpha)\mathbb{Y}(T)}{\mathbb{Y}(T)^3 T/2n^2}\right)^2 - \frac{1}{\mathbb{Y}(T)^2}\right],\end{aligned}$$

for small T, we deduce (2.37). □

Once the asymptotic formulas for $\mathbb{R}_n(T)$, $\alpha_n(T/2n^2)$ are obtained, we have the following theorem.

Theorem 2.8. *Let $n \to \infty, t \to 0^+$ and $T = 2n^2 t$ be fixed. For small T, the orthogonal polynomials $\widehat{P}_n(u)$ satisfy the confluent Heun equation,*

$$\widehat{P}_n''(u) + \left(\frac{\widetilde{\gamma}}{u} + \frac{\widetilde{\delta}}{u-1} + \widetilde{\epsilon}\right)\widehat{P}_n'(u) + \frac{\widetilde{\alpha}u - \widetilde{q}}{u(u-1)}\widehat{P}_n(u) = 0 \tag{2.42}$$

with parameters

$$\widetilde{\gamma} = -\alpha-\beta+2\lambda, \quad \widetilde{\delta} = \beta+1, \quad \widetilde{\epsilon} = -s, \quad \widetilde{\alpha} = -s\lambda, \quad \widetilde{q} = \lambda(\alpha-\lambda-s) - a.$$

Here, $\widehat{P}_n(u) := u^{-\lambda}P_n(1/u)$, $u := 1/z$. The parameters a, b, s, λ are given by

$$s = \frac{[(\alpha-1)\alpha(\alpha+1)^2 - T]T}{2n^2\alpha^2(\alpha^2-1)}, \qquad a = \frac{(\alpha^2+3)T^2 - 2\alpha^3(\alpha^2-1)T}{4\alpha^4(\alpha^2-1)},$$

$$b = n(n+\alpha+\beta+1), \qquad \lambda = \frac{\alpha+\beta+1 \pm \sqrt{(\alpha+\beta+1)^2+4b}}{2}.$$

Proof. Substituting (2.28)–(2.32) into (2.27) and taking $T = 2n^2 t$, we find that the coefficients of the second order linear ordinary differential equation (2.27), $Q_n(z,T)$ and $S_n(z,T)$, are given in terms of $\mathbb{R}_n(T)$, $\mathbb{R}_n'(T)$ and $\alpha_n(T/(2n^2))$.

Sending $T \to 0^+$, $n \to \infty$ and combining with the asymptotic expressions (2.35) and (2.36) yield

$$\begin{aligned}\mathbb{Q}_n(z,T) &= \frac{s}{z^2} + \frac{\alpha+1}{z} + \frac{\beta+1}{z-1} + \mathcal{O}(T^3), \quad &T \to 0^+,\\ \mathbb{S}_n(z,T) &= -\frac{bz+a}{(z-1)z^2} + \mathcal{O}(T^3), \quad &T \to 0^+.\end{aligned}$$

Namely, we have

$$P_n''(z) + \left(\frac{s}{z^2} + \frac{\alpha+1}{z} + \frac{\beta+1}{z-1}\right)P_n'(z) - \frac{bz+a}{(z-1)z^2}P_n(z) = 0. \tag{2.43}$$

Let $u = 1/z$, then

$$\widehat{P}_n(u) := u^{-\lambda}P_n(1/u)$$

satisfy the confluent Heun equation (2.42). □

Corollary 2.9. The confluent Heun equation (2.43) with $t = 0$ reduces to the same Jacobi differential equation as in Corollary 2.3,

$$P_n''(z) + \left(\frac{\alpha+1}{z} + \frac{\beta+1}{z-1}\right)P_n'(z) - \frac{n(n+\alpha+\beta+1)}{(z-1)z}P_n(z) = 0. \tag{2.44}$$

Proof. When $t = 0$, we have $T = 0$, $a = s = 0$, and Eq. (2.43) reduces to (2.44).

Actually, the weight $w(t,x) = x^\alpha(1-x)^\beta e^{-t/x}$ is the deformed Jacobi weight. When $t = 0$, the weight reduces to the classical Jacobi weight $w(0,x) = x^\alpha(1-x)^\beta$. See the Proof of Corollary 2.3 for ladder operators approach to deduce the Jacobi differential equation. □

C. $(1-x^2)^\alpha(1-k^2x^2)^\beta$, $x\in[-1,1]$, $\alpha,\beta>0$, $k^2\in(0,1)$

The weight $(1-x^2)^\alpha(1-k^2x^2)^\beta$ is the generalization of the weight function $\left[(1-x^2)(1-k^2x^2)\right]^{-1/2}$ studied by Rees in 1945.[46] It has a strong relation with the famous string theory; see the work of Basor, Chen, and Haq.[5] In Ref. 5, the asymptotic expressions of recurrence coefficients $\beta_n(k^2)$ and of the second coefficients $p(n)$ of monic polynomials $P_n(z)$ were obtained. The large n asymptotics of Hankel determinants was also studied.

The second order linear ordinary differential equation for $P_n(z)$, associated with the weight $(1-x^2)^\alpha(1-k^2x^2)^\beta$, reads

$$P_n''(z) + \left(\frac{X'(z)}{2X(z)} - \frac{M_n'(z)}{M_n(z)}\right)P_n'(z) + \left(\frac{L_n(z)M_n'(z)}{Y(z)M_n(z)} + \frac{U_n(z)}{Y(z)}\right)P_n(z) = 0, \tag{2.45}$$

where

$$\begin{aligned}
X(z) &:= (1-z^2)^{2\alpha+2}(1-k^2z^2)^{2\beta+2},\\
Y(z) &:= (1-z^2)(1-k^2z^2),\\
M_n(z) &:= -2(n+1/2+\alpha+\beta)k^2z^2 - C_n,\\
L_n(z) &:= z\left[nk^2z^2 - n(k^2+1) + 2k^2(n+1/2+\alpha+\beta)\beta_n - 2k^2p(n)\right],\\
U_n(z) &:= -k^2z^2n(n+2\alpha+2\beta+3) + 2k^2(2n+1+2\alpha+2\beta)(p(n)-\beta_n)\\
&\quad + nk^2(n+1+2\beta) + n(n+1+2\alpha),
\end{aligned}$$

and

$$C_n = 2k^2(n+3/2+\alpha+\beta)(\beta_n+\beta_{n+1}) - 2\left[(n+\beta+1/2)k^2 + n+\alpha+1/2\right] - 4k^2p(n).$$

The coefficients of the second order differential equation (2.45) depend on $p(n)$ and $\beta_n(k^2)$ after substituting the equalities above into it. From Secs. II A and II B, we know that in order to reduce Eq. (2.45) to the Heun equation, we first need to know the asymptotic expressions of $p(n)$ and $\beta_n(k^2)$.

Theorem 2.10. *Let $P_n(x)$ be the monic orthogonal polynomials with respect to the weight $(1-x^2)^\alpha(1-k^2x^2)^\beta$, $n\to\infty, k^2\to 0^+, \beta\to\infty$ and $k^2\beta = t$ be fixed. Under these assumptions, the weight reduces to $(1-x^2)^\alpha e^{-tx^2}$. Then, if $\widehat{P}_n(u) = P_n(z^2)$, then $\widehat{P}_n(u)$ satisfies the confluent Heun equation*

$$\widehat{P}_n''(u) + \left(\frac{\widetilde{\gamma}}{u} + \frac{\widetilde{\delta}}{u-1} + \widetilde{\epsilon}\right)\widehat{P}_n'(u) + \left(\frac{\widetilde{\alpha}u - \widetilde{q}}{u(u-1)}\right)\widehat{P}_n(u) = 0 \tag{2.46}$$

with parameters

$$\widetilde{\gamma} = 1/2,\quad \widetilde{\delta} = \alpha+1,\quad \widetilde{\epsilon} = -t,\quad \widetilde{\alpha} = nt/2,\quad \widetilde{q} = n(n+2\alpha+t+1)/4.$$

Proof. From the work of Kuijlaars, McLaughlin, Van Assche, and Vanlessen,[34] and Basor, Chen, and Haq,[5] the asymptotic expressions as $n\to\infty$ of $p(n), \beta_n(k^2)$ are known,

$$\begin{aligned}
\beta_n(k^2) &= \frac{1}{4} - \frac{4\alpha^2-1}{16n^2} + \frac{4\alpha^2-1}{8n^3}\left(\alpha+\beta-\frac{\beta}{\sqrt{1-k^2}}\right) + \mathcal{O}\left(\frac{1}{n^4}\right),\\
p(n) &:= -\frac{n}{4} + \frac{\alpha-\beta}{4} + \frac{1}{8} + \frac{\beta(1-\sqrt{1-k^2})}{2k^2} - \frac{4\alpha^2-1}{16n^3}\left(\alpha+\beta-\frac{1}{2}-\frac{\beta}{\sqrt{1-k^2}}\right)^2\\
&\quad - \frac{4\alpha^2-1}{16n} + \frac{4\alpha^2-1}{16n^2}\left(\alpha+\beta-\frac{1}{2}-\frac{\beta}{\sqrt{1-k^2}}\right) + \mathcal{O}\left(\frac{1}{n^4}\right).
\end{aligned}$$

Since the coefficients of (2.45) are represented by $p(n), \beta_n(k^2)$, we have the following equation as $n \to \infty$:

$$P_n''(z) + \left(\frac{\beta k}{kz-1} + \frac{\beta k}{kz+1} + \frac{\alpha+1}{z-1} + \frac{\alpha+1}{z+1}\right)P_n'(z) + \left(\frac{n^2}{1-z^2} + \frac{n\left(2\alpha+2\beta-2\beta\sqrt{1-k^2}-2\alpha k^2z^2-2\beta k^2z^2-k^2z^2+1\right)}{(z^2-1)(k^2z^2-1)}\right)P_n(z) = 0. \tag{2.47}$$

Let $k^2 \to 0^+, \beta \to \infty$ and $k^2\beta = t$ be fixed. The weight $(1-x^2)^\alpha(1-k^2x^2)^\beta$ reduces to the weight $(1-x^2)^\alpha e^{-tx^2}$. Assuming $k = t^{1/4}/\sqrt{n}$, $\beta = n\sqrt{t}$, Eq. (2.47) reduces to the following equation as $n \to \infty$:

$$P_n''(z) + \left(\frac{\alpha+1}{z-1} + \frac{\alpha+1}{z+1} - 2tz\right)P_n'(z) - \frac{n\left(n+2\alpha+1-\left(2z^2-1\right)t\right)}{z^2-1}P_n(z) = 0. \tag{2.48}$$

Let $u = z^2$ and $\widehat{P}_n(u) := P_n(\sqrt{u})$. Substituting this into (2.48), after some direct calculations, we obtain the confluent Heun equation (2.46). □

Corollary 2.11. When $k = 0$, the orthogonal polynomials reduce to the Jacobi polynomials, and the confluent Heun equation reduces to the Jacobi differential equation

$$P_n''(z) + \left(\frac{\alpha+1}{z-1} + \frac{\alpha+1}{z+1}\right)P_n'(z) - \frac{n(n+2\alpha+1)}{z^2-1}P_n(z) = 0. \tag{2.49}$$

Proof. If $k = 0$, then the weight $(1-x^2)^\alpha(1-k^2x^2)^\beta$ reduces to the classical Jacobi weight $(1-x)^\alpha(1+x)^\alpha$ (see also the work of Basor and Chen[3]). We can proceed similarly as in Corollary 2.3. Also put $k = 0$ in (2.47) and $t = k^2\beta = 0$ in (2.48). □

Remark. Recall from Theorem 2.10 that when $k^2 \to 0^+, \beta \to \infty$ and $k^2\beta = t$ is fixed, the weight $(1-x^2)^\alpha(1-k^2x^2)^\beta$ reduces to $(1-x^2)^\alpha e^{-tx^2}$. Then, the normalization constant for orthogonal polynomials reads

$$h_j(t) = \int_{-1}^{1} P_j^2(x,t)(1-x^2)^\alpha e^{-tx^2}dx.$$

Changing the variable $x^2 = s$, we see that the even case and the odd case of j have different normalization, in particular,

$$\begin{aligned} h_{2n}(t) &= \int_{-1}^{1} P_{2n}^2(x,t)(1-x^2)^\alpha e^{-tx^2}dx \\ &= 2\int_0^1 P_{2n}^2(\sqrt{s},t)(1-s)^\alpha \frac{e^{-ts}}{2\sqrt{s}}ds \\ &= \int_0^1 \widetilde{P}_n^2(s,t)s^{-\frac{1}{2}}(1-s)^\alpha e^{-ts}ds =: \widetilde{h}_n(t) \end{aligned}$$

and

$$\begin{aligned} h_{2n+1}(t) &= \int_{-1}^{1} P_{2n+1}^2(x,t)(1-x^2)^\alpha e^{-tx^2}dx \\ &= 2\int_0^1 P_{2n+1}^2(\sqrt{s},t)(1-s)^\alpha \frac{e^{-ts}}{2\sqrt{s}}ds \\ &= \int_0^1 \widehat{P}_n^2(s,t)s^{\frac{1}{2}}(1-s)^\alpha e^{-ts}ds =: \widehat{h}_n(t). \end{aligned}$$

Here,

$$\begin{aligned} P_{2n}(\sqrt{s},t) &= (\sqrt{s})^{2n} + p(2n,t)(\sqrt{s})^{2n-2} + \cdots + P_{2n}(0,t) \\ &= s^n + \widetilde{p}(n,t)s^{n-1} + \cdots + \widetilde{P}_n(0,t) := \widetilde{P}_n(s,t) \end{aligned}$$

and

$$P_{2n+1}(\sqrt{s},t) = (\sqrt{s})^{2n+1} + p(2n,t)(\sqrt{s})^{2n-1} + \cdots + const\cdot\sqrt{s}$$
$$= \sqrt{s}\big(s^n + \widehat{p}(n,t)s^{n-1} + \cdots + const\big) := \sqrt{s}\widehat{P}_n(s,t).$$

The polynomials $\widetilde{P}_n(s,t)$ and $\widehat{P}_n(s,t)$ are monic polynomials of degree n in the variable s, and they are orthogonal with respect to $s^{-\frac{1}{2}}(1-s)^{\alpha}\mathrm{e}^{-ts}$ and $s^{\frac{1}{2}}(1-s)^{\alpha}\mathrm{e}^{-ts}$ over $(0,1]$, respectively. These weights are the special cases of the weight $s^{\alpha}(1-s)^{\beta}e^{-ts}$ for $\beta=-\frac{1}{2}$ and $\beta=\frac{1}{2}$, respectively; see Ref. 55.

The Hankel determinants generated by $s^{-\frac{1}{2}}(1-s)^{\alpha}\mathrm{e}^{-ts}$ and $s^{\frac{1}{2}}(1-s)^{\alpha}\mathrm{e}^{-ts}$, $0 < s \le 1$ are defined by

$$\widetilde{D}_m(t) := \det\left(\int_0^{\infty} s^{i+j-\frac{1}{2}}(1-s)^{\alpha}\mathrm{e}^{-ts}\mathrm{d}s\right)_{i,j=0}^{n-1} = \prod_{l=0}^{m-1}\widetilde{h}_l(s),$$
$$\widehat{D}_m(t) := \det\left(\int_0^{\infty} s^{i+j+\frac{1}{2}}(1-s)^{\alpha}\mathrm{e}^{-ts}\mathrm{d}s\right)_{i,j=0}^{n-1} = \prod_{l=0}^{m-1}\widehat{h}_l(s),$$

respectively. Hence,

$$D_n(t) = \prod_{j=0}^{n-1}h_j(t) = \begin{cases}\widetilde{D}_{k+1}\widehat{D}_k, & n=2k+1,\\ \widetilde{D}_k\widehat{D}_k, & n=2k.\end{cases}$$

The Hankel determinants for large n are very interesting; we shall not pursue this subject further as it lies beyond the scope of this paper. See the work of Lyu, Chen, and Fan[39] for the Gaussian weight, also the monograph by Szegö.[51]

III. LAGUERRE TYPE WEIGHTS

In this section, we consider two deformed Laguerre type weights: $x^{\alpha}(x+t)^{\lambda}\mathrm{e}^{-x}$ (see Refs. 4 and 17) and $x^{\alpha}\mathrm{e}^{-x-t/x}$ (see Ref. 15). The weight $x^{\alpha}(x+t)^{\lambda}\mathrm{e}^{-x}$ appear in multiple-input multiple-output (MIMO) wireless communication systems. The technique of ladder operators was used to study Hankel determinants and to show connection to the Jimbo-Miwa-Okamoto σ-form of the fifth Painlevé equation in Refs. 4 and 17. The deformed Laguerre weight $x^{\alpha}\mathrm{e}^{-x-t/x}$ (see Ref. 15) appears in mathematical physics and integrable quantum field theory of finite temperature (see Ref. 37). The technique of ladder operators and the Riemann-Hilbert approach gives connection of recurrence coefficients and of the logarithmic derivative of the Hankel determinant to the solutions of the third Painlevé equation (or the σ-form of it).

A. $x^{\alpha}(x+t)^{\lambda}\mathrm{e}^{-x}$, $x \in [0,\infty)$, $t,\alpha,\lambda > 0$

From Ref. 4, the second order differential equation satisfied by monic polynomials $P_n(x)$ orthogonal with respect to the weight $w(x,t,\lambda) = x^{\alpha}(x+t)^{\lambda}\mathrm{e}^{-x}$ is of the following form:

$$P_n''(z) + Q_n(z,t)P_n'(z) + S_n(z,t)P_n(z) = 0, \tag{3.1}$$

where

$$Q_n(z,t) = \frac{\alpha+1}{z} + \frac{\lambda+1}{z+t} - 1 - \frac{1}{z+t[1-R_n(t)]},$$
$$S_n(z,t) = \frac{t(r_n(t)+nR_n(t))}{z(t+z)(z+t(1-R_n(t)))} + \frac{n-\sum_{j=0}^{n-1}R_j(t)}{z} + \frac{\sum_{j=0}^{n-1}R_j(t)}{t+z}$$

with $R_n(t)$, $r_n(t)$ defined by

$$r_n(t)\frac{tR_n'(t)+\lambda-R_n(t)(t+2n+\alpha+\lambda-tR_n(t))}{2}, \tag{3.2}$$

$$\sum_{j=0}^{n-1}R_j(t)\frac{n(\alpha+\lambda+n)-tr_n(t)-\beta_n(t)}{t}, \tag{3.3}$$

and

$$\beta_n(t) = \frac{1}{1-R_n(t)}\left(n(\alpha+n) + \frac{r_n(t)^2-\lambda r_n(t)}{R_n(t)} + (\alpha+\lambda+2n)r_n(t)\right). \tag{3.4}$$

Obviously, the coefficients of (3.1) are given in terms of n, λ, t, $R_n(t)$ and its derivative. First, we will obtain the asymptotic formula for $R_n(t)$ and then show how the second order differential equation (3.1) can be reduced to the Heun equation.

Proposition 3.1. For $t > 0$ and $n \to \infty$, we have

$$R_n(t) = \frac{\lambda}{2(nt)^{1/2}} - \frac{4\lambda\left(\alpha^2 + t^2 + 2t(\alpha+\lambda+1)\right) - \lambda}{64(nt)^{3/2}} + \frac{\lambda^2\left(4t^2 - 4\alpha^2 + 1\right)}{64t^2n^2} + \mathcal{O}\left(\frac{1}{n^{5/2}}\right). \tag{3.5}$$

Proof. From Ref. 4, the auxiliary quantity $R_n(t)$ is given by

$$R_n(t) := \frac{\lambda}{h_n}\int_0^\infty \frac{P_n(x)^2}{x+t} x^\alpha (x+t)^\lambda \mathrm{e}^{-x}\mathrm{d}x, \qquad t, \alpha, \lambda > 0.$$

It satisfies

$$\begin{aligned} R_n''(t) = \frac{1}{2t^2(R_n(t)-1)R_n(t)} \Big\{ &\lambda^2 + 4t(\alpha+\lambda+2n+t+1)R_n(t)^3 \\ &- t^2R_n'(t)^2 + 2t^2R_n(t)^5 - t(2\alpha+2\lambda+2+4n+5t)R_n(t)^4 \\ &- \left(2tR_n'(t) + \alpha^2 - \lambda^2 + 2t(\alpha+\lambda+2n+1) + t^2\right)R_n(t)^2 \\ &+ 2\left(t^2R_n'(t)^2 + tR_n'(t) - \lambda^2\right)R_n(t)\Big\}. \end{aligned} \tag{3.6}$$

Disregarding the derivative part of this nonlinear second order differential equation, for small α, we have

$$(\widetilde{R}_n(t)-1)^2\left(\lambda^2 - t(2\lambda+4n+t+2)\widetilde{R}_n(t)^2 + 2t^2\widetilde{R}_n(t)^3\right) = 0.$$

The solutions are given by

$$\begin{aligned} \widetilde{R}_{n1,2}(t) &= 1, \\ \widetilde{R}_{n3,4}(t) &= \pm\frac{\lambda}{2\sqrt{nt}} \mp \frac{\lambda(2\lambda+t+2)}{16\sqrt{t}n^{-3/2}} + \frac{\lambda^2}{16n^2} \pm \frac{3\lambda(2\lambda+t+2)^2}{256\sqrt{t}n^{-5/2}} + \mathcal{O}\left(n^{-3}\right), \\ \widetilde{R}_{n5}(t) &= \frac{2n}{t} + \frac{2\lambda+t+2}{2t} - \frac{\lambda^2}{8n^2} + \frac{\lambda^2(2\lambda+t+2)}{16n^3} + \mathcal{O}\left(n^{-7/2}\right). \end{aligned}$$

Assuming that $R_n(t)$ has the form

$$R_n(t) = \sum_{j=1}^\infty b_j(t)n^{-j/2}, \qquad n\to\infty,$$

and substituting into (3.6), we obtain (3.5). □

Theorem 3.2. *As $n \to \infty$, the monic polynomials $P_n(x)$ orthogonal with respect to $x^\alpha(x+t)^\lambda e^{-x}$ over $[0,\infty)$ satisfy the confluent Heun equation*

$$\widehat{P}_n''(u) + \left(\frac{\widetilde{\gamma}}{u} + \frac{\widetilde{\delta}}{u-1} + \widetilde{\epsilon}\right)\widehat{P}_n'(u) + \frac{\widetilde{\alpha}u - \widetilde{q}}{u(u-1)}\widehat{P}_n(u) = 0 \tag{3.7}$$

with parameters

$$\widetilde{\gamma} = \alpha+1, \quad \widetilde{\delta} = 2\eta+\lambda, \quad \widetilde{\epsilon} = t, \quad \widetilde{\alpha} = t(\eta - n),$$

$$\widetilde{q} = -\frac{(2\eta+\lambda)(2(\alpha+\eta)+\lambda) - 4\lambda\sqrt{nt} + 4nt + 2\lambda t}{4}.$$

Here, $\widehat{P}_n(u) := (u-1)^{-\eta}P_n(-tu)$.

Proof. Substituting (3.2)–(3.4) into (3.1), the coefficients of (3.1) are given in terms of $R_n(t)$ and $R_n'(t)$. Substituting (3.5) and taking $n\to\infty$, we obtain

$$\begin{aligned}P_n''(z)&+\left(\frac{\alpha+1}{z}+\frac{\lambda}{z+t}-1\right)P_n'(z)\\&+\left(\frac{n}{z}-\frac{\lambda(4\sqrt{nt}-2\alpha-2t-\lambda)}{4z(t+z)}+\frac{\lambda t}{2z(t+z)^2}\right)P_n(z)=0.\end{aligned}\tag{3.8}$$

Let

$$z=-tu.$$

Then, $\widehat{P}_n(u):=(u-1)^{-\eta}P_n(-tu)$, where

$$\eta=\frac{1-\lambda\pm\sqrt{\lambda^2+1}}{2}$$

satisfies the confluent Heun equation (3.7). □

Corollary 3.3. When $\lambda=0$, the weight $x^{\alpha}(x+t)^{\lambda}e^{-x}$ reduces to the classical Laguerre weight $x^{\alpha}e^{-x}$, the polynomials $P_n(z)$ reduce to the Laguerre polynomials $L_n^{\alpha}(z)$, and Eq. (3.8) reduces to the Laguerre equation

$$P_n''(z)+\left(\frac{\alpha+1}{z}-1\right)P_n'(z)+\frac{n}{z}P_n(z)=0.\tag{3.9}$$

Proof. Let us use first ladder operator approach to derive (3.9). For the weight function $x^{\alpha}e^{-x}$, we have $v(x)=-\alpha\ln x+x$ and $v'(x)=-\frac{\alpha}{x}+1$. From (1.14) and (1.15), we have

$$\begin{aligned}A_n(z)&=\frac{1}{h_n}\int_0^{\infty}\frac{\alpha}{yz}P_n^2(y)y^{\alpha}\mathrm{e}^{-y}\,\mathrm{d}y=\frac{1}{z},\\B_n(z)&=\frac{1}{h_n}\int_0^{\infty}\frac{\alpha}{yz}P_n(y)P_{n-1}(y)y^{\alpha}\mathrm{e}^{-y}\,\mathrm{d}y=-\frac{n}{z}.\end{aligned}$$

Recalling (1.16), we obtain

$$\begin{aligned}-(v'(z)+\frac{A_n'(z)}{A_n(z)})&=\frac{\alpha+1}{z}-1,\\B_n'(z)-B_n(z)\frac{A_n'(z)}{A_n(z)}+\sum_{j=0}^{n-1}A_j(z)&=\frac{n}{z},\end{aligned}$$

which gives (3.9). □

From the expressions for $r_n(t)$, $\beta_n(t)$, $\sum_{j=0}^{n-1}R_j(t)$ in terms of $R_n(t)$, it is easy to obtain their asymptotic expressions.

Remark. The auxiliary quantities $r_n(t)$, $\beta_n(t)$, and $\sum_{j=0}^{n-1}R_j(t)$ have the following asymptotic expressions when n is large:

$$\begin{aligned}r_n(t)=&\frac{\lambda}{2}-\frac{\lambda n^{1/2}}{2t^{1/2}}-\frac{\lambda\left[4t(3t+2\alpha+2\lambda)-4\alpha^2+1\right]}{64t^{3/2}n^{1/2}}\\&+\frac{\lambda^2\left(4\alpha^2+4t^2-1\right)}{64nt^2}+\mathcal{O}\left(\frac{1}{n^{3/2}}\right),\end{aligned}\tag{3.10}$$

$$\beta_n(t)=n^2+n(\alpha+\lambda)-\frac{\lambda(tn)^{1/2}}{2}+\frac{1}{4}\lambda(2\alpha+\lambda)+\frac{\lambda\left(4t(t-2\alpha-2\lambda)-12\alpha^2+3\right)}{64\sqrt{nt}}+\frac{\left(1-4\alpha^2\right)\lambda^2}{32nt}+\mathcal{O}\left(\frac{1}{n^{3/2}}\right),\tag{3.11}$$

$$\begin{aligned}\sum_{j=0}^{n-1}R_j(t)=&\frac{\lambda n^{1/2}}{t^{1/2}}-\frac{\lambda(\lambda+2(\alpha+t))}{4t}+\frac{\lambda(4t(4\alpha+2\lambda+3t)+1)}{64t^{3/2}n^{1/2}}\\&-\frac{\lambda^2\left(1-4\alpha^2+4t^2\right)}{64nt^2}+\mathcal{O}\left(\frac{1}{n^{3/2}}\right).\end{aligned}\tag{3.12}$$

Remark. *From the relation $\alpha_n(t) = 2n + 1 + \alpha + \lambda - tR_n(t)$ (see Ref. 4) and asymptotic expression (3.5), we obtain*

$$\alpha_n(t) = 2n + 1 + \alpha + \lambda - \frac{\lambda\sqrt{t}}{2\sqrt{n}} + \frac{\lambda\left[4\left(\alpha^2 + t^2 + 2t(\alpha + \lambda + 1)\right) - 1\right]}{64\sqrt{t}n^{3/2}} + \mathrm{O}\left(\frac{1}{n^2}\right). \tag{3.13}$$

Next we will use Dyson's Coulomb fluid method to check the correctness of this result.

Proposition 3.4. *From Dyson's Coulomb fluid approximation theory, we obtain*

$$\alpha_n(t) \sim 2n + \alpha + \lambda - \frac{\lambda\sqrt{t}}{2\sqrt{n}} + \frac{\lambda\left((\alpha + t)^2 + 2\lambda t\right)}{16\sqrt{t}n^{3/2}} + \frac{\lambda^2\left(\alpha^2 - t^2\right)}{16n^2 t} + \mathcal{O}\left(\frac{1}{n^{5/2}}\right), \quad n \to \infty, \tag{3.14}$$

$$\beta_n(t) \sim n^2 + n(\alpha + \lambda) - \frac{\lambda(nt)^{1/2}}{2} + \frac{\lambda(2\alpha + \lambda)}{4} + \mathcal{O}\left(\frac{1}{n^{1/2}}\right), \quad n \to \infty. \tag{3.15}$$

Proof. For weight function $w(x) = x^\alpha(x+t)^\lambda e^{-x}$, we have

$$v(x) = -\alpha \ln x - \lambda \ln(x + t) + x \quad \text{and} \quad v'(x) = -\frac{\alpha}{x} - \frac{\lambda}{x + t} + 1.$$

Substituting $v'(x)$ into (1.19), combining with formulas (A1)–(A3) and (A5), we obtain the following algebraic equations:

$$\frac{\alpha}{\sqrt{ab}} + \frac{\lambda}{\sqrt{(a + t)(b + t)}} - 1 = 0, \tag{3.16}$$

$$2n + \alpha + \lambda - \frac{\lambda t}{\sqrt{(a + t)(b + t)}} - \frac{a + b}{2} = 0. \tag{3.17}$$

Let $Y(t) := (a + b)/2$ and $Y(0) = 2n + \alpha + \lambda$. Solving for $1/\sqrt{ab}$ from the sum of (3.17) and (3.16) multiplied by t, we get

$$\frac{1}{\sqrt{ab}} = \frac{Y(t) - (2n + \alpha + \lambda - t)}{t\alpha}. \tag{3.18}$$

Substituting $1/\sqrt{ab}$ into (3.16) and using the expression for $Y(t)$, we obtain

$$\frac{Y(t) - (2n + \alpha + \lambda)}{t} + \lambda/\sqrt{t^2 + 2tY(t) + \left(\frac{t\alpha}{Y(t) - (2n + \alpha + \lambda - t)}\right)^2} = 0.$$

Eliminating the square root, we get

$$\begin{aligned}(Y(t) - \alpha - \lambda - 2n)^2 &\left[\left(t^2 + 2tY(t)\right)(Y(t) - \alpha - \lambda - 2n + t)^2 + \alpha^2 t^2\right] \\ &= \lambda^2 t^2 (Y(t) - \alpha - \lambda - 2n + t)^2.\end{aligned} \tag{3.19}$$

For small α and $n \to \infty$, we consider the following equation:

$$t(-\lambda - 2n + t + \widetilde{Y}(t))^2\left((t + 2\widetilde{Y}(t))(\lambda + 2n - \widetilde{Y}(t))^2 - \lambda^2 t\right) = 0,$$

which is solved by

$$\widetilde{Y}_{1,2}(t) = 2n + \lambda - t,$$
$$\widetilde{Y}_{3,4}(t) = 2n + \lambda + \frac{\lambda}{2}\left(\frac{t}{n}\right)^{1/2} - \frac{\sqrt{t}(2\lambda + t)\lambda}{16}\left(\frac{1}{n}\right)^{3/2} - \frac{\lambda^2 t}{16n^2} + \mathcal{O}\left(\frac{1}{n^{5/2}}\right),$$
$$\widetilde{Y}_5(t) = -\frac{t}{2} + \frac{\lambda^2 t}{8n^2} + \mathcal{O}\left(\frac{1}{n^{5/2}}\right).$$

Assuming that $Y(t)$ has the form

$$Y(t) = \sum_{j=0}^{\infty} a_j(t) n^{1-j/2}, \quad n \to \infty,$$

and substituting the expression above into (3.19), we obtain $Y(t)$, which is the same as $\alpha_n(t)$ in (3.14). Then, combining with (3.18), we obtain

$$\beta_n(t) \sim \frac{((a+b)/2)^2 - ab}{4} = \frac{1}{4}\left(Y(t)^2 - \left(\frac{\alpha t}{Y(t) - (\alpha + \lambda + 2n - t)}\right)^2\right),$$

which gives (3.15).

Comparing (3.13) and (3.11) with (3.14) and (3.15), respectively, we see that the ladder operator approach and the Coulomb fluid approximation method yield the same asymptotic expressions for $\alpha_n(t)$ and $\beta_n(t)$ when n is large. □

Next, we consider Eq. (3.1) with coefficients depending on $R_n(t)$ and its derivative, without any reference to orthogonal polynomials.

Proposition 3.5. If $R_n(t)$ satisfies the Riccati equation

$$tR_n'(t) = tR_n(t)^2 - (\alpha + \lambda + t)R_n(t) + \lambda$$

with the solution

$$R_n(t) = \frac{\alpha C_2 U(\alpha + 1, \alpha + \lambda + 1, t) + L_{-\alpha-1}^{\alpha+\lambda}(t)}{C_2 U(\alpha, \alpha + \lambda, t) + L_{-\alpha}^{\alpha+\lambda-1}(t)} + 1, \tag{3.20}$$

then $\widetilde{P}_n(u) := P_n(-tu)$ satisfy the equation for the derivative of the confluent Heun function (1.5) with parameters

$$\widetilde{\gamma} = \alpha,\ \widetilde{\delta} = \lambda,\ \widetilde{\epsilon} = t,\ \widetilde{q} = -(n+1)t(1 - R_n(t)),\ \widetilde{\alpha} = -(n+1)t.$$

Proof. Let $z = -tu$, then $\widetilde{P}_n(u) := P_n(-tu)$, which gives

$$\widetilde{P}_n''(u) + \left(\frac{\alpha+1}{u} + \frac{\lambda+1}{u-1} + t - \frac{1}{u - [1 - R_n(t)]}\right)\widetilde{P}_n'(u)$$
$$-\left(\frac{r_n(t) + R_n(t)}{u(u-1)[u - (1 - R_n(t))]} + \frac{nt(u-1) + \sum_{j=0}^{n-1} tR_j(t)}{u(u-1)}\right)\widetilde{P}_n(u) = 0.$$

This equation is the equation for the derivative of the confluent Heun function when

$$tR_n'(t) = tR_n(t)^2 - (\alpha + \lambda + t)R_n(t) + \lambda.$$

Solving this differential equation, we obtain (3.20), and in the case when the constant $C_2 = 0$, we obtain

$$R_n(t) = \frac{\lambda M(\alpha; \alpha + \lambda + 1; t)}{(\alpha + \lambda)M(\alpha; \alpha + \lambda; t)}.$$

Note that if $R_n(t)$ satisfies both the Riccati equation and (3.6), then $n = -1$. □

B. $x^\alpha e^{-x-t/x}, x \in (0, \infty), \alpha, t > 0$

The weight $w(x, \alpha, t) = x^\alpha e^{-x-t/x}$ was studied by Chen and Its[15] for finite n and Chen and Chen[9] for $n \to \infty$.
The second order linear differential equation satisfied by $P_n(z)$ (see the work of Chen and Its[15]) is given by

$$P_n''(z) + Q_n(z,t)P_n'(z) + S_n(z,t)P_n(z) = 0, \tag{3.21}$$

where

$$Q_n(z,t) = \frac{\alpha+2}{z} + \frac{t}{z^2} - \frac{1}{z+R_n(t)} - 1,$$
$$S_n(z,t) = \sum_{j=0}^{n-1}\frac{R_j(t)}{z^2} + \frac{n\big((z-1)R_n(t)+z^2\big)-r_n(t)}{z^2(R_n(t)+z)}.$$

Auxiliary quantities $R_n(t)$ and $r_n(t)$ satisfy

$$r_n(t) = \frac{t + tR_n'(t) - (2n+1+\alpha+R_n(t))R_n(t)}{2}, \tag{3.22}$$

$$\sum_{j=0}^{n-1} R_j(t) = -n(n+\alpha) - r_n(t) + \beta_n(t), \tag{3.23}$$

where

$$\beta_n(t) = \frac{1}{R_n(t)}\left[nt - (2n+\alpha)r_n(t) - \frac{r_n^2(t) - tr_n(t)}{R_n(t)}\right]. \tag{3.24}$$

To obtain the asymptotic expression of $R_n(t)$, the method of double scaling will be used. Let $n\to\infty, t\to 0^+$, and let $s=(2n+\alpha+1)t$ be fixed. It should be pointed out that the asymptotic expression of $\mathbb{R}_n(s)$ was given in Ref. 9; see the following proposition. For convenience of the reader, we use the hollow symbol to define a new function of s, that is, $\mathbb{R}_n(s) = R_n(s/(2n+\alpha+1))$.

Proposition 3.6 (Ref. 9). *Let $n\to\infty$, $t\to 0^+$, and $s=(2n+\alpha+1)t$ be fixed.*
Case I: for large s, we have

$$\mathbb{R}_n(s) = \frac{1}{2n+\alpha+1}\left(s^{\frac{2}{3}} - \frac{\alpha}{3}s^{\frac{1}{3}} + \frac{\alpha(\alpha^2-1)}{81}s^{-\frac{1}{3}} + \frac{\alpha^2(\alpha^2-1)}{243}s^{-\frac{2}{3}}\right) + \mathcal{O}\big(s^{-2}\big). \tag{3.25}$$

Case II: for small s, we have

$$\mathbb{R}_n(s) = \frac{1}{2n+\alpha+1}\left(\frac{s}{\alpha} - \frac{s^2}{\alpha^2(\alpha^2-1)} + \frac{3s^3}{\alpha^3(\alpha^2-1)(\alpha^2-4)}\right) + \mathcal{O}\big(s^4\big), \tag{3.26}$$

where $\alpha \neq \mathbb{Z}$.

Proof. Define

$$\widetilde{R}(s) = \lim_{n\to\infty}\frac{R_n(s/(2n+\alpha+1))}{s/(2n+\alpha+1)}.$$

Then, $\widetilde{R}(s)$ satisfies

$$\widetilde{R}''(s) = \frac{\widetilde{R}'(s)^2}{\widetilde{R}(s)} - \frac{\widetilde{R}'(s)}{s} + \frac{\widetilde{R}^2(s)}{s} + \frac{\alpha}{s^2} - \frac{1}{s^2\widetilde{R}(s)}$$

with the initial conditions $\widetilde{R}(0) = 1/\alpha$, $\widetilde{R}'(0) = -1/(\alpha^2(\alpha^2-1))$.

For large s, we have

$$\widetilde{R}(s) = s^{-\frac{1}{3}} - \frac{\alpha}{3}s^{-\frac{2}{3}} + \frac{\alpha(\alpha^2-1)}{81}s^{-\frac{4}{3}} + \frac{\alpha^2(\alpha^2-1)}{243}s^{-\frac{5}{3}} + \mathcal{O}\big(s^{-2}\big).$$

For small s, we have

$$\widetilde{R}(s) = \frac{1}{\alpha} - \frac{s}{\alpha^2(\alpha^2-1)} + \frac{3s^2}{\alpha^3(\alpha^2-1)(\alpha^2-4)} - \frac{6(2\alpha^2-3)s^3}{\alpha^4(\alpha^2-1)^2(\alpha^2-4)(\alpha^2-9)} + \mathcal{O}\big(s^4\big);$$

see the work of Chen and Chen[9] for details.

From

$$\widetilde{R}(s) = \lim_{n\to\infty} \frac{\mathbb{R}_n(s)}{s/(2n+\alpha+1)},$$

we deduce

$$\mathbb{R}_n(s) = \frac{s}{2n+\alpha+1}\widetilde{R}(s), \quad n \to \infty.$$

This gives (3.25) and (3.26). □

The coefficients of (3.21) are given in terms of $R_n(t)$. Next, we show that (3.21) reduces to the double confluent Heun equations, both for small and for large s.

Theorem 3.7. *Let $n \to \infty$, $t \to 0^+$, and $s = (2n+\alpha+1)t$ be fixed. Equation (3.21) reduces to the double confluent Heun equation*

$$P_n''(z) + \left(\frac{\widetilde{\gamma}}{z^2} + \frac{\widetilde{\delta}}{z} + \widetilde{\epsilon}\right)P_n'(z) + \left(\frac{\widetilde{\alpha}z - \widetilde{q}}{z^2}\right)P_n(z) = 0 \tag{3.27}$$

with the following parameters.

Case I: for large s,

$$\widetilde{\gamma} = \frac{3s + 3s^{2/3} - \alpha s^{1/3}}{6n}, \quad \widetilde{\delta} = \alpha + 1, \quad \widetilde{\epsilon} = -1,$$
$$\widetilde{\alpha} = n, \quad \widetilde{q} = \frac{-6\alpha^2 - 27s^{2/3} + 18\alpha s^{1/3} + 1}{36}.$$

Case II: for small s,

$$\widetilde{\gamma} = \frac{s(\alpha+1)}{2n\alpha}, \quad \widetilde{\delta} = \alpha + 1, \quad \widetilde{\epsilon} = -1, \quad \widetilde{\alpha} = n, \quad \widetilde{q} = -\frac{s}{2\alpha}.$$

Proof. Substituting (3.22)–(3.24) into (3.21), the coefficients of (3.21) can be expressed in terms of $R_n(t)$ and $R_n'(t)$. Setting $s = (2n+\alpha+1)t$ and taking $n \to \infty$, we substitute (3.25) and (3.26) and obtain the results. □

The following asymptotic expansions hold.

Remark. Let $n \to \infty$, $t \to 0^+$, and $s = (2n+\alpha+1)t$ be fixed.

Case I: for large s, we have

$$r_n(s) = \frac{s}{4n} + \frac{1}{6}\left(\frac{1}{n} - 3\right)s^{2/3} + \frac{\alpha}{6}\left(1 - \frac{1}{6n}\right)s^{1/3} + \mathcal{O}\left(s^{-1/3}\right), \tag{3.28}$$

$$\beta_n(s) = n^2 + \alpha n + \frac{1}{12}\left(3 - \frac{1}{n}\right)s^{2/3} - \frac{\alpha}{18}\left(6 - \frac{1}{n}\right)s^{1/3} + \frac{6\alpha^2 - 1}{36} + \mathcal{O}\left(s^{-1/3}\right), \tag{3.29}$$

$$\sum_{j=0}^{n-1}\mathbb{R}_j(s) = -\frac{s}{4n} + \frac{1}{4}\left(3 - \frac{1}{n}\right)s^{2/3} - \frac{\alpha}{12}\left(6 - \frac{1}{n}\right)s^{1/3} + \frac{6\alpha^2 - 1}{36} + \mathcal{O}\left(s^{-1/3}\right). \tag{3.30}$$

Case II: for small s, we have

$$r_n(s) = -\frac{s(2n-\alpha-1)}{4\alpha n} + \frac{(n-1)s^2}{2\alpha^2(\alpha^2-1)n} - \frac{3(2n-3)s^3}{4\alpha^3(\alpha^2-4)(\alpha^2-1)n} + \mathcal{O}\left(s^4\right), \tag{3.31}$$

$$\beta_n(s) = n(n+\alpha) + \frac{(n-1)s^2}{4\alpha^2(\alpha^2-1)n} - \frac{(2n-3)s^3}{2\alpha^3(\alpha^2-4)(\alpha^2-1)n} + \mathcal{O}(s^3), \tag{3.32}$$

$$\sum_{j=0}^{n-1}\mathbb{R}_j(s) = \frac{s(2n-\alpha-1)}{4\alpha n} - \frac{(n-1)s^2}{4\alpha^2(\alpha^2-1)n} - \frac{(2n-3)s^3}{4\alpha^3(\alpha^2-4)(\alpha^2-1)n} + \mathcal{O}(s^4). \tag{3.33}$$

Proof. Since (3.22)–(3.24) are expressed in terms of $R_n(t)$ and $R_n'(t)$, setting $s = (2n+\alpha+1)t$ and combining with Proposition 3.6, we obtain the results as s goes to ∞ and 0^+, respectively. □

Corollary 3.8. When $t = 0$, the weight $x^\alpha e^{-x-t/x}$ reduces to the classical Laguerre weight $x^\alpha e^{-x}$. Equation (3.27) for small s reduces to the Laguerre differential equation

$$P_n''(z) + \left(\frac{\alpha+1}{z} - 1\right)P_n'(z) + \frac{n}{z}P_n(z) = 0. \tag{3.34}$$

Proof. See the Proof of Corollary 3.3. □

IV. WEIGHTS WITH A GAP

In this section, we consider weights with a gap. The weight $e^{-x^2}(1-\chi_{(-a,a)}(x))$, $x \in \mathbb{R}$, $a > 0$ was studied in Ref. 39).

It was shown that the Gaussian gap probabilities may be determined as the product of the smallest distributions of the Laguerre unitary ensemble with some special parameters.

It was shown that auxiliary quantities satisfy certain second order differential equations.

Also the connection to the Jimbo-Miwa-Okamoto σ-form of the fifth Painlevé equation was obtained.

For the weight $w(x) = (A + B\theta(x-t))x^\alpha e^{-x}$ (see Ref. 38), the largest eigenvalue distribution with finite n and large n was studied. Moreover, the asymptotic solution after soft edge scaling was derived and the second order differential equations for auxiliary quantities related to recurrence coefficients were obtained. The connection to the second Painlevé equation, the σ-form, and a particular case of Chazy's equation were also shown.

A. $e^{-x^2}(1-\chi_{(-a,a)}(x))$, $x \in \mathbb{R}$, $a > 0$

In Refs. 8 and 39, the second order differential equation for polynomials $P_n(x)$ orthogonal with respect to the weight $e^{-x^2}(1-\chi_{(-a,a)}(x))$, $x \in \mathbb{R}$, $a > 0$ was obtained. It is of the following form:

$$P_n''(z) + Q_n(z,a)P_n'(z) + S_n(z,a)P_n(z) = 0, \tag{4.1}$$

where

$$Q_n(z,a) = \frac{2aR_n(a)z}{(z^2-a^2)[2(z^2-a^2)+aR_n(a)]} - 2z,$$
$$S_n(z,a) = -\frac{r_n(a)\left(2\left(a^2+z^2\right) - aR_n(a)\right)}{(a-z)(a+z)(2a^2 - aR_n(a) - 2z^2)} + 2n + \frac{a\sum_{j=0}^{n-1}R_j(a)}{z^2-a^2},$$

with $\sum_{j=0}^{n-1}R_j(a)$ and $r_n(a)$ satisfying

$$r_n(a) = \frac{[R_n'(a) - R_n(a)^2 + 2aR_n(a)]a}{4a - 2R_n(a)}, \tag{4.2}$$

$$\sum_{j=0}^{n-1}R_j(a) = \frac{2r_n(a)^2}{R_n(a)} + (n + r_n(a))R_n(a) - (2a + \frac{r_n(a)}{a})r_n(a). \tag{4.3}$$

As usually, we will use the asymptotic expressions for auxiliary quantities to reduce the second order differential equation to a simpler form, which turns out to be one of the Heun equations. We consider the case $a > 0$ not tending to 0 so that the parameter a appears in the denominator of the asymptotic expansions.

Theorem 4.1. *Let $n \to \infty$ and $a > 0$. The polynomials $P_n(z)$ orthogonal with respect to the weight $\mathrm{e}^{-x^2}(1-\chi_{(-a,a)}(x))$, $x \in \mathbb{R}$, $a > 0$ satisfy the confluent Heun equation*

$$\widehat{P}_n''(u) + \left(\frac{1}{u-1} - \frac{1}{2u} - t\right)\widehat{P}_n'(u) + \frac{2ntu + \sqrt{2nt}}{4u(u-1)}\widehat{P}_n(u) = 0. \tag{4.4}$$

Here, $z = a\sqrt{u}$, $t = a^2$, and $\widehat{P}_n(u) := P_n(a\sqrt{u})$.

Proof. As shown in Ref. 39, the auxiliary quantity $R_n(a)$ satisfies the following second order differential equation:

$$R_n'' = \frac{R_n - a}{R_n - 2a}\frac{(R_n')^2}{R_n} - \frac{R_n R_n'}{a(R_n - 2a)} + \frac{R_n}{a}(R_n - 2a)(aR_n - a^2 + 2n + 1). \tag{4.5}$$

Disregarding the derivative parts of the equation above, we obtain

$$\widetilde{R}_n(a)^2(\widetilde{R}_n(a) - 2a)^2\left(-a^2 + a\widetilde{R}_n(a) + 2n + 1\right) = 0,$$

which is solved by

$$\widetilde{R}_{n1,2}(a) = 2a, \quad \widetilde{R}_{n3}(a) = \frac{a^2 - 2n - 1}{a}.$$

Assuming that $R_n(a)$ has the form

$$R_n(a) = \sum_{j=0}^{\infty} b_j(a) n^{-j/2}, \quad n \to \infty,$$

substituting the series above into (4.5), with $R_n(a) \geq 0$ and sending $n \to \infty$, we obtain

$$R_n(a) = 2a + \frac{1}{\sqrt{2n}} - \frac{4a^4 + 4a^2 - 1}{16a^2\sqrt{2}n^{3/2}} - \frac{4a^4 + 1}{32a^3 n^2} + \mathcal{O}\left(\frac{1}{n^{5/2}}\right). \tag{4.6}$$

Plugging (4.2) and (4.3) into (4.1), sending $n \to \infty$ and combining with (4.6), we obtain

$$P_n''(z) + \left(\frac{2a^2}{z(z^2 - a^2)} - 2z\right)P_n'(z) + \frac{2nz^2 + \sqrt{2n}a}{z^2 - a^2}P_n(z) = 0, \tag{4.7}$$

which can be reduced to a confluent Heun equation.

Let

$$z = a\sqrt{u}, \quad t = a^2.$$

Then, $\widehat{P}_n(u) := P_n(a\sqrt{u})$ satisfies the confluent Heun equation (4.4) with parameters

$$\widetilde{\gamma} = -1/2, \quad \widetilde{\delta} = 1, \quad \widetilde{\epsilon} = -t, \quad \widetilde{a} = nt/2, \quad q = -\sqrt{2nt}/4.$$

□

Corollary 4.2. The gap disappear when $a = 0$. The weight $w(x)$ reduces to the classical Gaussian weight e^{-x^2} for $x \in \mathbb{R}$. The orthogonal polynomials $P_n(z)$ reduce to the Hermite polynomials $H_n(z)$ and (4.7) reduces to the Hermite differential equation (http://mathworld.wolfram.com/HermiteDifferentialEquation.html),

$$P_n''(z) - 2zP_n'(z) + 2nP_n(z) = 0. \tag{4.8}$$

Proof. Let us use ladder operators. For the weight $w(x) = \mathrm{e}^{-x^2}$, we have $v(x) = x^2$ and $v'(x) = 2x$. From (1.14) and (1.15), we have

$$A_n(z) = \frac{1}{h_n}\int_0^{\infty} 2P_n^2(y)\mathrm{e}^{-y^2}\,\mathrm{d}y = 2,$$
$$B_n(z) = \frac{1}{h_n}\int_0^{\infty} 2P_n(y)P_{n-1}(y)\mathrm{e}^{-y^2}\,\mathrm{d}y = 0.$$

Recalling (1.16), we obtain

$$-(v'(z)+\frac{A_n'(z)}{A_n(z)})=-2z,$$
$$B_n'(z)-B_n(z)\frac{A_n'(z)}{A_n(z)}+\sum_{j=0}^{n-1}A_j(z)=2n,$$

which produces (4.8). □

Remark. The auxiliary quantities $\sum_{j=0}^{n-1}R_j(a)$ *and* $r_n(a)$ *have the following expansions when n is large and* $a>0$:

$$r_n(a)=-\sqrt{2n}a+a^2+\frac{1-4a^4}{8\sqrt{2n}a}+\mathcal{O}\Big(\frac{1}{n}\Big),$$
$$\sum_{j=0}^{n-1}R_j(a)=2an+\frac{1}{4a}-\frac{1}{2\sqrt{2n}}+\frac{4a^4+1}{32a^3n}+\mathcal{O}\Big(\frac{1}{n^{3/2}}\Big).$$

B. $(1-x^2)^\alpha(1-\chi_{(-a,a)}(x))$, $x\in[-1,1]$, $a\in(0,1)$, $\alpha>0$

From the definition of $R_n(a)$ and $r_n(a)$ (see the work of Min and Chen[43]), we have

$$P_n''(z)+Q_n(z,a)P_n'(z)+S_n(z,a)P_n(z)=0, \tag{4.9}$$

where

$$Q_n(z,a)=\frac{2(\alpha+1)z}{z^2-1}+\frac{2z}{z^2-a^2}-\frac{2z(2\alpha+2n+1)}{(a-a^3)R_n(a)-(a^2-z^2)(2\alpha+2n+1)},$$
$$\begin{aligned}S_n(z,a)=&\frac{(1-a^2)r_n(a)\big((a^2+z^2)(2\alpha+2n+1)+a(a^2-1)R_n(a)\big)}{(z^2-1)(a^2-z^2)\big((a^2-z^2)(2\alpha+2n+1)+a(a^2-1)R_n(a)\big)}\\&-\frac{n\Big(\big(a^2-z^2\big)^2(2\alpha+2n+1)+a\big(a^2-1\big)R_n(a)\big(a^2-3z^2\big)\Big)}{(z^2-1)(z^2-a^2)\big((a-a^3)R_n(a)+(z^2-a^2)(2\alpha+2n+1)\big)}\\&+\frac{(n^2+2\alpha n)(a^2-z^2)-a(1-a^2)\sum_{j=0}^{n-1}R_j(a)}{(z^2-a^2)(z^2-1)},\end{aligned}$$

with $r_n(a)$, $\sum_{j=0}^{n-1}R_j(a)$, and $\beta_n(a)$ given by

$$r_n(a)=\frac{a\big(-\big(a^2-1\big)R_n'(a)+\big(a^2-1\big)R_n(a)^2+2a(\alpha+n)R_n(a)\big)}{2\big((a^2-1)R_n(a)+a(2\alpha+2n+1)\big)}, \tag{4.10}$$

$$\sum_{j=0}^{n-1}R_j(a)=\frac{2\big(a^2-1\big)(\alpha+n)r_n(a)+\big(4(\alpha+n)^2-1\big)\beta_n(a)-n(2\alpha+n)}{a(a^2-1)}, \tag{4.11}$$

and

$$\beta_n(a)=\frac{1}{2n+2\alpha-1}\bigg[\frac{(r_n(a)+n)(r_n(a)+2\alpha+n)}{aR_n(a)+2\alpha+2n+1}-\frac{ar_n(a)^2}{R_n(a)}\bigg]. \tag{4.12}$$

Hence, the coefficients of (4.9) depend only on $R_n(a)$ and $R_n'(a)$.

Proposition 4.3. For large n and $a>0$, *we have*

$$\begin{aligned}R_n(a)=&\frac{2an}{1-a^2}+\frac{2\alpha a+a+1}{1-a^2}+\frac{4\alpha^2a^2-a^2+1}{8a^2n^2}\\&-\frac{a(2\alpha+1)\big[(2\alpha-1)a^2(a+2\alpha+1)+1\big]+1}{8a^3n^3}+\mathcal{O}\Big(\frac{1}{n^4}\Big),\quad n\to\infty.\end{aligned} \tag{4.13}$$

Proof. The second order differential equation for $R_n(a)$ can be obtained from the following system (see Ref. 43):

$$R_n'(a) = R_n^2(a) + \frac{2a^2(n+\alpha) - 2(\alpha^2-1)r_n(a)}{a(a^2-1)}R_n(a) - \frac{2(2n+2\alpha+1)}{a^2-1}r_n(a),$$

$$r_n'(a) = \frac{\left[(1-a^2)r_n^2(a) + 2(n+\alpha)r_n(a) + n^2 + 2n\alpha\right]R_n(a) - a(2n+2\alpha+1)r_n^2(a)}{(2n+2\alpha-1)R_n(a)(aR_n(a)+2n+2\alpha+1)} - \frac{2(n+\alpha)R_n(a)r_n(a) + n(n+2\alpha)R_n(a)}{(1-a^2)(aR_n(a)+2n+2\alpha+1)}.$$

If we consider the nonderivative part and terms with n^4, n^3, and n^2, then we have

$$4n^2\widetilde{R}_n(a)^2\left(a^2\left(4\left(6a^2-5\right)\alpha^2 + 8\left(a^2-1\right)\alpha(2n+3) + \left(a^2-1\right)(4n(n+2)+5)\right)\right.$$
$$\left. + \widetilde{R}_n(a)\left(2\left(2a^5-3a^3+a\right)(6\alpha+2n+3) + \left(6a^6-12a^4+7a^2-1\right)\widetilde{R}_n(a)\right)\right) = 0,$$

with the solution

$$\widetilde{R}_{n1,2}(a) = -\frac{2an}{a\left(2a \pm \sqrt{2-2a^2}\right)-1} + \frac{\left(\sqrt{2} \pm 6a\sqrt{1-a^2}\left(1-2a^2\right)\right)(2\alpha+1)}{2\sqrt{1-a^2}(6a^4-6a^2+1)} \mp \frac{2a^2\left(a^2-4\alpha^2-1\right)+(2\alpha+1)^2}{8\sqrt{2}a^2(1-a^2)^{3/2}n} + \mathcal{O}\left(\frac{1}{n^2}\right),\ n\to\infty.$$

Assuming that $R_n(a)$ is of the following form:

$$R_n(a) = \sum_{j=0}^{\infty} b_j(a)n^{1-j}, \quad n\to\infty,$$

and substituting the expression above into the second order differential equation for $R_n(a)$, we obtain (4.13) as $n\to\infty$. □

Theorem 4.4. *Let $n\to\infty$. The differential equation for polynomials $P_n(x)$ orthogonal with respect to $(1-x^2)^\alpha(1-\chi(-a,a)(x))$ over $[-1,1]$ reduces to the general Heun equation*

$$\widehat{P}_n''(u) + \left(-\frac{1}{2u} + \frac{\alpha+1}{u-1} + \frac{1}{u-t}\right)\widehat{P}_n'(u) - \frac{u(2\alpha+n+1)n + n\sqrt{t}}{4u(u-1)(u-t)}\widehat{P}_n(u) = 0, \tag{4.14}$$

where $\widehat{P}_n(u) := P_n(\sqrt{u})$.

Proof. Substituting (4.10)–(4.12) into (4.9) and sending $n\to\infty$, we plug in the asymptotic expression for $R_n(a)$ (4.13) and find that the polynomials $P_n(z)$ satisfy

$$P_n''(z) + \left(\frac{2z}{z^2-a^2} + \frac{2(\alpha+1)z}{z^2-1} - \frac{2}{z}\right)P_n'(z) - \frac{na + z^2(2\alpha+n+1)n}{(z^2-1)(z^2-a^2)}P_n(z) = 0. \tag{4.15}$$

Let

$$z = \sqrt{u}, \quad a^2 = t.$$

Then, $\widehat{P}_n(u) := P_n(\sqrt{u})$ satisfy the general Heun equation (4.14) with parameters

$$\widetilde{\gamma} = -1/2, \quad \widetilde{\delta} = \alpha+1, \quad \widetilde{\epsilon} = 1, \quad \widetilde{\alpha}\widetilde{\beta} = -n(n+2\alpha+1)/4, \quad \widetilde{q} = \sqrt{t}n/4.$$

□

Corollary 4.5. Equation (4.15) reduces to the Jacobi differential equation when $a = 0$,

$$P_n''(z) + \frac{2(\alpha+1)z}{z^2-1}P_n'(z) - \frac{(2\alpha+n+1)n}{z^2-1}P_n(z) = 0. \tag{4.16}$$

Proof. The weight $(1-x^2)^\alpha$ is the classical Jacobi weight in case $\beta = \alpha$. See the Proof of Corollary 2.3 for $x^\alpha(1-x)^\beta$ and the work of Chen and Ismail[14] for $(1-x)^\alpha(1+x)^\beta$. □

Remark. When n is large and $a > 0$, *we have*

$$\begin{aligned}
r_n(a) &= -\frac{an}{a+1} - \frac{a\alpha}{a+1} + \frac{4\alpha^2a^2 - a^2 + 1}{8an} + \mathcal{O}\left(\frac{1}{n^2}\right),\\
\beta_n(a) &= \frac{1}{4}(a-1)^2 - \frac{(a-1)^2(a+1)\left(a\left(4\alpha^2-1\right)-1\right)}{16an^2} + \mathcal{O}\left(\frac{1}{n^3}\right),\\
\sum_{j=0}^{n-1} R_j(a) &= \frac{an^2}{1-a^2} + \frac{2na\alpha}{1-a^2} + \frac{1-a}{4a(a+1)} + \frac{(a-1)\left(a^3\left(4\alpha^2-1\right)-1\right)}{16a^3n^2} + \mathcal{O}\left(\frac{1}{n^3}\right).
\end{aligned}$$

C. $x^\alpha e^{-x}(A + B\theta(x - t))$, $x \in [0, \infty)$, α, $t > 0$, $A \geq 0$, $A + B \geq 0$

For the weight $w(x) = (A + B\theta(x-t))x^\alpha e^{-x}$, the second order differential equation for $P_n(z,t)$ reads

$$P_n''(z) + Q_n(z,t)P_n'(z) + S_n(z,t)P_n(z) = 0, \tag{4.17}$$

where

$$\begin{aligned}
Q_n(z,t) &= \frac{\alpha+1}{z} + \frac{1}{z-t} - \frac{1}{z-t+tR_n(t)} - 1,\\
S_n(z,t) &= \frac{n}{z} - \frac{t\left[r_n(t) + nR_n(t)\right]}{z(z-t)(tR_n(t)-t+z)} + \frac{t}{z(z-t)}\sum_{j=0}^{n-1}R_j(t).
\end{aligned}$$

The auxiliary quantities satisfy $\sum_{j=0}^{n-1}R_j(t)$, and $r_n(t)$ satisfy

$$r_n(t) = \frac{tR_n'(t) - (2n+\alpha-t+tR_n(t))R_n(t)}{2}, \tag{4.18}$$

$$\sum_{j=0}^{n-1}R_j(t) = \frac{\beta_n(t) - tr_n(t) - n(n+\alpha)}{t}, \tag{4.19}$$

and

$$\beta_n(t) = \frac{1}{1-R_n(t)}\left[(2n+\alpha)r_n(t) + n(n+\alpha) + \frac{r_n(t)^2}{R_n(t)}\right]. \tag{4.20}$$

Here,

$$\begin{aligned}
R_n(t) &:= B\frac{P_n(t,t)^2t^\alpha e^{-t}}{h_n(t)},\\
r_n(t) &:= B\frac{P_n(t,t)P_{n-1}(t,t)t^\alpha e^{-t}}{h_n(t)},
\end{aligned}$$

and $R_n(t)$ satisfies

$$\begin{aligned}
R_n'' &= \frac{1}{2}\left(\frac{1}{R_n-1} + \frac{1}{R_n}\right)R_n'^2 - \frac{R_n'}{t} - \frac{\alpha^2}{2t^2}\frac{R_n}{R_n-1}\\
&\quad + (2n+\alpha+1)\frac{R_n(R_n-1)}{t} + \frac{R_n(R_n-1)(2R_n-1)}{2}.
\end{aligned} \tag{4.21}$$

See the work of Basor and Chen.[2]

In this paper, we consider two cases, the work of Lyu and Chen[39] with $A = 0$, $B = 1$ and Lyu and Chen[38] with $A = 1, B = -1$.
The case $A = 0$, $B = 1$.
To study the large n behavior of $R_n(t)$, we first recall some results from Ref .39.

Proposition 4.6 (Ref. 39). *The function*

$$R(s) := \lim_{n\to\infty} R_n\left(\frac{s}{4n}\right)$$

satisfies the following second order differential equation:

$$R''(s) = \left(\frac{1}{R(s)-1} + \frac{1}{R(s)}\right)\frac{R'(s)^2}{2} - \frac{R'(s)}{s} + \frac{R(s)(R(s)-1)}{2s} - \frac{\alpha^2 R(s)}{2s^2(R(s)-1)},$$

and it has the following expansion:

$$\begin{aligned} R(s) = 1 - \alpha s^{-\frac{1}{2}} - \frac{\alpha}{8}s^{-\frac{3}{2}} - \frac{\alpha^2}{4}s^{-2} \\ - \left(\frac{3\alpha^3}{8} + \frac{27\alpha}{128}\right)s^{-\frac{5}{2}} + \mathcal{O}\left(s^{-3}\right), \quad s\to\infty. \end{aligned} \tag{4.22}$$

Theorem 4.7. *Let $n\to\infty$ and $s = 4nt$ be fixed. Then, for large s, the polynomials $P_n(x)$ orthogonal with respect to $\theta(x-t)x^\alpha e^{-x}$ over $[0,\infty)$ satisfy the double confluent Heun equation*

$$P_n''(z) + \left(\frac{s-\alpha\sqrt{s}}{4nz^2} + \frac{\alpha+1}{z} - 1\right)P_n'(z) + \frac{4nz + (\sqrt{s}-\alpha)^2}{4z^2}P_n(z) = 0. \tag{4.23}$$

Proof. Substituting (4.18)–(4.20) into (4.17), sending $n\to\infty$ and combining with (4.22), for large s, we have

$$\begin{aligned} \mathbb{Q}_n(z,s) &= \frac{s-\alpha\sqrt{s}}{4nz^2} + \frac{\alpha+1}{z} - 1 + \mathcal{O}\left(s^{-1/2}\right), \quad s\to\infty, \\ \mathbb{S}_n(z,s) &= \frac{4nz + (\sqrt{s}-\alpha)^2}{4z^2} + \mathcal{O}\left(s^{-1/2}\right), \quad s\to\infty. \end{aligned}$$

Then, (4.17) is a double confluent Heun equation with parameters

$$\widetilde{\gamma} = \frac{s-\alpha\sqrt{s}}{4n}, \quad \widetilde{\delta} = \alpha+1, \quad \widetilde{\epsilon} = -1, \quad \widetilde{a} = n, \quad \widetilde{q} = -\frac{(\sqrt{s}-\alpha)^2}{4}.$$

□

Remark. For large s, we have

$$r_n(s/(4n)) = -n - \frac{\alpha}{2} + \frac{2\alpha^2 + 4n\alpha + \alpha}{4\sqrt{s}} + \frac{\alpha(2\alpha+4n+3)}{32s^{3/2}} + \frac{\alpha^2(\alpha+2n+2)}{8s^2} + \mathcal{O}\left(s^{-5/2}\right), \tag{4.24}$$

$$\beta_n(s/(4n)) = n(n+\alpha) + \frac{\alpha^2}{4} - \frac{\alpha\sqrt{s}}{4} + \frac{3\alpha}{32\sqrt{s}} + \frac{\alpha^2}{8s} + \frac{5\alpha(16\alpha^2+9)}{512s^{3/2}} + \mathcal{O}\left(s^{-3/2}\right), \tag{4.25}$$

$$\sum_{j=0}^{n-1}\mathbb{R}_j(s/(4n)) = n + \frac{\alpha}{2} - \frac{\alpha(4n+\alpha)}{2\sqrt{s}} + \frac{\alpha^2 n}{s} + \frac{\alpha(4n-\alpha)}{16s^{3/2}} + \frac{\alpha^2(2n-\alpha)}{8s^2} + \mathcal{O}\left(s^{-5/2}\right). \tag{4.26}$$

The case $A = 1$, $B = -1$.

Proposition 4.8. As $n\to\infty$, the quantity $R_n(t)$ has the following asymptotic expression:

$$R_n(t) = -\frac{2n}{t} + \frac{t-2(\alpha+1)}{2t} + \frac{\alpha^2}{8n^2} + \frac{-4\alpha^3 - 2\alpha^2(t+2) + t}{32n^3} + \mathcal{O}\left(n^{-4}\right). \tag{4.27}$$

Proof. Neglecting the derivative terms in (4.21) and replacing $R_n(t)$ by $\widetilde{R}_n(t)$, we obtain

$$2t^2\widetilde{R}_n(t)^3 + t(2\alpha + 4n - 5t + 2)\widetilde{R}_n(t)^2 - 4t(2n + \alpha + 1 - t)\widetilde{R}_n(t) \\ - \alpha^2 + 2t(\alpha + 2n + 1) - t^2 = 0.$$

The solution to the equation above when $n \to \infty$ is

$$\widetilde{R}_{n1}(t) = -\frac{2n}{t} + \frac{t - 2(\alpha + 1)}{2t} + \frac{\alpha^2}{8n^2} - \frac{\alpha^2(2\alpha + t + 2)}{16n^3} + \mathcal{O}\left(n^{-4}\right),$$
$$\widetilde{R}_{n2}(t) = 1 \pm \frac{\alpha}{2\sqrt{nt}} \mp \frac{\alpha(2\alpha + t + 2)}{16\sqrt{t}n^{3/2}} - \frac{\alpha^2}{16n^2} + \mathcal{O}\left(n^{-5/2}\right).$$

Since $R_n(t) = -P_n(t,t)^2 t^\alpha e^{-t}/h_n(t) < 0$, we assume that $R_n(t)$ has the following expression:

$$R_n(t) = \sum_{j=0}^{\infty} a_j n^{1-j}, \quad n \to \infty.$$

Substituting the expression above into (4.21), we obtain (4.27). □

Theorem 4.9. *Sending n to infinity, the polynomials* $\widetilde{P}_n(u) := P_n(tu)$ *satisfy the confluent Heun equation*

$$\widetilde{P}_n''(u) + \left(\frac{\alpha + 1}{u} + \frac{1}{u - 1} - t\right)\widetilde{P}_n'(u) + \frac{ntu - n(n + \alpha + 1 + t/2)}{u(u - 1)}\widetilde{P}_n(u) = 0. \tag{4.28}$$

Here, $P_n(x)$ *are orthogonal with respect to* $(1 - \theta(x - t))x^\alpha e^{-x}$ *over* $[0, \infty)$.

Proof. Substituting (4.18)–(4.20) into (4.17), sending $n \to \infty$, combining with (4.27) and setting $\widetilde{P}_n(u) = P_n(tu)$, we obtain the confluent Heun equation with parameters

$$\widetilde{\gamma} = \alpha + 1, \quad \widetilde{\delta} = 1, \quad \widetilde{\epsilon} = -t, \quad \widetilde{a} = nt, \quad \widetilde{q} = n(n + \alpha + 1 + t/2).$$

□

Remark. For large n, we have

$$r_n(t) = -\frac{n}{2} + \frac{t - 2\alpha}{8} + \frac{\alpha^2}{8n} + \frac{t - 2\alpha^2(\alpha + t)}{32n^2} + \mathcal{O}\left(n^{-3}\right),$$
$$\beta_n(t) = \frac{t^2}{16} + \frac{t^2(1 - 2\alpha^2)}{64n^2} + \mathcal{O}\left(n^{-3}\right),$$
$$\sum_{j=0}^{n-1}\mathbb{R}_j(t) = -\frac{n^2}{t} + n\left(\frac{1}{2} - \frac{\alpha}{t}\right) + \frac{\alpha}{4} - \frac{t}{16} - \frac{\alpha^2}{8n} + \frac{4\alpha^3 + 2\alpha^2 t - t}{64n^2} + \mathcal{O}\left(n^{-3}\right).$$

V. CONCLUSION

In this paper, we considered eight kinds of weight functions for monic orthogonal polynomials $P_n(x)$. These polynomials satisfy linear second order differential equations, and we showed that they reduce to Heun equations as $n \to \infty$. In this way, we obtained six confluent Heun equations, three double confluent Heun equations, and a general Heun equation.

Remark. For the deformed Freud weight, we will obtain the biconfluent Heun equation

$$\frac{d^2u}{dz^2} + \left(\frac{\gamma}{z} + \delta + \epsilon z\right)\frac{du}{dz} + \left(\frac{\alpha z - q}{z}\right)u = 0. \tag{5.1}$$

See the work of Clarkson and Jordaan[18] for the deformed Freud weight,

$$|x|^{2\lambda+1}e^{-x^4+tx^2}, \lambda > -1, x \in \mathbb{R}.$$

The biconfluent Heun equation was obtained in Ref. 18, p. 165 with parameters

$$\gamma = 1 + \lambda, \quad \delta = \frac{\sqrt{2}t}{2}, \quad \epsilon = -1, \quad \alpha = 0, \quad q = -\frac{\sqrt{6}n^{3/2}}{9}.$$

Also see the work of Zhu and Chen[56] *for*

$$|x|^{\alpha}\mathrm{e}^{-N[x^2+s(x^4-x^2)]}, \quad x \in \mathbb{R},$$

where the biconfluent Heun equation [Ref. 56, Eq. (6.22)] was obtained with parameters

$$\gamma = -\frac{\alpha+1}{2}, \quad \delta = \frac{\sqrt{2}N(1-s)}{2}, \quad \epsilon = 1, \quad \alpha = 0, \quad q = -\frac{\sqrt{6}k^{3/2}}{9}.$$

At present, we do not know the examples of weights that would lead to the triconfluent Heun equation.

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APPENDIX: INTEGRAL IDENTITIES

Using the Coulomb fluid method requires numerous integral formulas. Here, we list some integrals used in the main text, which can be found in Ref. 17. For the case of $0 < a < b$, we have

$$\int_a^b \frac{\mathrm{d}x}{\sqrt{(b-x)(x-a)}}\pi, \tag{A1}$$

$$\int_a^b \frac{x\mathrm{d}x}{\sqrt{(b-x)(x-a)}}\frac{a+b}{2}\pi, \tag{A2}$$

$$\int_a^b \frac{\mathrm{d}x}{x\sqrt{(b-x)(x-a)}}\frac{\pi}{\sqrt{ab}}, \tag{A3}$$

$$\int_a^b \frac{\mathrm{d}x}{x^2\sqrt{(b-x)(x-a)}}x\frac{a+b}{2(ab)^{3/2}}\pi, \tag{A4}$$

$$\int_a^b \frac{\mathrm{d}x}{(x+t)\sqrt{(b-x)(x-a)}}\frac{\pi}{\sqrt{(b+t)(t+a)}}. \tag{A5}$$

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