A characterization theorem for semi-classical orthogonal polynomials on non-uniform lattices

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Abstract

It is proved a characterization theorem for semi-classical orthogonal polynomials on non-uniform lattices that states the equivalence between the Pearson equation for the weight and some systems involving the orthogonal polynomials as well as the functions of the second kind. As a consequence, it is deduced the analogue of the so-called compatibility conditions in the ladder operator scheme. The classical orthogonal polynomials on non-uniform lattices are then recovered under such compatibility conditions, through a closed formula for the recurrence relation coefficients.

Key words: Orthogonal polynomials; Divided-difference operator; Non-uniform lattices; Askey-Wilson operator; semi-classical class.

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1 Introduction

Semi-classical orthogonal polynomials on special non-uniform lattices (snul) are related to a divided difference operator, say $D$, whose support is the so-called $q$-quadratic lattice $[15,19]$. Under some specifications, $D$ is the Askey-Wilson operator $[1]$. Such families of orthogonal polynomials are well-known within the theory of discrete orthogonal polynomials, and find many applications within a vast list of topics from Mathematical Physics (see, amongst many others, $[9,15,16,17,19]$).

In the classification of lattices and corresponding divided difference operators (see $[19$, Sec. 2, Table 2] and $[12]$), the $q$-quadratic lattices are a generalization of other lower complexity lattices, such as the quadratic, $q$-linear and linear lattices. Such a hierarchy of lattices is related to the well-known $q$-Askey scheme $[10]$.

The main motivation for this paper comes from some properties that characterize semi-classical orthogonal polynomials in the continuous setting, the so-called structure relations, that is, difference-differential relations connecting two consecutive polynomials,

\[ AP_n' = L_n P_n + M_n P_{n-1}, \]  

(1)

or, in view of the three-term recurrence relation,

\[ AP_n' = \tilde{L}_n P_n + \tilde{M}_n P_{n+1}, \]  

(2)

with $A, L_n, M_n, \tilde{L}_n, \tilde{M}_n$ polynomials of degree independent of $n$ (the degree of $P_n$). The classification of orthogonal polynomials via such kind of equations has a long history, see, for instance, $[14]$. On a more general framework, (1)–(2) are the lowering and raising relations, deduced in the ladder operator scheme $[6]$. Similar equations to (1)–(2), with the derivative replaced by difference operators, are well-known in the literature (see, for instance, the introduction of $[11]$ and references therein). For the snul case, see $[12,19]$.

In the present paper we give a characterization of semi-classical orthogonal polynomials on snul via difference systems that involve the polynomials as well some related functions, the so-called functions of the second kind (see Section 2 for more details). Combining those systems with the three-term recurrence relation we then deduce difference equations in the matrix form, giving some fundamental relations that we regard as the discrete analogue of the ones appearing in the ladder operator scheme $[6]$. Here, we would like to put emphasis on the formula for the determinant in Corollary 2. Through such a formula, we obtain a closed form equation for the recurrence relation coefficients of the classical families of orthogonal polynomials on snul $[8]$.
The remainder of the paper is organized as follows. In Section 2 we give the definitions and state the basic results which will be used in the forthcoming sections. In Section 3 we show the main results of the paper, Theorem 1 and Corollaries 1 and 2. The compatibility conditions are discussed in Sub-section 3.1. In Section 4 we show the formulae for the recurrence relation coefficients of the classical orthogonal polynomials on snul.

2 Preliminary results

We consider the divided difference operator $\mathbb{D}$ given as in [12, Eq.(1.1)], with the property that $\mathbb{D}$ leaves a polynomial of degree $n-1$ when applied to a polynomial of degree $n$. The operator $\mathbb{D}$, defined on the space of arbitrary functions, is given in terms of two functions $y_1, y_2$ (at this stage, unknown),

$$(\mathbb{D} f)(x) = \frac{f(y_2(x)) - f(y_1(x))}{y_2(x) - y_1(x)}. \quad (3)$$

The functions $y_1, y_2$ may be obtained as follows: applying $\mathbb{D}$ to $f(x) = x^2$ and $f(x) = x^3$, one obtains, respectively,

$$y_1(x) + y_2(x) = \text{polynomial of degree 1}, \quad (4)$$

$$(y_1(x))^2 + y_1(x)y_2(x) + (y_2(x))^2 = \text{polynomial of degree 2}, \quad (5)$$

the later condition being equivalent to $y_1(x)y_2(x) = \text{polynomial of degree less or equal than 2}$. Hence, conditions (4)–(5) define $y_1$ and $y_2$ as the two $y$-roots of a quadratic equation

$$\hat{a}y^2 + 2\hat{b}xy + \hat{c}x^2 + 2\hat{d}y + 2\hat{e}x + \hat{f} = 0, \quad \hat{a} \neq 0. \quad (6)$$

Set $\lambda = \hat{b}^2 - \hat{a}\hat{c}$, $\tau = \left((\hat{b}^2 - \hat{a}\hat{c})(\hat{d}^2 - \hat{a}\hat{f}) - (\hat{b}\hat{d} - \hat{a}\hat{e})^2\right)/\hat{a}$. If $\lambda \neq 0$, as $y_1, y_2$ are the roots of (6), we have

$$y_1(x) = p(x) - \sqrt{r(x)}, \quad y_2(x) = p(x) + \sqrt{r(x)}, \quad (7)$$

with $p, r$ polynomials given by

$$p(x) = -\frac{\hat{b}x + \hat{d}}{\hat{a}}, \quad r(x) = \frac{\lambda}{\hat{a}^2} \left(x + \frac{\hat{b}\hat{d} - \hat{a}\hat{e}}{\lambda}\right)^2 + \frac{\tau}{\hat{a}\lambda}. \quad (8)$$

The $q$-quadratic lattices correspond to the case $\lambda \tau \neq 0$ [2,12,15,16]. There is a well-known parametrization of the conic (6), say $x = x(s), y = y(s)$, such that

$$y_1(x) = x(s - 1/2), \quad y_2(x) = x(s + 1/2), \quad (9)$$

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given as [2,15,12]
\[ x(s) = \kappa_1 q^s + \kappa_2 q^{-s} + \kappa_3, \]
for some appropriate constants \( \kappa \)'s, and \( q \) defined through
\[ q + q^{-1} = \frac{4b^2}{ac} - 2, \quad q \neq 1. \tag{9} \]

Note that, in this case, we have the divided-difference operator (3) given as
\[ \mathbb{D} f(x(s)) = \frac{f(x(s + 1/2)) - f(x(s - 1/2))}{x(s + 1/2) - x(s - 1/2)}. \tag{10} \]

In such a case, the polynomials \( p, r \) are then recovered under
\[ x(s + 1/2) + x(s - 1/2) = 2p(x(s)), \quad (x(s + 1/2) - x(s - 1/2))^2 = 4r(x(s)). \tag{11} \]

The fundamental quantities to be used in the sequel depend on the data \( p(x), r(x), q \) previously defined. Throughout the paper we shall use the notation \( \Delta_y = y_2 - y_1 \). From (7), there follows
\[ \Delta_y = 2\sqrt{r}. \]

In this paper we will consider the general case \( \lambda \tau \neq 0 \), and the divided difference operator given in its general form (3). We will also use some operators defined [12], as follow. The operators \( \mathbb{E}_1 \) and \( \mathbb{E}_2 \), acting on arbitrary functions \( f \) as
\[ (\mathbb{E}_1 f)(x) = f(y_1(x)), \quad (\mathbb{E}_2 f)(x) = f(y_2(x)), \]
which gives us (3) defined as
\[ (\mathbb{D} f)(x) = \frac{(\mathbb{E}_2 f)(x) - (\mathbb{E}_1 f)(x)}{(\mathbb{E}_2 x)(x) - (\mathbb{E}_1 x)(x)}. \]

The companion operator of \( \mathbb{D} \), defined as
\[ (\mathbb{M} f)(x) = \frac{(\mathbb{E}_1 f)(x) + (\mathbb{E}_2 f)(x)}{2}. \tag{12} \]

Some useful identities involving \( \mathbb{D} \) and \( \mathbb{M} \) are listed below (see [12]):
\[ \mathbb{D}(gf) = \mathbb{D}g \mathbb{M} f + \mathbb{M} g \mathbb{D} f, \tag{13} \]
\[ \mathbb{M}(gf) = \mathbb{M} g \mathbb{M} f + \frac{\Delta_y^2}{4} \mathbb{D} g \mathbb{D} f, \tag{14} \]
\[ \mathbb{D}(1/f) = -\frac{\mathbb{D} f}{\mathbb{E}_1 f \mathbb{E}_2 f}, \tag{15} \]
\[ \mathbb{M}(1/f) = \frac{\mathbb{M} f}{\mathbb{E}_1 f \mathbb{E}_2 f}. \tag{16} \]
Eq. (13) has the equivalent forms
\[ D(gf) = DgE_1f + DfE_2g, \quad D(gf) = DgE_2f + DfE_1g. \] (17)

Also, we have the two equivalent forms
\[ D(g/f) = DgE_1f - DfE_1g, \quad D(g/f) = DgE_2f - DfE_2g. \] (18)

Note that \( Mf \) is a polynomial whenever \( f \) is a polynomial. Furthermore, if \( \deg(f) = n \), then \( \deg(Mf) = n \).

We shall consider orthogonal polynomials related to a (formal) Stieltjes function defined by
\[ S(x) = \sum_{n=0}^{+\infty} u_n x^{-n-1} \] (19)
where \( (u_n) \), the sequence of moments, is such that
\[ \det \begin{bmatrix} u_{i+j} \end{bmatrix}_{i,j=0}^n \neq 0, \quad n \geq 0. \] (20)

Indeed, it is well-known that (20) is a necessary and sufficient condition for the existence of a sequence of orthogonal polynomials related to \( S \) [18]. Furthermore, if the corresponding polynomial family, \( \{P_n\}_{n \geq 0} \), is orthogonal with respect to a weight, \( w \), i.e.,
\[ \int_{\text{supp } w} P_n(x) P_m w(x) dx = h_n \delta_{n,m}, \quad n, m = 0, 1, \ldots, \]
where \( h_n \neq 0 \) and \( \delta_{n,m} \) is the Kronecker’s delta, then \( S \) reads
\[ S(x) = \int_{\text{supp } w} \frac{w(t)}{x-t} dt. \]

The orthogonal polynomials \( P_n, n \geq 0, \) are taken to be monic, and we will denote the sequence \( \{P_n\}_{n \geq 0} \) by SMOP.

Monic orthogonal polynomials satisfy a three-term recurrence relation [18]
\[ P_{n+1}(x) = (x - \beta_n) P_n(x) - \gamma_n P_{n-1}(x), \quad n = 0, 1, 2, \ldots, \] (21)
with \( P_{-1}(x) = 0, \ P_0(x) = 1, \) and \( \gamma_n \neq 0, \ n \geq 1, \ \gamma_0 = 1. \)

Associated with \( \{P_n\}_{n \geq 0} \) we define two objects: the sequence of associated polynomials of the first kind, \( \{P_n^{(1)}\}_{n \geq 0} \), which satisfies the recurrence relation
\[ P_n^{(1)}(x) = (x - \beta_n) P_{n-1}^{(1)}(x) - \gamma_n P_{n-2}^{(1)}(x), \quad n = 1, 2, \ldots. \]
with \( P^{(1)}_{-1}(x) = 0, \ P^{(1)}_0(x) = 1, \) and the sequence of functions of the second kind, \( \{q_n\}_{n \geq 0}, \) defined by

\[
q_{n+1} = P_{n+1}S - P_n^{(1)}, \quad n \geq 0, \quad q_0 = S. 
\]

The sequence \( \{q_n\}_{n \geq 0} \) also satisfies a three-term recurrence relation,

\[
q_{n+1}(x) = (x - \beta_n)q_n(x) - \gamma_n q_{n-1}(x), \quad n = 0, 1, 2, \ldots . \tag{22}
\]

Semi-classical orthogonal polynomials on non-uniform lattices are defined through a difference equation for the corresponding Stieltjes function [12],

\[
A(x)D S(x) = C(x)M S(x) + D(x), \tag{23}
\]

where \( A(x), C(x), D(x) \) are polynomials in \( x, A \neq 0. \)

From (3), (12) and (19), there follows that \( A, C, D \) are polynomials such that

\[
\deg(A) \leq m + 2, \quad \deg(C) \leq m + 1, \quad \deg(D) \leq m, \tag{24}
\]

where \( m \) is some nonnegative integer.

In the sequel we will use the following matrices (see [13,19]):

\[
\mathcal{Y}_n = \begin{bmatrix} P_{n+1} & q_{n+1}/w \\ P_n & q_n/w \end{bmatrix}, \quad n \geq 0, \tag{25}
\]

where \( w \) is the orthogonality weight related to \( \{P_n\}_{n \geq 0}. \) In the account of (21) and (22), \( \mathcal{Y}_n \) satisfies the difference equation

\[
\mathcal{Y}_n = \mathcal{A}_n \mathcal{Y}_{n-1}, \quad \mathcal{A}_n = \begin{bmatrix} x - \beta_n - \gamma_n \\ 1 \\ 0 \end{bmatrix}, \quad n \geq 1,
\]

with initial condition \( \mathcal{Y}_0 = \begin{bmatrix} x - \beta_0 q_1/w \\ 1 \\ S/w \end{bmatrix}. \) The matrix \( \mathcal{A}_n \) is, as usual, called the transfer matrix.

Orthogonal polynomials related to Stieltjes functions such that (23) holds satisfy the difference equations, for all \( n \geq 0 \) (put \( B \equiv 0 \) in [4, Theorem 1] or in [12])

\[
\begin{align*}
AD P_{n+1} &= (l_n + \Delta y \pi_n) E_1 P_{n+1} - C/2 E_2 P_{n+1} + \Theta_n E_1 P_n, \\
AD q_{n+1} &= (l_n + \Delta y \pi_n) E_1 q_{n+1} + C/2 E_2 q_{n+1} + \Theta_n E_1 q_n. 
\end{align*} \tag{26}
\]
Equivalently, we have
\[
\begin{align*}
A D P_{n+1} &= (l_n - \Delta_y \pi_n) E_2 P_{n+1} - C/2 E_1 P_{n+1} + \Theta_n E_2 P_n, \\
A D q_{n+1} &= (l_n - \Delta_y \pi_n) E_2 q_{n+1} + C/2 E_1 q_{n+1} + \Theta_n E_2 q_n.
\end{align*}
\] (27)

Here, \( l_n, \pi_n, \Theta_n \) are polynomials in \( x \) of degrees bounded by
\[
\begin{align*}
\deg(\Theta_n) &\leq \max\{\deg(A) - 2, \deg(C) - 1\}, \\
\deg(l_n) &\leq \max\{\deg(A) - 1, \deg(C)\}, \\
\deg(\pi_n) &\leq \deg(C) - 1,
\end{align*}
\] (28)

with initial conditions
\[
\begin{align*}
\pi_{-1} &= 0, \quad \pi_0 = -D/2, \\
\Theta_{-1} &= D, \quad \Theta_0 = A - \frac{\Delta^2}{4}D - (l_0 - C/2)M(x - \beta_0), \\
l_{-1} &= C/2, \quad l_0 = -M(x - \beta_0)D - C/2.
\end{align*}
\] (30)

3 Characterization of semi-classical orthogonal polynomials on snul

**Theorem 1** Let \( S \) be a Stieltjes function related to a weight \( w \), and let \( \{\gamma_n\}_{n \geq 0} \) be the corresponding sequence defined by (25), \( \gamma_n = \begin{pmatrix} P_{n+1}/w \\ P_n q_n/w \end{pmatrix} \). \( \{\gamma_n\}_{n \geq 0} \).

The following statements are equivalent:

(a) the weight \( w \) satisfies a Pearson-type equation,
\[
A D w = C M w;
\] (33)

(b) the Stieltjes function satisfies \( A D S = C M S + D \);

(c) \( \gamma_n \) satisfies the matrix equations
\[
A D \gamma_n = B_{n,1} E_1 \gamma_n - \frac{C}{2} E_2 \gamma_n, \quad n \geq 1,
\] (34)

and
\[
A D \gamma_n = B_{n,2} E_2 \gamma_n - \frac{C}{2} E_1 \gamma_n, \quad n \geq 1,
\] (35)

with the matrices \( B_{n,j}, j = 1, 2 \), given by
\[
B_{n,j} = \begin{bmatrix} l_n + (-1)^{j+1} \Delta_y \pi_n & \Theta_n \\ -\frac{\Theta_{n-1}}{\gamma_n} & l_{n-1} + (-1)^{j+1} \Delta_y \pi_{n-1} + \frac{\Theta_{n-1}}{\gamma_n} E_j(x - \beta_n) \end{bmatrix},
\] (36)
where the \( l_n, \pi_n \) and \( \Theta_n \)'s are the polynomials in (26)–(27), under the initial conditions (30)–(32).

**Proof.** The equivalence between (a) and (b) is known. Indeed, (a) \(\Rightarrow\) (b): see [19, Prop. 4.1] or (a) \(\Rightarrow\) (g) in [8, Th. 5]; (b) \(\Rightarrow\) (a): see the proof of (g) \(\Rightarrow\) (a) of [8, Th. 5].

Let us now prove the equivalence between (a) and (c).

(a) \(\Rightarrow\) (c). Assuming (a), then \( S \) satisfies \( A \mathcal{D} S = C M S + D \). Thus, we have relations (26)–(27).

Take the first equation in (26) for \( n \) and use the recurrence relation (21), \( P_{n-1} = -\frac{P_{n+1}}{T_n} + \frac{(x-\beta_n)}{T_n} P_n \), thus getting

\[
A \mathcal{D} P_n = (l_{n-1} + \Delta y \pi_{n-1} + \Theta_{n-1}/\gamma_n E_1(x - \beta_n)) E_1 P_n - C/2 E_2 P_n
- \Theta_{n-1}/\gamma_n E_1 P_{n+1}. \quad (37)
\]

Let us now compute \( A \mathcal{D} \left( \frac{q_{n+1}}{w} \right) \).

From (18) we have

\[
\mathcal{D} \left( \frac{q_{n+1}}{w} \right) = \frac{\mathcal{D} q_{n+1} E_2 w - \mathcal{D} w E_2 q_{n+1}}{E_1 w E_2 w}.
\]

If we multiply the above equation by \( A \) and use the second equation in (26) as well as (33), we obtain, after some cancelations,

\[
A \mathcal{D} \left( \frac{q_{n+1}}{w} \right) = (l_n + \Delta y \pi_n) E_1 \left( \frac{q_{n+1}}{w} \right) - \frac{C}{2} E_2 \left( \frac{q_{n+1}}{w} \right) + \Theta_n E_1 \left( \frac{q_n}{w} \right). \quad (38)
\]

Now let us write (38) for \( n - 1 \) and use the recurrence relation (22), \( q_{n-1} = -\frac{q_{n+1}}{\gamma_n} + \frac{(x-\beta_n)}{\gamma_n} q_n \). We obtain

\[
A \mathcal{D} \left( \frac{q_n}{w} \right) = (l_{n-1} + \Delta y \pi_{n-1} + \Theta_{n-1}/\gamma_n E_1(x - \beta_n)) E_1 \left( \frac{q_n}{w} \right) - C/2 E_2 \left( \frac{q_n}{w} \right)
- \Theta_{n-1}/\gamma_n E_1 \left( \frac{q_{n+1}}{w} \right). \quad (39)
\]

Writing the first equation in (26) together with (37), (38) and (39) in the matrix form, we obtain (34).

The proof of (35) proceeds on a similar way, starting with relations (27).
We split the proof of \((c) \Rightarrow (a)\) into the following two parts.
Part one: we prove that, given the equations (34)–(35), then we get relations (42)–(43) below (to be used in Part two).

Indeed, from (34)–(35) we obtain the so-called compatibility conditions involving the transfer matrices \(A_n\), as given in [7]: for all \(n \geq 1\),
\[
A \prod A_n = B_{n,1} E_1 A_n - E_2 A_n B_{n-1,1}, \quad \text{(40)}
\]
\[
A \prod A_n = B_{n,2} E_2 A_n - E_1 A_n B_{n-1,2}. \quad \text{(41)}
\]

In turn, equations (40)–(41) yield the following relations, for all \(n \geq 0\) (see a proof in [7]):
\[
\pi_{n+1} = -\frac{1}{2} \sum_{k=0}^{n+1} \frac{\Theta_k}{\gamma_k}, \quad \text{(42)}
\]
\[
l_{n+1} + l_n + M(x - \beta_{n+1}) \frac{\Theta_n}{\gamma_{n+1}} = 0. \quad \text{(43)}
\]

Part two: from (34)–(35), using conditions (42)–(43), we deduce the Pearson-type equation (33).

Indeed, we have
\[
A \prod (\det Y_n) = A \prod \left( P_{n+1} \frac{q_n}{w} \right) - A \prod \left( \frac{q_{n+1}}{w} P_n \right). \quad \text{(44)}
\]

Let us first compute \(A \prod \left( P_{n+1} \frac{q_n}{w} \right)\).

Using (17), we have
\[
A \prod \left( P_{n+1} \frac{q_n}{w} \right) = A \prod (P_{n+1}) E_1 \left( \frac{q_n}{w} \right) + A \prod \left( \frac{q_n}{w} \right) E_2 P_{n+1}.
\]

The use of the equations resulting from positions (2, 2) of (34) and (1, 1) of (35) in the above formula yields
\[
A \prod (P_{n+1} \frac{q_n}{w}) = (l_n - \Delta_y \pi_n) E_2 P_{n+1} E_1 \left( \frac{q_n}{w} \right) + \Theta_n E_2 P_n E_1 \left( \frac{q_n}{w} \right) - \frac{C}{2} \left( P_{n+1} \frac{q_n}{w} \right) + \Theta_{n-1} E_2 P_{n+1} E_1 \left( \frac{q_{n+1}}{w} \right)
\]
\[
- \frac{C}{2} \left( E_1 \left( P_{n+1} \frac{q_n}{w} \right) + E_2 \left( P_{n+1} \frac{q_n}{w} \right) \right) - \frac{\Theta_{n-1}}{\gamma_n} E_2 P_{n+1} E_1 \left( \frac{q_{n+1}}{w} \right)
\]
\[
+ \left( l_{n-1} + \Delta_y \pi_{n-1} + \Theta_{n-1} E_1(x - \beta_n) \right) E_1 \left( \frac{q_n}{w} \right) E_2 P_{n+1}. \quad \text{(45)}
\]
Now let us compute \( A \mathbb{D} \left( \frac{q_{n+1}}{w} P_n \right) \).

Using (17), we have

\[
A \mathbb{D} \left( \frac{q_{n+1}}{w} P_n \right) = A \mathbb{D} \left( \frac{q_{n+1}}{w} \right) E_2 P_n + A \mathbb{D} E_1 \left( \frac{q_{n+1}}{w} \right).
\]

The use of the equations resulting from positions (1, 2) of (34) and (2, 1) of (35) in the above formula yields

\[
A \mathbb{D} \left( \frac{q_{n+1}}{w} P_n \right) = (l_n + \Delta y \pi_n) E_2 P_n E_1 \left( \frac{q_{n+1}}{w} \right) + \Theta_n E_2 \gamma_n E_1 \left( \frac{q_{n}}{w} \right) - \frac{C}{2} \left( E_1 \left( P_n \frac{q_{n+1}}{w} - P_n \frac{q_{n+1}}{w} \right) + E_2 \left( P_n \frac{q_{n+1}}{w} - P_n \frac{q_{n+1}}{w} \right) \right)
- \left( -\Delta y (n - \pi_{n-1}) + l_n + l_{n-1} + \frac{\Theta_n}{\gamma_n} E_1 (x - \beta_n) \right) E_2 \gamma_n E_1 \left( \frac{q_n}{w} \right)
- \left( \Delta y (n - \pi_{n-1}) + l_n + l_{n-1} + \frac{\Theta_n}{\gamma_n} E_2 (x - \beta_n) \right) E_2 \gamma_n E_1 \left( \frac{q_n}{w} \right),
\]

that is,

\[
A \mathbb{D} (\det \mathcal{Y}_n) = -C \mathcal{M} (\det \mathcal{Y}_n)
+ \left( -\Delta y (n - \pi_{n-1}) + l_n + l_{n-1} + \frac{\Theta_n}{\gamma_n} E_1 (x - \beta_n) \right) E_2 \gamma_n E_1 \left( \frac{q_n}{w} \right)
- \left( \Delta y (n - \pi_{n-1}) + l_n + l_{n-1} + \frac{\Theta_n}{\gamma_n} E_2 (x - \beta_n) \right) E_2 \gamma_n E_1 \left( \frac{q_n}{w} \right).
\]

Taking into account (43) we obtain

\[
A \mathbb{D} (\det \mathcal{Y}_n) = -C \mathcal{M} (\det \mathcal{Y}_n) - \xi_n \left( E_2 \gamma_n E_1 \left( \frac{q_n}{w} \right) + E_2 \gamma_n E_1 \left( \frac{q_n}{w} \right) \right),
\]  

(47)

where

\[
\xi_n = \Delta y (n - \pi_{n-1}) + \frac{\Theta_n}{2 \gamma_n} (E_2 (x - \beta_n) - E_1 (x - \beta_n)),
\]

thus,

\[
\xi_n = \Delta y (n - \pi_{n-1}) + \frac{\Theta_n}{2 \gamma_n}.
\]
In the account of (42) we get \( \xi_n = 0 \). As \( \det Y_n = (\prod_{k=0}^{n} \gamma_k) / w \) (see [5]), from (47) with \( \xi_n = 0 \) we get \( A \mathcal{D} \left( \frac{1}{w} \right) = -C M(\frac{1}{w}) \). Using \( \mathcal{D}(1/w) = -\frac{Dw}{E_1 w E_2 w} \) and \( M(1/w) = \frac{Mw}{E_1 w E_2 w} \) we obtain \( A \mathcal{D} w = C M w \), as required.

As a consequence of Theorem 1 we obtain the results that follow.

**Corollary 1** Let \( S \) be a Stieltjes function related to a semi-classical weight, \( w \), satisfying \( A \mathcal{D} w = C M w \). The following equation holds:

\[
A_{n+1} \mathcal{D} Y_n = (B_n - C/2 I) M Y_n, \quad n \geq 1, \tag{48}
\]

where

\[
A_{n+1} = A + \frac{\Delta y}{2} I_n,
\]

\( I \) is the identity matrix, and \( B_n \) is given as

\[
B_n = \begin{bmatrix}
l_n & \Theta_n \\
-\Theta_{n-1}/\gamma_n & l_{n-1} + \Theta_{n-1}/\gamma_n M(x - \beta_n)
\end{bmatrix}. \tag{49}
\]

**PROOF.** Let us write the matrices \( B_{n,j} \) given in (36) as

\[
B_{n,1} = \hat{B}_n + \Delta y \Pi_n + \Theta_{n-1}/\gamma_n E_1 K_n, \tag{50}
\]

\[
B_{n,2} = \hat{B}_n - \Delta y \Pi_n + \Theta_{n-1}/\gamma_n E_2 K_n, \tag{51}
\]

where

\[
\hat{B}_n = \begin{bmatrix}
l_n & \Theta_n \\
-\Theta_{n-1}/\gamma_n & l_{n-1}
\end{bmatrix}, \quad \Pi_n = \begin{bmatrix}
\pi_n & 0 \\
0 & \pi_{n-1}
\end{bmatrix}, \quad K_n = \begin{bmatrix}
0 & 0 \\
0 & (x - \beta_n)
\end{bmatrix}.
\]

The sum of (34) with (35) gives us

\[
A \mathcal{D} Y_n = (\hat{B}_n - C/2 I) M Y_n - \Pi_n \frac{\Delta y}{2} (E_2 Y_n - E_1 Y_n) + \Theta_{n-1}/2 \gamma_n (E_1 K_n E_1 Y_n + E_2 K_n E_2 Y_n). \tag{52}
\]

Taking into account the property (14), we have

\[
\frac{1}{2} (E_1 K_n E_1 Y_n + E_2 K_n E_2 Y_n) = MK_n MY_n + \frac{\Delta y}{4} K_n \mathcal{D} Y_n. \tag{53}
\]

Also, we have

\[
\Delta y (E_2 Y_n - E_1 Y_n) = \Delta y \mathcal{D} Y_n. \tag{54}
\]
The substitution of (53) and (54) into (52) yields

\[ A \mathcal{D} \mathcal{Y}_n = (\mathcal{B}_n - C/2 I)\mathcal{M} \mathcal{Y}_n - \Pi_n \frac{\Delta^2}{2} \mathcal{D} \mathcal{Y}_n + \frac{\Theta_{n-1}}{\gamma_n} \left( \mathcal{M} \mathcal{K}_n \mathcal{M} \mathcal{Y}_n + \frac{\Delta^2}{4} \mathcal{F} \mathcal{D} \mathcal{Y}_n \right), \]

with \( \mathcal{F} = \mathcal{D} \mathcal{K}_n = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \). Thus, we get

\[ \left( A I + \frac{\Delta^2}{2} \Pi_n - \frac{\Delta^2 \Theta_{n-1}}{4 \gamma_n} \right) \mathcal{D} \mathcal{Y}_n = \left( \mathcal{B}_n - C/2 I + \frac{\Theta_{n-1}}{\gamma_n} \mathcal{M} \mathcal{K}_n \right) \mathcal{M} \mathcal{Y}_n. \] (55)

Taking into account (42), equation (55) yields (48).

### 3.1 Compatibility conditions

Let us emphasize the so-called compatibility conditions (40)–(41) involving the transfer matrices,

\[
A \mathcal{D} \mathcal{A}_n = \mathcal{B}_{n,1} \mathcal{E}_1 \mathcal{A}_n - \mathcal{E}_2 \mathcal{A}_n \mathcal{B}_{n-1,1}, \\
A \mathcal{D} \mathcal{A}_n = \mathcal{B}_{n,2} \mathcal{E}_2 \mathcal{A}_n - \mathcal{E}_1 \mathcal{A}_n \mathcal{B}_{n-1,2}.
\]

As mentioned in the proof of Theorem 1, these equations yield relations (42) and (43) involving the polynomials \( l_n, \pi_n, \Theta_n \). Furthermore, they also yield

\[- A + M(x - \beta_{n+1})(l_{n+1} - l_n) - \frac{\Delta^2}{2} (\pi_{n+1} + \pi_n) + \Theta_{n+1} = \frac{\gamma_{n+1}}{\gamma_n} \Theta_{n-1}, \quad n \geq 0.\] (56)

Further compatibility relations are deduced in the following corollary.

We will use the notation \( X_{(i,j)} \) to denote the element in position \((i, j)\) of a matrix \( X \).

**Corollary 2** The matrix \( \mathcal{B}_n \) given in (49) satisfies the following identities:

\[ \text{tr} \mathcal{B}_n = 0, \quad n \geq 0, \quad (57) \]

\[ \det \mathcal{B}_n = -\Delta^2 n^2 + \det \mathcal{B}_{0,1} + A \sum_{k=1}^{n} \frac{\Theta_{k-1}}{\gamma_k}, \quad n \geq 1. \quad (58) \]

Here, \( \det \mathcal{B}_{0,1} = \det \mathcal{B}_{0,2} = AD - C^2/4 \).

**Proof.** Equation (57) is (43), which combined with the initial conditions (32) also holds for \( n = 0 \).
To deduce (58) we start by using (40), thus obtaining
\[
\det(B_{n,1}) \det(E_1A_n) = \det(E_2A_n) \det(B_{n-1,1}) + A \det(E_2A_n B_{n-1,1})_{(2,2)}.
\]
As \(\det(E_jA_n) = \gamma_n\), \(j = 1, 2\), and \((E_2A_n B_{n-1,1})_{(2,2)} = (B_{n-1,1})_{(1,2)}\), we get
\[
\det(B_{n,1}) = \det(B_{n-1,1}) + A \frac{\Theta_{n-1}}{\gamma_n}.
\] (59)

Similarly, we get
\[
\det(B_{n,2}) = \det(B_{n-1,2}) + A \frac{\Theta_{n-1}}{\gamma_n}.
\] (60)

Iteration on (59) as well as on (60) yields, for all \(n \geq 1\),
\[
\begin{align*}
\det(B_{n,1}) &= \det(B_{0,1}) + A \sum_{k=1}^{n} \frac{\Theta_{k-1}}{\gamma_k}, \\
\det(B_{n,2}) &= \det(B_{0,2}) + A \sum_{k=1}^{n} \frac{\Theta_{k-1}}{\gamma_k}.
\end{align*}
\] (61) (62)

On the other hand, we have
\[
\det(B_{0,1}) = \det(B_{0,2}).
\] (63)

A direct use of (36) gives us, after some simplifications where we use \(l_{n-1} + \frac{\Theta_{n-1}}{\gamma_n} M (x - \beta_n) = -l_n\) and \(\pi_{n-1} - \frac{\Theta_{n-1}}{\gamma_n} = \pi_n\) (cf. (42) and (43)),
\[
\frac{1}{2} (\det(B_{n,1}) + \det(B_{n,2})) = -l_n^2 + \Theta_n \frac{\Theta_{n-1}}{\gamma_n} + \Delta_2^2 \pi_n^2.
\] (64)

Thus, in the account of (61)–(63), and taking into account that \(B_n\) can be written as
\[
B_n = \begin{bmatrix} l_n & \Theta_n \\ -\Theta_{n-1}/\gamma_n & -l_n \end{bmatrix},
\] (65)
then (64) gives us
\[
\det(B_{0,1}) + A \sum_{k=1}^{n} \frac{\Theta_{k-1}}{\gamma_k} = \det(B_n) + \Delta_2^2 \pi_n^2.
\]

Hence, we get (58) as required.

Note that in the account of \(\Theta_{-1}/\gamma_0 = D\) (cf. (31)) and (42), an equivalent equation for (58) is
\[
\det B_n = -\Delta_2^2 \pi_n^2 - C^2 \pi_n^2 - 2A_\pi_n,
\] (66)
that is,

\[-l_n^2 + \Theta_n \frac{\Theta_{n-1}}{\gamma_n} = -\Delta y \pi_n^2 - \frac{C^2}{4} - 2A \pi_n.\]  

(67)

**Remark 1** Equation (58) is the analogue of Magnus’ summation formula [13] (see also [3, Cor.1]).

Let us emphasize that (43), (56), and (58) can be regarded as the analogue of 
(S_1), (S_2), and (S_2'), respectively, in the ladder operator approach [6].

### 4 The recurrence relation coefficients of classical orthogonal polynomials on snul from compatibility conditions

We consider the families of classical orthogonal polynomials (see, amongst others, [2,16] and [8, Th. 5]). We have \(A(x)Dw(x) = C(x)Mw(x)\) with

\[\deg(A) \leq 2, \quad \deg(C) \leq 1.\]

Therefore, in the account of (28)–(29), \(\deg(l_n) = 1, \deg(\pi_n) = \deg(\Theta_n) = 0.\)

We will use the following notations:

\[A(x) = a_2 x^2 + a_1 x + a_0, \quad C(x) = c_1 x + c_0, \quad D(x) = d_0,\]

\[l_n(x) = \ell_{n,1} x + \ell_{n,0}, \quad \pi_n(x) = \pi_n, \quad \Theta_n(x) = \Theta_n, \quad \pi_n, \Theta_n \text{ constants.}\]

In the next lemma we show that some quantities, to be used in the sequel, depend only on the lattice as well as on the coefficients of the Pearson equation.

**Lemma 1** Under the previous notations, the quantities \(\ell_{n,1}, \ell_{n,0}, \Theta_n/\gamma_{n+1}\) and \(\pi_n\) are given, for all \(n \geq 0\), by
Let \( \ell_{n+1,0} = \frac{2r_1 \pi^2 + c_0 c_1 + a_1 \pi n + 1}{\ell_{n+1,1}}, \)

with the initial conditions

\[
\ell_{0,1} = -p_1 d_0 - \frac{c_1}{2}, \quad \Theta_0 = \frac{-a_2 + c_1 p_1 + 2 d_0 (r_2 + p^2)}{p_1^2 - r_2}, \quad \pi_0 = -\frac{d_0}{2}, \quad \ell_{0,0} = -(p_0 - \beta_0) d_0 - \frac{c_0}{2}.
\]

Here, \( d_0 = -(a_2 + c_1 p_1)/(p_1^2 - r_2), \ p_1, r_2 \) are the leading coefficients of \( p(x), r(x), \) respectively, defined in (8), and \( q \) is defined through (9).

**PROOF.** Equations (68)–(70), together with the initial conditions (72), are deduced from \( B \equiv 0 \) in [7, Lemma1]. The \( x \)-coefficient of (67) gives us (71).

**Theorem 2** Let \( A(x) D w(x) = C(x) \hat{M} w(x) \) with \( \deg(A) \leq 2, \ \deg(C) \leq 1. \) Let \( \{P_n\}_{n \geq 0} \) be the SMOP related to \( w, \) satisfying the recurrence relation (21). Under the notations of the previous lemma, the recurrence relation coefficients are given by the following equations:

\[
\gamma_{n+1} = \frac{\ell_{n,0}^2 - 4 r_0 \pi_0^2 - c_0^2/4 - 2 a_0 \pi_0}{\Theta_n/\gamma_{n+1} \Theta_{n-1}/\gamma_n}, \quad n \geq 1,
\]

\[
\beta_{n+1} = \frac{\ell_{n+1,0} + \ell_{n,0} + p_0 \Theta_n/\gamma_{n+1}}{\Theta_n/\gamma_{n+1}}, \quad n \geq 0,
\]

with \( \gamma_1 \) and \( \beta_0 \) given by

\[
\gamma_1 = \frac{(a_0 + p_0 c_0 + p_0^2 d_0 - r_0 d_0 - c_0 \beta_0 - 2 p_0 d_0 \beta_0 + d_0 \beta_0^2) (p_1^2 - r_2)}{-a_2 + c_1 p_1 + 2 d_0 (r_2 + p_1^2)},
\]

\[
\beta_0 = \frac{a_1 + p_1 c_0 + p_0 c_1 + 2 p_0 p_1 d_0 - r_1 d_0}{c_1 + 2 p_1 d_0}.
\]
**PROOF.** Equation (73) follows from the independent coefficient of (67),

\[-\ell_{n,0}^2 + \gamma_{n+1} \frac{\Theta_n}{\gamma_{n+1}} \frac{\Theta_{n-1}}{\gamma_n} = -4r_0\pi_n^2 - \frac{c_0^2}{4} - 2a_0\pi_n, \quad n \geq 1.\]

Equation (74) is obtained from the independent term of (43),

\[\ell_{n+1,0} + \ell_{n,0} + (p_0 - \beta_{n+1}) \frac{\Theta_n}{\gamma_{n+1}} = 0, \quad n \geq 0.\]

To obtain \(\beta_0\) and \(\gamma_1\) we equate coefficients in (31) and (32), thus getting

\[
\begin{align*}
\ell_{0,1} &= -p_1d_0 - c_1/2, \quad (77) \\
\ell_{0,0} &= -(p_0 - \beta_0)d_0 - c_0/2. \quad (78) \\
0 &= a_1 - r_1d_0 - (\ell_{0,1} - c_1/2)(p_0 - \beta_0) - (\ell_{0,0} - c_0/2)p_1, \quad (79) \\
\Theta_0 &= a_0 - r_0d_0 - (\ell_{0,0} - c_0/2)(p_0 - \beta_0). \quad (80)
\end{align*}
\]

The use of (77) and (78) in (79) yields \(\beta_0\). From (80) we have, using (78), \(\Theta_0\) given by

\[\Theta_0 = a_0 - r_0d_0 + ((p_0 - \beta_0)d_0 + c_0)(p_0 - \beta_0). \quad (81)\]

From \(\Theta_0/\gamma_1\) given by (72) combined with (81) we get \(\gamma_1\).

### 4.1 Askey-Wilson polynomials from compatibility conditions

The Askey-Wilson operator [1] is obtained under the following specializations. Let us define the base \(q = e^{2i\eta}\) and consider the projection map from the unit circle \(\{z = e^{i\theta}, \theta \in [-\pi,\pi]\}\) onto \([-1,1]\) by \(x = \frac{1}{2}(z + z^{-1})\). Consider the symmetrised and canonical form of the lattice defined through (6) (see, e.g., [19, Sec. 2])

\[\hat{a} = \hat{c}, \text{ arbitrary and non-zero}, \quad \hat{b} = -\hat{a}\cos(\eta), \quad \hat{d} = \hat{c} = 0, \quad \hat{f} = -\hat{a}\sin^2(\eta), \quad (82)\]

and \(\theta = 2s\eta\). Then we get the parametrization \(x(s)\) given by

\[x(s) = \frac{1}{2}(q^s + q^{-s}), \quad (83)\]

and we obtain, from (10), the Askey-Wilson operator (see [9, Eq. (12.1.12)])

\[
\mathbb{D}f(x) = \frac{f(\frac{1}{2}(q^{1/2}z + q^{-1/2}z^{-1})) - f(\frac{1}{2}(q^{-1/2}z + q^{1/2}z^{-1}))}{\frac{1}{2}(q^{1/2} - q^{-1/2})(z - z^{-1})}. \quad (84)
\]
Using (11) combined with (83) or by plugging the data (82) into the definition of \( p(x), r(x) \) in (8), we get
\[
p(x) = \frac{1}{2}(q^{1/2} + q^{-1/2})x, \quad r(x) = \frac{1}{4}(q^{1/2} - q^{-1/2})^2(x^2 - 1). \tag{84}
\]

Let us take the Askey-Wilson weight [1] (see also [10])
\[
w(x; \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}) = \frac{h(x, 1)h(x, -1)h(x, q^{1/2})h(x, -q^{1/2})}{\sqrt{1 - x^2}h(x, \alpha_1)h(x, \alpha_2)h(x, \alpha_3)h(x, \alpha_4)},
\]
where
\[
h(x, \alpha) = \prod_{k=0}^{+\infty} (1 - 2\alpha xq^k + \alpha^2 q^{2k}), \quad x = \cos(\theta).
\]
Let us denote by \( \sigma_j \) the \( j \)-th elementary symmetric polynomial of \( \alpha_1, \ldots, \alpha_4 \), that is,
\[
\sigma_1 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \quad \sigma_2 = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_1\alpha_4 + \alpha_2\alpha_3 + \alpha_2\alpha_4 + \alpha_3\alpha_4, \quad \\
\sigma_3 = \alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_2\alpha_4 + \alpha_2\alpha_3\alpha_4 + \alpha_1\alpha_3\alpha_4, \quad \sigma_4 = \alpha_1\alpha_2\alpha_3\alpha_4.
\]
We have \( A\mathbb{D}w = C\mathbb{M}w \), with the polynomials
\[
A(x) = a_2x^2 + a_1x + a_0, \quad C(x) = c_1x + c_0
\]
where [19, Prop. 5.1]
\[
a_2 = 2(1 + \sigma_4 q^{-2}), \quad a_1 = -(q^{-1/2}\sigma_1 + q^{-3/2}\sigma_3), \quad a_0 = -1 + q^{-1}\sigma_2 - q^{-2}\sigma_4, \tag{85}
\]
\[
c_1 = 4q^{-2}\sigma_1 - 1, \quad c_0 = 2\frac{q^{-1/2}\sigma_1 - q^{-3/2}\sigma_3}{q^{1/2} - q^{-1/2}}. \tag{86}
\]
The recurrence coefficients of the SMOP \( \{P_n\}_{n \geq 0} \) orthogonal with respect to \( w \) are determined through (73)–(74). Thus, we recover the recurrence coefficients for the monic Askey-Wilson polynomials,
\[
\beta_n = \left[\sigma_1(q + \sigma_4(q^{2n} - q^n - q^{-n-1})) + \sigma_3(1 - q^n - q^{n+1} + \sigma_4 q^{2n-1})\right] \\
\times \frac{q^{n-1}}{2(1 - \sigma_4 q^{2n})(1 - \sigma_4 q^{2n-2})}, \quad n \geq 0, \tag{87}
\]
\[
\gamma_n = \frac{1}{4} \frac{(1 - q^n)(1 - \sigma_4 q^{n-2})G_n}{(1 - \sigma_4 q^{2n-3})(1 - \sigma_4 q^{2n-2})(1 - \sigma_4 q^{2n-1})}, \quad n \geq 1, \tag{88}
\]
where
\[
G_n = (1 - \alpha_1\alpha_2 q^{n-1})(1 - \alpha_1\alpha_3 q^{n-1})(1 - \alpha_1\alpha_4 q^{n-1}) \times (1 - \alpha_2\alpha_3 q^{n-1})(1 - \alpha_2\alpha_4 q^{n-1})(1 - \alpha_3\alpha_4 q^{n-1}).
\]
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