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The smallest eigenvalue of large Hankel matrices

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ABSTRACT

We investigate the large N behavior of the smallest eigenvalue, λ_N, of an $(N+1)\times(N+1)$ Hankel (or moments) matrix $\mathcal{H}_N$, generated by the weight $w(x)=x^{\alpha}(1-x)^{\beta}$, $x\in[0,1]$, $\alpha>-1$, $\beta>-1$. By applying the arguments of Szegö, Widom and Wilf, we establish the asymptotic formula for the orthonormal polynomials $P_n(z), z\in\mathbb{C}\setminus[0,1]$, associated with $w(x)$, which are required in the determination of λ_N. Based on this formula, we produce the expressions for λ_N, for large N.

Using the parallel algorithm presented by Emmart, Chen and Weems, we show that the theoretical results are in close proximity to the numerical results for sufficiently large N.

1. Introduction

Let $\mu(x)$ be a positive measure with the bounded support $I(\subseteq\mathbb{R})$ and define the moment sequence of $\mu(x)$ by

$$h_k := \int_I x^k d\mu(x) \qquad k=0,1,2,\dots. \tag{1.1}$$

We know that the Hankel determinant plays a significant role in the theory of random Hermitian matrices. Associated with $\mu(x)$, the $(N+1)\times(N+1)$ Hankel matrix $\mathcal{H}_N$, is defined by

$$\mathcal{H}_N := (h_{m+n})_{m,n=0}^{N}. \tag{1.2}$$

It is known that the smallest eigenvalue of the Hankel matrix is intimately related to the distribution function $\mu(x)$. We are motivated by the fact that the smallest eigenvalue depends $\mu'(x)$ in a non-trivial way.

Let $I=[a,b]$, where a and b are fixed constants, such that the Szegö condition,

$$\int_a^b \frac{\ln w(x)}{\sqrt{(b-x)(x-a)}}dx > -\infty, \tag{1.3}$$

with $w(x)=\mu'(x)$ is satisfied. The asymptotic behavior of the Hankel determinants for large enough N is given by Szegö [2–5].

Let λ_N denote the smallest eigenvalue of $\mathcal{H}_N$. The behavior of λ_N, N large, has attracted a lot of attention. See e.g. Szegö [3], Widom and Wilf [7–9], Chen et al. [10,11,15], Berg et al. [15,16], etc. Szegö [3] studied the special cases for $w(x)$, defined

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on I, which can either be a finite or infinite. For finite cases, if $w(x)=1, x\in(-1,1)$ or $w(x)=1, x\in(0,1)$, the smallest eigenvalues for large N are given, respectively, by*

$$\lambda_N \simeq 2^{\frac{9}{4}}\pi^{\frac{3}{2}}\sqrt{N}\left(\sqrt{2}+1\right)^{-2N-3},$$

$$\lambda_N \simeq 2^{\frac{15}{4}}\pi^{\frac{3}{2}}\sqrt{N}\left(\sqrt{2}+1\right)^{-4N-4}.$$

Widom and Wilf [7] found a kind of 'universal' law, where they show that if $w(x)>0, x\in[a,b]$, and the Szegö condition (1.3) is satisfied, then

$$\lambda_N \simeq A\sqrt{N}B^{-N},$$

where A and B are computable constants depending on $w(x)$, a, b, and are independent of N.

For cases of an infinite interval, Szegö [3] chose the Gaussion weight ($w(x)=\mathrm{e}^{-x^2},\ x\in\mathbb{R}$) and Laguerre weight ($w(x)=\mathrm{e}^{-x},\ x\geq 0$). The corresponding smallest eigenvalues are approximated, respectively, by

$$\lambda_N \simeq \mathrm{e}2^{\frac{13}{4}}\pi^{\frac{3}{2}}N^{\frac{1}{4}}\mathrm{e}^{-2\sqrt{2N}},$$
$$\lambda_N \simeq \mathrm{e}2^{\frac{7}{2}}\pi^{\frac{3}{2}}N^{\frac{1}{4}}\mathrm{e}^{-4\sqrt{N}}.$$

Chen and Lawrence generalized the results of Szegö in [3]. By means of Dyson's Coulomb fluid method, they deduced the case for $w(x)=\mathrm{e}^{-x^{\beta}},\ x\in[0,+\infty),\ \beta>\frac{1}{2}$ and then gave two asymptotic formulas of λ_N for $\beta=n+1/2$ and $\beta\neq n+1/2$, $n=1,2,3,\ldots,$ respectively. See [10] for details.

Recently, Zhu, Chen, Emmart and Weems [13] studied the asymptotic behavior of λ_N where they chose the weight $w(x)=x^{\alpha}\mathrm{e}^{-x^{\beta}}, x\in[0,\infty),\ \alpha>-1,\ \beta>\frac{1}{2}$. This generalized the work in [3,10]. In specially, they obtained the approximation formula of λ_N for the Laguerre weight $w(x)=x^{\alpha}\mathrm{e}^{-x},\ x\in[0,\infty),\ \alpha>-1$.

We note that the smallest eigenvalues of the examples given above are exponentially small. Hence, it's hard to determine the smallest eigenvalues of the Hankel matrices associated with these weights by numerical techniques.

This paper is organized as follows, firstly, we establish the asymptotic formula for the orthonormal polynomials $P_n(z)$ associated with the weight $w(x)$ in Theorem 2.1. Then in Theorem 2.2, we give the specific asymptotic expression of λ_N. Finally, we present some numerical results compared with our theoretical results in Section 3.

In order to meet the demands of some proofs in our results, we define the whole complex plane by $\mathbb{C}\cup\{\infty\}$, and the unit disc by

$$D:=\{z\in\mathbb{C}\big||z|\leq 1\},$$

with its boundary (unit circle)

$$\partial D:=\{z\in\mathbb{C}\big||z|=1\}.$$

2. Main results

In this section, we shall produce the asymptotic expression for λ_N, the smallest eigenvalue of the $(N+1)\times(N+1)$ Hankel matrix $\mathcal{H}_N$. We consider the weight

$$w(x)=x^{\alpha}(1-x)^{\beta}, x\in[0,1],\ \alpha>-1,\ \beta>-1, \tag{2.1}$$

which satisfies

$$\int_0^1\frac{\ln w(x)}{\sqrt{x(1-x)}}dx=-2\pi(\alpha+\beta)\ln 2>-\infty.$$

The $N+1$ by $N+1$ Hankel matrix $\mathcal{H}_N$ is defined by

$$\mathcal{H}_N:=(h_{m+n})_{m,n=0}^{N},$$

where h_{m+n} is the $(m+n)$th moment with respected to $w(x)$, reads

$$h_{m+n}:=\int_0^1 x^{m+n}w(x)dx=\int_0^1 x^{m+n+\alpha}(1-x)^{\beta}dx,\ \ m,n=0,1,2,\ldots.$$

By the definition of the Gamma function

$$\Gamma(x)=\int_0^{\infty}t^{x-1}\mathrm{e}^{-t}dt,\quad \Re\, x>0,$$

* Throughout this paper, the relation $a_n\simeq b_n$ means $\lim_{n\to\infty}a_n/b_n$=1.

and the Beta function

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt,\quad \Re\, x>0, \Re\, y>0,$$

with the relationship

$$B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

we have

$$h_{m+n}=\frac{\Gamma(\alpha+m+n+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+m+n+2)},\quad m,n=0,1,2,\dots,N. \tag{2.2}$$

The Hankel matrix for $\alpha=\beta=0$ is the Hilbert matrix $\mathcal{H}_N=\left(\frac{1}{m+n+1}\right)_{m,n=0}^{N}$, for which some partial results were obtained in [3–7] (in which the factor $-1/2\pi$ of Lemma 2 should be changed to $-1/4\pi$). The following two examples give $\mathcal{H}_N$ for some special choices of α and β.

Example 2.1. For $\alpha>-1$,and $\beta=0$, i.e. $w(x)=x^{\alpha}, x\in[0,1]$, the Hankel matrix reads

$$\mathcal{H}_N=\left(\frac{1}{1+m+n+\alpha}\right)_{m,n=0}^{N}.$$

Example 2.2. For $\alpha=\beta=-\frac{1}{2}$, i.e. $w(x)=\frac{1}{\sqrt{x(1-x)}}$, $x\in[0,1]$, the Hankel matrix is given by

$$\mathcal{H}_N=\left(\frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}+m+n\right)}{\Gamma(1+m+n)}\right)_{m,n=0}^{N}.$$

For generic α and β, we shall show that there is an asymptotic formula for λ_N, the smallest eigenvalues of the Hankel matrices, in the following form:

$$\lambda_N\simeq\frac{8\sqrt{N}}{\psi(\alpha,\beta)\left(1+\sqrt{2}\right)^{4N+2}},$$

where

$$\psi(\alpha,\beta)=2^{-\frac{3}{4}}\pi^{-\frac{3}{2}}\left(1+2^{\frac{1}{2}}\right)^{2\alpha+2}\left(1+2^{-\frac{1}{2}}\right)^{2\beta}.$$

See details in the proof of the Theorem 2.2.

Let $\{P_n(x)\}_{n=0}^{\infty}$ be the orthonormal polynomials associated with our weight $w(x)$, i.e.,

$$\int_0^1 P_m(x)P_n(x)w(x)dx=\delta_{m,n},\quad m,n=0,1,\dots,N.$$

We define $P_n(x)$ and the kth moment h_k of $w(x)$ to be

$$P_n(x):=\sum_{k=0}^{n}a_{n,k}x^k,$$

$$h_k:=\int_0^1 x^k w(x)dx.$$

Then, the orthogonality relation can be rewritten as

$$\delta_{m,n}=\sum_{i,j=0}^{N}a_{m,i}h_{i+j}a_{n,j},\quad m,n=0,1,\dots,N,$$

which, in matrix form, reads

$$\mathcal{I}=\mathcal{A}_N\mathcal{H}_N\mathcal{A}_N^T, \tag{2.3}$$

where

$$\mathcal{A}_N:=\begin{bmatrix} a_{0,0} & 0 & 0 & \cdots & 0\\ a_{1,0} & a_{1,1} & 0 & \cdots & 0\\ a_{2,0} & a_{2,1} & a_{2,2} & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ a_{N,0} & a_{N,1} & a_{N,2} & \cdots & a_{N,N}\end{bmatrix}.$$

From (2.3), we find

$$\mathcal{H}_N^{-1} = \mathcal{A}_N^T\left(\mathcal{A}_N\mathcal{A}_N^T\right)\left(\mathcal{A}_N^T\right)^{-1},$$

which shows that $\frac{1}{\lambda_N}$ is the largest eigenvalue of $\mathcal{A}_N\mathcal{A}_N^T$. This is not a new result, for more details, see [3,5,7]. Denoting by $\sigma_{m,\,n}$ the $(m,\,n)$th entry of $\mathcal{A}_N\mathcal{A}_N^T$, we have

$$\sigma_{m,n} := \left(\mathcal{A}_N\mathcal{A}_N^T\right)_{m,n} = \sum_{k=0}^{N} a_{m,k}a_{n,k} = \frac{1}{2\pi}\int_0^{2\pi} P_m\left(\mathrm{e}^{\mathrm{i}\theta}\right)P_n\left(\mathrm{e}^{-\mathrm{i}\theta}\right)d\theta.$$

We shall make use of $P_n(z)$ to study the behavior of $\sigma_{m,\,n}$ and thus of λ_N. Szegö [2] has proved the case of the interval $[-1, 1]$ if $\mu(x)$ is absolutely continuous and, Geronimus has proved that case in [1] for general $\mu(x)$. We can deduce the case [0,1] by a linear transformation $\varphi : x \mapsto \frac{1}{2}(x+1), x \in [-1, 1]$. Since if $\Phi_n(x)$ are the orthonormal polynomials associated with the weight $w(x) = \mu'(\varphi(x))$ then $P_n(x) = \Phi_n(2x-1), x \in [0, 1]$. Based on the discussion in [1, Thm.9.3], [2, Thm.12.1.2], and [7, Lem.2], etc., we obtain an asymptotic expression for $P_n(z),\ z \in \mathbb{C}\setminus[0, 1]$.

Theorem 2.1. *The asymptotic behavior of the orthonormal polynomials* $P_n(z)$ *with respect to the weight* $w(x) = x^\alpha(1-x)^\beta,\ x \in [0, 1],\ \alpha > -1,\ \beta > -1,$ *uniformly for* z *on compact subsets of* $\mathbb{C}\setminus[0, 1]$, *satisfies*

$$P_n(z) \simeq \frac{1}{\sqrt{\pi}}\zeta^n(z)A(\zeta(z)),$$

Here

$$\zeta(z) = \left(\sqrt{z} + \sqrt{z-1}\right)^2, \tag{2.4}$$

with the square roots taking the positive values as $\Re z \to \infty$, *and noting that* $|\zeta(z)| > 1$ *in* $z \in \mathbb{C}\setminus[0, 1]$, *then* $|A(\zeta(z))|$ *is given by*

$$\left|A\left(r\mathrm{e}^{\mathrm{i}\theta}\right)\right| = \exp\left[\frac{1}{4\pi}\int_{-\pi}^{\pi}\ln\left(\cos^{2\alpha}\frac{t}{2}\sin^{2\beta}\frac{t}{2}|\sin t|\right)\frac{1-r^2}{1-2r\cos(\theta-t)+r^2}dt\right]. \tag{2.5}$$

Proof. See [7, Lemma 2]. □

Lemma 1. *The maximum of* $|\zeta(\mathrm{e}^{\mathrm{i}\theta})|$, *for* $0 \le \theta < 2\pi$, *is attained at* $\theta = \pi$, *and*

$$\max_{0\le\theta<2\pi}\left|\zeta\left(\mathrm{e}^{\mathrm{i}\theta}\right)\right| = \left(1+\sqrt{2}\right)^2.$$

Proof. From (2.4), we have

$$|\zeta(z)| = |z| + |z-1| + 2\Re\left\{\sqrt{z(\bar{z}-1)}\right\},$$

substituting $z = \mathrm{e}^{\mathrm{i}\theta}, \theta \in [0, 2\pi]$, we find,

$$\left|\zeta\left(\mathrm{e}^{\mathrm{i}\theta}\right)\right| = 1 + \sqrt{2-2\cos\theta} + \sqrt{2-2\cos\theta + 2\sqrt{2-2\cos\theta}}, \tag{2.6}$$

the upper bound for $|\zeta(\mathrm{e}^{\mathrm{i}\theta})|$ follows immediately. □

Lemma 2. *The entries of* $A_NA_N^T$, $\{\sigma_{m,n}\}_{m,n=0}^N$, *have the following upper bound:*

$$|\sigma_{m,n}| \le C\cdot\frac{|1+\sqrt{2}|^{2(m+n)}}{\sqrt{m+n+1}}.$$

Proof. Recall that

$$\sigma_{m,n} = \frac{1}{2\pi}\int_0^{2\pi} P_m\left(\mathrm{e}^{\mathrm{i}\theta}\right)P_n\left(\mathrm{e}^{-\mathrm{i}\theta}\right)d\theta.$$

According to Theorem 2.1, the asymptotic behavior of $\sigma_{m,\,n}$ depends on the factor $\zeta^n(z)$. Note that $\partial D \cap [0, 1] = 1$, so for any $\varepsilon > 0$, we have

$$\begin{aligned}
|\sigma_{m,n}| &\le \frac{1}{2\pi}\int_{-\varepsilon}^{\varepsilon}\left|P_m\left(\mathrm{e}^{\mathrm{i}\theta}\right)P_n\left(\mathrm{e}^{\mathrm{i}\theta}\right)\right|d\theta + \frac{1}{2\pi}\int_{\varepsilon}^{2\pi-\varepsilon}\left|P_m\left(\mathrm{e}^{\mathrm{i}\theta}\right)P_n\left(\mathrm{e}^{\mathrm{i}\theta}\right)\right|d\theta \\
&\le \frac{1}{2\pi}\int_{-\varepsilon}^{\varepsilon}\left|P_m\left(\mathrm{e}^{\mathrm{i}\theta}\right)P_n\left(\mathrm{e}^{\mathrm{i}\theta}\right)\right|d\theta + C_0\cdot\int_{\varepsilon}^{2\pi-\varepsilon}\left|\zeta\left(\mathrm{e}^{\mathrm{i}\theta}\right)\right|^{m+n}d\theta \\
&= \frac{1}{2\pi}\int_{-\varepsilon}^{\varepsilon}\left|P_m\left(\mathrm{e}^{\mathrm{i}\theta}\right)P_n\left(\mathrm{e}^{\mathrm{i}\theta}\right)\right|d\theta + C_0\cdot\int_{\varepsilon}^{2\pi-\varepsilon}\mathrm{e}^{-(m+n)\left(-\ln\left|\zeta\left(\mathrm{e}^{\mathrm{i}\theta}\right)\right|\right)}d\theta.
\end{aligned} \tag{2.7}$$

Here, we have used $P_n(\mathrm{e}^{-\mathrm{i}\theta}) = \overline{P_n(\mathrm{e}^{\mathrm{i}\theta})}$, and C_0 is a constant.

We first deal with the second integral of (2.7). Since $\frac{d}{d\theta}|\zeta(e^{i\theta})||_{\theta=\pi}=0$, $\frac{d}{d\theta}\ln|\zeta(e^{i\theta})||_{\theta=\pi}=0$. Applying the formula (2.6), we have

$$\frac{d^2}{d\theta^2}\left|\zeta\left(e^{i\theta}\right)\right| = -\frac{1}{2}\sin\frac{\theta}{2} + \frac{1+\cos\theta\left(3+\csc\frac{\theta}{2}\right)}{4\sqrt{\sin^2\frac{\theta}{2}+\sin\frac{\theta}{2}}} - \frac{\left(2+\sqrt{2}\right)\sin\theta\left(\sin\theta+\cos\frac{\theta}{2}\right)}{8\left(\sin^2\frac{\theta}{2}+\sin\frac{\theta}{2}\right)^{\frac{3}{2}}}, \tag{2.8}$$

and it is immediate that,

$$-\frac{d^2}{d\theta^2}\ln\left|\zeta\left(e^{i\theta}\right)\right|\Big|_{\theta=\pi} = \frac{\sqrt{2}}{8} > 0.$$

Applying the Laplace method when $m+n$ large enough, we get

$$C_0\cdot\int_{\varepsilon}^{2\pi-\varepsilon} e^{-(m+n)\left(-\ln\left|\zeta\left(e^{i\theta}\right)\right|\right)}d\theta \le C_1\cdot\frac{|\zeta(-1)|^{m+n}}{\sqrt{m+n}},$$

where C_1 is also a constant. So for all m, n there is another suitable constant C such that

$$C_1\cdot\frac{|\zeta(-1)|^{m+n}}{\sqrt{m+n}} \le C\cdot\frac{|\zeta(-1)|^{m+n}}{\sqrt{m+n+1}}, \quad m,n=0,1,2,\ldots. \tag{2.9}$$

To estimate the first integral in (2.7), let R_ε be a rectangle with its four vertices $\cos\varepsilon \pm i\sin\varepsilon$, $1\pm i\tan\varepsilon$. The arc of ∂D given by $|\theta|\le\varepsilon$ is contained within R_ε. Applying the Theorem 3.3.1 of [2], the polynomials $P_m(z)P_n(z)$ has only real zeros, so its maximum absolute value on R_ε must be attained on the horizontal sides of R_ε. Hence, according to the Theorem 2.1, we have

$$\limsup_{m+n\to+\infty}\max_{z\in R_\varepsilon}|P_m(z)P_n(z)|^{1/(m+n)} = \left|\zeta\left(e^{i\left(\varepsilon+O\left(\varepsilon^2\right)\right)}\right)\right|.$$

So as $m+n\to\infty$,

$$\int_{-\varepsilon}^{\varepsilon}\left|P_m\left(e^{i\theta}\right)P_n\left(e^{i\theta}\right)\right|d\theta = O\left(\left|\zeta\left(e^{i(2\varepsilon)}\right)\right|^{m+n}\right),$$

since $\left|\zeta\left(e^{i(2\varepsilon)}\right)\right| > \left|\zeta\left(e^{i\left(\varepsilon+O\left(\varepsilon^2\right)\right)}\right)\right|$ if $\varepsilon\to0$. Hence, as $\varepsilon\to0$, we have

$$\int_{-\varepsilon}^{\varepsilon}\left|P_m\left(e^{i\theta}\right)P_n\left(e^{i\theta}\right)\right|d\theta = o\left(\frac{|\zeta(-1)|^{m+n}}{\sqrt{m+n}}\right),\ m+n\to+\infty, \tag{2.10}$$

since by Lemma 1, $\zeta(e^{i\theta})$ attains its maximum modulus at $\theta=\pi$, and not at $\theta=0$. Thus the Lemma 2 is proved based on formulas (2.7), (2.9) and (2.10). □

Remark 2.1. The Laplace method [4, p.96, Problem 201] gives

$$\int_a^b f(t)e^{-\lambda g(t)}dt \simeq e^{-\lambda g(c)}f(c)\sqrt{\frac{2\pi}{\lambda g''(c)}}, \qquad \text{as} \quad \lambda\to\infty,$$

where g assumes a strict minimum over $[a, b]$ at an interior critical point c, such that

$$g'(c)=0, \qquad g''(c)>0 \qquad \text{and} \qquad f(c)\ne0.$$

Lemma 3. *For $z=e^{i\pi}=-1$, we find*

$$|A(\zeta(-1))| = 2^{-\frac{3}{4}}\left(1+2^{\frac{1}{2}}\right)^{\alpha+1}\left(1+2^{-\frac{1}{2}}\right)^{\beta}. \tag{2.11}$$

Proof. From (2.5), we obtain

$$\begin{aligned}
\ln|A(\zeta(-1))| &= \frac{1}{4\pi}\int_{-\pi}^{\pi}\ln\left(\cos^{2\alpha}\frac{t}{2}\sin^{2\beta}\frac{t}{2}|\sin t|\right)\frac{1-\eta^2}{1-2\eta\cos(\pi-t)+\eta^2}dt \\
&= \frac{\ln 2}{2\pi}\int_0^{\pi}\frac{1-\eta^2}{1+2\eta\cos t+\eta^2}dt && (a)\\
&\quad + \frac{2\alpha+1}{2\pi}\int_0^{\pi}\ln\left(\cos\frac{t}{2}\right)\frac{1-\eta^2}{1+2\eta\cos t+\eta^2}dt && (b)\\
&\quad + \frac{2\beta+1}{2\pi}\int_0^{\pi}\ln\left(\sin\frac{t}{2}\right)\frac{1-\eta^2}{1+2\eta\cos t+\eta^2}dt, && (c)
\end{aligned}$$

where $\eta := \left(1+\sqrt{2}\right)^2$. Note that if $\alpha = \beta = -\frac{1}{2}$, then (b) and (c) are zero, this is Example 2.2. Applying the Residue theorem, the integral (a) can be rewritten as

$$\frac{\ln 2}{4\pi \mathrm{i}} \int_{|z|=1} \frac{1-\left(1+\sqrt{2}\right)^4}{\left(1+\sqrt{2}\right)^2\left[z+\left(1+\sqrt{2}\right)^{-2}\right]\left[z+\left(1+\sqrt{2}\right)^2\right]} dz = -\frac{\ln 2}{2}. \tag{2.12}$$

For the integral (b), we have

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{\pi} \ln\left(\cos\frac{t}{2}\right)\frac{1-\eta^2}{1+2\eta\cos t+\eta^2}dt \\
&= \frac{1-\eta^2}{4\pi}\int_0^{\pi}\frac{\ln\left(\frac{1+\cos t}{2}\right)}{1+2\eta\cos t+\eta^2}dt \qquad (x := \cos t) \\
&= \frac{1-\eta^2}{4\pi}\int_{-1}^{1}\frac{\ln\left(\frac{1+x}{2}\right)}{1+2\eta x+\eta^2}\frac{dx}{\sqrt{1-x^2}} \qquad \left(y := \frac{1+x}{2}\right) \\
&= \frac{1-\eta^2}{4\pi}\int_0^1 \frac{\ln y}{1+2\eta(2y-1)+\eta^2}\frac{dy}{\sqrt{y(1-y)}} \qquad (\eta = (1+\sqrt{2})^2) \\
&= \frac{\sqrt{2}}{4\pi}\int_0^1\frac{\ln(1-y)}{\sqrt{y(1-y)}(y-2)}dy \qquad \text{(See the Appendix)} \\
&= \frac{\ln\left(1+\sqrt{2}\right)}{2}.
\end{aligned} \tag{2.13}$$

For the integral (c), we find

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{\pi}\ln\left(\sin\frac{t}{2}\right)\frac{1-\eta^2}{1+2\eta\cos t+\eta^2}dt \\
&= -\frac{\sqrt{2}}{4\pi}\int_0^1\frac{\ln(1-y)}{\sqrt{y(1-y)}(y+1)}dy \\
&= \frac{\ln\left(1+\sqrt{2}\right)}{2} - \frac{\ln 2}{4}.
\end{aligned} \tag{2.14}$$

So from (2.12)–(2.14), we have

$$\begin{aligned}
|A(\zeta(-1))| &= \exp\left[-\frac{\ln 2}{2} + \frac{(2\alpha+1)\ln\left(1+\sqrt{2}\right)}{2} + \frac{(2\beta+1)\ln\frac{(1+\sqrt{2})^2}{2}}{4}\right] \\
&= 2^{-\frac{3}{4}}\left(1+2^{\frac{1}{2}}\right)^{\alpha+1}\left(1+2^{-\frac{1}{2}}\right)^{\beta}.
\end{aligned}$$

□

Theorem 2.2. *For λ_N, we have*

$$\lambda_N \simeq 2^{\frac{15}{4}}\pi^{\frac{3}{2}}\left(1+2^{\frac{1}{2}}\right)^{-2\alpha}\left(1+2^{-\frac{1}{2}}\right)^{-2\beta}N^{\frac{1}{2}}\left(1+2^{\frac{1}{2}}\right)^{-4(N+1)}.$$

Proof. Based on the discussion in Theorem 2.1 and Lemma 2, we find

$$\begin{aligned}
\sigma_{m,n} =& \frac{1}{2\pi}\int_{-\varepsilon}^{\varepsilon}P_m\left(e^{\mathrm{i}\theta}\right)P_n\left(e^{-\mathrm{i}\theta}\right)d\theta + \frac{1}{2\pi}\int_{\varepsilon}^{2\pi-\varepsilon}P_m\left(e^{\mathrm{i}\theta}\right)P_n\left(e^{-\mathrm{i}\theta}\right)d\theta \\
=& \frac{1}{2\pi^2}\int_{\varepsilon}^{2\pi-\varepsilon}\left|\zeta\left(e^{\mathrm{i}\theta}\right)\right|^{m+n}\left[\operatorname{sgn}\zeta\left(e^{\mathrm{i}\theta}\right)\right]^{m-n}\left|A\left(\zeta\left(e^{\mathrm{i}\theta}\right)\right)\right|^2 d\theta \\
&+ o\left(\frac{|\zeta(-1)|^{m+n}}{\sqrt{m+n}}\right), \quad (m,n\to+\infty)
\end{aligned}$$

where $\operatorname{sgn}(z) := \frac{z}{|z|}, z \in \mathbb{C}$. It should now be easy to determine the asymptotic behavior of the entries, $\{\sigma_{m,n}\}_{m,n=0}^N$, as $m, n \to +\infty$ with $m-n$ bounded. We know that the maximum of $|\zeta(e^{\mathrm{i}\theta})|$ occurs at $\theta = \pi$, and by the Laplace method for asymptotic expansion of an integral, combined with Lemma 1., we get

$$\begin{aligned}\sigma_{m,n} &\simeq \frac{(-1)^{m-n}|A(\zeta(-1))|^2|\zeta(-1)|^{m+n}}{\sqrt[4]{2}\sqrt{\pi^3}\sqrt{m+n}} \\ &= (-1)^{m-n}2^{-\frac{1}{4}}\pi^{-\frac{3}{2}}|A(\zeta(-1))|^2\left(1+\sqrt{2}\right)^{2(m+n)}(m+n)^{-\frac{1}{2}}.\end{aligned} \tag{2.15}$$

where $m, n \to +\infty$ with $m-n$ bounded.

We will now find the behavior of the eigenvalue λ_N, for large N.

Let

$$g(\theta) := \left|\zeta\left(\mathrm{e}^{\mathrm{i}\theta}\right)\right| \quad \text{and} \quad \psi(\alpha,\beta) := \frac{|A(\zeta(-1))|^2\sqrt{\eta}}{\sqrt{2}\sqrt{\pi^3}\sqrt{|g''(\pi)|}}.$$

From (2.4), we can get

$$\zeta(-1) = -\left(1+\sqrt{2}\right)^2 \quad \text{and} \quad \frac{d^2g(\theta)}{d\theta^2}\bigg|_{\theta=\pi} = -\frac{1}{2} - \frac{3\sqrt{2}}{8}.$$

Hence, from (2.11), and an easy computation gives

$$\psi(\alpha,\beta) = 2^{-\frac{3}{4}}\pi^{-\frac{3}{2}}\left(1+2^{\frac{1}{2}}\right)^{2\alpha+2}\left(1+2^{-\frac{1}{2}}\right)^{2\beta}. \tag{2.16}$$

For the sake of the completeness, we give the standard method, following closely in the footsteps Widom and Wilf's [7, P. 342, Thm.], applied to our weight function $w(x)$. We define,

$$\rho_{m,n} := (-1)^{m-n}\eta^{m+n}, \quad \eta = \left(1+\sqrt{2}\right)^2, \tag{2.17}$$

and

$$\tau_{m,n} := \sigma_{m,n} - \left(\frac{\psi(\alpha,\beta)}{\sqrt{2N}}\right)\rho_{m,n}. \tag{2.18}$$

Fixing an ε and a sufficiently large N_ε. It follows from (2.15) that if m and n are sufficiently large, but $|m-n| \le N_\varepsilon$, we shall have

$$\left|\sigma_{m,n} - \psi(\alpha,\beta)\frac{(-1)^{m-n}\eta^{m+n}}{\sqrt{m+n}}\right| \le \varepsilon\frac{\eta^{m+n}}{\sqrt{m+n}}.$$

Therefore if N is sufficiently large and much larger than N_ε with $N - N_\varepsilon \le m, n \le N$,

$$\begin{aligned}|\tau_{m,n}| &= \left|\sigma_{m,n} - \psi(\alpha,\beta)\frac{(-1)^{m-n}\eta^{m+n}}{\sqrt{2N}}\right| \\ &\le \left|\sigma_{m,n} - \psi(\alpha,\beta)\frac{(-1)^{m-n}\eta^{m+n}}{\sqrt{m+n}}\right| + \psi(\alpha,\beta)\eta^{m+n}\left[\frac{1}{\sqrt{m+n}} - \frac{1}{\sqrt{2N}}\right] \\ &\le \varepsilon_1\frac{\eta^{m+n}}{\sqrt{2N-2N_\varepsilon}} + \varepsilon_2\frac{\eta^{m+n}}{\sqrt{2N-2N_\varepsilon}} \\ &\le \varepsilon\frac{\eta^{m+n}}{\sqrt{N}} = \frac{\left(1+\sqrt{2}\right)^{2(m+n)}\varepsilon}{\sqrt{N}}.\end{aligned} \tag{2.19}$$

where ε_1, ε_2 are arbitrary small.

It follows from Lemma 2 that for all m, n,

$$|\tau_{m,n}| \le C_1\frac{\left(1+\sqrt{2}\right)^{2(m+n)}}{\sqrt{m+n+1}}, \tag{2.20}$$

where C_1 is a constant. Denote by p_N the maximum modulus of the eigenvalues of $(\tau_{m,n})_{m,n=0}^N$. Then from (2.19) and (2.20) we obtain

$$
\begin{aligned}
p_N^2 &\le \sum_{m,n=0}^{N} \tau_{m,n}^2 \\
&\le \frac{\varepsilon^2}{N}\sum_{m,n=N-N_\varepsilon}^{N} \eta^{2(m+n)} + \left(\sum_{m=N-N_\varepsilon}^{N}\sum_{n=0}^{N-N_\varepsilon} + \sum_{m=0}^{N-N_\varepsilon}\sum_{n=0}^{N}\right)\frac{C_1^2\eta^{2(m+n)}}{m+n+1} \\
&\le \frac{\varepsilon^2\eta^{4N+4}}{\left(\eta^2-1\right)^2 N} + C_2\frac{\eta^{2(2N-N_\varepsilon)}}{2N-N_\varepsilon},
\end{aligned}
$$

where C_2 is another constant. Assuming N_ε to be sufficiently large in comparison to ε, this will simply for large enough N to N_ε

$$
|p_N| \le \left(\frac{\varepsilon^2\eta^{4N+4}}{\left(\eta^2-1\right)^2 N} + \frac{3\varepsilon^2\eta^{4N+4}}{\left(\eta^2-1\right)^2 N}\right)^{\frac{1}{2}} \le \frac{2\varepsilon\eta^{2N+2}}{\left(\eta^2-1\right)\sqrt{N}} = \frac{\left(1+\sqrt{2}\right)^{4N+2}\varepsilon}{2\sqrt{2}\sqrt{N}}. \tag{2.21}
$$

Let λ_N^{-1} denote the largest eigenvalue of the matrix $(\sigma_{m,n})_{m,n=0}^N$. It follows from (2.17) , (2.18) and (2.21) that if q_N is the largest eigenvalue of the matrix $(\rho_{m,n})_{m,n=0}^N$, then

$$
\frac{q_N\psi(\alpha,\beta)}{\sqrt{2N}} - p_N \le \frac{1}{\lambda_N} \le \frac{q_N\psi(\alpha,\beta)}{\sqrt{2N}} + p_N,
$$

and so for sufficiently large N, we have

$$
\frac{q_N\psi(\alpha,\beta)}{\sqrt{2N}} - \frac{2\varepsilon\eta^{2N+2}}{\left(\eta^2-1\right)\sqrt{N}} \le \frac{1}{\lambda_N} \le \frac{q_N\psi(\alpha,\beta)}{\sqrt{2N}} + \frac{2\varepsilon\eta^{2N+2}}{\left(\eta^2-1\right)\sqrt{N}},
$$

where $\eta = \left(1+\sqrt{2}\right)^2$ and $\psi(\alpha,\beta) = 2^{-\frac{3}{4}}\pi^{-\frac{3}{2}}\left(1+2^{\frac{1}{2}}\right)^{2\alpha+2}\left(1+2^{-\frac{1}{2}}\right)^{2\beta}$.

But $(\rho_{m,n})_{m,n=0}^N$ is of rank 1, so its only nonzero eigenvalue (and certainly the largest) is given by the trace of $(\rho_{m,n})_{m,n=0}^N$, i.e.

$$
q_N = \sum_{m=0}^{N}\eta^{2m} = \frac{\eta^{2N+2}-1}{\eta^2-1} \simeq \frac{\left(1+\sqrt{2}\right)^{4N+2}}{4\sqrt{2}}.
$$

Hence,

$$
\begin{aligned}
\lambda_N &\simeq \frac{\sqrt{2N}}{q_N\psi(\alpha,\beta)} \simeq \frac{\left(\eta^2-1\right)\sqrt{2N}}{\psi(\alpha,\beta)\eta^{2N+2}} \\
&= 2^{\frac{15}{4}}\pi^{\frac{3}{2}}\left(1+2^{\frac{1}{2}}\right)^{-2\alpha}\left(1+2^{-\frac{1}{2}}\right)^{-2\beta}N^{\frac{1}{2}}\left(1+2^{\frac{1}{2}}\right)^{-4(N+1)}.
\end{aligned}
$$

□

Remark 2.2. Putting $\alpha=\beta=0$, Szegö's classical result [3] for the weight function $w(x)=1$ is recovered:

$$
\lambda_N \simeq 2^{\frac{15}{4}}\pi^{\frac{3}{2}}N^{\frac{1}{2}}\left(\sqrt{2}-1\right)^{4N+4}.
$$

3. Comparing with numerical results

It is well known that Hankel matrices (moment matrices) of this form are extremely ill-conditioned. This can also been seen from the analytic formula, where the dominant term of λ_N is $\left(1+\sqrt{2}\right)^{-4(N+1)}$. Due to the ill-conditioned nature of these matrices, standard eigensolver packages based on double precision floating values can only solve for small values of N, for example $N<20$, before the available precision is exhausted (53 bit in the mantissa, 11 bits in the exponent).

In [14], Emmart, Chen and Weems developed an efficient parallel algorithm based on arbitrary precision arithmetic and the Secant method that can handle the extreme ill-conditioning and we employ their algorithms here for our numeric computations. We use the numeric results to test the convergence of our asymptotic formulas to the actual smallest eigenvalues for various N and several values of the parameters α and β. Even with efficient software, the computation times for the largest size, $N=1000$, require almost 10 hours of CPU time on a modern Core i7 processor.

Tables 1–3 give samples of numerical λ_N compared with theoretical λ_N. The errors of λ_N for some special choices of α and β are illustrated by Figs. 1–4.

In the following, we will present some of our other numerical results.

From Tables 1–3 and Figs. 1, 2, We note that the Theoretical λ_N is slightly less than the Numerical λ_N for the the cases $\alpha\beta>0$ and $\alpha\beta<0$. And the other way around for $\alpha=\beta=0$. What is more, for the case $\alpha<0, \beta=0$, we can find an interesting point $\alpha=-\frac{1}{2}, \beta=0$ where the errors for virous N approximate to zero, see Fig. 2.

Table 1
Numerical vs. theoretical values of λ_N for $\alpha = \beta = -\frac{1}{2}$, see Example 2.2.

Size $N+1$	Numerical λ_N	Theoretical λ_N	Error (%)
25	8.0295×10^{-36}	7.9829×10^{-36}	0.5804
50	6.0370×10^{-74}	6.0198×10^{-74}	0.2849
100	2.3866×10^{-150}	2.3832×10^{-150}	0.1409
150	8.1510×10^{-227}	8.1434×10^{-227}	0.0932
200	2.6230×10^{-303}	2.6212×10^{-303}	0.0701
250	8.1711×10^{-380}	8.1665×10^{-380}	0.0563
300	2.4937×10^{-456}	2.4925×10^{-456}	0.0467
350	7.5033×10^{-533}	7.5003×10^{-533}	0.0400
400	2.2344×10^{-609}	2.2336×10^{-609}	0.0350
450	6.6016×10^{-686}	6.5995×10^{-686}	0.0318
500	1.9383×10^{-762}	1.9378×10^{-762}	0.0280
550	5.6626×10^{-839}	5.6611×10^{-839}	0.0265
600	1.6474×10^{-915}	1.6470×10^{-915}	0.0233
700	1.3805×10^{-1068}	1.3802×10^{-1068}	0.0200
800	1.1449×10^{-1221}	1.1447×10^{-1221}	0.0175
900	9.4211×10^{-1375}	9.4197×10^{-1375}	0.0155
1000	7.7042×10^{-1528}	7.7031×10^{-1528}	0.0140

Table 2
List of numerical results compared with theoretical values.

α	β	Size $N+1$	Numerical λ_N	Theoretical λ_N	Error (%)
$-\frac{7}{8}$	$-\frac{1}{8}$	100	3.0997×10^{-150}	3.0907×10^{-150}	0.2911
		200	3.4042×10^{-303}	3.3993×10^{-303}	0.1449
		300	3.2355×10^{-456}	3.2324×10^{-456}	0.0965
		400	2.8988×10^{-609}	2.8967×10^{-609}	0.0723
		500	2.5144×10^{-762}	2.5130×10^{-762}	0.0578
		600	2.1369×10^{-915}	2.1359×10^{-915}	0.0482
		700	1.7906×10^{-1068}	1.7899×10^{-1068}	0.0413
		800	1.4851×10^{-1221}	1.4845×10^{-1221}	0.0361
		900	1.2220×10^{-1374}	1.2216×10^{-1374}	0.0321
		1000	9.9926×10^{-1528}	9.9897×10^{-1528}	0.0289
$-\frac{1}{2}$	$-\frac{7}{8}$	100	3.5723×10^{-150}	3.5593×10^{-150}	0.3640
		200	3.9218×10^{-303}	3.9147×10^{-303}	0.1812
		300	3.7270×10^{-456}	3.7225×10^{-456}	0.1206
		400	3.3389×10^{-609}	3.3359×10^{-609}	0.0904
		500	2.8961×10^{-762}	2.8940×10^{-762}	0.0723
		600	2.4612×10^{-915}	2.4597×10^{-915}	0.0602
		700	2.0624×10^{-1068}	2.0613×10^{-1068}	0.0516
		800	1.7104×10^{-1221}	1.7096×10^{-1221}	0.0451
		900	1.4074×10^{-1374}	1.4068×10^{-1374}	0.0401
		1000	1.1508×10^{-1527}	1.1504×10^{-1527}	0.0361
$-\frac{1}{2}$	2	100	1.6637×10^{-151}	1.6439×10^{-151}	1.1930
		200	1.8189×10^{-304}	1.8080×10^{-304}	0.5998
		300	1.7261×10^{-457}	1.7192×10^{-457}	0.4006
		400	1.5453×10^{-610}	1.5407×10^{-610}	0.3008
		500	1.3398×10^{-763}	1.3366×10^{-763}	0.2407
		600	1.1383×10^{-916}	1.1360×10^{-916}	0.2007
		700	9.5365×10^{-1070}	9.5201×10^{-1070}	0.1721
		800	7.9078×10^{-1223}	7.8959×10^{-1223}	0.1506
		900	6.5060×10^{-1376}	6.4973×10^{-1376}	0.1339
		1000	5.3197×10^{-1529}	5.3133×10^{-1529}	0.1205
2	$-\frac{1}{2}$	100	3.0068×10^{-152}	2.9060×10^{-152}	3.3526
		200	3.2511×10^{-305}	3.1961×10^{-305}	1.6921
		300	3.0740×10^{-458}	3.0392×10^{-458}	1.1316
		400	2.7469×10^{-611}	2.7235×10^{-611}	0.8500
		500	2.3790×10^{-764}	2.3628×10^{-764}	0.6807
		600	2.0197×10^{-917}	2.0082×10^{-917}	0.5676
		700	1.6911×10^{-1070}	1.6829×10^{-1070}	0.4867
		800	1.4018×10^{-1223}	1.3958×10^{-1223}	0.4260
		900	1.1529×10^{-1376}	1.1486×10^{-1376}	0.3788
		1000	9.4248×10^{-1530}	9.3926×10^{-1530}	0.3410

Table 3
List of numerical results compared with theoretical values.

α	β	Size $N+1$	Numerical λ_N	Theoretical λ_N	Error (%)
0	0	100	5.7797×10^{-151}	5.7827×10^{-151}	0.0519
		200	6.3585×10^{-304}	6.3601×10^{-304}	0.0261
		300	6.0468×10^{-457}	6.0478×10^{-457}	0.0174
		400	5.4190×10^{-610}	5.4197×10^{-610}	0.0131
		500	4.7013×10^{-763}	4.7018×10^{-763}	0.0105
		600	3.9959×10^{-916}	3.9963×10^{-916}	0.0087
		700	3.3487×10^{-1069}	3.3489×10^{-1069}	0.0075
		800	2.7774×10^{-1222}	2.7776×10^{-1222}	0.0065
		900	2.2855×10^{-1375}	2.2856×10^{-1375}	0.0058
		1000	1.8690×10^{-1528}	1.8691×10^{-1528}	0.0052
1	2	100	1.1930×10^{-152}	1.1683×10^{-152}	2.0716
		200	1.2985×10^{-305}	1.2849×10^{-305}	1.0458
		300	1.2304×10^{-458}	1.2218×10^{-458}	0.6994
		400	1.1101×10^{-611}	1.0949×10^{-611}	0.5254
		500	9.5390×10^{-765}	9.4989×10^{-765}	0.4207
		600	8.1019×10^{-918}	8.0735×10^{-918}	0.3508
		700	6.7861×10^{-1071}	6.7657×10^{-1071}	0.3009
		800	5.6262×10^{-1224}	5.6114×10^{-1224}	0.2633
		900	4.6283×10^{-1377}	4.6175×10^{-1377}	0.2341
		1000	3.7840×10^{-1530}	3.7760×10^{-1530}	0.2108
2	1	100	6.0492×10^{-153}	5.8413×10^{-153}	3.4369
		200	6.5383×10^{-306}	6.4245×10^{-306}	1.7405
		300	6.1811×10^{-459}	6.1091×10^{-459}	1.1653
		400	5.5230×10^{-612}	5.4746×10^{-612}	0.8758
		500	4.7830×10^{-765}	4.7494×10^{-765}	0.7016
		600	4.0605×10^{-918}	4.0367×10^{-918}	0.5851
		700	3.3999×10^{-1071}	3.3829×10^{-1071}	0.5019
		800	2.8181×10^{-1224}	2.8057×10^{-1224}	0.4393
		900	2.3178×10^{-1377}	2.3087×10^{-1377}	0.3907
		1000	1.8947×10^{-1530}	1.8880×10^{-1530}	0.3517
10	10	100	7.7362×10^{-163}	2.8934×10^{-163}	62.5995
		200	5.3174×10^{-316}	3.1823×10^{-316}	40.1536
		300	4.2830×10^{-469}	3.0260×10^{-469}	29.3481
		400	3.5259×10^{-622}	2.7118×10^{-622}	23.0900
		500	2.9052×10^{-775}	2.3526×10^{-775}	19.0220
		600	2.3852×10^{-928}	1.9995×10^{-928}	16.1690
		700	1.9497×10^{-1081}	1.6756×10^{-1081}	14.0587
		800	1.5871×10^{-1234}	1.3898×10^{-1234}	12.4348
		900	1.2871×10^{-1387}	1.1436×10^{-1387}	11.1468
		1000	1.0403×10^{-1540}	9.3519×10^{-1541}	10.1004

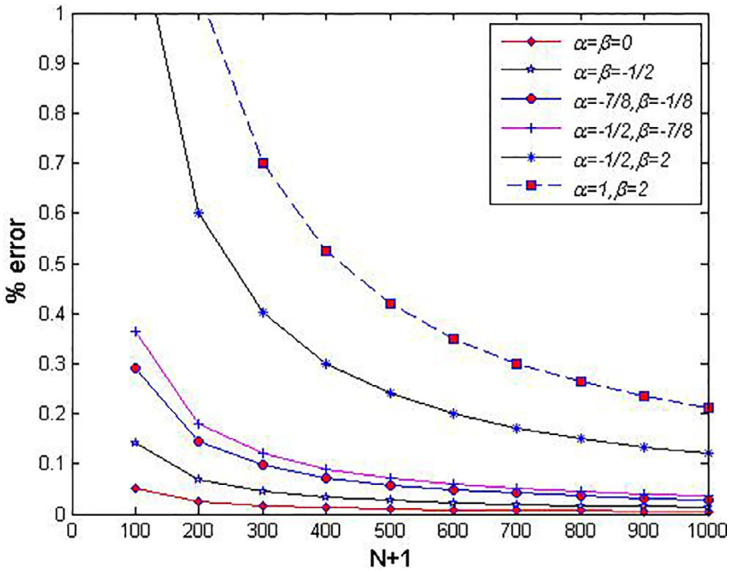

Fig. 1. The percentage error of the theoretical values of λ_N when compared with those obtained numerically, for various α and β.

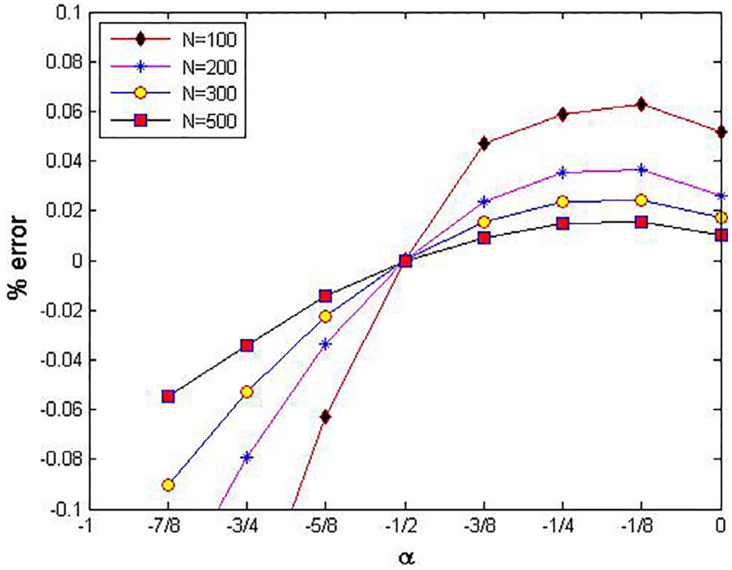

Fig. 2. The percentage error of the theoretical values of λ_N when compared with those obtained numerically, for different values of N with $-1 < \alpha < 0, \beta = 0$.

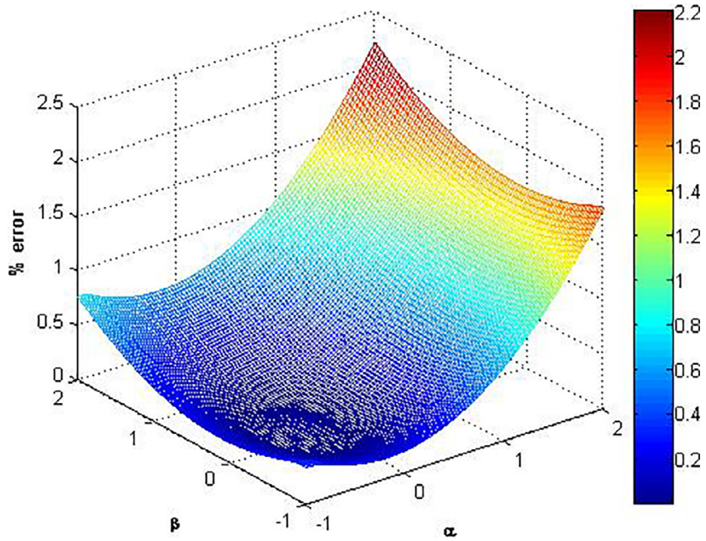

Fig. 3. The percentage error of the theoretical values of $\lambda_N, N+1 = 200$ when compared with those obtained numerically, for $-1 < \alpha \leq 2, -1 < \beta \leq 2$.

Remark 3.1. In Fig. 2,

$$\%\ error = \frac{Theoretical\ \lambda_N - \ Numerical\ \lambda_N}{Numerical\ \lambda_N} \times 100, \tag{3.1}$$

and the values $\%\,error$ in Fig. 1, 3, 4 and Tables 1–3 are given by the absolute values of (3.1).

Remark 3.2. Figs. 3 and 4 (given by 576 points) show that the contour lines of the error have elliptical shapes, because the value of the integral (b) is approximately double the value of (c), which can be seen from (2.19) and (2.20).

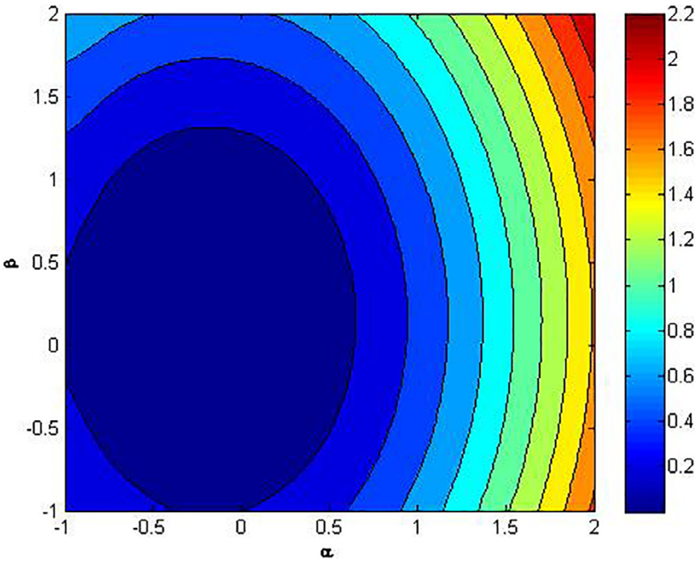

Fig. 4. The contour graph of the Fig. 3.

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Appendix

The integrals identities listed below, can be found in [12] and [17].

For $t \geq 1$,

$$\int_0^1 \frac{dx}{(x+t)\sqrt{x(1-x)}} = \frac{\pi}{\sqrt{t(t+1)}}. \tag{A.1}$$

$$\int_0^1 \frac{\ln(1-x)}{(x+t)\sqrt{x(1-x)}} dx = \frac{\pi \ln\left[\frac{t+1}{(\sqrt{t}+\sqrt{t+1})^2}\right]}{\sqrt{t(t+1)}}. \tag{A.2}$$

For $t < -1$ (here, we take $\sqrt{c} = \mathrm{i}\sqrt{-\mathrm{c}}$, $\mathrm{c} < 0$, with $\mathrm{i}^2 = -1$),

$$\int_0^1 \frac{dx}{(x+t)\sqrt{x(1-x)}} = -\frac{\pi}{\sqrt{t(t+1)}}. \tag{A.3}$$

$$\int_0^1 \frac{\ln(1-x)}{(x+t)\sqrt{x(1-x)}} dx = -\frac{\pi \ln\left[\frac{t+1}{(\sqrt{t}+\sqrt{t+1})^2}\right]}{\sqrt{t(t+1)}}. \tag{A.4}$$

References

[1] Y.L. Geronimus, Polynomials Orthogonal on a Circle and Interval, Pergamon, New York, 1960.

[2] G. Szegö, Orthogonal polynomials, in: American Mathematical Society Colloquium Publications, 23, Providence, R. I., 1959. Rev. ed.

[3] G. Szegö, On some Hermitian forms associated with two given curves of the complex plane, Trans. Am. Math. Soc. 40 (1936) 450–461. In: Collected papers (volume 2), 666–678. Birkhaüser, Boston, Basel, Stuttgart, 1982

[4] G. Pólya, G. Szegö, Problems and Theorems in Analysis I, Springer-Verlag, Berlin Heidelberg, New York, 1978.

[5] G. Szegö, Hankel Forms (English translation of A Hankel-féle formákról) Collected papers, 1, Birkhäuser, Basle, 1982, p. P111.

[6] J. Todd, Contributions to the solution of systems of linear equations and the determination of eigenvalues, Nat. Bur. Stand. Appl. Math. Ser. 39 (1959) 109–116.

[7] H. Widom, H.S. Wilf, Small eigenvalues of large Hankel matrices, Proc. Am. Math. Soc. 17 (1966) 338–344.

[8] H. Widom, H.S. Wilf, Errata: small eigenvalues of large Hankel matrices, Proc. Am. Math. Soc. 19 (1968) 1508.

[9] H.S. Wilf, Finite Sections of Some Classical Inequalities, Springer, Berlin, Heidelberg, New York, 1970.
[10] Y. Chen, N.D. Lawrence, Small eigenvalues of large Hankel matrices, J. Phys. A: Math. Gen. 32 (1999) 7305–7315.
[11] Y. Chen, D.S. Lubinsky, Smallest eigenvalues of Hankel matrices for exponential weights, J. Math. Anal. Appl. 293 (2004) 476–495.
[12] Y. Chen, M.R. McKay, Coulumb fluid, painlevé transcendents, and the information theory of MIMO systems, IEEE Trans. Inform. Theory 58 (2012) 4594–4634.
[13] M. Zhu, Y. Chen, N. Emmart, C. Weems, The smallest eigenvalue of large Hankel matrices generated by a deformed Laguerre weight. ArXiv:1803.113221803.11322 [math-ph].
[14] N. Emmart, Y. Chen, C.C. Weems, Computing the smallest eigenvalue of large ill-conditioned Hankel matrices, Commun. Comput. Phys. 18 (2015) 104–124.
[15] C. Berg, Y. Chen, M.E.H. Ismail, Small eigenvalues of large Hankel matrices: the indeterminate case, Math. Scand. 91 (2002) 67–81.
[16] C. Berg, R. Szwarc, The smallest eigenvalue of Hankel matrices, Constr. Approx. 34 (2011) 107–133.
[17] I.S. Gradshteyn, I.M. Ryzhik, Table of Integrals, Series, and Products, seventh ed., Elsevier/Academic Press, Amsterdam, 2007. Translated from the Russian, Translation edited and with a preface by Alan Jeffrey and Daniel Zwillinger, With one CD-ROM (Windows, Macintosh and UNIX)