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Explicitly analytical self-similar solutions for the Euler/Navier-Stokes-Korteweg equations in R^N

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Explicitly analytical self-similar solutions for the Euler/Navier-Stokes-Korteweg equations in R^N

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Abstract

In this paper, we present an unified formulae for explicit self-similar solutions of the Euler/Navier-Stokes-Korteweg equations arising in the modeling of capillary fluids. The technique used here is to reduce Euler/Navier-Stokes-Korteweg equations into a series of solvable ordinary differential equations by making use of multi-dimensional self-similar ansatz and variable separation method.

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1 Introduction

We consider the Euler/ Navier-Stokes-Korteweg equations arising in the modeling of capillary fluids, which read

$$\rho_t + \nabla \cdot (\rho u) = 0, \tag{1.1}$$

$$(\rho u)_t + (\rho u \cdot \nabla)u + \nabla p + \mu \triangle u = \rho \nabla \left(K(\rho) \triangle \rho + \frac{1}{2} K'(\rho) |\nabla \rho|^2 \right), \tag{1.2}$$

where $x = (x_1, x_2, \cdots, x_N)^T \in R^N$ and $u = (u_1, u_2, \cdots, u_N)^T$ are the components of the N-dimensional velocity field, $\rho(x,t)$ is the density and $p(x,t)$ is the pressure of the fluid. The right-hand side of the second equation modelizes capillarity forces. It is a dispersive perturbation of the classical Euler equations. If $\mu = 0$, the system (1.1)-(1.2) is called the Euler-Korteweg equations [1]-[4]. If $\mu > 0$, the system (1.1)-(1.2) is the Navier-Stokes-Korteweg equations [5]-[8]. So we call the system (1.1)-(1.2) as Euler/Navier-Stokes-Korteweg equations in this paper.

For special case when $K(\rho) = k_2 > 0$, the system (1.1)-(1.2) reduce to [9, 10]

$$\rho_t + \nabla \cdot (\rho u) = 0, \tag{1.3}$$

$$(\rho u)_t + (\rho u \cdot \nabla)u + \nabla p + \mu \triangle u = k_2 \rho \nabla \triangle \rho, \tag{1.4}$$

which further reduce to compressible Euler/Navier-Stokes equations when $k_2 = 0$ [11]-[24]

$$\rho_t + \nabla \cdot (\rho u) = 0, \tag{1.5}$$

$$(\rho u)_t + (\rho u \cdot \nabla)u + \nabla p + \mu \triangle u = 0. \tag{1.6}$$

The compressible Euler/Navier-Stokes equations (1.5)-(1.6) have been extensively investigated in the term of both weak solutions and exact analytical solutions. For instance, the global existence in critical spaces for compressible Navier-Stokes equations was shown by Danchin [11]. The existence and explicit structure of global solutions for Euler equations with antisymmetry continuous initial data was constructed by Zhang [12]. Chen proved the existence of global weak solution for Euler equation with symmetry outside of a circular core with the center at the origin [13]. For more results and details, see references [14]-[24]. There are also much work on the exact solutions of the Euler/Navie-Stokes equations. For example, In 1995, Zhang and Zheng obtained spiral solutions for the 2D compressible Euler equations with $p = K\rho^{\gamma}$ and $\gamma = 2$ [25]. Recently, Li found Lax pairs for 2D and 3D incompressible Euler equations [26]. Lou et al proposed Backlund transformation, Darboux transformation and exact solutions for the 2D Euler equations in vorticity form [27]. Ludlow, Clarkson and Bassom found similarity reductions and exact solutions for the two-dimensional incompressible Navier-Stokes equations [28]. Yuen further obtained a class of self-similar solutions for the compressible Euler/Navier-Stokes equations in R^N [29, 30]. Based on matrix theory and decomposition technique, we theoretically show the existence of the Cartesian rotational solutions for the general N-dimensional compressible Euler/Navier-Stokes equations [31].

However, related to the Euler/Navier-Stokes-Korteweg equations (1.1)-(1.2) or (1.3)-(1.4), except some mathematical results on the existence and well-posedness of weak solutions in Sobolev space [1]-[10], the exact solutions, especially self-similar solution of the Euler/Navier-Stokes-Korteweg equations have been still unknown to our knowledge.

The self-similar solution is a kind of important analytical solution, which describes the intermediate asymptotics of a problem. They are much simpler than the full solutions, so they are easier to understand and to study in different regions of parameter space. In some cases self-similar solutions helps to understand diffusion-like properties or the existence of compact supports of the solution. There some work on self-similar solutions of Euler and Navier-Stokes equations. For example, with a straightforward generalization of the self-similar ansatz a partial differential equation system of the three-dimensional Navier-Stokes equations was successfully investigated [40]. Bana et al obtained analytic self-similar solutions for a one-dimensional compressible Euler equations with heat conduction and the Navier-Stokes equations [32, 33]. Recently, some similarity solutions of the two dimensional incompressible NS equation was presented [42]. Though analytic self-similar for a N-dimensional compressible Euler equations can be constructed, these solutions are still not explicit [29, 30].

Therefore, in present paper we would like to present explicit self-similar solutions for the N-dimensional Euler/Navier-Stokes-Korteweg equations (1.3)-(1.4). Here we not only generalize self-similar solutions to N-dimensional case, but also provide an explicitly unified formulae for the self-similar solutions. The technique used here is to reduce Euler/Navier-Stokes-Korteweg equations into a series of solvable ordinary differential equations by making use of generalization of self-similar ansatz and variable separation method.

2 Unified formulae of self-similar solutions

To solve the Euler equations (1.5)-(1.6), Yuen showed that the conservation equation (1.5) there exist a solution in the form [29]

$$u_i = \frac{\dot{a}_i(t)}{a_i(t)} x_i, \quad \rho = \frac{h(\xi)}{\prod_{i=1}^{N} a_i(t)}, \quad \xi = \sum_{i=1}^{N} \frac{x_i^2}{a_i^2(t)}. \tag{2.1}$$

Through their the conservation equations (1.3) and (1.5) are the same between the Euler/Navier-Stokes-Korteweg equation (1.3)-(1.4) and Euler equations (1.5)-(1.6), by checking computation, we find that similar transformation (2.1) is not compatible solution of the second equation (1.4) of the system (1.3)-(1.4). So it is the key in our present paper to search for a suitable similar transformation from the conservation equation (1.3), so that it can be used to construct explicit self-similar solutions for the Euler-Korteweg equations (1.3)-(1.4).

In many situations in gas dynamics, we may consider the case where the density ρ and pressure p satisfy a relation

$$p(\rho) = k_1 \rho^{\gamma}, \tag{2.2}$$

with $k_1 > 0$ and the constant $\gamma = c_p/c_\gamma$, where c_p, c_γ are the specific heat capacities under constant pressure and constant volume respectively.

We can rewrite (1.3) and (1.4) in the form

$$\rho_t + \sum_{i=1}^{N} (\rho_{x_i} u_i + \rho u_{i,x_i}) = 0, \tag{2.3}$$

$$u_{i,t} + \sum_{j=1}^{N} u_j u_{i,x_j} + \frac{k_1\gamma}{\gamma - 1} (\rho^{\gamma-1})_{x_i} + \mu\rho^{-1} \triangle u_i = k_2(\Delta\rho)_{x_i}, \quad i = 1, \cdots, N. \tag{2.4}$$

In the following we we search for solutions of the system (2.3)-(2.4) in the form of self-similar ansatz

$$\begin{aligned} &u_i = f_i(t)x_i, \quad i = 1, \cdots, N, \\ &\rho = g(t)h(\xi), \quad \xi = \sum_{i=1}^{N} a_i(t)x_i, \end{aligned} \tag{2.5}$$

where $a_i(t),\ f_i(t),\ i = 1, \cdots, N$, $g(t)$ and $h(\xi)$ are functions to be determined.

Substituting (2.5) into the first equation (2.3) yields

$$\begin{aligned} &\rho_t + \sum_{i=1}^{N} (\rho_{x_i} u_i + \rho u_{i,x_i}) \\ &= \dot{g}(t)h(t) + g(t)h'(\xi) \sum_{i=1}^{N} \dot{a}_i(t)x_i + \sum_{i=1}^{N} (gh'(\xi) f_i a_i x_i + ghf_i), \\ &= h(\xi) \left(\dot{g}(t) + g(t) \sum_{i=1}^{N} f_i(t) \right) + g(t)h'(\xi) \sum_{i=1}^{N} \left[\dot{a}_i(t) + f_i(t)a_i(t)\right] x_i = 0. \end{aligned} \tag{2.6}$$

where the notations dot $\cdot = \frac{\partial}{\partial t}$ denotes derivative with respect to t and the prime $' = \frac{\partial}{\partial \xi}$ denotes derivative with respect to ξ.

It is obvious that the equation (2.6) is satisfied if we set

$$\dot{g}(t) + g(t) \sum_{i=1}^{N} f_i(t) = 0, \tag{2.7}$$

$$\dot{a}_i(t) + f_i(t)a_i(t) = 0, \quad i = 1, \cdots, N. \tag{2.8}$$

The above system (2.7)-(2.8) is a system with $N + 1$ ordinary differential equations and $2N + 1$ unknown functions, which always have solutions.

From (2.8), we directly obtain that

$$f_i(t) = -\frac{\dot{a}_i(t)}{a_i(t)}, \quad i = 1, \cdots, N. \tag{2.9}$$

Substituting (2.9) back to (2.7) yields

$$
\begin{aligned}
&\dot{g}(t)+g(t)\sum_{i=1}^{N} f_i(t)=\dot{g}(t)-g(t)\sum_{i=1}^{N}\frac{\dot{a}_i(t)}{a_i(t)}\\
&=\dot{g}(t)-g(t)\frac{d}{dt}\sum_{i=1}^{N}\ln a_i(t)=\dot{g}(t)-g(t)\frac{d}{dt}\ln\prod_{i=1}^{N}a_i(t)=0,
\end{aligned}
$$

which has a solution

$$
g(t)=\exp\left(\int\frac{d}{dt}\ln\prod_{i=1}^{N}a_i(t)dt\right)=\prod_{i=1}^{N}a_i(t). \tag{2.10}
$$

By virtue of (2.9) and (2.10), the self-similar solution (2.5) becomes

$$
u_i=-\frac{\dot{a}_i(t)}{a_i(t)}x_i,\quad \rho=h(\xi)\prod_{i=1}^{N}a_i(t),\quad \xi=\sum_{i=1}^{N}a_i(t)x_i, \tag{2.11}
$$

which implies that it a solution of the conservation equation of mass (2.3). Here we notice that in the expression (2.11), the functions $a_i(t),\ i=1,\cdots,N$ and $h(\xi)$ are $N+1$ arbitrary functions to be used in solving the second equation (2.4).

Now inserting (2.11) back into (2.4), one can write

$$
\begin{aligned}
&\frac{2\dot{a}_i^2(t)-a_i(t)\ddot{a}_i(t)}{a_i^2(t)}x_i+k_1\gamma a_i(t)\left(\prod_{j=1}^{N}a_j(t)\right)^{\gamma-1}h^{\gamma-2}(\xi)h'(\xi)\\
&-k_2a_i(t)\prod_{j=1}^{N}a_j(t)\sum_{j=1}^{N}a_j^2(t)h'''(\xi)=0,\quad i=1,\cdots,N.
\end{aligned} \tag{2.12}
$$

Observe that, the system (2.12) contains N equations and $N+1$ unknown functions $a_i(t)$ and $h(\xi)$ with respect to variables t and ξ respectively. This motivate us to split the system (2.12) into two systems.

It is obvious that the system (2.12) hold if $a_i(t),\ \ i=1,\cdots,N$ and $h(\xi)$ satisfy the following equations

$$
2\dot{a}_i^2(t)-a_i(t)\ddot{a}_i(t)=0,\quad i=1,\cdots,N, \tag{2.13}
$$

$$
k_1\gamma\left(\prod_{j=1}^{N}a_j(t)\right)^{\gamma-2}h^{\gamma-2}(\xi)h'(\xi)-k_2\sum_{j=1}^{N}a_j^2(t)h'''(\xi)=0. \tag{2.14}
$$

The system (2.13) is a system of ordinary differential equations, which can be rewritten into

$$\frac{d\ln \dot{a}_i(t)}{dt} = \frac{d\ln a_i^2(t)}{dt}, \quad i = 1, \cdots, N.$$

Integration twice gives

$$a_i(t) = \frac{1}{c_{i,1}t + c_{i,2}}, \quad i = 1, \cdots, N, \tag{2.15}$$

where $c_{i,1}, c_{i,2}$ are arbitrary constants.

Substituting (2.15) into the equation (2.14) yields

$$k_1\gamma \prod_{j=1}^{N}(b_j t + c_j)^{2-\gamma} h^{\gamma-2}(\xi)h'(\xi) - k_2 \sum_{j=1}^{N}(b_j t + c_j)^{-2} h'''(\xi) = 0. \tag{2.16}$$

We make a constraint

$$\gamma = \frac{2}{N} + 2, \quad c_{i,1} = c_1, \quad c_{i,2} = c_2, \quad i = 1, \cdots, N,$$

where c_1 and c_2 are arbitrary constants, then (2.16) becomes an nonlinear ordinary differential equation with respect to ξ

$$\frac{2k_1(N+1)}{N} h^{\frac{2}{N}}(\xi)h'(\xi) - k_2 N h'''(\xi) = 0. \tag{2.17}$$

which has a solution

$$h(\xi) = \nu \xi^{-N},$$

where $\nu = \left(\frac{k_2 N^2(N+2)}{2k_1}\right)^{N/2}$.

In this way, we find an unified formulae for analytical self-similar solutions of the N-dimensional Euler/Navier-Stokes-Korteweg equations

$$\begin{aligned} u_i &= -\frac{c_1}{c_1 t + c_2} x_i, \quad i = 1, \cdots, N, \\ \rho &= \nu \left(\frac{c_1 t + c_2}{x_1 + \cdots + x_N}\right)^N \end{aligned} \tag{2.18}$$

The expression (2.18) is an unified formula for the self-similar solutions of Euler/Navier-Stokes-Korteweg equations in R^N. For example, when $N=1$, the expression (2.17) gives a kind of self-similar solutions for one-dimensional Euler/Navier-Stokes-Korteweg equations

$$u_1 = -\frac{c_1}{c_1 t + c_2} x_1, \quad \rho = \sqrt{\frac{3k_2}{2k_1}} \left(\frac{c_1 t + c_2}{x_1} \right). \tag{2.19}$$

When $N=2$, the expression (2.17) gives a kind of self-similar solutions for two-dimensional Euler/Navier-Stokes-Korteweg equations

$$u_1 = -\frac{c_1}{c_1 t + c_2} x_1, \quad u_2 = -\frac{c_1}{c_1 t + c_2} x_2, \quad \rho = \frac{8k_2}{k_1} \left(\frac{c_1 t + c_2}{x_1 + x_2} \right)^2.$$

3 Further discussion on one-dimensional case

Let us go back to observe the the system (2.12), where we split it into $N+1$ solvable ordinary differentials equations (2.13) and (2.14). It is still difficult to directly split to N ordinary differentials equations. However, it is can done for one-dimensional case. In this way, we may obtain more general self-similar solutions than (2.19).

In fact, for $N=1$, then the system (2.12) become a simple partial differential equation

$$\frac{2\dot{a}_1^2(t) - a_1(t)\ddot{a}_1(t)}{a_1^3(t)} \xi + k_1 \gamma a_1^{\gamma}(t) h^{\gamma-2}(\xi) h'(\xi) - k_2 a_1^4(t) h'''(\xi) = 0, \tag{3.20}$$

here we have used the key relation $\xi = a_1(t) x_1$. But there is no such relation to be sued for N-dimensional case.

We let $\gamma = 4$ and

$$\frac{2\dot{a}_1^2(t) - a_1(t)\ddot{a}_1(t)}{a_1^7(t)} = \alpha, \tag{3.21}$$

where α is an arbitrary constant. In this way, the equation (3.20) becomes an ordinary differential equation with respect to ξ

$$k_2 h'''(\xi) - 4k_1 h^2(\xi) h'(\xi) + \alpha \xi = 0. \tag{3.22}$$

So we can find more general self-similar solutions than (2.19)

$$u_i = -\frac{\dot{a}_1(t)}{a_1(t)} x_1, \quad \rho = a_1(t) h(\xi), \quad \xi = a_1(t) x_1, \tag{3.23}$$

where $a_1(t)$ and $h(\xi)$ satisfy equations (3.21) and (3.22) respectively.

For $\alpha = 0$, solving (3.21) and (3.22), and then (3.23) is exactly the solution (2.19).

For $\alpha \neq 0$, we can get another kind of solutions. For example, we take a solution of (3.21) as

$$a_1(t) = (c_1 t + c_2)^{-2/5}, \quad \alpha = -\frac{6c_1^2}{25}.$$

We then get a solution

$$u(x,t) = \frac{2c_1}{5}(c_1 t + c_2)^{-1} x, \quad \rho = (c_1 t + c_2)^{-2/5} h(\xi), \quad \xi = (c_1 t + c_2)^{-2/5} x,$$

where c_1 and c_2 are arbitrary constants and $h(\xi)$ is given by

$$k_2 h'''(\xi) - 4k_1 h^2(\xi) h'(\xi) + \frac{6c_1^2}{25}\xi = 0.$$

which can be integrated once and yields

$$k_2 h''(\xi) - \frac{4}{3}k_1 h^3(\xi) + \frac{3}{25}c_1^2 \xi^2 = 0. \tag{3.24}$$

This is a variable-coefficient nonlinear ordinary differential equation, whose explicit solutions are still difficult to be constructed here.

4 Conclusions and Remarks

The current analysis presents a procedure that establishes explicit analytical self-similar solutions of the N-dimensional Euler/Navier-Stokes-Korteweg equations (1.3)-(1.4). Besides the intrinsic mathematical interests, these solutions will be applicable to many physical disciplines, e.g. fluid mechanics and astrophysics. High speed flows from the release of a localized amount of energy typically lead to a velocity field in similarity variables for large time and large distance from the origin. These similarity variable solutions usually match the form investigated in this paper. Despite the progress made here and other works in the literature, many challenges still remain ahead:

(1) How to construct analytical self-solutions for the more general Euler/Navier-Stokes-Korteweg equations (1.1)-(1.2) in R^N?

(2) For N-dimensional Euler/Navier-Stokes-Korteweg equations (1.1)-(1.2) or (1.3)-(1.4), how to utilize matrix technique for constructing more general Cartesian solutions of the form

$$u = b(t) + Ax,$$

which have been obtained for the compressible Euler/Navier-Stokes equations (1.5)-(1.6) [31].

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