AN OPTIMAL TWO-LEVEL OVERLAPPING DOMAIN DECOMPOSITION METHOD FOR ELLIPTIC PROBLEMS IN TWO AND THREE DIMENSIONS *

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Abstract. We consider the solution of linear systems of algebraic equations that arise from elliptic finite element problems. We study a two-level overlapping domain decomposition method that can be viewed as a combination of the additive and multiplicative Schwarz methods. This method combines the advantages of the two methods. It converges faster than the additive Schwarz algorithm and is more parallelizable than the multiplicative Schwarz algorithm, and works for general, not necessarily selfadjoint, linear, second order, elliptic equations. We use the GMRES method to solve the resulting preconditioned linear system of equations and we show that the algorithm is optimal in the sense that the rate of convergence is independent of the mesh size and the number of subregions in both R^2 and R^3. A numerical comparison with the additive and multiplicative Schwarz preconditioned GMRES is reported.

Key words. overlapping domain decomposition, elliptic equations, finite elements, iterative method

AMS(MOS) subject classifications. 65F10, 65N30.

1. Introduction. Domain decomposition technique is a class of preconditioned iterative methods for solving partial differential equations and has been proved to be very effective for parallel computing. In this paper, we study a new class of methods based on the Dryja-Widlund decomposition [7], in which the usual finite element space is optimally decomposed into the sum of a finite number of uniformly overlapped two-level subspaces. Based on this decomposition two methods, the additive Schwarz (ASM) [3, 5, 7] and the multiplicative Schwarz methods (MSM) [2, 6], have been studied. A recent paper [4] shows that MSM, despite of its less parallelism, is substantially faster than ASM in terms of their algebraic convergence rates. In this paper, we develop a new method that can be viewed as a combination of ASM and MSM and it converges faster than the additive Schwarz method and is more parallelizable than the multiplicative Schwarz method. If the number of processors is about the same as the number of subdomains that have the same color, which will be described in detail later, then the parallelism of the new method is as well as that of ASM. We show that the new method, accelerated by certain Krylov space based iterative method, such as GMRES, has an optimal convergence rate independent of the mesh size and the number of subproblems for general elliptic problems, not necessarily symmetric, in both two and three dimensional spaces. The main difference between the new method and MSM is the treatment of the coarse grid operator. There are other recently developed iterative methods that make special use of the coarse grid operator; see e.g. [1, 11, 12].

The paper is organized as follows. In § 2, we briefly introduce the elliptic finite element problem and the Dryja-Widlund decomposition. Then, we discuss the idea of transformed systems with certain well-known examples in § 3. The new method is introduced in § 4, in which the convergence rate of the new method is also analyzed. In § 5, we provide a numerical comparison of the new method with the additive and multiplicative Schwarz methods. We conclude the paper with a few remarks in § 6.

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2. Model problem and Dryja-Widlund decomposition. Let Ω be a bounded polygonal region in R^d ($d = 2$ or 3) with boundary $\partial\Omega$. We consider the weak form of the homogeneous Dirichlet boundary value problem: Find $u \in H_0^1(\Omega)$ such that

$$b(u,v) = (f,v), \quad \forall v \in H_0^1(\Omega), \tag{1}$$

where the bilinear form $b(u,v) = a(u,v) + s(u,v)$ and

$$a(u,v) \;=\; \sum_{i,j=1}^{d} \int_\Omega a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx \;\text{ and }\; s(u,v) = \sum_{i=1}^{d} \int_\Omega b_i \frac{\partial u}{\partial x_i} v dx \;+ \int_\Omega cuvdx.$$

We assume that all coefficients are sufficiently smooth, the matrix $\{a_{ij}(x)\}$ is symmetric and uniformly positive definite, and $f \in H^{-1}(\Omega)$. We also assume that the equation has a unique solution and that $b(\cdot,\cdot)$ satisfies, for some positive constants c and C,

- $c\,\|u\|_a^2 \le b(u,u), \;\; \forall u \in H_0^1(\Omega)$
- $|\, b(u,v)\,| \le C\,\|u\|_a \|v\|_a, \;\; \forall u,v \in H_0^1(\Omega).$

Here $\|\cdot\|_a = a(\cdot,\cdot)^{1/2}$ is the energy norm of $H_0^1(\Omega)$. We solve equation (1) by the Galerkin conformal finite element method. For simplicity, we use piecewise linear triangular elements in R^2 and the corresponding tetrahedral elements in R^3. Following Dryja and Widlund [7], we describe a two-level triangulation of Ω and the corresponding finite element spaces. We define $\{\Omega_i, i = 1,\cdots,N\}$ to be a shape regular finite element triangulation of Ω, where the diameter of Ω_i is of order $O(H)$. We call Ω_i a substructure and $\{\Omega_i\}$ the coarse grid or H-level triangulation of Ω. In our second step, we further divide each Ω_i into smaller simplices of diameter $O(h)$, and the union of these forms a shape regular finite element triangulation of Ω. We call it the fine mesh or h-level triangulation of Ω. We denote by V^H and V^h the continuous, piecewise linear finite element function spaces over the H-level and h-level triangulations of Ω, respectively. The Galerkin approximation of equation (1) is formulated as follows: Find $u_h^* \in V^h$, such that

$$b(u_h^*, v_h) \;=\; (f, v_h), \;\; \forall v_h \in V^h. \tag{2}$$

We next describe the Dryja-Widlund decomposition of V^h. To decompose Ω into overlapping subregions, we extend each Ω_i to a larger subregion Ω_i', i.e. $\Omega_i \subset \Omega_i' \subset \Omega$. The overlap is of size $O(H)$, or more precisely $\mathrm{dist}(\partial\Omega_i' \cap \Omega, \partial\Omega_i \cap \Omega) \ge \alpha H, \;\; \forall i$, for a constant $\alpha > 0$. We assume that $\partial\Omega_i'$ aligns with the h-level elements and denote $\Omega_0' \;=\; \Omega$. For each $\Omega_i', i > 0$, we define $V_i^h = \{v_h \in V^h |\; v_h(x) = 0,\, x \bar{\in} \Omega_i'\} \subset V^h$. We also use the subspace $V_0^h = V^H$. It is easy to see that V^h can be represented as the sum of the $N+1$ subspaces,

$$V^h \;=\; V_0^h \;+\; V_1^h \;+\cdots+\; V_N^h.$$

We now regroup the subregions in terms of the following coloring strategy. Associated with the decomposition $\{\Omega_i'\}$, we define an undirected graph in which nodes represent the extended subregions and the edges intersections of the extended subregions. This graph can be colored, using colors $0,\cdots,J$, such that no connected nodes have the same color. We note that Ω_0' needs its own color. It is obvious that the coloring is not unique.

3. Transformed linear system. Let $b_i(\cdot,\cdot)$ be a bilinear form, defined on the subspace V_i^h, which we will refer as the subspace preconditioner for $b(\cdot,\cdot)$. In this paper, we only consider two cases

(I) $b_i(\cdot,\cdot) = b(\cdot,\cdot)$, for $i = 0, 1, \cdots N$;

(II) $b_0(\cdot,\cdot) = b(\cdot,\cdot)$ and $b_i(\cdot,\cdot) = a(\cdot,\cdot)$ for $i = 1, \cdots, N$.

We introduce the operator $T_i : V^h \longrightarrow V_i^h$ by

$$b_i(T_i u_h, v_h) = b(u_h, v_h), \quad \forall u_h \in V^h \text{ and } \forall v_h \in V_i^h.$$

We note that among all these operators, T_0 is the only global operator and all the others are local. We recall that $u_h^* \in V^h$ denote the exact solution of the Galerkin equation (2). It is easy to see that the vector $T_i u_h^* \in V_i^h$ can be computed, without knowing u_h^*, by using the definition of T_i and the equation (2). As an immediate consequence, if we define

$$T = poly(T_0, T_1, \cdots, T_N)$$

as a polynomial of these $T_i^{'}$s such that $poly(0, \cdots, 0) = 0$, then $Tu_h^* \in V^h$ can also be computed without knowing u_h^* itself. By denoting $g = Tu_h^*$, we refer

$$Tu_h^* = g \tag{3}$$

as the *transformed system* of (2). It is not difficult to prove that

THEOREM 3.1. *If T is invertible, then the equation (3) has the same solution as the Galerkin equation (2).*

We now group these maps T_i in terms of the color that the subregion was assigned. For $j = 0, 1, \cdots, J$, we denote Q_j as the sum of all $T_i^{'}$s that correspond to the subregions with the j^{th} color. In fact $Q_0 = T_0$. We remark that N (the number of subregions) may be large, J (the number of colors) can still be small. We next look at two special examples. The first one, which is the simplest case and the degree of $poly(\cdots)$ is one, is the additive Schwarz method, in which the operator has the form

$$T_{asm} = Q_0 + Q_1 + \cdots + Q_J.$$

The second example is the so-called multiplicative Schwarz operator

$$T_{msm} = I - E_{J+1},$$

where I is the identity map and $E_{J+1} = (I - Q_0)(I - Q_1)\cdots(I - Q_J)$. The degree of this polynomial depends on the number of colors, and the exact form of the polynomial depends on how the subregions are colored.

It is important to note that even if the original equation (2) is not well-conditioned, the transformed systems can be uniformly well-conditioned and more importantly the transformed system can be so arranged that a highly parallelizable algorithm can be developed for solving it. To build such a well-conditioned and easily parallelizable transformed system is the main purpose of this paper.

4. A new transformed system and its spectral bounds. The parallelism of MSM results mainly from the fact that, for $j \neq 0$, Q_j is a sum of some local independent subproblems, that can be handled in parallel. However, the global operator $Q_0 v_h = T_0 v_h$ is very special and it can not be handled in parallel with other local subproblems. It is not the case for ASM in which all subproblems, including T_0, can be solved in parallel.

Motivated by the above observation, we now define an operator in which the global operator T_0 is made to be additive to the rest of local operators.

$$T_{new} = \omega T_0 + I - E_J, \tag{4}$$

where $E_J = (I - Q_1)\cdots(I - Q_J)$ and $0 < \omega \in R$ is a balancing parameter. If we define $f_{new} = T_{new} u_h^*$, then our new algorithm can be described as

ALGORITHM: *Find the solution of equation (2) by solving the transformed system*

$$T_{new} u_h^* = f_{new} \tag{5}$$

with an iterative method.

We show in the next theorem that the operator T_{new} is, under certain assumptions, uniformly well-conditioned. In other words, its spectral bounds are independent of the mesh parameter as well as the number of subproblems. The symmetric part of T_{new} is uniformly positive definite, which guarantees the convergence of a class of Krylov space based iterative methods, such as the GMRES method [8, 9].

THEOREM 4.1. *There exist constants $H_0 > 0$ and $\omega > 0$, independent of h and H, such that if $H \le H_0$ then,*

$$\|T_{new}\|_a \le C$$

and

$$a(T_{new} u_h, u_h) \ge \frac{c}{(J+1)^2} \|u_h\|_a^2, \quad \forall u_h \in V^h,$$

where $C = C(H_0)$ and $c = c(H_0)$ are positive constants independent of H and h.

In order to prove the main theorem, we need to quote some known results for the well-conditionness of T_{msm}.

THEOREM 4.2 (CAI AND WIDLUND [6]). *There exist constants $H_0 > 0$, $\gamma_i > 0$, $i = 1, 2$, such that if $H \le H_0$, then*

$$\|E_J\|_a \le \sqrt{1 - \frac{\gamma_1 H^2}{J^2}} \quad \text{and} \quad \|E_{J+1}\|_a \le \sqrt{1 - \frac{\gamma_2}{(J+1)^2}},$$

where $\gamma_i = \gamma_i(H_0)$ are independent of H and h.

LEMMA 4.3 (CAI AND WIDLUND [5]). *There exists a constants $H_0 > 0$, such that if $H \le H_0$, then for any $u_h \in V^h$*

$$\|T_0 u_h\|_a \le C \|u_h\|_a, \quad \|T_0 u_h - u_h\|_{L^2} \le CH \|u_h\|_a$$

and

$$a(T_0 u_h, u_h) \ge \|T_0 u_h\|_a^2 - cH \|u_h\|_a^2,$$

where $c = c(H_0)$ and $C = C(H_0)$ are positive constants independent of H and h.

Proof. (**of Theorem 4.1**): It is easy to see that the following identity holds.

$$T_{new} = \omega T_0 - T_0 E_J + I - E_{J+1}. \tag{6}$$

The upper bound part of this theorem can be trivially proved by using Theorem 4.2 and Lemma 4.3.

For the lower bound part, we only prove the case where $b_i(\cdot,\cdot) = b(\cdot,\cdot)$ for $i = 0, \cdots, N$. The proof for the other case can be obtained in a similar way. Directly from the identity (6), we have that

$$a(T_{new} u_h, u_h) = \omega a(T_0 u_u, u_h) + a(u_h, u_h) - a(E_{J+1} u_h, u_h) - a(T_0 E_J u_h, u_h). \tag{7}$$

We now estimate the right-hand-side of the above equality term-by-term. Following Theorem 4.2, we obtain

$$a(E_{J+1}u_h, u_h) \leq (1-\tilde{c})\|u_h\|_a^2, \tag{8}$$

where the constant $\tilde{c} = 1 - \sqrt{1-\gamma_2/(J+1)^2} > 0$. It is easy to verify that

$$\begin{aligned} a(T_0E_Ju_h, u_h) &= b(T_0u_h, T_0E_Ju_h) - s(u_h, T_0E_Ju_h) \\ &= a(T_0u_h, T_0E_Ju_h) + s(T_0u_h - u_h, T_0E_Ju_h) \end{aligned} \tag{9}$$

By using the fact that $|s(u,v)| \leq C\|u\|_{L^2}\|v\|_a,\ \forall u, v \in H_0^1(\Omega)$, Lemma 4.3 and Theorem 4.2, we have

$$|s(T_0u_h - u_h, T_0E_Ju_h)| \leq CH\|u_h\|_a^2$$

and hence

$$\begin{aligned} a(T_0E_Ju_h, u_h) &\leq \|T_0u_h\|_a\|T_0E_Ju_h\|_a + CH\|u_h\|_a^2 \\ &\leq C_1\|u_h\|_a\|T_0u_h\|_a + CH\|u_h\|_a^2 \\ &\leq \frac{C_1\lambda}{2}\|u_h\|_a^2 + \frac{C_1}{2\lambda}\|T_0u_h\|_a^2 + CH\|u_h\|_a^2, \end{aligned} \tag{10}$$

where λ is an arbitrary positive constant. By taking $\lambda = \tilde{c}/C_1$, we have

$$a(T_0E_Ju_h, u_h) \leq \frac{\tilde{c}}{2}\|u_h\|_a^2 + \frac{C_1^2}{2\tilde{c}}\|T_0u_h\|_a^2 + CH\|u_h\|_a^2. \tag{11}$$

Taking all the above estimates (7), (8), (11) and the last inequality of Lemma 4.3 into account, we have

$$\begin{aligned} a(T_{new}u_h, u_h) &\geq \tilde{c}\|u_h\|_a^2 + \omega\|T_0u_h\|_a^2 - CH\omega\|u_h\|_a^2 \\ &\quad -\frac{\tilde{c}}{2}\|u_h\|_a^2 - \frac{C_1^2}{2\tilde{c}}\|T_0u_h\|_a^2 - CH\|u_h\|_a^2. \end{aligned} \tag{12}$$

Therefore, if we choose $\omega = C_1^2/(2\tilde{c})$, then

$$a(T_{new}u_h, u_h) \geq \frac{\tilde{c}}{2}\|u_h\|_a^2 - CH\|u_h\|_a^2. \tag{13}$$

Thus, if H is small enough, we have

$$a(T_{new}u_h, u_h) \geq \frac{\tilde{c}}{4}\|u_h\|_a^2 \geq \frac{\gamma_2}{8(J+1)^2}\|u_h\|_a^2, \tag{14}$$

which completes the proof of the main theorem. □

A remark is in order here about the choice of ω. ω does not depend on the size of the linear system, nor the number of subproblems. Our numerical experiments, cf. the next section, show that the algorithm is not very sensitive to ω. In fact $\omega = 1$ has always given us better convergence than ASM.

TABLE 1
The parallel complexity of the algorithms with p processors

Method	# of iterations	p= # of subproblems	p = (max # of subdomains with the same color + 1)
MSM	$O(1)$	$Jt_i + t_c$	$Jt_i + t_c$
ASM	$O(1)$	$\max\{t_i, t_c\}$	$\max\{Jt_i, t_c\}$
NEW	$O(1)$	$\max\{Jt_i, t_c\}$	$\max\{Jt_i, t_c\}$

TABLE 2
Iteration counts for solving the Poisson equation($\delta = 0$) with different h , H and overlap sizes. Here $\omega = 1.0$.

$h^{-1} =$	32	64	128	32	64	128	64	128
overlapping size	$H = 1/4$			$H = 1/8$			$H = 1/16$	
ovlp=h	8	8	10	8	7	7	6	6
ovlp=$2h$	8	8	8	7	7	7	6	6
ovlp=$4h$	6	7	8		7	7		6

5. Numerical experiments and comparison with ASM and MSM. In this section, we first briefly discuss the parallel complexity of the new algorithm as compared with ASM and MSM and then present some numerical results.

Let us make some basic assumptions before providing a parallel complexity analysis with p parallel processors. In this paper, we only focus on these computer architecture independent factors. We assume that the communication, synchronization and load balancing costs can be ignored; and also that each subproblem is solved by using only one processor.

Furthermore, we assume that all interior problems, defined on any extended substructures, are of relatively the same size and need t_i unit time (or number of arithmetic operations) to solve. Of course, t_i depends not only on how many unknowns each subregion has but also the method used to solve the interior problem. Similarly, t_c is for the coarse mesh problem. Table 1 shows the parallel complexity of performing the preconditioner-vector multiply by using multiplicative, additive and the new Schwarz type methods.

We next present some numerical results for solving this equation where

$$-\triangle u + \delta u_x + \delta u_y = f, \text{ in } \Omega \tag{15}$$

with $u = 0$ on $\partial\Omega$ and $\Omega = [0,1]\times[0,1]$. In all cases, the exact solution $u = e^{xy}\sin(\pi x)\sin(\pi y)$, and f can thus be set accordingly.

The unit square is subdivided into two-level uniform meshes, with h and H representing the fine and coarse mesh sizes. The elliptic operator is then discretized by the usual five-point central or upwinding difference methods over both meshes. The full GMRES method, without restarting, with zero initial guess is used for all of the transformed linear systems, in the usual Euclidean norm, and the stopping criterion is the reduction of the initial preconditioned residual by five orders of magnitude in the L^2 norm.

We first test a special case $\delta = 0$. Although this is a symmetric problem, we still use GMRES as the outer iterative method. The iteration counts are given in Table 2.

Our second test problem is a nonsymmetric, constant coefficient problem. We specify the constant $\delta > 0$ in Table 3. The elliptic operator is discretized by two schemes, namely,

TABLE 3

Iteration counts for solving the nonsymmetric model equation with various values of δ and two discretizations. The parameter $h = 1/128$. Here $\omega = 1.0$.

	$H = 1/4$						$H = 1/8$					
Central-difference Method												
$\delta =$	1	5	10	50	100	150	1	5	10	50	100	150
ovlp=h	10	12	12	16	16	14	8	9	10	16	23	25
ovlp=$2h$	9	10	10	14	12	12	7	9	9	15	20	23
ovlp=$4h$	8	9	9	11	12	12	7	8	9	13	17	20
Upwind-difference Method												
$\delta =$	10	50	100	500	1000	10000	10	50	100	500	1000	10000
ovlp=h	12	13	13	11	11	11	11	14	15	16	16	17
ovlp=$2h$	10	11	11	11	11	11	10	13	14	15	15	15
ovlp=$4h$	9	10	10	10	10	10	9	11	12	12	12	12

TABLE 4

Iteration counts for different balancing parameter ω's Here $\delta = 10$, $h = 1/128$, $H = 1/8$, $ovlp = 2h$ and central-differencing is used.

ω	0.0	0.25	0.5	0.75	1.0	1.25	1.5	1.75	2.0	2.5
Iteration	17	12	10	9	9	9	10	10	10	10

the central-difference method, for relatively small δ and the upwind-difference method, for relatively large δ.

The optimal choice of ω is not unique and $\omega = 1.0$ seems among the optimal choices for the example that we tested. An example can be found in Table 4.

We finally compare the new algorithm with ASM and MSM by listing the convergence history in Table 5. It is clear that the convergence rate of the new algorithm is faster than that of ASM but slower than MSM. Some results for the same test problems by using other domain decomposition methods can be found in [4].

6. Concluding remarks. In this paper, we introduced a new member in the class of Schwarz type overlapping domain decomposition methods. This class of methods has been shown to be fast, even in the case involving boundary layers, see e.g. the recent paper of Tang [10]. The new method shares the robustness of other Schwarz methods with added parallelism.

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TABLE 5

The maximum norm of the error, defined as the difference between the computed solution and the true solution of the continuous problem, at each step of iteration. The parameters are $h = 1/128$, $H = 1/4$, *overlap* $= 4h$ *and* $\delta = 50.0$. *The central differencing is used here. For the new algorithm,* $\omega = 1.0$.

Iteration	MSM	NEW	ASM
1	1.126987e-01	6.025081e-01	5.994051e-01
2	3.011373e-02	3.440657e-01	5.605597e-01
3	5.950362e-03	1.932006e-01	3.647781e-01
4	1.467230e-03	7.443918e-02	3.019285e-01
5	4.354542e-04	3.493269e-02	1.113954e-01
6	2.405614e-04	1.581771e-02	9.212396e-02
7	1.969721e-04	7.474377e-03	3.602628e-02
8		3.709754e-03	1.901591e-02
9		1.234765e-03	1.255937e-02
10		3.609956e-04	7.544490e-03
11		2.050532e-04	4.329650e-03
12			2.030623e-03
13			9.756193e-04
14			6.124153e-04
15			5.179665e-04
16			2.999394e-04
17			2.096750e-04
18			2.000241e-04

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