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Nonlinear difference equations for a modified Laguerre weight: Laguerre-Freud equations and asymptotics

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Abstract

In this paper we derive second and third order nonlinear difference equations for one of the recurrence coefficients in the three term recurrence relation of polynomials orthogonal with respect to a modified Laguerre weight. We show how these equations can be obtained from the Bäcklund transformations of the third Painlevé equation. We also show how to use nonlinear difference equations to derive a few terms in the formal asymptotic expansions in n of the recurrence coefficients.

Keywords: orthogonal polynomials, difference equations, Painlevé equations, Bäcklund transformations, asymptotic expansions.

MSC: 33C47, 39A99, 34E05, 42C05.

§1. Introduction

In recent years there has been a considerable interest to derive linear difference-differential equations for polynomials which are orthogonal with respect to a weight having a rational logarithmic derivative - the semi-classical weights, and nonlinear difference equations for their recurrence coefficients (e.g., [1, 2, 4, 3] and the references therein). In the present paper we will use two methods, commonly used in the literature of special functions:

the so-called Laguerre method [11, 12], and the Ladder operator scheme [5]. In general terms, the nonlinear difference equations for coefficients are very cumbersome. For some cases, the recurrence relation coefficients are related to solutions of the famous Painlevé equations. We refer the interested reader to [10], for basic information on the Painlevé equations, and [6, 9, 12] for examples of weights leading to the Painlevé equations for the recurrence coefficients. As we will show in this paper on the example of the generalized Laguerre polynomials, it is possible to relate nonlinear difference equations for the recurrence coefficients to the Bäcklund transformations of solutions of the corresponding Painlevé equation. Some of the so-called discrete Painlevé equations are obtained from the Bäcklund transformations of the (differential) Painlevé equations and they possess numerous nice properties. In general, the derivation of the so-called integrable (either discrete or differential) equations is very important, hence new examples of non-standard integrable discrete equations coming from the theory of orthogonal polynomials are interesting in their own right for further studies.

In this paper we take the modified Laguerre weight

$$w(x) = w(x,s) := x^{\alpha} e^{-x} e^{-s/x}, \tag{1.1}$$

where $0 < x < \infty$, $\alpha > -1$, $s > 0$. It can be represented as $w(x,s) = w_0(x) e^{-s/x}$, where $w_0(x) = x^{\alpha} e^{-x}$ is the Laguerre weight. We take the sequence of monic orthogonal polynomials with respect to (1.1), that is,

$$P_n(x,s) = x^n + \mathrm{p}_1(n,s) x^{n-1} + \ldots + P_n(0,s),$$

satisfying a three-term recurrence relation [7, 13]

$$xP_n(x) = P_{n+1}(x) + \alpha_n(s) P_n(x) + \beta_n(s) P_{n-1}(x) \tag{1.2}$$

with initial conditions $P_0(x) = 1$, $\beta_0 P_{-1}(x) = 0$. The so-called recurrence relation coefficients, that is, the parameters $\alpha_n(s)$ and $\beta_n(s)$, satisfy $\alpha_n(s) \in \mathbb{R}$, $n = 0, 1, \ldots$, and $\beta_n(s) > 0$, $n = 1, 2, \ldots$ The coefficient $\mathrm{p}_1(n,s)$ satisfies $\mathrm{p}_1(n,s) - \mathrm{p}_1(n+1,s) = \alpha_n(s)$, or, equivalently, $\mathrm{p}_1(n,s) = -\sum_{j=0}^{n-1} \alpha_j(s)$, $\mathrm{p}_1(0,s) := 0$.

The deformed (by a factor $e^{-s/x}$) Laguerre weight (1.1) was extensively studied in [2, 6, 14] (see other references therein). In [6] nonlinear difference equations for recurrence coefficients α_n and β_n were obtained. Moreover, a connection to the third Painlevé equation was established. In particular, in [2] it is shown that, for fixed n, the recurrence coefficient α_n is related to the solutions of the third Painlevé equation (as the function of s). In the present paper we will use nonlinear difference equations for α_n and β_n that were obtained via the ladder operator scheme in [6] and add new equations via the above mentioned Laguerre method. Such equations are commonly known as Laguerre-Freud equations [12]. The main objective is to derive a single nonlinear difference equation for one of the recurrence coefficients α_n in case $s \neq 0$. In case $s = 0$ the recurrence coefficients can be found explicitly. We will show that the coefficient β_n is expressed in terms of α_{n+1}, α_n and α_{n-1}. In particular, we will obtain a second order equation

$$F(s) = F(\alpha_{n+1}, \alpha_n, \alpha_{n-1}, s) = 0$$

and a third order equation

$$G(s) = G(\alpha_{n+2}, \alpha_{n+1}, \alpha_n, \alpha_{n-1}, s) = 0.$$

Next, we will use the connection of $\alpha_n(s)$ to the solutions of the third Painlevé equation to derive the explicit formulas for $\alpha_{n+2}(s)$, $\alpha_{n+1}(s)$ and $\alpha_n(s)$ in terms of the $\alpha_n(s)$ and its derivative via the Bäcklund transformations and hence show that equations $F(s) = 0$ and $G(s) = 0$ can be regarded as generalized (or non-standard) discrete Painlevé equations. We will use these equations (or the coupled equations for α_n and β_n) to obtain a few terms in the formal asymptotic expansions for the recurrence coefficients α_n and β_n and of the coefficient $\mathrm{p}_1(n, s)$ as $n \to \infty$ in case $s \neq 0$. Note that certain asymptotic expansions with particular conditions are given in [14] via the Riemann-Hilbert problems. In this paper the formal asymptotic expansions are obtained from only difference equations. We will also present some expansions for $F(0) = 0$ and $G(0) = 0$.

§2. Nonlinear difference equations

First, we will briefly recall some necessary formulas. In this section we use the notation $' = d/dx$.

The weight (1.1) is a member of the so-called semi-classical class, as it satisfies a homogeneous differential equation with polynomial coefficients [12]. Indeed, for the weight (1.1) we have $Aw' = Cw$ with

$$A(x) = x^2, \quad C(x) = -x^2 + \alpha x + s. \tag{2.1}$$

According to [12] (see also [8]), the polynomials P_n satisfy the structure relations, for all $n \geq 1$,

$$AP_n' = (l_{n-1} - C/2)P_n + \Theta_{n-1}P_{n-1}, , \tag{2.2}$$

$$AP_{n-1}' = -\frac{\Theta_{n-2}}{\beta_{n-1}}P_n + \left(l_{n-2} + (x - \alpha_{n-1})\frac{\Theta_{n-2}}{\beta_{n-1}} - \frac{C}{2}\right)P_{n-1}, \tag{2.3}$$

where l_n, Θ_n are polynomials of degree two and one, respectively, satisfying the initial conditions

$$\Theta_{-1} = D, \qquad \Theta_0 = A + (x - \alpha_0)(C/2 - l_0),$$
$$l_{-1} = C/2, \qquad l_0 = -C/2 - (x - \alpha_0)D,$$

where $D = x + \alpha_0 - 1 - \alpha$. Here, α_0 is given by the ratio of the first two moments, that is, $\alpha_0 = \mu_1/\mu_0$, where

$$\mu_1 = \int_0^\infty xw(x)dx, \quad \mu_0 = \int_0^\infty w(x)dx.$$

We will also use later on the following notation:

$$l_n(x) = l_{n,2}x^2 + l_{n,1}x + l_{n,0}, , \quad \Theta_n(x) = \Theta_{n,1}x + \Theta_{n,0}.$$

Similar to (2.2), (2.3) expressions are given in [6]. They are as follows:

$$P_n' = -B_n P_n + \beta_n A_n P_{n-1}, \tag{2.4}$$
$$P_{n-1}' = -A_{n-1} P_n + (B_n + v')P_{n-1}, \tag{2.5}$$

where $v' = -w'/w = -C/A$. Here, A_n and B_n are functions that satisfy the following fundamental compatibility conditions (see also [5]):

$$B_{n+1}(x) + B_n(x) = (x - \alpha_n)A_n(x) - v'(x), \tag{S_1}$$
$$1 + (x - \alpha_n)(B_{n+1}(x) - B_n(x)) = \beta_{n+1}A_{n+1}(x) - \beta_n A_{n-1}(x), \tag{S_2}$$
$$B_n^2(x) + v'(x)B_n(x) + \sum_{j=0}^{n-1} A_j(x) = \beta_n A_n(x)A_{n-1}(x). \tag{S_2'}$$

Using (2.2), (2.4) and (2.3), (2.5), we have

$$\beta_n A A_n = \Theta_{n-1}, \quad AB_n = C/2 - l_{n-1}.$$

We also have $A(B_n + v') = l_{n-2} + (x - \alpha_{n-1})\Theta_{n-2}/\beta_{n-1} - C/2$, which holds identically by using the following formulas from [8]:

$$A = (x - \alpha_n)(l_n - l_{n-1}) + \Theta_n - \beta_n \frac{\Theta_{n-2}}{\beta_{n-1}}, \quad n \geq 1, \tag{2.6}$$

$$l_n + (x - \alpha_n)\frac{\Theta_{n-1}}{\beta_n} = l_{n-2} + (x - \alpha_{n-1})\frac{\Theta_{n-2}}{\beta_{n-1}}, \quad n \geq 1, \tag{2.7}$$

$$l_n + l_{n-1} + (x - \alpha_n)\frac{\Theta_{n-1}}{\beta_n} = 0, \quad n \geq 0. \tag{2.8}$$

Remark 2.1. Let us define $\mathcal{P}_n = \begin{bmatrix} P_{n+1} \\ P_n \end{bmatrix}$ and write the recurrence relation (1.2) in the following matrix form:

$$\mathcal{P}_n = \mathcal{A}_n \mathcal{P}_{n-1}, \quad \mathcal{A}_n = \begin{bmatrix} x-\alpha_n & -\beta_n \\ 1 & 0 \end{bmatrix}, \quad n \geq 1, \tag{2.9}$$

with initial conditions $\mathcal{P}_0 = \begin{bmatrix} x-\alpha_0 \\ 1 \end{bmatrix}$. Also, let us write (2.2), (2.3) in the matrix form

$$A\mathcal{P}_n' = \mathcal{B}_n \mathcal{P}_n, \quad n \geq 0, \tag{2.10}$$

where

$$\mathcal{B}_n = \begin{bmatrix} l_n - C/2 & \Theta_n \\ -\Theta_{n-1}/\beta_n & l_{n-1} + (x-\alpha_n)\Theta_{n-1}/\beta_n - C/2 \end{bmatrix}. \tag{2.11}$$

From the compatibility relation for the Lax pair (2.9), (2.10) we get

$$A\mathcal{A}_n' = \mathcal{B}_n \mathcal{A}_n - \mathcal{A}_n \mathcal{B}_{n-1}, \quad n \geq 1. \tag{2.12}$$

Now, (2.12) yields two non-trivial equations from positions $(1,1)$ and $(1,2)$. They are given by equations (2.6) and (2.7), respectively.

Note, that after basic computations, (2.7) implies (2.8). In turn, (2.8) gives $\operatorname{tr}(\mathcal{B}_n) = -C$. Additionally, (2.6) implies, after some basic computations,

$$\det \mathcal{B}_n = -\frac{C^2}{4} + A \sum_{k=0}^{n} \frac{\Theta_{k-1}}{\beta_k}, \quad n \geq 1. \tag{2.13}$$

It can also be shown that relations (S_1) and (S_2) hold by using (2.6), (2.7) and (2.8). Furthermore, (S_2') agrees with (2.13).

Using the notation from [6] we have

$$A_n(x) = \frac{1}{x} + \frac{a_n}{x^2}, \quad B_n = -\frac{n}{x} + \frac{b_n}{x^2},$$

and

$$\Theta_n = \beta_{n+1}(x + a_{n+1}), \quad l_n = C/2 + (n+1)x - b_{n+1}.$$

Using [8, Lemma 3.1], we obtain

$$a_n \quad = \quad -2n - 1 + \alpha_n - \alpha, \tag{2.14}$$

$$b_n \quad = \quad s/2 - l_{n-1,0} = \beta_n - \alpha_0 - \eta_{n-1},$$

where $\mathrm{p}_1(n,s) = -(\eta_{n-1} + \alpha_0)$.

It is shown in [6] that a_n and b_n satisfy

$$\begin{aligned} b_{n+1} + b_n &= s - (2n+1+\alpha+a_n)a_n, & (2.15)\\ (b_n^2 - sb_n)(a_n + a_{n-1}) &= (ns - (2n+\alpha)b_n)a_n a_{n-1}. & (2.16)\end{aligned}$$

In addition,

$$b_n^2 - sb_n = \beta_n a_n a_{n-1}. \tag{2.17}$$

From (2.16) and (2.17) we have

$$b_n = \frac{ns - (a_n + a_{n-1})\beta_n}{2n+\alpha}. \tag{2.18}$$

Hence, equations (2.15) and (2.16) give equations for the recurrence coefficients α_n and β_n by using (2.14) and (2.18). Explicitly,

$$\begin{aligned} & s + \alpha_n(2n+1+\alpha-\alpha_n) - \frac{ns + (4n+2\alpha-\alpha_{n-1}-\alpha_n)\beta_n}{2n+\alpha}\\ &\quad = \frac{(n+1)s + (4n+4+2\alpha-\alpha_n-\alpha_{n+1})\beta_{n+1}}{2n+2+\alpha}, & (2.19)\\ & (2n-1+\alpha-\alpha_{n-1})(2n+1+\alpha-\alpha_n)\beta_n\\ &\quad = -\frac{(ns + (4n+2\alpha-\alpha_{n-1}-\alpha_n)\beta_n)(s(n+\alpha) + (\alpha_{n-1}+\alpha_n-2\alpha-4n)\beta_n)}{(2n+\alpha)^2}. & (2.20)\end{aligned}$$

This system of equations, (2.19) and (2.20), will be used later on to calculate the formal asymptotics of the recurrence coefficients in n.

To derive other nonlinear difference equations for α_n and β_n, we will use equations (2.6), (2.7) and (2.8). Using [8, Lemma 3.1] and from (2.8), evaluated at $x = \alpha_n$, we get

$$\eta_n = (\alpha_{n+1}^2 + \beta_{n+1} + \beta_{n+2} - (2n+4+\alpha)\alpha_{n+1} - s - 2\alpha_0)/2,$$

hence,

$$\mathrm{p}_1(n,s) = -(\alpha_n^2 + \beta_n + \beta_{n+1} - (2n+2+\alpha)\alpha_n - s)/2. \tag{2.21}$$

Evaluating (2.6) and (2.7) at $x = \alpha_n$ we get, respectively,

$$(\alpha_{n-1} - 2n + 1 - \alpha)\beta_n + \alpha_n(\alpha_n + \beta_n) + (2n+3+\alpha)\beta_{n+1} = (\alpha_n + \alpha_{n+1})\beta_{n+1}, \tag{2.22}$$

$$\alpha_{n-1}^2-(2n-2+\alpha)\alpha_{n-1}+(2n+2+\alpha)\alpha_n-\alpha_n^2=\beta_{n+1}-\beta_{n-1}. \tag{2.23}$$

Next, we will use equations (2.22), (2.23), (2.19) and (2.20) to derive the third order difference equation for α_n. From (2.22) we find β_{n+1} and then from (2.19) (substituting the obtained expression for β_{n+1}) we find β_n. The compatibility between these two equations (that is, equality of two expressions for β_{n+1} and β_n with n replaced by $n+1$) give an expression of α_{n+2} in terms of α_{n+1}, α_n and α_{n-1}. Hence, the following statements hold.

Theorem 2.2. *The recurrence coefficient α_n satisfies the third order difference equation*

$$G(s)=G(\alpha_{n+2},\alpha_{n+1},\alpha_n,\alpha_{n-1},s)=0,$$

where

$$G(s)=\left(\sum_{p=0}^{3}\sum_{q=0}^{4}\sum_{r=0}^{1}d_{p,q,r}\alpha_{n+1}^p\alpha_n^q\alpha_{n-1}^r\right)\alpha_{n+2}-\sum_{p=0}^{4}\sum_{q=0}^{4}\sum_{r=0}^{1}c_{p,q,r}\alpha_{n+1}^p\alpha_n^q\alpha_{n-1}^r.$$

The list of non-zero coefficients $c_{p,q,r}$ and $d_{p,q,r}$ is given in the Appendix.

Theorem 2.3. *The recurrence coefficient β_n can be expressed in terms of α_{n+1}, α_n and α_{n-1} as*

$$\beta_n=\frac{f_1(\alpha_{n+1},\alpha_n)}{f_2(\alpha_{n+1},\alpha_n,\alpha_{n-1})}, \tag{2.24}$$

where

$$\begin{aligned}
f_1 \;=\;& \alpha_{n+1}\left((\alpha+2n)\alpha_n^2-\left(\alpha(\alpha+2)+4n^2+4(\alpha+1)n\right)\alpha_n-\alpha s\right)\\
&+\alpha_n\left(8n^3+4(3\alpha+5)n^2+2\left(3\alpha^2+10\alpha+6\right)n+\alpha\left(\alpha^2+5\alpha-s+6\right)\right)\\
&+(\alpha+2n)\alpha_n^3-(2\alpha(\alpha+2n)+4n(\alpha+2n)+4(\alpha+2n))\alpha_n^2+2\alpha ns+\alpha^2 s+3\alpha s
\end{aligned}$$

and

$$\begin{aligned}
f_2 \;=\;& 2\alpha_n^2+(2\alpha_{n-1}-12n-6\alpha-6)\,\alpha_n+8n(\alpha+2n)\\
&+4\alpha(\alpha+2n)+8(\alpha+2n)-3(\alpha+2n+2)\alpha_{n-1}+(2\alpha_{n-1}-3(\alpha+2n)+2\alpha_n)\,\alpha_{n+1}.
\end{aligned}$$

Substituting β_{n+1} and β_n found before into (2.23) and (2.20), the first one gives the third order difference equation for α_n and the second one, with n replaced by $n+1$, gives a second order difference equation for α_n.

Theorem 2.4. *The recurrence coefficient α_n satisfies a second order difference equation*

$$F(\alpha_{n+1},\alpha_n,\alpha_{n-1},s)=\sum_{p=0}^{2}\sum_{q=0}^{8}\sum_{r=0}^{2}h_{p,q,r}\alpha_{n+1}^{p}\alpha_n^{q}\alpha_{n-1}^{r}=0.$$

Due to very cumbersome expressions, the coefficients of F are not presented in this paper explicitly. However, we will present an equation for a_n using (2.14). The coefficient a_n satisfies a second order difference equation,

$$\tilde{F}(a_{n+1},a_n,a_{n-1},s)=\sum_{p=0}^{2}\sum_{q=0}^{8}\sum_{r=0}^{2}k_{p,q,r}a_{n+1}^{p}a_n^{q}a_{n-1}^{r}=0.$$

The list of non-zero coefficients $k_{p,q,r}$ is given in the Appendix.

Remark 2.5. Let us take (2.13) as

$$-l_n^2+\Theta_n\frac{\Theta_{n-1}}{\beta_n}=-\frac{C^2}{4}+A\sum_{k=0}^{n}\frac{\Theta_{k-1}}{\beta_k}. \tag{2.25}$$

Equating coefficients in (2.25) we get, for the coefficient of x^1 and x^0, respectively,

$$-2l_{n,1}l_{n,0}+\Theta_{n,0}\frac{\Theta_{n-1,1}}{\beta_n}+\Theta_{n,1}\frac{\Theta_{n-1,0}}{\beta_n}=-\frac{\alpha s}{2},\qquad -l_{n,0}^2+\Theta_{n,0}\frac{\Theta_{n-1,0}}{\beta_n}=-\frac{s^2}{4}.$$

Thus, after basic computations,

$$\beta_{n+1}(\alpha_{n+1}+\alpha_n-2-2n-\alpha)=(\alpha+2n+2)\left(\sum_{j=0}^{n}\alpha_j\right)+(n+1)s, \tag{2.26}$$

$$-\left(\sum_{j=0}^{n}\alpha_j-\beta_{n+1}+\frac{s}{2}\right)^2+\beta_{n+1}\left(2n+3-\alpha_{n+1}+\alpha\right)\left(2n+1-\alpha_n+\alpha\right)=-\frac{s^2}{4}. \tag{2.27}$$

Eliminating $\sum_{j=0}^{n}\alpha_j$ between these equations and using the expression for β_n from Theorem 2.3 and then the expression for α_{n+2} from Theorem 2.2, we get the second order difference equation for α_n from Theorem 2.4, which gives an alternative way to derive this equation.

Remark 2.6. When $s = 0$ we get the classical Laguerre weight. Using the same technique as above, it is not difficult to show that, in this case, $A(x) = x$ and $C(x) = -x + \alpha$. Equations (2.6), (2.7) and (2.8) give us

$$\alpha_n + \beta_n - \beta_{n-1} = 0,$$
$$(2 + \alpha_{n-1} - \alpha_n)\beta_n + n = 0,$$
$$\alpha_n = 2n + 1 + \alpha,$$

which yield $\beta_n = n(n + \alpha)$ and, hence, $\mathrm{p}_1(n, 0) = -n(n + \alpha)$. Accordingly, the recurrence relation coefficients can be found explicitly.

§3. Connection to the Bäcklund transformations of the third Painlevé equation

In this section we use the notation $' = d/ds$ (or sometimes $' = d/dt$ when we deal with the third Painlevé equation).

In [6] it was shown that as functions of s the coefficients a_n and b_n (related to the recurrence coefficients α_n and β_n via (2.14) and (2.18)) satisfy the following coupled differential system:

$$\begin{aligned} s\frac{da_n}{ds} &= 2b_n + (2n + 1 + \alpha + a_n)a_n - s, \\ s\frac{db_n}{ds} &= \frac{2}{a_n}(b_n^2 - sb_n) + (2n + \alpha + 1)b_n - ns. \end{aligned}$$

From this system it is easy to derive a second order differential equation for $a_n = a_n(s)$:

$$a_n'' = \frac{(a_n')^2}{a_n} - \frac{a_n'}{s} + (2n + 1 + \alpha)\frac{a_n^2}{s^2} + \frac{a_n^3}{s^2} + \frac{\alpha}{s} - \frac{1}{a_n}. \tag{3.1}$$

This equation can easily be transformed to the standard third Painlevé equation for $y = y(t)$:

$$y'' = \frac{(y')^2}{y} - \frac{y'}{t} + \frac{1}{t}(\alpha_p y^2 + \beta_p) + \gamma_p y^3 + \frac{\delta_p}{y}. \tag{PIII}$$

Indeed, by taking a bit different from [6] substitution $a_n(s) = t/(2y(t))$ with $t = 2\sqrt{s}$, we get from (3.1) equation (PIII) with parameters

$$\alpha_p = -2\alpha, \quad \beta_p = -2(2n + 1 + \alpha), \quad \gamma_p = -\delta_p = 1.$$

The third Painlevé equation possesses the following properties [10, p. 150]:

1. If $y(t)$ is a solution of (PIII) with parameters $(\alpha_p, \beta_p, \gamma_p, \delta_p)$, then $-y(t)$ is a solution of (PIII) with parameters $(-\alpha_p, -\beta_p, \gamma_p, \delta_p)$. We shall refer to this transformation as T_1 following the notation in [10].

2. If $y(t)$ is a solution of (PIII) with parameters $(\alpha_p, \beta_p, \gamma_p, \delta_p)$, then $1/y(t)$ is a solution of (PIII) with parameters $(-\beta_p, -\alpha_p, -\delta_p, -\gamma_p)$. We shall refer to this transformation as T_2.

Moreover, the third Painlevé equation with $\gamma_p = -\delta_p = 1$ possesses the following Bäcklund transformations [10, Theorem 34.4], relating solutions with different values of the parameters: if $y(t, \alpha_p, \beta_p)$ is a solution of (PIII), then the function $\tilde{y}(t, \tilde{\alpha}_p, \tilde{\beta}_p)$ is a solution of (PIII) with new values of the parameters

$$\tilde{\alpha}_p = \varepsilon_1(\varepsilon_2 B - \varepsilon_3 A + 4)/2, \quad \tilde{\beta}_p = \varepsilon_2 B/2 + \varepsilon_3 A/2,$$

where

$$B = \beta_p + \alpha_p\varepsilon_1 - 2, \quad A = \beta_p - \alpha_p\varepsilon_1 + 2, \quad \varepsilon_j^2 = 1, \ j = 1, 2, 3,$$

and

$$\tilde{y}(t, \tilde{\alpha}_p, \tilde{\beta}_p) = \frac{2tR(t)(R(t) - 2)}{2tR'(t) + (\varepsilon_3 A - \varepsilon_2 B)R(t) - 2\varepsilon_3 A},$$

with

$$R(t) = y' - \varepsilon_1 y^2 - (\alpha_p\varepsilon_1 - 1)y/t + 1,$$

provided that $R(t)(R(t) - 2) \neq 0$. Since this transformation depends on ε_j, $j = 1, 2, 3$, where ε_j takes values either 1 or -1, it is convenient to refer to this transformation as $T_{\varepsilon_1,\varepsilon_2,\varepsilon_3}$.

Theorem 3.1. *Let $y = y(t)$ be a solution of (PIII) with parameters*

$$\alpha_p = -2\alpha, \quad \beta_p = -2(2n + 1 + \alpha), \quad \gamma_p = -\delta_p = 1.$$

Then, the functions $y_1 = y_1(t)$ and $y_2 = y_2(t)$ with

$$\begin{aligned} y_1 &= 4y^2\left(\frac{n+1}{y(2\alpha + 4n - ty + 3) + t - ty'} + \frac{\alpha + n + 1}{y(2\alpha + 4n + ty + 3) + t - ty'}\right) - y, \\ y_2 &= 4y^2\left(\frac{n}{y(2\alpha + 4n - ty + 1) + t + ty'} + \frac{\alpha + n}{y(2\alpha + 4n + ty + 1) + t + ty'}\right) - y, \end{aligned}$$

are, respectively, solution of (PIII) with parameters

$$\alpha_p^{(1)} = -2\alpha, \qquad \beta_p^{(1)} = -2(2n+3+\alpha), \qquad \gamma_p^{(1)} = -\delta_p^{(1)} = 1,$$
$$\alpha_p^{(2)} = -2\alpha, \qquad \beta_p^{(2)} = -2(2n-1+\alpha), \qquad \gamma_p^{(2)} = -\delta_p^{(2)} = 1.$$

Indeed, to obtain y_1 we use the composition of transformations $T_1 \circ T_2 \circ T_{-1,-1,1} \circ T_2 \circ T_{1,-1,1}$ and for y_2 we use $T_1 \circ T_2 \circ T_{1,1,-1} \circ T_2 \circ T_{-1,1,-1}$. There might be other compositions of transformations leading to the same result.

Hence, using the connection to differential equation for $a_n(s)$ we can easily obtain formulas for $a_{n+1}(s)$ and $a_{n-1}(s)$ in terms of $a_n(s)$ and its derivative. We can verify in any computer algebra that equations $G(s) = 0$ and $\tilde{F}(s) = 0$ are automatically satisfied. Therefore, equations $F(s) = 0$ and $G(s) = 0$ can be viewed as (non-standard) discrete Painlevé equations.

§4. An application: asymptotics of recurrence coefficients

In this section we will obtain a few terms in the formal asymptotic expansions of α_n, β_n and $\mathrm{p}_1(n,s)$ via nonlinear difference equations (2.19) and (2.20). The formulas are found by substituting an appropriate Ansatz and by determining the unknown coefficients. Indeed, substituting

$$\alpha_n(s) = 2n+1+\alpha + \sum_{i=1}^{\infty} \frac{c_i}{n^{i/3}} \quad \text{and} \quad \beta_n(s) = n^2 + \alpha n + \sum_{i=-2}^{\infty} \frac{d_i}{n^{1/3}}$$

we can find the unknown coefficients step by step by expanding expressions in (2.19) and (2.20) in powers of $n^{1/3}$. The formula for $\mathrm{p}_1(n,s)$ follows from (2.21). Thus, we have the following statement.

Theorem 4.1. *For fixed s and $n \to \infty$ the following expansions hold:*

$$\begin{aligned}
\alpha_n(s) &= 2n+1+\alpha + \frac{s^{2/3}}{2^{1/3}}\frac{1}{n^{1/3}} - \frac{\alpha s^{1/3}}{3\cdot 2^{2/3} n^{2/3}} + O\left(\frac{1}{n^{4/3}}\right), \\
\beta_n &= n^2 + \alpha n + \frac{s^{2/3} n^{2/3}}{2\cdot 2^{1/3}} - \frac{1}{3} 2^{1/3} \alpha s^{1/3} + \frac{1}{36}(6\alpha^2 - 1) + O\left(\frac{1}{n^{1/3}}\right), \\
\mathrm{p}_1(n,s) &= -n^2 - \alpha n - \frac{3 s^{2/3} n^{2/3}}{2\cdot 2^{1/3}} + O\left(n^{1/3}\right).
\end{aligned}$$

Remark 4.2. The results in Theorem 4.1 for α_n and β_n are in agreement with [14] (see also [2]). The expression for $\mathrm{p}_1(n,s)$ is new to authors' knowledge. Here we use only nonlinear difference equations to find a formal expansion in n.

In general, asymptotic expansions of difference equations is an interesting subject which is being developed. For example, we can also find expansions of the form

$$\alpha_n = 2n+1+\alpha+\frac{s^{2/3}}{(2n+1+\alpha)^{1/3}}-\frac{\alpha s^{1/3}}{3(2n+1+\alpha)^{2/3}}+\dots$$

Using the second order difference equation $F(\alpha_{n+1},\alpha_n,\alpha_{n-1},0)=0$ (with $s=0$) or the third order equation $G(\alpha_{n+2},\alpha_{n+1},\alpha_n,\alpha_{n-1},0)=0$ we can obtain the expansion

$$\alpha_n = 2n+1+\alpha+\frac{c}{n^2}+\frac{c(2c-\alpha-1)}{n^3}+O\left(\frac{1}{n^4}\right),$$

where c is arbitrary. For $s=0$ there exist also algebraic expansions of the form

$$\alpha_n = 2n+1+\alpha+\frac{c}{n^{4/3}}+\frac{18c^2}{n^{5/3}}+\frac{243c^3}{n^2}+O\left(\frac{1}{n^{7/3}}\right),$$

where c is arbitrary. When $c=0$ we get $\alpha_n = 2n+1+\alpha$ for the last two expansions, which corresponds to the classical Laguerre weight. Our observations might be interesting for researchers dealing with asymptotic expansions of difference equations.

§Appendix

The list of non-zero coefficients of $G(\alpha_{n+2},\alpha_{n+1},\alpha_n,\alpha_{n-1},s)$ in Theorem 2.2:

$$\begin{aligned}
c_{0,0,0} &= 8\alpha s(\alpha+2n+2),\\
c_{0,0,1} &= \alpha s(\alpha+2n+1),\\
c_{0,1,0} &= -4\alpha^4-20\alpha^3-64n^4-32(4\alpha+5)n^3-16\left(6\alpha^2+15\alpha+2\right)n^2\\
&\quad +2n\left(-16\alpha^3-60\alpha^2+\alpha(s-16)+32\right)+\alpha^2(s-8)+\alpha(32-5s),\\
c_{0,1,1} &= 4\alpha^3+24\alpha^2+32n^3+48(\alpha+2)n^2+8\left(3\alpha^2+12\alpha+8\right)n-\alpha(s-32),\\
c_{0,2,0} &= 11\alpha^3+57\alpha^2+88n^3+12(11\alpha+19)n^2+2\left(33\alpha^2+114\alpha+52\right)n-\alpha(s-52),\\
c_{0,2,1} &= -7\alpha^2-36\alpha-28n^2-4(7\alpha+18)n-32,\\
c_{0,3,0} &= -10\alpha^2-48\alpha-40n^2-8(5\alpha+12)n-32,\\
c_{0,3,1} &= 3\alpha+6n+12,
\end{aligned}$$

$$
\begin{aligned}
c_{0,4,0} &= 3\alpha + 6n + 12, \\
c_{1,0,0} &= 4\alpha^4 + 44\alpha^3 + 64n^4 + 32(4\alpha+11)n^3 + 16\left(6\alpha^2 + 33\alpha + 38\right)n^2 \\
&\quad + n\left(32\alpha^3 + 264\alpha^2 - 2\alpha(s-304) + 320\right) - \alpha^2(s-152) + \alpha(160-9s), \\
c_{1,0,1} &= -3\alpha^3 - 33\alpha^2 - 24n^3 - 12(3\alpha+11)n^2 - 6\left(3\alpha^2 + 22\alpha + 38\right)n - \alpha(s+114) - 120, \\
c_{1,1,0} &= -48\alpha^2 - 192\alpha - 192n^2 - 192(\alpha+2)n - 120, \\
c_{1,1,1} &= -4\alpha^2 - 16n^2 - 16\alpha n + 40, \\
c_{1,2,0} &= -12\alpha^2 - 16\alpha - 48n^2 - 16(3\alpha+2)n + 40, \\
c_{1,2,1} &= 8\alpha + 16n + 18, \\
c_{1,3,0} &= 10\alpha + 20n + 18, \\
c_{1,3,1} &= -2, \\
c_{1,4,0} &= -2, \\
c_{2,0,0} &= -11\alpha^3 - 75\alpha^2 - 88n^3 - 12(11\alpha+25)n^2 - 2\left(33\alpha^2 + 150\alpha + 124\right)n + \alpha(s-124), \\
c_{2,0,1} &= 8\alpha^2 + 54\alpha + 32n^2 + 4(8\alpha+27)n + 88, \\
c_{2,1,0} &= 12\alpha^2 + 80\alpha + 48n^2 + 16(3\alpha+10)n + 88, \\
c_{2,1,1} &= -2\alpha - 4n - 16, \\
c_{2,2,0} &= -16, \\
c_{2,2,1} &= -2, \\
c_{2,3,0} &= -2, \\
c_{3,0,0} &= 10\alpha^2 + 32\alpha + 40n^2 + 8(5\alpha+8)n, \\
c_{3,0,1} &= -7\alpha - 14n - 22, \\
c_{3,1,0} &= -10\alpha - 20n - 22, \\
c_{3,1,1} &= 2, \\
c_{3,2,0} &= 2, \\
c_{4,0,0} &= -3\alpha - 6n, \\
c_{4,0,1} &= 2, \\
c_{4,1,0} &= 2, \\
d_{0,0,0} &= \alpha s(\alpha + 2n + 3),
\end{aligned}
$$

$$
\begin{aligned}
d_{0,1,0} &= -3\alpha^3 - 3\alpha^2 - 24n^3 - 12(3\alpha+1)n^2 - 6\left(3\alpha^2+2\alpha-2\right)n - \alpha(s-6),\\
d_{0,1,1} &= 3\alpha^2 + 6\alpha + 12n^2 + 12(\alpha+1)n,\\
d_{0,2,0} &= 8\alpha^2 + 10\alpha + 32n^2 + 4(8\alpha+5)n,\\
d_{0,2,1} &= -5\alpha - 10n - 6,\\
d_{0,3,0} &= -7\alpha - 14n - 6,\\
d_{0,3,1} &= 2,\\
d_{0,4,0} &= 2,\\
d_{1,0,0} &= 4\alpha^3 + 24\alpha^2 + 32n^3 + 48(\alpha+2)n^2 + 8\left(3\alpha^2+12\alpha+8\right)n - \alpha(s-32),\\
d_{1,0,1} &= -3\alpha^2 - 18\alpha - 12n^2 - 12(\alpha+3)n - 24,\\
d_{1,1,0} &= -4\alpha^2 - 32\alpha - 16n^2 - 16(\alpha+4)n - 24,\\
d_{1,1,1} &= 8,\\
d_{1,2,0} &= -2\alpha - 4n + 8,\\
d_{1,2,1} &= 2,\\
d_{1,3,0} &= 2,\\
d_{2,0,0} &= -7\alpha^2 - 20\alpha - 28n^2 - 4(7\alpha+10)n,\\
d_{2,0,1} &= 5\alpha + 10n + 14,\\
d_{2,1,0} &= 8\alpha + 16n + 14,\\
d_{2,1,1} &= -2,\\
d_{2,2,0} &= -2,\\
d_{3,0,0} &= 3\alpha + 6n,\\
d_{3,0,1} &= -2,\\
d_{3,1,0} &= -2.
\end{aligned}
$$

The list of non-zero coefficients of $\tilde{F}(a_{n+1}, a_n, a_{n-1}, s)$ after Theorem 2.4:

$$
\begin{aligned}
k_{0,0,2} &= (n+1)s^2(n+\alpha+1),\\
k_{0,1,1} &= s^2(2n+\alpha+1)(2n+\alpha+2),\\
k_{0,1,2} &= s(s(2n+\alpha+2) - \alpha(2n+\alpha+1)),
\end{aligned}
$$

$$
\begin{aligned}
k_{0,2,0} &= s^2(2n+\alpha+1)^2,\\
k_{0,0,2} &= (n+1)s^2(n+\alpha+1),\\
k_{0,1,1} &= s^2(2n+\alpha+1)(2n+\alpha+2),\\
k_{0,1,2} &= s(s(2n+\alpha+2)-\alpha(2n+\alpha+1)),\\
k_{0,2,0} &= s^2(2n+\alpha+1)^2,\\
k_{0,2,1} &= s\left(s(6n+3\alpha+4)-\alpha(2n+\alpha+1)^2\right),\\
k_{0,2,2} &= -4n^2-2((s+2)\alpha+2)n+(s+1)\left(s-(\alpha+1)^2\right),\\
k_{0,3,0} &= 2s^2(2n+\alpha+1),\\
k_{0,3,1} &= -2\left(8n^3+12(\alpha+1)n^2+2(\alpha(s+3\alpha+6)+3)n-(s+\alpha+1)\left(s-(\alpha+1)^2\right)\right),\\
k_{0,3,2} &= -8n^2-4(2\alpha+3)n-\alpha(s+2\alpha+6)-4,\\
k_{0,4,0} &= \left(-4n^2-4(\alpha+1)n-(\alpha+1)^2+s\right)\left(4n^2+4(\alpha+1)n+(\alpha+1)^2+s\right),\\
k_{0,4,1} &= -16n^3-24(\alpha+2)n^2-12(\alpha+1)(\alpha+3)n-\alpha\left(2(\alpha+3)^2+s\right)-8,\\
k_{0,4,2} &= -4n^2-4(\alpha+3)n-\alpha(\alpha+6)-6,\\
k_{0,5,0} &= -4(2n+\alpha+1)^3,\\
k_{0,5,1} &= -6(2n+\alpha+1)(2n+\alpha+2),\\
k_{0,5,2} &= -2(2n+\alpha+2),\\
k_{0,6,0} &= -6(2n+\alpha+1)^2,\\
k_{0,6,1} &= -2(6n+3\alpha+4),\\
k_{0,6,2} &= -1,\\
k_{0,7,0} &= -4(2n+\alpha+1),\\
k_{0,7,1} &= -2,\\
k_{0,8,0} &= -1,\\
k_{1,0,1} &= s^2\left(\alpha^2+2n\alpha+\alpha+2n(n+1)\right),\\
k_{1,0,2} &= s(s(2n+\alpha+2)-\alpha(2n+\alpha+1)),\\
k_{1,1,0} &= s^2(2n+\alpha)(2n+\alpha+1),\\
k_{1,1,1} &= 2s(2n+\alpha+1)(2s-\alpha(2n+\alpha+1)),\\
k_{1,1,2} &= \alpha^3+(6n-2s+3)\alpha^2+4(n+1)(3n-s)\alpha+4n(n(2n+3)+2)+2\left(s^2+2\alpha+1\right),
\end{aligned}
$$

$$
\begin{aligned}
k_{1,2,0} &= s\left(s(6n+3\alpha+2)-\alpha(2n+\alpha+1)^2\right), \\
k_{1,2,1} &= \alpha^4+4(2n+1)\alpha^3+3(8n(n+1)-2s+3)\alpha^2+2(2n+1)(8n(n+1)-3s+5)\alpha \\
&\quad +4\left(\left(n^2+n+1\right)(2n+1)^2+s^2\right), \\
k_{1,2,2} &= 8n^3+4(3\alpha+4)n^2+2(\alpha(3\alpha+8)+6)n+\alpha(-3s+\alpha(\alpha+4)+6)+4, \\
k_{1,3,0} &= 2\left(8n^3+12(\alpha+1)n^2+2(\alpha(-s+3\alpha+6)+3)n+(\alpha+1)^3+s^2-s\alpha(\alpha+1)\right), \\
k_{1,3,1} &= 2\left(8n^3+12(\alpha+1)n^2+6(\alpha(\alpha+2)+2)n+\alpha(-2s+\alpha(\alpha+3)+6)+4\right), \\
k_{1,3,2} &= (2n+\alpha)^2, \\
k_{1,4,0} &= -16n^3-24\alpha n^2-12\left(\alpha^2-1\right)n-\alpha\left(2\alpha^2+s-6\right)+4, \\
k_{1,4,1} &= -3(2n+\alpha)(2n+\alpha+2), \\
k_{1,4,2} &= -2(2n+\alpha+2), \\
k_{1,5,0} &= -6(2n+\alpha)(2n+\alpha+1), \\
k_{1,5,1} &= -8(2n+\alpha+1), \\
k_{1,5,2} &= -2, \\
k_{1,6,0} &= -2(6n+3\alpha+2), \\
k_{1,6,1} &= -4, \\
k_{1,7,0} &= -2, \\
k_{2,0,0} &= ns^2(n+\alpha), \\
k_{2,0,1} &= s(\alpha+(2n+\alpha)(s+\alpha)), \\
k_{2,0,2} &= (-2n+s-\alpha-1)(2n+s+\alpha+1), \\
k_{2,1,0} &= s(\alpha+(2n+\alpha)(s+\alpha)), \\
k_{2,1,1} &= -\alpha^3-(6n+2s+3)\alpha^2-4(n(3n+s+3)+1)\alpha+2s^2-4n(n(2n+3)+2)-2, \\
k_{2,1,2} &= -2s\alpha, \\
k_{2,2,0} &= s^2-(s+1)\alpha^2-4n(n+1)-2n(s+2)\alpha-2\alpha-1, \\
k_{2,2,1} &= 8n^3+4(3\alpha+2)n^2+\left(6\alpha^2+8\alpha+4\right)n+\alpha(-3s+\alpha(\alpha+2)+2), \\
k_{2,2,2} &= 4n^2+4(\alpha+1)n+\alpha(\alpha+2)+2, \\
k_{2,3,0} &= 8n^2+(8\alpha+4)n+\alpha(-s+2\alpha+2), \\
k_{2,3,1} &= (2n+\alpha+2)^2,
\end{aligned}
$$

$$
\begin{aligned}
k_{2,4,0} &= -4n^2 - 4(\alpha-1)n - (\alpha-2)\alpha + 2,\\
k_{2,4,1} &= -2(2n+\alpha),\\
k_{2,4,2} &= -1,\\
k_{2,5,0} &= -2(2n+\alpha),\\
k_{2,5,1} &= -2,\\
k_{2,6,0} &= -1.
\end{aligned}
$$

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