Asymptotics of the Recurrence Coefficient of a Pollaczek-Jacobi Type Orthogonal Polynomials and the Associated Hankel Determinant

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Abstract

In this paper, we study the orthogonal polynomials with respect to a singularly perturbed Pollaczek-Jacobi type weight

$$w(x,t) := (1-x^2)^{\alpha} \mathrm{e}^{-\frac{t}{1-x^2}}, \qquad x \in [-1,1], \;\; \alpha > 0, \;\; t > 0.$$

By using the ladder operator approach, we establish the second-order difference equations satisfied by the recurrence coefficient $\beta_n(t)$ and the sub-leading coefficient $\mathrm{p}(n,t)$ of the monic orthogonal polynomials, respectively. We show that the logarithmic derivative of $\beta_n(t)$ can be expressed in terms of a particular Painlevé V transcendent. The large n asymptotic expansions of $\beta_n(t)$ and $\mathrm{p}(n,t)$ are obtained by using Dyson's Coulomb fluid method together with the related difference equations.

Furthermore, we study the associated Hankel determinant $D_n(t)$ and show that a quantity $\sigma_n(t)$, allied to the logarithmic derivative of $D_n(t)$, can be expressed in terms of the σ-function of a particular Painlevé V. The second-order differential and difference equations for $\sigma_n(t)$ are also obtained. In the end, we derive the large n asymptotics of $\sigma_n(t)$ and $D_n(t)$ from the relations with $\beta_n(t)$ and $\mathrm{p}(n,t)$.

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1 Introduction

Orthogonal polynomials are of great importance in mathematical physics, random matrix theory, approximation theory, etc. For orthogonal polynomials with weight $w(x)$ supported on $[-1,1]$ and satisfying the Szegö condition

$$\int_{-1}^{1} \frac{\ln w(x)}{\sqrt{1-x^2}} dx > -\infty, \tag{1.1}$$

the classical theory of Szegö [46] gives a comprehensive description of the large n behavior of the recurrence coefficients and the polynomials.

A weight satisfying the condition (1.1) is often said to be of the Szegö class. The Jacobi weight, $w(x) = (1-x)^{\alpha}(1+x)^{\beta},\ x \in [-1,1],\ \alpha,\ \beta > -1$, is a typical example of the Szegö class. Kuijlaars et al. [36] considered a modified Jacobi weight

$$w(x) = (1-x)^{\alpha}(1+x)^{\beta}h(x), \qquad x \in [-1,1], \quad \alpha,\ \beta > -1,$$

where $h(x)$ is real analytic and strictly positive on $[-1,1]$. They obtained the large n asymptotics of the orthogonal polynomials, the recurrence coefficients and the associated Hankel determinant by using the steepest descent analysis for Riemann-Hilbert problems.

Zeng, Xu and Zhao [53] studied the asymptotic behavior of the leading coefficients and the recurrence coefficients of the orthonormal polynomials and the Hankel determinant associated with the perturbed Jacobi weight

$$w(x) = (1-x^2)^{\beta}(t^2-x^2)^{\alpha}h(x), \qquad x \in [-1,1], \quad \beta > -1,\ \alpha+\beta > -1,\ t > 1,$$

in the sense of a double scaling limit. Here the function $h(x)$ satisfies the same condition as above.

However, there are some weights that violate the Szegö condition. For example, Chen and Dai [14] considered a Pollaczek-Jacobi type weight

$$w(x) = x^{\alpha}(1-x)^{\beta}\mathrm{e}^{-\frac{t}{x}}, \qquad x \in [0,1], \quad \alpha,\ \beta > 0,\ t \geq 0, \tag{1.2}$$

and showed that the logarithmic derivative of the Hankel determinant satisfies the Jimbo-Miwa-Okamoto σ-form of a particular Painlevé V. Later, Chen et al. [13] studied the asymptotic behavior of the orthogonal polynomials, the recurrence coefficients and associated Hankel determinant under a suitable double scaling.

Very recently, by using the Riemann-Hilbert approach, Wang and Fan [49] studied the large n asymptotics of the monic orthogonal polynomials with respect to another singularly perturbed Pollaczek-Jacobi type weight

$$w(x) = x^{\alpha}(1-x)^{\beta}\mathrm{e}^{-\frac{t}{x(1-x)}}, \qquad x \in [0,1], \quad \alpha,\ \beta > 0,\ t \geq 0.$$

Compared to the weight (1.2), this weight has one more singularity at the edge. But they did not consider the asymptotics of the recurrence coefficients and the Hankel determinant.

The orthogonal polynomials and Hankel determinants with the singularly perturbed weights have attracted a lot of interests over the past few years, due to the applications in random matrix theory; see [1, 5, 23, 45, 51, 52] for reference. This is because the Hankel determinants are closely related to the partition functions of the unitary ensembles. The asymptotics of the partition functions usually can be expressed in terms of a particular solution of the Painlevé equations. In addition, the weights with jump discontinuities and Fisher-Hartwig singularities have also been studied in recent years; cf., e.g., [8, 9, 10, 41, 42, 43, 50]. See also [26] on the asymptotics of Toeplitz, Hankel, and Toeplitz+Hankel determinants with Fisher-Hartwig singularities.

In this paper, we consider the following symmetric Pollaczek-Jacobi type weight with two singularities at the edge, namely,

$$w(x,t) := (1-x^2)^{\alpha}\mathrm{e}^{-\frac{t}{1-x^2}}, \qquad x \in [-1,1], \quad \alpha > 0, \quad t > 0. \tag{1.3}$$

It is easy to see that this weight vanishes infinitely fast at $x = \pm 1$.

Our main purpose is to obtain the large n asymptotic expansions of the recurrence coefficients and the sub-leading coefficients of the monic orthogonal polynomials, and the associated Hankel determinant. We also would like to establish the relation of our problem with the Painlevé equations.

Let $P_n(x,t),\ n = 0,1,2,\ldots,$ be the monic polynomials of degree n orthogonal with respect to the weight (1.3), i.e.,

$$\int_{-1}^{1} P_m(x,t)P_n(x,t)w(x,t)dx = h_n(t)\delta_{mn}, \qquad m,n = 0,1,2,\ldots. \tag{1.4}$$

Since the weight $w(x,t)$ is even, we have $P_n(x,t) = P_n(-x,t)$; see [18, Theorem 4.3]. Then $P_n(x,t)$ has the following monomial expansion,

$$P_n(x,t) = x^n + \mathrm{p}(n,t)x^{n-2} + \cdots, \qquad n = 0,1,2,\ldots, \tag{1.5}$$

Here $\mathrm{p}(n,t)$ denotes the coefficient of x^{n-2}, and we will see that it plays a significant role in our problem. We set the initial values of $\mathrm{p}(n,t)$ to be $\mathrm{p}(0,t) = 0,\ \mathrm{p}(1,t) = 0$.

The orthogonal polynomials $P_n(x,t),\ n = 0,1,2,\ldots$, satisfy the following three-term recurrence relation [18, Theorem 4.1 and 4.3]

$$xP_n(x,t) = P_{n+1}(x,t) + \beta_n(t)P_{n-1}(x,t), \tag{1.6}$$

supplemented by the initial conditions

$$P_0(x,t) = 1, \quad P_{-1}(x,t) = 0.$$

From (1.6) we know that the monic orthogonal polynomials are completely determined by the recurrence coefficient $\beta_n(t)$.

The combination of (1.4), (1.5) and (1.6) show that the recurrence coefficient $\beta_n(t)$ has two alternative representations:

$$\beta_n(t) = \mathrm{p}(n,t) - \mathrm{p}(n+1,t) \tag{1.7}$$

$$= \frac{h_n(t)}{h_{n-1}(t)}. \tag{1.8}$$

A telescopic sum of (1.7) produces an important identity

$$\sum_{j=0}^{n-1} \beta_j(t) = -\mathrm{p}(n,t). \tag{1.9}$$

In addition, from (1.7) we have $\beta_0(t) = 0$.

We introduce the Hankel determinant generated by the weight (1.3),

$$D_n(t) := \det\left(\int_{-1}^{1} x^{i+j} w(x,t)dx\right)_{i,j=0}^{n-1}.$$

It is well known that $D_n(t)$ can be expressed as the product of $h_j(t)$ (see [34, (2.1.6)]),

$$D_n(t) = \prod_{j=0}^{n-1} h_j(t), \tag{1.10}$$

where $h_j(t)$ is defined from the orthogonality (1.4).

Furthermore, the Hankel determinants play an important role in random matrix theory [40]. Our Hankel determinant $D_n(t)$ can be viewed as the partition function of the singularly perturbed Jacobi unitary ensemble [34, Corollary 2.1.3], i.e.,

$$D_n(t) = \frac{1}{n!} \int_{[-1,1]^n} \prod_{1 \le i < j \le n} (x_i - x_j)^2 \prod_{k=1}^{n} w(x_k, t) dx_k.$$

Here $x_1, x_2, \ldots, x_n$, are the eigenvalues of $n \times n$ Hermitian matrices from the ensemble, and the joint probability density function reads,

$$p(x_1, x_2, \ldots, x_n) \prod_{k=1}^{n} dx_k = \frac{1}{n!\, D_n(t)} \prod_{1 \le i < j \le n} (x_i - x_j)^2 \prod_{k=1}^{n} w(x_k, t) dx_k.$$

We will show that $\sigma_n(t)$, a quantity allied to the logarithmic derivative of $D_n(t)$ and defined in (4.1), can be expressed in terms of the σ-function of a Painlevé V. In addition, from (1.8) we have the following relation:

$$\beta_n(t) = \frac{D_{n+1}(t) D_{n-1}(t)}{D_n^2(t)}. \tag{1.11}$$

To achieve our main target on the asymptotics of $\beta_n(t)$, $\mathrm{p}(n,t)$, $\sigma_n(t)$ and $D_n(t)$, first, we derive the second-order difference equations satisfied by β_n and $\mathrm{p}(n,t)$ respectively by utilizing the ladder operator approach. Then, we make use of the Coulomb fluid method to obtain the form of the large n asymptotic expansion of $\beta_n(t)$ with the known leading term. The combination gives a full asymptotic expansion of $\beta_n(t)$. The asymptotics of $\mathrm{p}(n,t)$ follows from the corresponding difference equation and the important relation (1.7). We also find the asymptotics of $\sigma_n(t)$ from the fact that it can be expressed in terms of $\mathrm{p}(n,t)$. Finally, we derive the asymptotics of the Hankel determinant by connecting it with the free energy and taking advantage of formula (1.11).

The rest of the paper is arranged as follows. In sec. 2, we apply the ladder operators to our Pollaczek-Jacobi type weight and obtain two auxiliary quantities $R_n(t)$ and $r_n(t)$. From the compatibility conditions (S_1), (S_2) and (S_2'), we obtain some important identities. Then, we derive the second-order difference equations satisfied by $\beta_n(t)$ and $\mathrm{p}(n,t)$, respectively. Finally, we show the second-order differential equation for the monic orthogonal polynomials $P_n(x,t)$. In Sec. 3, we prove that the auxiliary quantities $R_n(t)$ and $r_n(t)$ satisfy the coupled Riccati equations, from which we obtain the second-order differential equations for $R_n(t)$ and $r_n(t)$, respectively. We find that $R_n(t)$ is intimately related to a particular Painlevé V transcendent. In Sec. 4, we derive the

second-order differential and difference equations satisfied by $\sigma_n(t)$. We also show that this quantity can be expressed in terms of the σ-function of a particular Painlevé V. In Sec. 5, we study the large n asymptotic expansions of the recurrence coefficient $\beta_n(t)$, the sub-leading coefficient $\mathrm{p}(n,t)$, the log-derivative of the Hankel determinant $\sigma_n(t)$ and the Hankel determinant $D_n(t)$.

2 Ladder Operators and Second-Order Difference Equations

The ladder operator approach has been applied to solve many problems on orthogonal polynomials and Hankel determinants; see, e.g., [3, 14, 16, 21, 24, 29, 42, 43]. Following Chen and Its [16], we have the lowering and raising operators for our Pollaczek-Jacobi type orthogonal polynomials:

$$\left(\frac{d}{dz}+B_n(z)\right)P_n(z)=\beta_nA_n(z)P_{n-1}(z), \tag{2.1}$$

$$\left(\frac{d}{dz}-B_n(z)-\mathrm{v}'(z)\right)P_{n-1}(z)=-A_{n-1}(z)P_n(z), \tag{2.2}$$

where $\mathrm{v}(z):=-\ln w(z)$ and

$$A_n(z):=\frac{1}{h_n}\int_{-1}^{1}\frac{\mathrm{v}'(z)-\mathrm{v}'(y)}{z-y}P_n^2(y)w(y)dy, \tag{2.3}$$

$$B_n(z):=\frac{1}{h_{n-1}}\int_{-1}^{1}\frac{\mathrm{v}'(z)-\mathrm{v}'(y)}{z-y}P_n(y)P_{n-1}(y)w(y)dy. \tag{2.4}$$

Note that we often suppress the t-dependence of $P_n(x)$, $w(x)$, β_n and h_n for brevity. We believe that this will not lead to any confusion.

The functions $A_n(z)$ and $B_n(z)$ satisfy the following compatibility conditions valid for $z\in\mathbb{C}\cup\{\infty\}$:

$$B_{n+1}(z)+B_n(z)=zA_n(z)-\mathrm{v}'(z), \tag{S_1}$$

$$1+z(B_{n+1}(z)-B_n(z))=\beta_{n+1}A_{n+1}(z)-\beta_nA_{n-1}(z), \tag{S_2}$$

$$B_n^2(z)+\mathrm{v}'(z)B_n(z)+\sum_{j=0}^{n-1}A_j(z)=\beta_nA_n(z)A_{n-1}(z). \tag{S_2'}$$

In addition, eliminating $P_{n-1}(z)$ from (2.1) and (2.2) show that $P_n(z)$ satisfies the second-order linear ordinary differential equation

$$P_n''(z)-\left(\mathrm{v}'(z)+\frac{A_n'(z)}{A_n(z)}\right)P_n'(z)+\left(B_n'(z)-B_n(z)\frac{A_n'(z)}{A_n(z)}+\sum_{j=0}^{n-1}A_j(z)\right)P_n(z)=0, \tag{2.5}$$

where use has been made of (S_2').

For the weight function given in (1.3), we find

$$\mathrm{v}(z) = -\ln w(z) = \frac{t}{1-z^2} - \alpha \ln(1-z^2).$$

Hence,

$$\mathrm{v}'(z) = \frac{2\alpha z}{1-z^2} + \frac{2tz}{(1-z^2)^2}$$

and

$$\frac{\mathrm{v}'(z)-\mathrm{v}'(y)}{z-y} = \frac{2\alpha(1+zy)}{(1-z^2)(1-y^2)} + \frac{2t\left[1-(z^2-2)zy-z^2y^2-zy^3\right]}{(1-z^2)^2(1-y^2)^2}. \tag{2.6}$$

Since the right hand side of (2.6) is rational, $A_n(z)$ and $B_n(z)$ should be also rational from their definitions in (2.3) and (2.4).

Proposition 2.1. *We have*

$$A_n(z) = \frac{2n+1+2\alpha}{1-z^2} + \frac{R_n(t)}{(1-z^2)^2}, \tag{2.7}$$

$$B_n(z) = \frac{nz}{1-z^2} + \frac{z\, r_n(t)}{(1-z^2)^2}, \tag{2.8}$$

where

$$R_n(t) := \frac{2t}{h_n}\int_{-1}^{1} \frac{1}{1-y^2} P_n^2(y) w(y) dy, \tag{2.9}$$

$$r_n(t) := \frac{2t}{h_{n-1}}\int_{-1}^{1} \frac{y}{1-y^2} P_n(y) P_{n-1}(y) w(y) dy. \tag{2.10}$$

Proof. From the definition of $A_n(z)$ in (2.3), and taking account of the parity, we have

$$\begin{aligned} A_n(z) &= \frac{2\alpha}{(1-z^2)h_n}\int_{-1}^{1}\frac{1}{1-y^2}P_n^2(y)w(y)dy + \frac{2t}{(1-z^2)^2 h_n}\int_{-1}^{1}\frac{1}{(1-y^2)^2}P_n^2(y)w(y)dy \\ &- \frac{2tz^2}{(1-z^2)^2 h_n}\int_{-1}^{1}\frac{y^2}{(1-y^2)^2}P_n^2(y)w(y)dy \\ &= \left[\frac{2(\alpha-t)}{(1-z^2)h_n} + \frac{2t}{(1-z^2)^2 h_n}\right]\int_{-1}^{1}\frac{1}{1-y^2}P_n^2(y)w(y)dy \\ &+ \frac{2t}{(1-z^2)h_n}\int_{-1}^{1}\frac{1}{(1-y^2)^2}P_n^2(y)w(y)dy. \end{aligned} \tag{2.11}$$

We will show that the above two integrals have a simple relation. Since

$$\begin{aligned}
&\frac{2t}{h_n}\int_{-1}^{1}\frac{1}{(1-y^2)^2}P_n^2(y)w(y)dy-\frac{2t}{h_n}\int_{-1}^{1}\frac{1}{1-y^2}P_n^2(y)w(y)dy\\
=\ &\frac{2t}{h_n}\int_{-1}^{1}\frac{y^2}{(1-y^2)^2}P_n^2(y)w(y)dy=-\frac{1}{h_n}\int_{-1}^{1}yP_n^2(y)(1-y^2)^{\alpha}d\mathrm{e}^{-\frac{t}{1-y^2}}\\
=\ &\frac{1}{h_n}\int_{-1}^{1}\mathrm{e}^{-\frac{t}{1-y^2}}\ \frac{d}{dy}\left[yP_n^2(y)(1-y^2)^{\alpha}\right]dy\\
=\ &2n+1+2\alpha-\frac{2\alpha}{h_n}\int_{-1}^{1}\frac{1}{1-y^2}P_n^2(y)w(y)dy,
\end{aligned}$$

we get

$$\frac{2t}{h_n}\int_{-1}^{1}\frac{1}{(1-y^2)^2}P_n^2(y)w(y)dy=2n+1+2\alpha+\frac{2(t-\alpha)}{h_n}\int_{-1}^{1}\frac{1}{1-y^2}P_n^2(y)w(y)dy.$$

Substituting it into (2.11), we obtain (2.7).

Similarly, from the definition of $B_n(z)$ in (2.4), we have

$$\begin{aligned}
B_n(z)\ =\ &\frac{2\alpha z}{(1-z^2)h_{n-1}}\int_{-1}^{1}\frac{y}{1-y^2}P_n(y)P_{n-1}(y)w(y)dy\\
+\ &\frac{2tz(2-z^2)}{(1-z^2)^2h_{n-1}}\int_{-1}^{1}\frac{y}{(1-y^2)^2}P_n(y)P_{n-1}(y)w(y)dy\\
-\ &\frac{2tz}{(1-z^2)^2h_{n-1}}\int_{-1}^{1}\frac{y^3}{(1-y^2)^2}P_n(y)P_{n-1}(y)w(y)dy\\
=\ &\left[\frac{2\alpha z}{(1-z^2)h_{n-1}}+\frac{2tz}{(1-z^2)^2h_{n-1}}\right]\int_{-1}^{1}\frac{y}{1-y^2}P_n(y)P_{n-1}(y)w(y)dy\\
+\ &\frac{2tz}{(1-z^2)h_{n-1}}\int_{-1}^{1}\frac{y}{(1-y^2)^2}P_n(y)P_{n-1}(y)w(y)dy. \qquad (2.12)
\end{aligned}$$

Through integration by parts, we find

$$\frac{2t}{h_{n-1}}\int_{-1}^{1}\frac{y}{(1-y^2)^2}P_n(y)P_{n-1}(y)w(y)dy=n-\frac{2\alpha}{h_{n-1}}\int_{-1}^{1}\frac{y}{1-y^2}P_n(y)P_{n-1}(y)w(y)dy.$$

Inserting it into (2.12), we arrive at (2.8). This completes the proof. □

Substituting (2.7) and (2.8) into (S_1), we obtain

$$r_{n+1}(t)+r_n(t)=R_n(t)-2t. \qquad (2.13)$$

From (S_2) we have the following two equations:

$$r_{n+1}(t)-r_n(t)=\beta_{n+1}R_{n+1}(t)-\beta_nR_{n-1}(t), \qquad (2.14)$$

$$r_{n+1}(t) - r_n(t) - 1 = (2n - 1 + 2\alpha)\beta_n - (2n + 3 + 2\alpha)\beta_{n+1}. \tag{2.15}$$

We write (2.15) in another form

$$r_{n+1}(t) - r_n(t) - 1 = (2n - 1 + 2\alpha)\beta_n - (2n + 1 + 2\alpha)\beta_{n+1} - 2\beta_{n+1}.$$

Replacing n by j in the above and taking a telescopic sum from $j = 0$ to $j = n - 1$ produces an important identity

$$r_n(t) = n - (2n + 1 + 2\alpha)\beta_n + 2\mathrm{p}(n, t), \tag{2.16}$$

where we have used (1.9) and the initial conditions $\beta_0 = 0, r_0(t) = 0$.

Finally, from (S_2'), we obtain the following three equations:

$$r_n^2(t) + 2tr_n(t) = \beta_n R_n(t) R_{n-1}(t), \tag{2.17}$$

$$r_n^2(t) + 2(t - n - \alpha)r_n(t) - 2nt + (2n + 1 + 2\alpha)\beta_n R_{n-1}(t) + (2n - 1 + 2\alpha)\beta_n R_n(t) = 0, \tag{2.18}$$

$$n(n + 2\alpha - 2t) - 2(n + \alpha)r_n(t) + \sum_{j=0}^{n-1} R_j(t) = (2n + 1 + 2\alpha)(2n - 1 + 2\alpha)\beta_n. \tag{2.19}$$

The combination of (2.17) and (2.18) gives the expression of β_n in terms of $r_n(t)$ and $R_n(t)$:

$$\beta_n = \frac{2nt + 2(n + \alpha - t)r_n(t) - r_n^2(t)}{(2n - 1 + 2\alpha)R_n(t)} - \frac{(2n + 1 + 2\alpha)(r_n^2(t) + 2t\, r_n(t))}{(2n - 1 + 2\alpha)R_n^2(t)}. \tag{2.20}$$

Theorem 2.2. *The recurrence coefficient $\beta_n(t)$ satisfies the following second-order nonlinear difference equation:*

$$\begin{aligned}
& \Big\{ \big[68(n+\alpha)^2 - 9\big]\beta_n^3 + \big[12 - 80(n+\alpha)^2 + (2n - 3 + 2\alpha)(14n + 5 + 14\alpha)\beta_{n-1} \\
+ \;& (2n + 3 + 2\alpha)(14n - 5 + 14\alpha)\beta_{n+1}\big]\beta_n^2 + \big[24(n+\alpha)^2 + 4t(t - 2\alpha) - 3 \\
- \;& 2(2n + 1 + 2\alpha)(2n - 3 + 2\alpha)\beta_{n-1} - 2(2n - 1 + 2\alpha)(2n + 3 + 2\alpha)\beta_{n+1} \\
+ \;& (2n - 3 + 2\alpha)(2n + 3 + 2\alpha)\beta_{n-1}\beta_{n+1}\big]\beta_n - 2\left[(n+\alpha)^2 - t\alpha\right] \Big\}^2 \\
= \;& 4\Big\{2(2n - 1 + 2\alpha)(2n + 1 + 2\alpha)\beta_n^2 + \big[(2n + 1 + 2\alpha)(2n - 3 + 2\alpha)\beta_{n-1} \\
+ \;& (2n - 1 + 2\alpha)(2n + 3 + 2\alpha)\beta_{n+1} - 2(2n - 1 + 2\alpha)(2n + 1 + 2\alpha)\big]\beta_n \\
+ \;& (n+\alpha)^2 + t(t - 2\alpha)\Big\}\Big\{12(n+\alpha)\beta_n^2 + \big[(2n - 3 + 2\alpha)\beta_{n-1} + (2n + 3 + 2\alpha)\beta_{n+1} \\
- \;& 8(n+\alpha)\big]\beta_n + n + \alpha\Big\}^2.
\end{aligned} \tag{2.21}$$

Proof. From (2.16), we have

$$r_{n+1}(t) = n + 1 - (2n + 3 + 2\alpha)\beta_{n+1} + 2\mathrm{p}(n + 1, t), \tag{2.22}$$

$$r_{n-1}(t) = n - 1 - (2n - 1 + 2\alpha)\beta_{n-1} + 2\mathrm{p}(n - 1, t). \tag{2.23}$$

Using (1.7), it follows that

$$\mathrm{p}(n + 1, t) = \mathrm{p}(n, t) - \beta_n, \tag{2.24}$$

$$\mathrm{p}(n - 1, t) = \mathrm{p}(n, t) + \beta_{n-1}. \tag{2.25}$$

Substituting (2.24) into (2.22) and (2.25) into (2.23) respectively, we get

$$r_{n+1}(t) = n + 1 - (2n + 3 + 2\alpha)\beta_{n+1} - 2\beta_n + 2\mathrm{p}(n, t), \tag{2.26}$$

$$r_{n-1}(t) = n - 1 - (2n - 3 + 2\alpha)\beta_{n-1} + 2\mathrm{p}(n, t). \tag{2.27}$$

In view of (2.13), we have

$$R_n(t) = 2t + r_{n+1}(t) + r_n(t), \tag{2.28}$$

$$R_{n-1}(t) = 2t + r_n(t) + r_{n-1}(t). \tag{2.29}$$

Inserting (2.26) into (2.28) and (2.27) into (2.29) respectively, we find

$$R_n(t) = n + 1 + 2t + r_n(t) + 2\mathrm{p}(n, t) - (2n + 3 + 2\alpha)\beta_{n+1} - 2\beta_n, \tag{2.30}$$

$$R_{n-1}(t) = n - 1 + 2t + r_n(t) + 2\mathrm{p}(n, t) - (2n - 3 + 2\alpha)\beta_{n-1}. \tag{2.31}$$

Substituting (2.30) and (2.31) into (2.17) and (2.18), we obtain

$$\begin{aligned} r_n^2(t) + 2t\, r_n(t) &= \beta_n \left[n + 1 + 2t + r_n(t) + 2\mathrm{p}(n, t) - (2n + 3 + 2\alpha)\beta_{n+1} - 2\beta_n\right] \\ &\times \left[n - 1 + 2t + r_n(t) + 2\mathrm{p}(n, t) - (2n - 3 + 2\alpha)\beta_{n-1}\right], \end{aligned} \tag{2.32}$$

$$\begin{aligned} & r_n^2(t) + 2(t - n - \alpha)r_n(t) - 2nt + (2n + 1 + 2\alpha)\beta_n\big[n - 1 + 2t + r_n(t) + 2\mathrm{p}(n, t) \\ - \; & (2n - 3 + 2\alpha)\beta_{n-1}\big] + (2n - 1 + 2\alpha)\beta_n\big[n + 1 + 2t + r_n(t) + 2\mathrm{p}(n, t) \\ - \; & (2n + 3 + 2\alpha)\beta_{n+1} - 2\beta_n\big] = 0. \end{aligned} \tag{2.33}$$

Equations (2.16), (2.32) and (2.33) can be regarded as a system of nonlinear equations satisfied by β_n, $r_n(t)$ and $\mathrm{p}(n,t)$. Now we are ready to derive the second-order difference equation for β_n from this system. We begin with expressing $\mathrm{p}(n,t)$ in terms of β_n and $r_n(t)$ from (2.16):

$$2\mathrm{p}(n,t) = (2n+1+2\alpha)\beta_n + r_n(t) - n.$$

Inserting it into (2.32) and (2.33) respectively, we get the following two equations:

$$\begin{aligned} & r_n^2 + 2t\, r_n - \beta_n \left[(2n+1+2\alpha)\beta_n - (2n-3+2\alpha)\beta_{n-1} + 2r_n + 2t - 1\right] \\ \times\ & \left[(2n-1+2\alpha)\beta_n - (2n+3+2\alpha)\beta_{n+1} + 2r_n + 2t + 1\right] = 0, \end{aligned} \tag{2.34}$$

$$\begin{aligned} & r_n^2 + 2r_n\left[t - (n+\alpha)(1-4\beta_n)\right] + 2\beta_n^2\left[4(n+\alpha)^2+1\right] - \beta_n\big[2 - 8t(n+\alpha) \\ +\ & (2n+1+2\alpha)(2n-3+2\alpha)\beta_{n-1} + (2n-1+2\alpha)(2n+3+2\alpha)\beta_{n+1}\big] - 2nt = 0. \end{aligned} \tag{2.35}$$

Note that equation (2.35) may be put down as a quadratic equation for $r_n(t)$. Substituting either solution into (2.34), we obtain (2.21) after clearing the square root. □

Theorem 2.3. *The sub-leading coefficient of the monic orthogonal polynomials,* $\mathrm{p}(n) := \mathrm{p}(n,t)$, *satisfies the following second-order nonlinear difference equation:*

$$\begin{aligned} & [n+2\mathrm{p}(n) - (2n+1+2\alpha)(\mathrm{p}(n)-\mathrm{p}(n+1))]^3 + [n+2\mathrm{p}(n) - (2n+1+2\alpha)(\mathrm{p}(n)-\mathrm{p}(n+1))]^2 \\ \times\ & [n-2+4t-(2n-3+2\alpha)(\mathrm{p}(n-1)-\mathrm{p}(n)) + (2n+1+2\alpha)(\mathrm{p}(n)-\mathrm{p}(n+1))] \\ -\ & 2\,[n+2\mathrm{p}(n)-(2n+1+2\alpha)(\mathrm{p}(n)-\mathrm{p}(n+1))]\,\big\{n^2 - n(1-\alpha) - \alpha + 2t - 2t^2 \\ -\ & (2n-3+2\alpha)(\mathrm{p}(n-1)-\mathrm{p}(n))\,[n+\alpha-t-(2n+1+2\alpha)(\mathrm{p}(n)-\mathrm{p}(n+1))] \\ -\ & (2n+1+2\alpha)(n-1+2t)(\mathrm{p}(n)-\mathrm{p}(n+1))\big\} - [n-1+2t-(2n-3+2\alpha)(\mathrm{p}(n-1)-\mathrm{p}(n))] \\ \times\ & \big\{2nt - (2n+1+2\alpha)(\mathrm{p}(n)-\mathrm{p}(n+1))\,[n-1+2t-(2n-3+2\alpha)(\mathrm{p}(n-1)-\mathrm{p}(n))]\,\big\} \\ +\ & 4(2n+1+2\alpha)(\mathrm{p}(n))^2(\mathrm{p}(n)-\mathrm{p}(n+1)) + 2\mathrm{p}(n)\big\{\,[n+2\mathrm{p}(n)-(2n+1+2\alpha)(\mathrm{p}(n)-\mathrm{p}(n+1))]^2 \\ -\ & 2\,[n+2\mathrm{p}(n)-(2n+1+2\alpha)(\mathrm{p}(n)-\mathrm{p}(n+1))]\,[n+\alpha-t-(2n+1+2\alpha)(\mathrm{p}(n)-\mathrm{p}(n+1))] \\ -\ & 2nt + 2(2n+1+2\alpha)(\mathrm{p}(n)-\mathrm{p}(n+1))\,[n-1+2t-(2n-3+2\alpha)(\mathrm{p}(n-1)-\mathrm{p}(n))]\,\big\} = 0. \end{aligned} \tag{2.36}$$

Proof. Eliminating β_{n+1} from equations (2.32) and (2.33), and then replacing $r_n(t)$ by (2.16), we get an equation for β_n, β_{n-1} and $\mathrm{p}(n,t)$. Using the following relations from (1.7) to eliminate β_n and β_{n-1},

$$\beta_n = \mathrm{p}(n,t) - \mathrm{p}(n+1,t),$$

$$\beta_{n-1} = \mathrm{p}(n-1,t) - \mathrm{p}(n,t),$$

we finally obtain the desired result. Noting that, without the first step, we would obtain the third-order difference equation for $\mathrm{p}(n,t)$. □

Remark 1. *One can also derive the second-order difference equations satisfied by the auxiliary quantities $R_n(t)$ and $r_n(t)$ from the system of algebraic equations (2.13), (2.17) and (2.18), following the procedure in [44].*

In the end of this section, we show the second-order differential equation satisfied by $P_n(z)$, with the coefficients expressed in terms of β_n and $\mathrm{p}(n,t)$.

Theorem 2.4. *The monic orthogonal polynomials $P_n(z)$, $n = 0,1,2,\ldots$, satisfy the following second-order differential equation:*

$$P_n''(z) - \left(\mathrm{v}'(z) + \frac{A_n'(z)}{A_n(z)}\right) P_n'(z) + \left(B_n'(z) - B_n(z)\frac{A_n'(z)}{A_n(z)} + \sum_{j=0}^{n-1} A_j(z)\right) P_n(z) = 0,$$

where

$$A_n(z) = \frac{2n+1+2\alpha}{1-z^2} + \frac{2n+1+2t-(2n+3+2\alpha)(\beta_n+\beta_{n+1})+4\mathrm{p}(n,t)}{(1-z^2)^2}, \tag{2.37}$$

$$B_n(z) = \frac{nz}{1-z^2} + \frac{z\left[n-(2n+1+2\alpha)\beta_n+2\mathrm{p}(n,t)\right]}{(1-z^2)^2}, \tag{2.38}$$

$$\sum_{j=0}^{n-1} A_j(z) = \frac{n^2+2n\alpha}{1-z^2} + \frac{n(n+2t)-(2n+1+2\alpha)\beta_n+4(n+\alpha)\mathrm{p}(n,t)}{(1-z^2)^2}, \tag{2.39}$$

and

$$\mathrm{v}'(z) = \frac{2\alpha z}{1-z^2} + \frac{2tz}{(1-z^2)^2}.$$

Proof. The general form of the second-order differential equation satisfied by the monic orthogonal polynomials has been given in (2.5). The remaining task is to express the coefficients of $P_n(z)$ and $P_n'(z)$ in terms of β_n and $\mathrm{p}(n,t)$.

The combination of (2.13) and (2.16) gives the expression of $R_n(t)$ in terms of β_n and $\mathrm{p}(n,t)$:

$$R_n(t) = 2n+1+2t-(2n+3+2\alpha)(\beta_n+\beta_{n+1})+4\mathrm{p}(n,t), \tag{2.40}$$

where we have used the fact $\mathrm{p}(n+1,t) = \mathrm{p}(n,t) - \beta_n$. Substituting (2.40) into (2.7) and (2.16) into (2.8) respectively, we obtain (2.37) and (2.38).

From (2.7) we have

$$\sum_{j=0}^{n-1} A_j(z) = \frac{n^2+2n\alpha}{1-z^2} + \frac{\sum_{j=0}^{n-1} R_j(t)}{(1-z^2)^2}.$$

Using (2.19) to eliminate $\sum_{j=0}^{n-1} R_j(t)$ and in view of (2.16), we obtain (2.39). This completes the proof. We mention that $\mathrm{p}(n,t)$ can also be expressed in the sum of β_j, $\mathrm{p}(n,t) = -\sum_{j=0}^{n-1} \beta_j$. $\square$

3 t Evolution and Painlevé V

Recall that our weight function depends on t. As a consequence, the recurrence coefficient, the sub-leading coefficient and the auxiliary quantities all depend on t. The objective of this section is to establish the relationships between the auxiliary quantities and the derivatives with respect to t of the key quantities β_n, $\mathrm{p}(n,t)$ and $\ln h_n$. This, in turn, will allow us to obtain the coupled Riccati equations satisfied by the auxiliary quantities $R_n(t)$ and $r_n(t)$. Based on these results, we find that $R_n(t)$, up to a simple linear transformation, is the solution of a particular Painlevé V equation.

From (1.4) we have

$$\int_{-1}^{1} P_n^2(x,t)(1-x^2)^\alpha \mathrm{e}^{-\frac{t}{1-x^2}} dx = h_n(t), \qquad n = 0,1,2,\ldots.$$

Taking a derivative with respect to t gives

$$h_n'(t) = -\int_{-1}^{1} \frac{1}{1-x^2} P_n^2(x,t) w(x) dx.$$

It follows that

$$2t\frac{d}{dt}\ln h_n(t) = -R_n(t). \tag{3.1}$$

Using (1.8), we have

$$2t\frac{d}{dt}\ln \beta_n(t) = R_{n-1}(t) - R_n(t).$$

Hence

$$2t\beta_n'(t) = \beta_n R_{n-1}(t) - \beta_n R_n(t). \tag{3.2}$$

From (1.4) we also have

$$\int_{-1}^{1} P_n(x,t)P_{n-2}(x,t)(1-x^2)^\alpha \mathrm{e}^{-\frac{t}{1-x^2}} dx = 0, \quad n = 1,2,\ldots.$$

Differentiating the above formula with respect to t, we obtain

$$\frac{d}{dt}\mathrm{p}(n,t) = \frac{1}{h_{n-2}}\int_{-1}^{1}\frac{1}{1-x^2}P_n(x,t)P_{n-2}(x,t)w(x)dx. \tag{3.3}$$

Using (1.6) and (1.8), we have

$$\frac{P_{n-2}(x,t)}{h_{n-2}} = \frac{xP_{n-1}(x,t)}{h_{n-1}} - \frac{P_n(x,t)}{h_{n-1}}. \tag{3.4}$$

Inserting (3.4) into (3.3) produces an important relation:

$$2t\frac{d}{dt}\mathrm{p}(n,t) = r_n(t) - \beta_n R_n(t). \tag{3.5}$$

Proposition 3.1. *The auxiliary quantities $r_n(t)$ and $R_n(t)$ satisfy the coupled Riccati equations:*

$$2t\, r_n'(t) = 2nt - r_n^2(t) + 2(n+\alpha+1-t)r_n(t) - \frac{2(2n+1+2\alpha)(r_n^2(t)+2t\, r_n(t))}{R_n(t)}, \tag{3.6}$$

$$2tR_n'(t) = R_n^2(t) + 2(n+\alpha+1-t)R_n(t) - 2r_n(t)(2n+1+2\alpha+R_n(t)) - 2t(2n+1+2\alpha). \tag{3.7}$$

Proof. Taking a derivative in (2.16) and exploiting (3.2) and (3.5), we have

$$2t\, r_n'(t) = 2r_n(t) - (2n+1+2\alpha)\beta_n R_{n-1}(t) + (2n-1+2\alpha)\beta_n R_n(t). \tag{3.8}$$

With the aid of (2.17), equality (3.8) becomes

$$2t\, r_n'(t) = 2r_n(t) - (2n+1+2\alpha)\frac{r_n^2(t)+2tr_n(t)}{R_n(t)} + (2n-1+2\alpha)\beta_n R_n(t). \tag{3.9}$$

Substituting (2.20) into (3.9), we obtain (3.6).

Next, the combination of (3.2) and (2.17) gives us

$$2t\beta_n'(t) = \frac{r_n^2(t)+2tr_n(t)}{R_n(t)} - \beta_n R_n(t). \tag{3.10}$$

Substituting (2.20) into (3.10), and using (3.6) to eliminate the terms involving $r_n'(t)$, we obtain

$$\begin{aligned}
& \left[r_n^2(t)(4n+4\alpha+2+R_n(t)) + 4t(2n+1+2\alpha)r_n(t) - 2(n+\alpha-t)R_n(t)r_n(t) - 2ntR_n(t)\right] \\
\times\ & \left[2tR_n'(t) - R_n^2(t) - 2(n+\alpha+1-t)R_n(t) + 2r_n(t)(2n+1+2\alpha+R_n(t)) + 2t(2n+1+2\alpha)\right] \\
=\ & 0.
\end{aligned}$$

Clearly, the above formula yields two equations. One is an algebraic equation for $R_n(t)$ and $r_n(t)$, which does not hold and should be discarded. This can be checked by taking special values. So we obtain (3.7). □

Theorem 3.2. *The auxiliary quantities $R_n(t)$ and $r_n(t)$ satisfy the following second-order nonlinear ordinary differential equations:*

$$\begin{aligned} & 8t^2R_n(2n+1+2\alpha+R_n)R_n'' - 4t^2(4n+2+4\alpha+3R_n)(R_n')^2 + 8tR_n(2n+1+2\alpha+R_n)R_n' \\ - \quad & R_n^5 - 2(2n+1+2\alpha)R_n^4 - 4\left[(n+\alpha)(n+1+\alpha) - t(t-2\alpha)\right]R_n^3 + 16t(2n+1+2\alpha)(t-\alpha)R_n^2 \\ + \quad & 4t(2n+1+2\alpha)^2(5t-2\alpha)R_n + 8t^2(2n+1+2\alpha)^3 = 0, \end{aligned} \tag{3.11}$$

$$\begin{aligned} & 4t^2r_n(2t+r_n)r_n'' - 4t^2(t+r_n)(r_n')^2 + 4t\,r_n^2\,r_n' - r_n^5 - (2n+2\alpha+5t)r_n^4 - 8t(n+\alpha+t)r_n^3 \\ - \quad & 4t\left[(t+\alpha)^2 + n(2t+\alpha) - 1\right]r_n^2 + 4n^2t^2\,r_n + 4n^2t^3 = 0. \end{aligned} \tag{3.12}$$

Let

$$W_n(t) := \frac{2n+1+2\alpha+R_n(t)}{2n+1+2\alpha}.$$

Then $W_n(t)$ satisfies the Painlevé V equation [33]

$$W_n'' = \frac{(3W_n-1)(W_n')^2}{2W_n(W_n-1)} - \frac{W_n'}{t} + \frac{(W_n-1)^2}{t^2}\left(\mu_1W_n + \frac{\mu_2}{W_n}\right) + \frac{\mu_3W_n}{t} + \frac{\mu_4W_n(W_n+1)}{W_n-1}, \tag{3.13}$$

with

$$\mu_1 = \frac{(2n+1+2\alpha)^2}{8}, \qquad \mu_2 = -\frac{1}{8}, \qquad \mu_3 = \alpha, \qquad \mu_4 = -\frac{1}{2}.$$

Proof. Solving for $r_n(t)$ from (3.7) and substituting it into (3.6), we obtain (3.11). On the other hand, solving for $R_n(t)$ from (3.6) and substituting it into (3.7), we arrive at (3.12). With the given linear transformation, equation (3.11) turns into (3.13). □

4 Logarithmic Derivative of the Hankel Determinant

In this section, we define a quantity allied to the logarithmic derivative of the Hankel determinant,

$$\sigma_n(t) := 2t\frac{d}{dt}\ln D_n(t). \tag{4.1}$$

We will show that $\sigma_n(t)$ can be expressed in terms of the σ-function of a particular Painlevé V in the Jimbo-Miwa-Okamoto σ-form. In the following, we would like to derive the second-order differential equation satisfied by $\sigma_n(t)$ at first.

It is easy to see from (1.10) and (3.1) that

$$\sigma_n(t) = -\sum_{j=0}^{n-1} R_j(t). \tag{4.2}$$

Then, we have the following equation from (2.19):

$$n(n+2\alpha-2t) - 2(n+\alpha)r_n(t) - \sigma_n(t) - (2n+1+2\alpha)(2n-1+2\alpha)\beta_n = 0. \tag{4.3}$$

So we can express β_n in terms of $r_n(t)$ and $\sigma_n(t)$,

$$\beta_n = \frac{n(n+2\alpha-2t) - 2(n+\alpha)r_n(t) - \sigma_n(t)}{(2n+1+2\alpha)(2n-1+2\alpha)}. \tag{4.4}$$

Regarding (2.18) and (3.8) as a linear system for $\beta_n R_{n-1}(t)$ and $\beta_n R_n(t)$, we obtain

$$\beta_n R_{n-1}(t) = \frac{2t(n-r_n'(t)) + 2(n+1+\alpha-t)r_n(t) - r_n^2(t)}{2(2n+1+2\alpha)}, \tag{4.5}$$

$$\beta_n R_n(t) = \frac{2t(n+r_n'(t)) + 2(n-1+\alpha-t)r_n(t) - r_n^2(t)}{2(2n-1+2\alpha)}. \tag{4.6}$$

From (2.17) we have

$$\beta_n R_{n-1}(t) \cdot \beta_n R_n(t) = \beta_n(r_n^2(t) + 2t\, r_n(t)). \tag{4.7}$$

Substituting (4.5), (4.6) and (4.4) into (4.7), we obtain an equation satisfied by $r_n(t)$, $r_n'(t)$ and $\sigma_n(t)$:

$$\begin{aligned} & 4t^2\left[n^2 - (r_n'(t))^2\right] + 8t\, r_n(t)\left(nt - n\alpha + r_n'(t) + \sigma_n(t)\right) + r_n^4(t) + 4(n+\alpha+t)r_n^3(t) \\ + \quad & 4r_n^2(t)\left[(t+\alpha)^2 + 3nt - 1 + \sigma_n(t)\right] = 0. \end{aligned} \tag{4.8}$$

On the other hand, substituting (4.5), (4.6) and (4.4) into (3.2), we find a quadratic equation satisfied by $r_n(t)$,

$$r_n^2(t) + 2(n+\alpha+t)r_n(t) + 2t(n+\sigma_n'(t)) = 0.$$

Then we have two solutions:

$$r_n(t) = -n-\alpha-t \pm \sqrt{(n+\alpha)^2 + t(t+2\alpha-2\sigma_n'(t))}.$$

Substituting either solution into (4.8) and removing the square root, we obtain the second-order differential equation satisfied by $\sigma_n(t)$:

$$
\begin{aligned}
& \Big\{ t^4(\sigma_n'')^2 + 2t^2\left[(n+\alpha)^2 + t(\alpha-\sigma_n')\right]\sigma_n'' + 2t^3(\sigma_n')^3 - t^2\left[(t+\alpha)^2 - 3n(n+2\alpha) - 1 + 4\sigma_n\right](\sigma_n')^2 \\
- \;& 2t\left[3n^4 + 12n^3\alpha + n^2(t^2+4t\alpha+15\alpha^2+2) + 2n\alpha(t^2+4t\alpha+3\alpha^2+2) + \alpha(t+2\alpha)\right]\sigma_n' \\
+ \;& 2t\left[3(n+\alpha)^2 + t(t+4\alpha)\right]\sigma_n\sigma_n' - \big[2n^4 + 8n^3\alpha + 2n^2(t^2+3t\alpha+6\alpha^2) + 4n\alpha(t+\alpha)(t+2\alpha) \\
+ \;& 2\alpha(t+\alpha)^3\big]\sigma_n + 2n^6 + 12n^5\alpha + 2(t^2+3t\alpha+14\alpha^2+1)n^4 + 8n^3\alpha(t^2+3t\alpha+4\alpha^2+1) \\
+ \;& n^2\left[2t^3\alpha + (14\alpha^2+1)t^2 + 2t\alpha(15\alpha^2+2) + 6\alpha^2(3\alpha^2+2)\right] + n\big[4t^3\alpha^2 + 2t^2\alpha(6\alpha^2+1) \\
+ \;& 4t\alpha^2(3\alpha^2+2) + 4\alpha^3(\alpha^2+2)\big] + 2\alpha^2(t+\alpha)^2\Big\}^2 = 4(n+\alpha)^2\left[(n+\alpha)^2 + t(t+2\alpha) - 2t\sigma_n'\right] \\
\times \;& \Big\{ t^2\sigma_n'' - t(2n^2+4n\alpha+1-2\sigma_n)\sigma_n' - \left[(n+\alpha)^2 + t(t+2\alpha)\right]\sigma_n + n^4 + 4n^3\alpha + \alpha(t+\alpha) \\
+ \;& n^2(t^2+2t\alpha+5\alpha^2+1) + 2n\alpha\left[(t+\alpha)^2+1\right]\Big\}^2. \qquad (4.9)
\end{aligned}
$$

Theorem 4.1. *The quantity $\sigma_n(t)$ satisfies the second-order nonlinear ordinary differential equation (4.9), and also satisfies the following second-order nonlinear difference equation,*

$$
\begin{aligned}
& \Big\{ \left[n(n+2\alpha-2t) - \sigma_n\right]\left[(2n+1+2\alpha)\sigma_{n-1} - (2n-1+2\alpha)\sigma_{n+1} - \sigma_{n-1}\sigma_{n+1}\right. \\
- \;& \left.\sigma_n^2 + \sigma_n(\sigma_{n-1}+\sigma_{n+1}-2)\right] - 2nt(2n-1+2\alpha)(2n+1+2\alpha)\Big\}^2 \\
+ \;& 4t(n+\alpha)(2n-1+2\alpha+\sigma_{n-1}-\sigma_n)(2n+1+2\alpha+\sigma_n-\sigma_{n+1}) \\
\times \;& \Big\{ \left[n(n+2\alpha-2t) - \sigma_n\right]\left[(2n+1+2\alpha)\sigma_{n-1} - (2n-1+2\alpha)\sigma_{n+1} - \sigma_{n-1}\sigma_{n+1}\right. \\
- \;& \left.\sigma_n^2 + \sigma_n(\sigma_{n-1}+\sigma_{n+1}-2)\right] - 2nt(2n-1+2\alpha)(2n+1+2\alpha)\Big\} \\
- \;& 4(n+\alpha)^2(\sigma_{n-1}-\sigma_n)(\sigma_n-\sigma_{n+1})(n^2+2n\alpha-\sigma_n)(2n-1+2\alpha+\sigma_{n-1}-\sigma_n) \\
\times \;& (2n+1+2\alpha+\sigma_n-\sigma_{n+1}) = 0. \qquad (4.10)
\end{aligned}
$$

Proof. The first part of the theorem is proved above, so we only need to derive the difference equation satisfied by $\sigma_n(t)$. From (4.2) we have

$$R_n(t) = \sigma_n(t) - \sigma_{n+1}(t).$$

Substituting it into (2.17) and (2.18) gives

$$r_n^2(t) + 2t\, r_n(t) - \beta_n(\sigma_n(t) - \sigma_{n+1}(t))(\sigma_{n-1}(t) - \sigma_n(t)) = 0, \qquad (4.11)$$

$$
\begin{aligned}
& r_n^2(t) + 2(t-n-\alpha)r_n(t) - 2nt + (2n+1+2\alpha)\beta_n(\sigma_{n-1}(t)-\sigma_n(t)) \\
+ \quad & (2n-1+2\alpha)\beta_n(\sigma_n(t)-\sigma_{n+1}(t)) = 0. \qquad (4.12)
\end{aligned}
$$

Subtracting (4.12) from (4.11) to eliminate $r_n^2(t)$, we have

$$
\begin{aligned}
& 2nt + 2(n+\alpha)r_n(t) - \beta_n(\sigma_n(t)-\sigma_{n+1}(t))(\sigma_{n-1}(t)-\sigma_n(t)) \\
- \quad & (2n+1+2\alpha)\beta_n(\sigma_{n-1}(t)-\sigma_n(t)) - (2n-1+2\alpha)\beta_n(\sigma_n(t)-\sigma_{n+1}(t)) = 0. \qquad (4.13)
\end{aligned}
$$

Solving for β_n and $r_n(t)$ from the linear system (4.3) and (4.13), we obtain

$$
\beta_n = \frac{n^2+2n\alpha-\sigma_n}{(2n-1+2\alpha+\sigma_{n-1}-\sigma_n)(2n+1+2\alpha+\sigma_n-\sigma_{n+1})}, \qquad (4.14)
$$

$$
r_n(t) = \frac{N[r_n(t)]}{D[r_n(t)]}, \qquad (4.15)
$$

where

$$
\begin{aligned}
N[r_n(t)] : \; = \; & \big[n(n+2\alpha-2t)-\sigma_n\big]\big[(2n+1+2\alpha)\sigma_{n-1}-(2n-1+2\alpha)\sigma_{n+1}-\sigma_{n-1}\sigma_{n+1} \\
- \; & \sigma_n^2+\sigma_n(\sigma_{n-1}+\sigma_{n+1}-2)\big] - 2nt(2n-1+2\alpha)(2n+1+2\alpha),
\end{aligned}
$$

$$
D[r_n(t)] := 2(n+\alpha)(2n-1+2\alpha+\sigma_{n-1}-\sigma_n)(2n+1+2\alpha+\sigma_n-\sigma_{n+1}).
$$

Substituting (4.14) and (4.15) into (4.11), we arrive at the difference equation (4.10). This completes the proof. □

From the differential equation (4.9), we can not see the relation between $\sigma_n(t)$ and the σ-function of a Painlevé equation directly, but we have the following result.

Theorem 4.2. *The quantity $\sigma_n(t)$ can be expressed in terms of the σ-function of a Painlevé V as follows:*

$$
\sigma_{2n}(t) = 2\tilde{H}_n\left(t,\alpha,\frac{1}{2}\right) + 2\tilde{H}_n\left(t,\alpha,-\frac{1}{2}\right) + 4n(n+\alpha),
$$

$$
\sigma_{2n+1}(t) = 2\tilde{H}_n\left(t,\alpha,\frac{1}{2}\right) + 2\tilde{H}_{n+1}\left(t,\alpha,-\frac{1}{2}\right) + (2n+1)(2n+1+2\alpha).
$$

Here $\tilde{H}_n(t,\alpha,\beta)$ satisfies the Jimbo-Miwa-Okamoto σ-form of Painlevé V [35, (C.45)],

$$
\big(t\tilde{H}_n''\big)^2 = \big[\tilde{H}_n - t\tilde{H}_n' + 2\big(\tilde{H}_n'\big)^2 + (\nu_0+\nu_1+\nu_2+\nu_3)\tilde{H}_n'\big]^2 - 4\big(\nu_0+\tilde{H}_n'\big)\big(\nu_1+\tilde{H}_n'\big)\big(\nu_2+\tilde{H}_n'\big)\big(\nu_3+\tilde{H}_n'\big), \quad (4.16)
$$

with parameters $\nu_0 = 0,\ \nu_1 = -(n+\alpha+\beta),\ \nu_2 = n,\ \nu_3 = -\beta$.

Proof. The main idea of the proof is to establish the relation of $D_n(t)$ and the following Hankel determinant,

$$\tilde{D}_n(t,\alpha,\beta) := \det\left(\int_0^1 x^{j+k}x^\alpha(1-x)^\beta \mathrm{e}^{-\frac{t}{x}}\right)_{j,k=0}^{n-1}. \tag{4.17}$$

The Hankel determinant (4.17) was studied by Chen and Dai [14]. Let $\tilde{P}_n(x,\alpha,\beta),\ n=0,1,2,\ldots$ be the monic polynomials of degeree n orthogonal with respect to the weight $x^\alpha(1-x)^\beta \mathrm{e}^{-\frac{t}{x}}$, i.e.,

$$\int_0^1 \tilde{P}_m(x,\alpha,\beta)\tilde{P}_n(x,\alpha,\beta)x^\alpha(1-x)^\beta \mathrm{e}^{-\frac{t}{x}}dx = \tilde{h}_n(t,\alpha,\beta)\delta_{mn}.$$

From the orthogonality (1.4), we have

$$\begin{aligned}
h_{2n}(t)\delta_{2m,2n} &= \int_{-1}^1 P_{2m}(x)P_{2n}(x)(1-x^2)^\alpha \mathrm{e}^{-\frac{t}{1-x^2}}dx \\
&= 2\int_0^1 P_{2m}(x)P_{2n}(x)(1-x^2)^\alpha \mathrm{e}^{-\frac{t}{1-x^2}}dx \\
&= \int_0^1 P_{2m}\big(\sqrt{1-y}\big)P_{2n}\big(\sqrt{1-y}\big)y^\alpha(1-y)^{-\frac{1}{2}}\mathrm{e}^{-\frac{t}{y}}dy \\
&= \int_0^1 \big[(-1)^m P_{2m}\big(\sqrt{1-y}\big)\big]\big[(-1)^n P_{2n}\big(\sqrt{1-y}\big)\big]y^\alpha(1-y)^{-\frac{1}{2}}\mathrm{e}^{-\frac{t}{y}}dy,
\end{aligned}$$

which implies $(-1)^n P_{2n}\big(\sqrt{1-y}\big),\ n=0,1,2,\ldots$, are the *monic* polynomials of degree n orthogonal with respect to the weight $y^\alpha(1-y)^{-\frac{1}{2}}\mathrm{e}^{-\frac{t}{y}}$. Hence,

$$h_{2n}(t) = \tilde{h}_n\left(t,\alpha,-\frac{1}{2}\right). \tag{4.18}$$

Similarly, we find that $(-1)^n P_{2n+1}\big(\sqrt{1-y}\big)/\sqrt{1-y},\ n=0,1,2,\ldots$, are the *monic* polynomials of degree n orthogonal with respect to the weight $y^\alpha(1-y)^{\frac{1}{2}}\mathrm{e}^{-\frac{t}{y}}$, and have

$$h_{2n+1}(t) = \tilde{h}_n\left(t,\alpha,\frac{1}{2}\right). \tag{4.19}$$

Taking account of (1.10), we find the the following relations on the Hankel determinants from (4.18) and (4.19):

$$\begin{aligned}
D_{2n}(t) &= \tilde{D}_n\left(t,\alpha,\frac{1}{2}\right)\tilde{D}_n\left(t,\alpha,-\frac{1}{2}\right), \\
D_{2n+1}(t) &= \tilde{D}_n\left(t,\alpha,\frac{1}{2}\right)\tilde{D}_{n+1}\left(t,\alpha,-\frac{1}{2}\right).
\end{aligned}$$

It follows from (4.1) that

$$\sigma_{2n}(t) = 2\tilde{H}_n\left(t,\alpha,\frac{1}{2}\right) + 2\tilde{H}_n\left(t,\alpha,-\frac{1}{2}\right) + 4n(n+\alpha),$$

$$\sigma_{2n+1}(t) = 2\tilde{H}_n\left(t, \alpha, \frac{1}{2}\right) + 2\tilde{H}_{n+1}\left(t, \alpha, -\frac{1}{2}\right) + (2n+1)(2n+2\alpha+1),$$

where

$$\tilde{H}_n(t, \alpha, \beta) := t\frac{d}{dt}\ln \tilde{D}_n(t, \alpha, \beta) - n(n+\alpha+\beta),$$

and the function $\tilde{H}_n(t, \alpha, \beta)$, as shown in [14], satisfies the Jimbo-Miwa-Okamoto σ-form of a particular Painlevé V equation (4.16). The proof is complete. □

5 Asymptotics of the Recurrence Coefficient, Sub-leading Coefficient and the Hankel Determinant

Based on Dyson's Coulomb fluid approach [27], for sufficiently large n, the eigenvalues of the $n \times n$ Hermitian matrices from a unitary ensemble with weight $w(x)$ can be approximated as a continuous fluid with a density $\sigma(x)$ supported in J (a subset of $\mathbb{R}$).

Following Chen and Ismail [15], the equilibrium density $\sigma(x)$ is obtained by minimizing the free energy functional

$$F[\sigma] := \int_J \sigma(x)\mathrm{v}(x)dx - \int_J\int_J \sigma(x)\ln|x-y|\sigma(y)dxdy \tag{5.1}$$

subject to

$$\int_J \sigma(x)dx = n. \tag{5.2}$$

Here $\mathrm{v}(x) = -\ln w(x)$ is the potential.

Upon minimization, the density $\sigma(x)$ is found to satisfy the integral equation

$$\mathrm{v}(x) - 2\int_J \ln|x-y|\sigma(y)dy = A, \qquad x \in J, \tag{5.3}$$

where A is the Lagrange multiplier for the constraint (5.2). By taking a derivative with respect to x, equation (5.3) is converted into the following singular integral equation,

$$\mathrm{v}'(x) - 2P\int_J \frac{\sigma(y)}{x-y}dy = 0, \qquad x \in J, \tag{5.4}$$

where P denotes the Cauchy principal value.

When the potential $\mathrm{v}(x)$ is convex and $\mathrm{v}''(x) > 0$ in a set of positive measure, $\sigma(x)$ is supported in a single interval (a, b) [15]. In this case, the solution of (5.4) subject to the boundary condition

$\sigma(a) = \sigma(b) = 0$ reads,

$$\sigma(x) = \frac{\sqrt{(b-x)(x-a)}}{2\pi^2} P \int_a^b \frac{\mathrm{v}'(x) - \mathrm{v}'(y)}{(x-y)\sqrt{(b-y)(y-a)}} dy \tag{5.5}$$

with two supplementary conditions

$$\int_a^b \frac{\mathrm{v}'(x)}{\sqrt{(b-x)(x-a)}} dx = 0, \tag{5.6}$$

$$\int_a^b \frac{x\,\mathrm{v}'(x)}{\sqrt{(b-x)(x-a)}} dx = 2\pi n. \tag{5.7}$$

In addition, $F[\sigma]$ and A have the following relation [15, (2.14)]

$$\frac{\partial F}{\partial n} = A. \tag{5.8}$$

For our problem,

$$w(x,t) = (1-x^2)^\alpha \mathrm{e}^{-\frac{t}{1-x^2}},$$

$$\mathrm{v}(x) = \frac{t}{1-x^2} - \alpha \ln(1-x^2), \qquad \mathrm{v}'(x) = \frac{2\alpha x}{1-x^2} + \frac{2tx}{(1-x^2)^2},$$

$$\frac{\mathrm{v}'(x) - \mathrm{v}'(y)}{x-y} = \frac{2\alpha(1+xy)}{(1-x^2)(1-y^2)} + \frac{2t\,[1-(x^2-2)xy - x^2y^2 - xy^3]}{(1-x^2)^2(1-y^2)^2}.$$

Since our potential $\mathrm{v}(x)$ is even, the support interval should be symmetric, i.e., $a = -b,\ 0 < b < 1$. From (5.5), we find after some elementary computations,

$$\sigma(x) = \frac{\sqrt{b^2-x^2}\,[2t - b^2t(1+x^2) + 2\alpha(1-b^2)(1-x^2)]}{2\pi(1-b^2)^{3/2}(1-x^2)^2}.$$

Lemma 5.1. *We have*

$$A = \frac{t}{\sqrt{1-b^2}} - 2n \ln\frac{b}{2} + 2\alpha \ln\frac{2}{1+\sqrt{1-b^2}}. \tag{5.9}$$

Proof. We start from writing (5.3) as

$$\mathrm{v}(x) - 2\int_{-b}^b \ln|x-y|\sigma(y)dy = A, \qquad x \in [-b,b].$$

Multiplying both sides by $\frac{1}{\sqrt{b^2-x^2}}$ and integrating from $-b$ to b gives rise to

$$\int_{-b}^b \frac{\mathrm{v}(x)}{\sqrt{b^2-x^2}} dx - 2\int_{-b}^b dy\,\sigma(y) \int_{-b}^b \frac{\ln|x-y|}{\sqrt{b^2-x^2}} dx = \pi A, \tag{5.10}$$

where use has been made of the integral formula

$$\int_{-b}^b \frac{dx}{\sqrt{b^2-x^2}} = \pi.$$

Note that

$$\int_{-b}^{b} \frac{\ln|x-y|}{\sqrt{b^2-x^2}}dx = C, \tag{5.11}$$

where C is a constant independent of y. This is because we have the formula

$$P\int_{-b}^{b} \frac{1}{(x-y)\sqrt{b^2-x^2}}dx = 0.$$

Hence, we can replace y by b in (5.11) to compute C:

$$\begin{aligned} C &= \int_{-b}^{b} \frac{\ln(b-x)}{\sqrt{b^2-x^2}}dx \\ &= \int_{0}^{1} \frac{\ln[2b(1-t)]}{\sqrt{t(1-t)}}dt \\ &= \pi \ln\frac{b}{2}, \end{aligned} \tag{5.12}$$

where we have used the formula

$$\int_{0}^{1} \frac{\ln(1-t)}{\sqrt{t(1-t)}}dt = -2\pi\ln 2.$$

With $\mathrm{v}(x) = \frac{t}{1-x^2} - \alpha\ln(1-x^2)$, we find after some elementary computations,

$$\int_{-b}^{b} \frac{\mathrm{v}(x)}{\sqrt{b^2-x^2}}dx = \frac{\pi\, t}{\sqrt{1-b^2}} + 2\pi\alpha\, \ln\left(\frac{2}{1+\sqrt{1-b^2}}\right). \tag{5.13}$$

Substituting (5.11), (5.12), (5.13) into (5.10), and using the fact (5.2), we establish the lemma. □

To derive the result of b, note that equation (5.6) holds automatically in our even potential case, and (5.7) becomes

$$\int_{-b}^{b} \frac{2x^2\left[t+\alpha(1-x^2)\right]}{(1-x^2)^2\sqrt{b^2-x^2}}dx = 2\pi n.$$

This gives an equation satisfied by b:

$$\frac{b^2 t + 2\alpha(1-b^2)\left(1-\sqrt{1-b^2}\right)}{(1-b^2)^{\frac{3}{2}}} = 2n.$$

Let

$$u = \sqrt{1-b^2}. \tag{5.14}$$

Then u satisfies the following cubic equation:

$$2(n+\alpha)u^3 + (t-2\alpha)u^2 - t = 0.$$

It has only one real solution,

$$u = \frac{1}{6(n+\alpha)}\left[2\alpha - t + \sqrt[3]{\xi} + \frac{(2\alpha - t)^2}{\sqrt[3]{\xi}}\right], \tag{5.15}$$

where

$$\xi = 8\alpha^3 + 6t\left(9n^2 + 18n\alpha + 7\alpha^2\right) + 6t^2\alpha - t^3 + 6(n+\alpha)\sqrt{3t\left[27n(n+2\alpha)t - (t-8\alpha)(t+\alpha)^2\right]}.$$

From (5.14) we have the expression of b in terms of n, t and α:

$$b = \sqrt{1-u^2}, \tag{5.16}$$

where u is given by (5.15).

Substituting (5.16) into (5.9) and letting $n \to \infty$, we find

$$\begin{aligned} A &= n\ln 4 + \frac{3t^{2/3}\sqrt[3]{n}}{2^{2/3}} + \alpha\ln 4 + \frac{\sqrt[3]{t}\,(t-8\alpha)}{4\sqrt[3]{2n}} + \frac{t^{2/3}\alpha}{(2n)^{2/3}} - \frac{\alpha^2}{3n} - \frac{\alpha\sqrt[3]{t}\,(t-8\alpha)}{12\sqrt[3]{2}\,n^{4/3}} \\ &- \frac{5t^3 - 48t^2\alpha + 960\alpha^2 t + 320\alpha^3}{2160\times 2^{2/3}\sqrt[3]{t}\,n^{5/3}} + \frac{\alpha^3}{3n^2} + O(n^{-7/3}). \end{aligned}$$

Note that, the above asymptotics is only valid for $t > 0$. If $t = 0$, we will find that the asymptotic expansion of A will be in a different form. Actually, all the asymptotic expansions in this paper hold only for $t > 0$.

Then, it follows from (5.8) that the free energy $F[\sigma]$ has the following asymptotic expansion as $n \to \infty$,

$$\begin{aligned} F[\sigma] &= n^2\ln 2 + \frac{9\,t^{2/3}n^{4/3}}{4\times 2^{2/3}} + n\alpha\ln 4 + \frac{3\sqrt[3]{t}\,(t-8\alpha)n^{2/3}}{8\sqrt[3]{2}} + \frac{3t^{2/3}\alpha\sqrt[3]{n}}{2^{2/3}} - \frac{\alpha^2}{3}\ln n \\ &+ C_0(t,\alpha) + \frac{\alpha\sqrt[3]{t}\,(t-8\alpha)}{4\sqrt[3]{2n}} + \frac{5t^3 - 48t^2\alpha + 960\alpha^2 t + 320\alpha^3}{1440\sqrt[3]{t}\,(2n)^{2/3}} + O(n^{-1}), \end{aligned} \tag{5.17}$$

where $C_0(t,\alpha)$ is a constant independent of n.

It is shown in [15] that

$$\beta_n \sim \left(\frac{b-a}{4}\right)^2, \qquad n \to \infty.$$

Hence, in our problem,

$$\beta_n \sim \frac{b^2}{4} = \frac{1-u^2}{4}, \qquad n \to \infty.$$

In view of (5.15), we have

$$\lim_{n\to\infty}\beta_n = \frac{1}{4}.$$

In order to obtain the complete asymptotic expansion of β_n, using (5.15), we find that

$$\begin{aligned}\frac{b^2}{4} &= \frac{1-u^2}{4} \\ &= \frac{1}{4} - \frac{t^{2/3}}{4\times(2n)^{2/3}} + \frac{\sqrt[3]{t}(t-2\alpha)}{12\sqrt[3]{2}\,n^{4/3}} + \frac{t^{2/3}\alpha}{6\times 2^{2/3}\,n^{5/3}} - \frac{(t-2\alpha)^2}{48n^2} \\ &- \frac{\alpha\sqrt[3]{t}(t-2\alpha)}{9\sqrt[3]{2}\,n^{7/3}} + \frac{5(t^3-6t^2\alpha-6t\alpha^2-8\alpha^3)}{648\times 2^{2/3}\sqrt[3]{t}\,n^{8/3}} + O\left(\frac{1}{n^3}\right), \qquad n\to\infty. \end{aligned} \tag{5.18}$$

Hence, we suppose that

$$\beta_n = a_0 + \sum_{j=1}^{\infty} \frac{a_j}{n^{j/3}}, \qquad n\to\infty, \tag{5.19}$$

where

$$a_0 = \frac{1}{4},$$

and $a_j,\ j=1,2,\ldots$, are the expansion coefficients to be determined. By using the second-order difference equation satisfied by β_n, we obtain the following theorem.

Theorem 5.2. *The recurrence coefficient $\beta_n(t)$ has the following large n expansion:*

$$\beta_n(t) = \frac{1}{4} + \sum_{j=1}^{\infty} \frac{a_j}{n^{j/3}}, \qquad n\to\infty, \tag{5.20}$$

where the first few terms of expansion coefficients are

$$\begin{aligned} a_1 &= 0, & a_2 &= -\frac{t^{2/3}}{4\times 2^{2/3}}, \\ a_3 &= 0, & a_4 &= \frac{\sqrt[3]{t}(t-2\alpha)}{12\sqrt[3]{2}}, \\ a_5 &= \frac{t^{2/3}\alpha}{6\times 2^{2/3}}, & a_6 &= \frac{5-3(t-2\alpha)^2}{144}, \\ a_7 &= -\frac{\alpha\sqrt[3]{t}(t-2\alpha)}{9\sqrt[3]{2}}, & a_8 &= \frac{5\left[2t^3-12t^2\alpha-t(12\alpha^2+17)-16\alpha(\alpha^2-1)\right]}{1296\times 2^{2/3}\sqrt[3]{t}}, \end{aligned}$$

with more terms easily computable.

Proof. Substituting the expansion (5.19) into the difference equation (2.21), and taking a large n limit, we have an expression of the form

$$e_2 n^2 + e_{5/3} n^{5/3} + e_{4/3} n^{4/3} + \sum_{j=-\infty}^{3} e_{j/3} n^{j/3} = 0,$$

where each $e_{j/3}$ depends on the expansion coefficients a_j, t and α. In order to satisfy the above equation, all the coefficients of powers of n are identically zero. The equation $e_2 = 0$ reads,

$$-4(4a_0 - 1)^3(16a_0^3 - 8a_0^2 + a_0 - t^2) = 0,$$

which holds identically by the fact $a_0 = \frac{1}{4}$.

Setting $a_0 = \frac{1}{4}$ leads to $e_{5/3}$ and $e_{4/3}$ vanishing identically. The equation $e_1 = 0$ gives rise to

$$256t^2a_1^3 = 0.$$

Since $t > 0$, we have

$$a_1 = 0.$$

Setting $a_0 = \frac{1}{4}$ and $a_1 = 0$ leads to $e_{2/3}$ and $e_{1/3}$ vanishing identically. The equation $e_0 = 0$ gives

$$256t^2a_2^3 + t^4 = 0,$$

we find

$$a_2 = -\frac{t^{2/3}}{4 \times 2^{2/3}}.$$

With the values of a_0, a_1 and a_2, the equation $e_{-1/3} = 0$ shows

$$12 \times 2^{2/3}t^{10/3}a_3 = 0,$$

we have

$$a_3 = 0.$$

With the above a_0, a_1, a_2 and a_3, the equation $e_{-2/3} = 0$ gives rise to

$$\sqrt[3]{2}\, t^{10/3}\left[12\sqrt[3]{2}\, a_4 - \sqrt[3]{t}(t - 2\alpha)\right] = 0,$$

we obtain

$$a_4 = \frac{\sqrt[3]{t}(t - 2\alpha)}{12\sqrt[3]{2}}.$$

This procedure can be easily extended to find higher coefficients $a_5, a_6, a_7, \ldots$ We only list some of them:

$$a_5 = \frac{t^{2/3}\alpha}{6 \times 2^{2/3}},$$

$$a_6 = \frac{5 - 3(t - 2\alpha)^2}{144},$$

$$a_7 = -\frac{\alpha\sqrt[3]{t}(t-2\alpha)}{9\sqrt[3]{2}},$$

$$a_8 = \frac{5\left[2t^3 - 12t^2\alpha - t(12\alpha^2+17) - 16\alpha(\alpha^2-1)\right]}{1296\times 2^{2/3}\sqrt[3]{t}}.$$

This completes the proof. □

Remark 2. *Note that the asymptotic expansion of $\beta_n(t)$ in (5.20) is only valid for $t > 0$. This can be seen from the proof of the above theorem. If $t = 0$, we will obtain a different form of the asymptotic expansion by using our approach, and it is consistent with the known result; see, e.g., [3, Lemma 6.1 and Table 1]. The large n asymptotic expansions of $\mathrm{p}(n,t)$, $\sigma_n(t)$ and $D_n(t)$ derived subsequently are all valid only for $t > 0$ since they are obtained based on the asymptotics of $\beta_n(t)$.*

Remark 3. *The difference equation method has also been used to derive the large n asymptotic expansion of the recurrence coefficient β_n in the generalized Freud weight problems; see Clarkson and Jordaan [19, 20].*

Since

$$\beta_n = \mathrm{p}(n,t) - \mathrm{p}(n+1,t), \tag{5.21}$$

and in view of the large n asymptotic expansion of β_n, we suppose that $\mathrm{p}(n,t)$ has the large n expansion

$$\mathrm{p}(n,t) = b_{-3}n + b_{-2}n^{2/3} + b_{-1}n^{1/3} + b_0 + \sum_{j=1}^{\infty}\frac{b_j}{n^{j/3}}. \tag{5.22}$$

Substituting (5.22) into (5.21) and taking a large n limit, we find

$$b_{-3} = -\frac{1}{4}.$$

Using the second-order difference equation satisfied by $\mathrm{p}(n,t)$, we obtain the following result.

Theorem 5.3. *The sub-leading coefficient $\mathrm{p}(n,t)$ has the following expansion as $n\to\infty$:*

$$\mathrm{p}(n,t) = b_{-3}n + b_{-2}n^{2/3} + b_{-1}n^{1/3} + b_0 + \sum_{j=1}^{\infty}\frac{b_j}{n^{j/3}}, \tag{5.23}$$

where

$$b_{-3} = -\frac{1}{4}, \qquad b_{-2} = 0,$$
$$b_{-1} = \frac{3\,t^{2/3}}{4\times 2^{2/3}}, \qquad b_0 = \frac{2\alpha+1-4t}{8},$$
$$b_1 = \frac{\sqrt[3]{t}\,(t-2\alpha)}{4\sqrt[3]{2}}, \qquad b_2 = \frac{(2\alpha-1)t^{2/3}}{8\times 2^{2/3}},$$
$$b_3 = \frac{5-3(t-2\alpha)^2}{144}, \qquad b_4 = \frac{(2\alpha-1)\sqrt[3]{t}\,(2\alpha-t)}{24\sqrt[3]{2}},$$

are the expansion coefficients of the first few terms.

Proof. Substituting (5.22) into the difference equation (2.36), and taking a large n limit, we have

$$l_{8/3}n^{8/3} + l_{7/3}n^{7/3} + \sum_{j=-\infty}^{6} l_{j/3}n^{j/3} = 0,$$

where the expressions of $l_{j/3},\ j = 8, 7, 6, \ldots$, depend on the expansion coefficients $b_j,\ t$ and α. Then each $l_{j/3}$ should be identically zero. The equation $l_{8/3} = 0$ reads,

$$\frac{4}{3}(4b_{-3}+1)^2\, b_{-2} = 0,$$

which holds identically by the fact that $b_{-3} = -\frac{1}{4}$. Setting $b_{-3} = -\frac{1}{4}$ leads to $l_{7/3} = 0$ identically. Then $l_2 = 0$ shows

$$\frac{400}{27}b_{-2}^3 = 0.$$

We have

$$b_{-2} = 0.$$

With $b_{-3} = -\frac{1}{4}$ and $b_{-2} = 0$, $l_{5/3}$ and $l_{4/3}$ vanish identically. The equation $l_1 = 0$ gives

$$\frac{512}{27}b_{-1}^3 - 2t^2 = 0.$$

We get

$$b_{-1} = \frac{3\,t^{2/3}}{4\times 2^{2/3}}.$$

Following the same procedure in the proof of the previous theorem, we can find higher coefficients easily. Finally we establish the theorem. □

Remark 4. *For consistency, substituting (5.23) into* $\beta_n = \mathrm{p}(n,t) - \mathrm{p}(n+1,t)$ *and taking a large* n *limit, we find that* β_n *has the same expansion as (5.20).*

Next, we consider $\sigma_n(t)$, which is related to the Hankel determinant in (4.1). Before deriving the large n asymptotics of $\sigma_n(t)$, we present a lemma below.

Lemma 5.4. *The quantity $\sigma_n(t)$ can be expressed in terms of $\mathrm{p}(n,t)$ and $\mathrm{p}(n+1,t)$ as follows:*

$$\sigma_n(t) = -n(n+2t) - (2n-1+2\alpha)\mathrm{p}(n,t) - (2n+1+2\alpha)\mathrm{p}(n+1,t). \tag{5.24}$$

Proof. Substituting (2.16) into (4.3) to eliminate $r_n(t)$, we have

$$\sigma_n(t) = -n(n+2t) + (2n+1+2\alpha)\beta_n - 4(n+\alpha)\mathrm{p}(n,t).$$

Using the fact that $\beta_n = \mathrm{p}(n,t) - \mathrm{p}(n+1,t)$, we arrive at (5.24). □

From the above lemma, we are ready to obtain the large n expansion of $\sigma_n(t)$.

Theorem 5.5. *The quantity $\sigma_n(t) = 2t\frac{d}{dt}\ln D_n(t)$ has the following large n asymptotic expansion:*

$$\begin{aligned}\sigma_n(t) &= -\frac{3t^{2/3}n^{4/3}}{2^{2/3}} - \frac{\sqrt[3]{t}(t-2\alpha)n^{2/3}}{\sqrt[3]{2}} - 2^{4/3}\alpha\, t^{2/3}n^{1/3} + \frac{3t^2+60t\alpha-24\alpha^2+4}{36} \\ &- \frac{2^{2/3}\alpha\sqrt[3]{t}\,(t-2\alpha)}{3\,n^{1/3}} - \frac{t^3-6t^2\alpha+48t\alpha^2+2t-8\alpha^3+8\alpha}{54\sqrt[3]{t}\,(2n)^{2/3}} + O(n^{-1}).\end{aligned} \tag{5.25}$$

Proof. Substituting (5.23), the expansion of $\mathrm{p}(n,t)$, into (5.24), we obtain the desired result by taking a large n limit. □

Remark 5. *If we assume $\sigma_n(t) = \sum_{j=-\infty}^{4} d_j n^{j/3}$ as $n\to\infty$, then we can obtain the expansion coefficients d_j from the second-order difference equation (4.10) satisfied by $\sigma_n(t)$. We find that this agrees precisely with the result in the above theorem.*

Finally, we have the large n asymptotic expansion of $D_n(t)$.

Theorem 5.6. *The Hankel determinant $D_n(t)$ has the following expansion as $n\to\infty$:*

$$\begin{aligned}\ln D_n(t) &= -n^2\ln 2 - \frac{9\,t^{2/3}n^{4/3}}{4\times 2^{2/3}} - \tilde{c}_1(\alpha)n - \frac{3\sqrt[3]{t}\,(t-8\alpha)n^{2/3}}{8\sqrt[3]{2}} - \frac{3\alpha\, t^{2/3}\sqrt[3]{n}}{2^{2/3}} \\ &+ \frac{(12\alpha^2-5)\ln n}{36} + \frac{3t^2+120t\alpha-8(6\alpha^2-1)\ln t}{144} - \tilde{c}_0(\alpha) - \frac{\alpha\sqrt[3]{t}\,(t-8\alpha)}{4\sqrt[3]{2n}} \\ &- \frac{5t^3-48t^2\alpha+40(24\alpha^2+1)t+320\alpha(\alpha^2-1)}{1440\sqrt[3]{t}\,(2n)^{2/3}} + O(n^{-1}),\end{aligned}$$

where $\tilde{c}_0(\alpha)$ and $\tilde{c}_1(\alpha)$ are constants depending on α only.

Proof. Let

$$F_n(t) := -\ln D_n(t)$$

be the "free energy". From (1.11) we have

$$-\ln \beta_n = F_{n+1}(t) + F_{n-1}(t) - 2F_n(t). \tag{5.26}$$

For sufficiently large n, Chen and Ismail [15] showed that $F_n(t)$ is approximated by the free energy $F[\sigma]$ defined in (5.1). Taking account of (5.17), we suppose that $F_n(t)$ has the following large n expansion:

$$F_n(t) = c(t,\alpha)\ln n + \sum_{j=-\infty}^{6} c_{j/3}(t,\alpha) n^{j/3}. \tag{5.27}$$

Substituting (5.20) and (5.27) into the difference equation (5.26) and letting $n \to \infty$, we obtain the asymptotic expansion of $F_n(t)$ by equating coefficients of powers of n,

$$\begin{aligned} F_n(t) &= n^2 \ln 2 + \frac{9\, t^{2/3} n^{4/3}}{4 \times 2^{2/3}} + c_1(t,\alpha) n + \frac{3\sqrt[3]{t}\,(t-8\alpha) n^{2/3}}{8\sqrt[3]{2}} + \frac{3\alpha\, t^{2/3}\sqrt[3]{n}}{2^{2/3}} - \frac{(12\alpha^2-5)\ln n}{36} \\ &+ c_0(t,\alpha) + \frac{\alpha\sqrt[3]{t}\,(t-8\alpha)}{4\sqrt[3]{2n}} + \frac{5t^3 - 48t^2\alpha + 40(24\alpha^2+1)t + 320\alpha(\alpha^2-1)}{1440\sqrt[3]{t}\,(2n)^{2/3}} + O(n^{-1}), \end{aligned} \tag{5.28}$$

where $c_1(t,\alpha)$ and $c_0(t,\alpha)$ are undetermined coefficients independent of n. Hence,

$$\begin{aligned} \ln D_n(t) &= -n^2 \ln 2 - \frac{9\, t^{2/3} n^{4/3}}{4 \times 2^{2/3}} - c_1(t,\alpha) n - \frac{3\sqrt[3]{t}\,(t-8\alpha) n^{2/3}}{8\sqrt[3]{2}} - \frac{3\alpha\, t^{2/3}\sqrt[3]{n}}{2^{2/3}} + \frac{(12\alpha^2-5)\ln n}{36} \\ &- c_0(t,\alpha) - \frac{\alpha\sqrt[3]{t}\,(t-8\alpha)}{4\sqrt[3]{2n}} - \frac{5t^3 - 48t^2\alpha + 40(24\alpha^2+1)t + 320\alpha(\alpha^2-1)}{1440\sqrt[3]{t}\,(2n)^{2/3}} + O(n^{-1}). \end{aligned} \tag{5.29}$$

In order to know more information of $c_1(t,\alpha)$ and $c_0(t,\alpha)$, we take a derivative with respect to t in (5.29) and substitute it into (4.1), to find

$$\begin{aligned} \sigma_n(t) &= -\frac{3t^{2/3} n^{4/3}}{2^{2/3}} - 2nt\frac{d}{dt}c_1(t,\alpha) - \frac{\sqrt[3]{t}(t-2\alpha) n^{2/3}}{\sqrt[3]{2}} - 2^{4/3}\alpha\, t^{2/3} n^{1/3} - 2t\frac{d}{dt}c_0(t,\alpha) \\ &- \frac{2^{2/3}\alpha\sqrt[3]{t}\,(t-2\alpha)}{3\, n^{1/3}} - \frac{t^3 - 6t^2\alpha + 48t\alpha^2 + 2t - 8\alpha^3 + 8\alpha}{54\sqrt[3]{t}\,(2n)^{2/3}} + O(n^{-1}). \end{aligned} \tag{5.30}$$

Comparing the above with (5.25), we have

$$\frac{d}{dt}c_1(t,\alpha) = 0,$$

$$-2t\frac{d}{dt}c_0(t,\alpha) = \frac{3t^2 + 60t\alpha - 24\alpha^2 + 4}{36}.$$

It follows that

$$c_1(t,\alpha) = \tilde{c}_1(\alpha), \tag{5.31}$$

$$c_0(t,\alpha) = \frac{8(6\alpha^2-1)\ln t - 3t^2 - 120t\alpha}{144} + \tilde{c}_0(\alpha), \tag{5.32}$$

where $\tilde{c}_0(\alpha)$ and $\tilde{c}_1(\alpha)$ are constants depending on α only. Substituting (5.31) and (5.32) into (5.29), we establish the theorem. □

Remark 6. *We can not evaluate explicitly the constants $\tilde{c}_0(\alpha)$ and $\tilde{c}_1(\alpha)$ with our method. However, if we compare (5.28) with (5.17), we conjecture that $\tilde{c}_1(\alpha) = c_1(t,\alpha) = \alpha \ln 4$.*

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