Domain Decomposition Algorithms for Indefinite Elliptic Problems*

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Abstract

Iterative methods for linear systems of algebraic equations arising from the finite element discretization of nonsymmetric and indefinite elliptic problems are considered. Methods previously known to work well for positive definite, symmetric problems are extended to certain nonsymmetric problems, which can also have some eigenvalues in the left half plane.

We first consider an additive Schwarz method applied to linear, second order, symmetric or nonsymmetric, indefinite elliptic boundary value problems in two and three dimensions. An alternative linear system, which has the same solution as the original problem, is derived and this system is then solved by using GMRES, an iterative method of conjugate gradient type. In each iteration step, a coarse mesh finite element problem and a number of local problems are solved on small, overlapping subregions into which the original region is subdivided. We show that the rate of convergence is independent of the number of degrees of freedom and the number of local problems if the coarse mesh is fine enough. The performance of the method in two dimensions is illustrated by results of several numerical experiments.

We also consider two other iterative method for solving the same class of elliptic problems in two dimensions. Using an observation of Dryja and Widlund, we show that the rate of convergence of certain iterative substructuring methods deteriorates only quite slowly when the local problems increase in size. A similar result is established for Yserentant's hierarchical basis method.

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1 Introduction

Domain decomposition techniques are powerful iterative methods for solving linear systems of equations that arise from finite element problems. In each iteration step, a coarse mesh finite element problem and a number of smaller linear systems, which correspond to the restriction of the original problem to subregions, are solved instead of the large original system. These algorithms can be regarded as divide and conquer methods. The number of subproblems can be large and these methods are therefore promising for parallel computation. The central mathematical question is to obtain estimates on the rate of convergence of the iteration by deriving bounds on the spectrum of the iteration operator. We are able to establish quite satisfactory bounds if the coarse mesh is fine enough.

We work with two triangulations of the region: 1) partitioning the region into subregions, also called substructures, which define a coarse, global model; 2) partitioning the region into elements of a finite element model. As in the positive definite case considered previously, see Cai [3],[4], Dryja [6] and Dryja and Widlund [7],[8], the coarse problem provides interchange of information among the different parts of the region; it is known that without such a coarse subproblem the rate of convergence is considerably slower; cf [24]. This part of the approximate solver plays an additional role in the indefinite case. We can interpret the main results of this paper by saying that if the eigenfunctions corresponding to the eigenvalues in the left half plane are approximated well enough on the coarse mesh, then the spectrum of the preconditioned linear system of equations lies in a fixed bounded subset of the right half plane. This is important for the rate of convergence of the iterative method. The least favorable situation for iterative methods of conjugate gradient type is the case where the origin of the complex plane is surrounded by eigenvalues of the iteration operator. Here we are able to avoid such a situation.

The additive Schwarz algorithms, introduced in [7], cf. also [6],[8],[9],[18], provide a means of constructing preconditioners for many problems in terms of a partition of a given finite element space into a sum of subspaces. The use of such a preconditioner involves solving, exactly or approximately, the restriction of the original problem to the different subspaces. The residual, which plays a central role in the iteration, is computed as a sum of terms from the different subspaces. These terms can be computed in parallel. We note that it has been shown in Dryja and Widlund [9] that many domain decomposition methods can be viewed as additive Schwarz methods. For recent work on the case of more than two levels of triangulation, see Dryja and Widlund [10] and Xu [25].

In the symmetric, positive definite case, the iterative method most commonly used to solve the transformed (preconditioned) equations is the conjugate gradient method. For the cases considered here symmetry is always lost. In our experiments, we have used a generalized conjugate residual method GMRES; see [22]. Since the spectrum of the operator is confined to the right half plane, Manteuffel's Chebyshev algorithm would also be successful; cf. [17]. Since we can show that the symmetric part of the operator is uniformly positive definite, with respect to a suitable inner product, and that the spectrum is uniformly bounded, we can guarantee a rate of convergence, which is independent of the mesh size and the number of subregions.

Other methods for indefinite, elliptic problems are discussed in [2],[14],[16], [27],[28].

The paper is organized as follows. In Section 2, we introduce a class of indefinite, elliptic boundary value problem, the two triangulations of the domain and a Galerkin finite element method. We briefly review the GMRES method in Section 3. In Section 4, we present two variants of the additive Schwarz method and a detailed analysis of their rates of convergence. Our analysis is based on previous work on the positive definite case, see [3],[4],[6],[7],[8], and a result due to Schatz [23]. Schatz's work, in turn, is based on Gårding's inequality and the Aubin-Nitsche trick; see Ciarlet [5] or Nitsche [21]. In Section 5, we discuss some numerical results. Finally, in Section 6, we show that, for problems in the plane, our result can be extended to iterative substructuring and hierarchical basis algorithms discussed in Dryja and Widlund [8],[9] and Yserentant [26], respectively.

2 The Elliptic Problems

Let Ω be an open, bounded polygonal region in $R^d, d = 2$ or 3, with boundary $\partial\Omega$. Consider the homogeneous Dirichlet boundary value problem:

$$\left\{ \begin{array}{rcll} Lu & = & f & \text{in} \quad \Omega, \\ u & = & 0 & \text{on} \quad \partial\Omega. \end{array} \right. \tag{1}$$

The elliptic operator L has the form

$$Lu(x) = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial u(x)}{\partial x_j}) + 2\sum_{i=1}^{d} b_i(x)\frac{\partial u(x)}{\partial x_i} + c(x)u(x).$$

All the coefficients are, by assumption, sufficiently smooth and the matrix $\{a_{ij}(x)\}$ is symmetric and uniformly positive definite for $\forall x \in \Omega$. The right hand side $f \in L^2(\Omega)$. We also assume that the equation has a unique solution in $H_0^1(\Omega)$.

Let $(\cdot,\cdot)$ denote the usual L^2 inner product and $\|\cdot\|$ or $\|\cdot\|_{L^2}$ the corresponding norm. The weak form of equation (1) is: Find $u \in H_0^1(\Omega)$ such that

$$B(u,v) = (f,v), \qquad \forall v \in H_0^1(\Omega). \tag{2}$$

The bilinear form $B(u,v)$ is defined by

$$B(u,v) \;=\; \sum_{i,j=1}^{d} \int_\Omega a_{ij} \frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i} dx + \sum_{i=1}^{d} 2\int_\Omega b_i \frac{\partial u}{\partial x_i} v dx + \int_\Omega cuv dx$$

or

$$B(u,v) \;=\; \sum_{i,j=1}^{d} \int_\Omega a_{ij} \frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i} dx + \sum_{i=1}^{d} \int_\Omega b_i \frac{\partial u}{\partial x_i} v + \frac{\partial (b_i u)}{\partial x_i} v dx \;+ \int_\Omega \tilde{c} uv dx.$$

Here, $\tilde{c}(x) = c(x) - \sum_{i=1}^{d} \partial b_i(x)/\partial x_i$.

We also use two other bilinear forms

$$A(u,v) \;=\; \sum_{i,j=1}^{d} \int_\Omega a_{ij} \frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i} dx$$

and

$$S(u,v) \;=\; \sum_{i=1}^{d} \int_\Omega b_i \frac{\partial u}{\partial x_i} v + \frac{\partial (b_i u)}{\partial x_i} v dx \;,$$

which correspond to the second order terms and the skew-symmetric part of L, respectively. The bilinear form A defines a norm, which we denote by $\|\cdot\|_A$. Under the assumptions on the coefficients a_{ij}, this norm is equivalent to the H_0^1 norm. It is also easy to verify that

$$S(u,v) \;=\; -S(v,u), \;\; \forall u,v \in H_0^1(\Omega).$$

Throughout this paper, c and C, with or without subscripts, denote generic, strictly positive constants. They are independent of the mesh parameters h and H, which will be introduced later in this section.

Using elementary, standard tools, it is easy to establish the following inequalities:

(i) $\mid B(u,v) \mid \leq C\|u\|_A \|v\|_A, \;\; \forall u,v \in H_0^1(\Omega).$

(ii) Gårding's inequality: There exists a constant C, such that

$$\|u\|_A^2 - C\|u\|_{L^2(\Omega)}^2 \leq B(u,u), \;\; \forall u \in H_0^1(\Omega) \;.$$

(iii) There exists a constant C, such that

$$| S(u,v) | \leq C \|u\|_A \|v\|_{L^2(\Omega)}, \quad \forall u,v \in H_0^1(\Omega),$$

$$| S(u,v) | \leq C \|v\|_A \|u\|_{L^2(\Omega)}, \quad \forall u,v \in H_0^1(\Omega).$$

We note that the bounds for $B(\cdot,\cdot)$ and $S(\cdot,\cdot)$ are different, since each of the terms in $S(\cdot,\cdot)$ contains a factor, which is of zero order. This enables us to control the skew-symmetric term and makes our analysis possible.

We also use the following regularity result; cf. Grisvard [13] and Nečas [19].

(iv) The solution w of the adjoint equation

$$B(\phi, w) \;=\; (g, \phi), \;\; \forall \phi \in H_0^1(\Omega)$$

satisfies

$$\|w\|_{H^{1+\gamma}(\Omega)} \leq C \|g\|_{L^2(\Omega)} \;,$$

where γ depends on the interior angles of $\partial\Omega$, is independent of g and is at least $1/2$.

We approximate equation (2) by a Galerkin conforming finite element method. For simplicity, we consider only continuous, piecewise linear, triangular elements in R^2 and tetrahedral elements in R^3.

To define the additive Schwarz algorithms, we need two levels of triangulation that have already been introduced in [3],[4],[6],[7],[8],[9]. We first partition Ω into substructures $\{\Omega_i\}, i = 1, \cdots, N$, which provide a regular finite element triangulation of Ω. The Ω_i are non-overlapping, d-dimensional simplices. They satisfy all the standard rules of finite elements; cf. Ciarlet [5]. This is the coarse mesh and it defines a mesh parameter $H = \max\{H_1, \cdots, H_N\}$. The triangulation is assumed to be shape regular, i.e. H_i, the diameter of Ω_i is bounded uniformly in terms of the diameter of the largest inscribed ball in Ω_i.

In a second step, we divide each substructure Ω_i into smaller simplices, denoted by $\{\ \tau_i^j,\ j = 1, \cdots\ \}$. They form a shape regular, fine mesh (h-level) finite element triangulation of Ω with the mesh parameter $h = \max_{i,j}\{h_i^j\}$. Here h_i^j is the diameter of τ_i^j.

We can now define the piecewise linear finite element spaces over the H-level and the h-level triangulations of Ω.

$$V^H = \{ v^H \mid \text{ continuous on } \Omega,\ v^H\,|_{\Omega_i} \text{ linear },\ v^H = 0 \text{ on } \partial\Omega \}$$

and

$$V^h = \{ v^h \mid \text{ continuous on } \Omega,\ v^h\,|_{\tau_i^j} \text{ linear },\ v^h = 0 \text{ on } \partial\Omega \}\ .$$

The Galerkin approximation of equation (2) is defined by: Find $u^h \in V^h$ such that

$$B(u^h, v^h) \;=\; (f, v^h), \;\; \forall v^h \in V^h. \tag{3}$$

If the mesh size h is small enough, it follows from a result by Schatz [23] that this problem has a unique solution. By using nodal basis functions to span the finite element space, equation (3) is transformed into a linear system of algebraic equations, which is large, sparse, nonsymmetric, indefinite and relatively ill-conditioned.

3 A Brief Discussion of the GMRES Method

Among the possible iterative methods to solve the linear system, we have only used one, the GMRES method; cf. Saad and Schultz [22] and Eisenstat, Elman and Schultz [11]. This is a generalized minimum residual method, which in practice has proven quite powerful for a large class of nonsymmetric problems. The GMRES method is described in [22] and the theory developed in $L^2(\Omega)$ can be found in [11]. Both the algorithm and the theory can easily be extended to an arbitrary Hilbert space; see Cai [3]. In developing our theory and in the numerical results that are discussed in Section 5, we have exclusively used the A-norm introduced in Section 2. Here we briefly describe the GMRES algorithm and state a theorem without proof.

Let P be a linear operator in the finite dimensional space R^n with an inner product $[\cdot,\cdot]$, and a corresponding norm $\|\cdot\|$, chosen to take advantage of the special properties of P. (In our applications, P is the preconditioned stiffness matrix and the A-norm is used.) P is not symmetric but is positive definite with respect to $[\cdot,\cdot]$. The GMRES method is used to solve the linear system of equations

$$Px \;=\; b,$$

where $b \in R^n$ is given. We begin from an initial approximation $x_0 \in R^n$ and the initial residual $r_0 \;=\; b \;-\; Px_0$. In the m^{th} iteration, a correction vector z_m is computed from the Krylov subspace

$$\mathcal{K}_m(r_0) \;=\; \text{span}\{r_0, Pr_0, \cdots, P^{m-1}r_0\},$$

which minimizes the norm of the residual. In other words, z_m solves

$$\min_{z \in \mathcal{K}_m(r_0)} \|b \;-\; P(x_0 + z)\| \;.$$

The m^{th} iterate is $x_m \;=\; x_0 + z_m$.

The exact solution would be reached in no more than n iterations if we use exact arithmetic.

Following Eisenstat, Elman and Schultz [11], the rate of convergence of the GMRES method can be characterized in terms of the minimal eigenvalue of the symmetric part of the operator and the norm of the operator. They are defined by

$$c_p \;=\; \inf_{x\neq 0} \frac{[x, Px]}{[x, x]} \quad \text{and} \quad C_p \;=\; \sup_{x\neq 0} \frac{\|Px\|}{\|x\|}.$$

By considering the decrease of the norm of the residual in a single step, the following theorem can be established.

Theorem*(Eisenstat, Elman and Schultz). If $c_p \;>\; 0$, then, the GMRES method converges and after m steps, the norm of the residual is bounded by*

$$\|r_m\| \;\leq\; (1 \;-\; \frac{c_p^2}{C_p^2})^{m/2} \; \|r_0\|.$$

4 Algorithms on Overlapping Subregions

In this section, we introduce two variants of an additive Schwarz algorithm and provide bounds on their convergence rates; see Theorem 1 in the following discussion. The analysis is valid for both two and three dimensions.

We first form a basic decomposition of the domain Ω into overlapping subregions and then introduce the projections which define our algorithms.

We use the H-level subdivision $\{\Omega_i\}$ of Ω. Each subregion Ω_i is extended to a larger region $\Omega_i^{'}$, i.e. $\Omega_i \subset \Omega_i^{'}$. The overlap is generous in the sense that there exists a constant $\alpha > 0$, such that

$$\text{distance}(\partial\Omega_i^{'} \cap \Omega, \partial\Omega_i \cap \Omega) \geq \; \alpha H_i, \;\; \forall i.$$

We assume that $\partial\Omega_i^{'}$ does not cut through any h-level elements. We use the same construction for the subregions that intersect the boundary $\partial\Omega$ except that we cut off the part that is outside Ω.

We also use the notation $\Omega_0^{'} \;=\; \Omega$.

We note that the larger α is, the fewer iterations can be expected. However, if we increase the overlap, the size, and hence the cost of the subproblems increases. It is an important practical issue to balance the total number of iterations and the cost of solving the subproblems.

For each $\Omega_i^{'}, i > 0$, a regular finite element subdivision is inherited from the h-level subdivision of Ω. The corresponding finite element space is defined by

$$V_i^h = H_0^1(\Omega_i^{'}) \cap V^h.$$

The elements of this subspace of V^h can be extended continuously by zero to the complement of Ω_i'. We also use the subspace

$$V_0^h = V^H.$$

It is easy to see that our finite element function space V^h can be represented as the sum of the $N+1$ subspaces,

$$V^h \; = \; V_0^h \; + \; V_1^h \; + \cdots + \; V_N^h.$$

We can now define the projection operators, which are the main building blocks of our algorithms. These operators map the finite element space V^h onto the subspaces V_i^h and are defined in terms of the bilinear forms $B(\cdot,\cdot)$ and $A(\cdot,\cdot)$.

Definition: *For* $i = 0, \cdots, N$:

For any $w^h \in V^h$, $Q_i w^h \in V_i^h$ *is the solution of the finite element equation*

$$B(Q_i w^h, v_i^h) \; = \; B(w^h, v_i^h), \;\; \forall v_i^h \in V_i^h.$$

For any $w^h \in V^h$, $P_i w^h \in V_i^h$ *is the solution of the finite element equation*

$$A(P_i w^h, v_i^h) \; = \; B(w^h, v_i^h), \;\; \forall v_i^h \in V_i^h.$$

We now introduce the two operators which define our transformed equations

$$Q^{(1)} \; = \; Q_0 \; + \; Q_1 \; + \cdots + \; Q_N$$

and

$$Q^{(2)} \; = \; Q_0 \; + \; P_1 \; + \cdots + \; P_N.$$

Our main effort goes into the study of the spectra of these two operators. The only difference between $Q^{(1)}$ and $Q^{(2)}$ is that, for $i > 0$, we replace the projection Q_i, corresponding to Ω_i', by P_i. The coarse mesh projection is not changed.

The computation of $Q_i w^h$ or $P_i w^h$, for $i > 0$ and for an arbitrary function $w^h \in V^h$, involves the solution of a standard finite element linear system of algebraic equations on the small subregion Ω_i'. The former gives rise to a nonsymmetric linear system of equations and the latter to a positive definite, symmetric problem. For $i = 0$, the problem is a standard finite element equation on the H-level, coarse space. One can view P_i as a preconditioner of Q_i in the subspace V_i^h; cf. the discussion in Dryja and Widlund [8],[9]. The cost of the computation can often be decreased by simplifying the local problems further. We can replace the given second order elliptic operator by the Laplacian. If it is possible to choose some of the Ω_i' to be rectangular and the corresponding mesh to be uniform, a Fast Poisson solver can then be used to compute the contribution from V_i^h. It is an easy exercise to modify our theory to cover such a case.

We will consider two additive Schwarz algorithms:

Algorithm 1: Obtain the solution of equation (3) by solving the equation

$$Q^{(1)}u^h \; = \; b^{(1)}, \tag{4}$$

and

Algorithm 2: Obtain the solution of equation (3) by solving the equation

$$Q^{(2)}u^h \; = \; b^{(2)}. \tag{5}$$

In order for equations (4) and (5) to have unique solutions, the operators $Q^{(1)}$ and $Q^{(2)}$ must be invertible. This follows from Theorem 1 given in the following discussion. To obtain the same solution as equation (3), the right hand sides $b^{(1)}$ and $b^{(2)}$ must be chosen correctly. The crucial observation is that these right hand sides can be computed without knowledge of the solution of equation (3). The following formulas are valid:

$$b^{(1)} \; = \; Q^{(1)}u^h \; = \; \sum_{i=0}^{N} Q_i u^h$$

and

$$b^{(2)} \; = \; Q^{(2)}u^h \; = \; Q_0 u^h + \sum_{i=1}^{N} P_i u^h.$$

Each of these terms can be computed by solving a problem in a subspace since, by equation (3) and the definitions of Q_i and P_i,

$$B(Q_i u^h, v_i^h) \; = \; B(u^h, v_i^h) \; = \; (f, v_i^h), \;\; \forall v_i^h \in V_i^h$$

and

$$A(P_i u^h, v_i^h) \; = \; B(u^h, v_i^h) \; = \; (f, v_i^h), \;\; \forall v_i^h \in V_i^h.$$

The main result of this study is Theorem 1. By combining it with the Theorem given in Section 3, we establish that the two algorithms converge at a rate which is independent of the mesh parameters h and H, if the coarse mesh is fine enough.

Theorem 1 *There exist constants $H_0 > 0$, $c(H_0) \; > \; 0$ and $C(H_0) \; > \; 0$, such that if $H \leq H_0$, then, for $i = 1, 2$*

$$c(H_0) C_0^{-2} A(u^h, u^h) \quad \leq \quad A(u^h, Q^{(i)}u^h)$$

and

$$A(Q^{(i)}u^h, Q^{(i)}u^h) \quad \leq \quad C(H_0) A(u^h, u^h).$$

The special constant C_0 is introduced in Lemma 1.

Remarks:

(a) The operator Q_0 is very important, since it provides global transportation of information. All the other projections are local mappings. Without using Q_0, information would travel only from one subregion to its neighbors in each iteration and it would take $O(1/H)$ iterations for the information to propagate across the region. For further details, see [24].

Without such a global mechanism, it would also be impossible to confine the spectrum to the right half plane. To see this, we consider a symmetric, indefinite case. If the subregions are small enough, all the local elliptic problems are positive definite, symmetric and, in the absence of a global part, the preconditioner defined by the Schwarz algorithm is positive definite symmetric. Therefore, by the inertia theorem, the operator P has as many negative eigenvalues as the original discrete elliptic problem.

(b) The constant H_0 determines the minimal size of the coarse mesh problem and it depends on the operator L. In general, H_0 decreases if we increase the coefficients of the skew-symmetric terms, it decreases with $\tilde{c}$, while it increases if we increase the overlap. H_0 also depends on the shape of the domain Ω. If the domain is not convex, the estimate of H_0, implicit in our proof of Lemma 5, depends on the parameter γ in (iv). We do not have an explicit formula for H_0 but we know from experience that it can be determined by numerical experiments.

If the operator L is positive definite, symmetric, there is no restriction on the coarse mesh size H, i.e. $H_0 = \infty$.

The proof of Theorem 1 is based on the following results.

Lemma 1 *There exists a constant C_0, which is independent of h and H, such that, for all $u^h \in V^h$, there exist $u_i^h \in V_i^h$ with*

$$u^h = \sum_{i=0}^{N} u_i^h$$

and

$$\sum_{i=0}^{N} A(u_i^h, u_i^h) \leq C_0^2 A(u^h, u^h).$$

This lemma is also central in the theory previously developed for positive definite, symmetric problems. For a proof see Dryja and Widlund [8]; cf. also Lions [15] or Nepomnyaschikh [20]. Note that this lemma is independent of the skew-symmetric and zero order terms of the elliptic operator. In the symmetric, positive definite case, Lemma 1 is combined with an abstract argument to give a lower bound for the spectrum of the iteration operator.

The next lemma is a variation of a result by Schatz; cf. [23]. In his proof, Gårding's inequality, (ii), and the regularity result, (iv), are used. The proof of Lemma 2 follows directly from Schatz's work by replacing the approximate solution by the coarse mesh solution and the exact solution of the continuous problem by the finite element solution in V^h.

Lemma 2 *There exist constants $H_0 > 0$ and $C(H_0) > 0$, such that if $H \leq H_0$, then,*

$$\|Q_0 u^h\|_A \leq C(H_0)\|u^h\|_A$$

and

$$\|Q_0 u^h - u^h\|_{L^2} \leq C(H_0) H^{\gamma} \|Q_0 u^h - u^h\|_A.$$

Lemma 3 *The restriction of the quadratic form $B(\cdot,\cdot)$ to the subspaces V_i^h, $i > 0$, is strictly positive definite for H sufficiently small, i.e. there exists a constant $c > 0$ such that*

$$cA(u^h, u^h) \leq B(u^h, u^h), \forall u^h \in V_i^h \ .$$

Proof of Lemma 3: We have to prove that the second order terms dominate the other symmetric term; the contribution from the skewsymmetric term vanishes. This follows from the fact that the smallest eigenvalue for the Dirichlet problem for $-\triangle$ on the region Ω_i' is on the order of H_i^{-2}.

Lemma 4 *Let $v^h = \sum v_i^h$, where $v_i^h \in V_i^h$. Then there exists a constant $C > 0$, such that*

$$\| \sum v_i^h \|_A^2 \leq C \sum \|v_i^h\|_A^2.$$

Proof of Lemma 4: The proof follows from the observation that for each $x \in \Omega$, the number of terms in the sum, which differ from zero, is uniformly bounded.

Lemma 5 *There exist constants $H_0 > 0$, $c(H_0) > 0$ and $C(H_0) > 0$, such that if $H \leq H_0$, then,*

$$c(H_0) C_0^{-2} A(u^h, u^h) \leq \sum_{i=0}^{N} A(Q_i u^h, Q_i u^h) \leq C(H_0) A(u^h, u^h)$$

and

$$\begin{aligned} c(H_0) C_0^{-2} A(u^h, u^h) &\leq A(Q_0 u^h, Q_0 u^h) + \textstyle\sum_{i=1}^{N} A(P_i u^h, P_i u^h) \\ &\leq C(H_0) A(u^h, u^h). \end{aligned}$$

Proof of Lemma 5: An upper bound for $A(Q_0 u^h, Q_0 u^h)$ is given in Lemma 2. To obtain an upper bound for the sum of the other terms, we use Lemma 3 and the formula

$$B(Q_i u^h, Q_i u^h) = B(u^h, Q_i u^h)$$

to show that

$$c\sum_{i=1}^{N} A(Q_i u^h, Q_i u^h) \leq B(u^h, \sum_{i=1}^{N} Q_i u^h).$$

The right hand side can be estimated by using inequality (i) and Lemma 4. The other upper bound is established in a similar way.

To prove the lower bounds, we begin by using Lemma 2 and the triangle inequality to obtain

$$\|u^h\|_{L^2}^2 \leq C(H^{2\gamma} A(u^h, u^h) + \|Q_0 u^h\|_{L^2}^2).$$

Since the eigenvalues of the Dirichlet problem for $-\triangle$ are bounded from below and Lemma 2 holds, the last term can be replaced by $C\|Q_0 u^h\|_A \|u^h\|_A$. By using Gårding's inequality, (ii), it follows that

$$(1 - CH^{2\gamma}) A(u^h, u^h) \leq B(u^h, u^h) + C\|Q_0 u^h\|_A \|u^h\|_A \ .$$

By the definition of the operators Q_i and Lemma 1, we find that

$$B(u^h, u^h) = \sum_{i=0}^{N} B(u^h, u_i^h) = \sum_{i=0}^{N} B(Q_i u^h, u_i^h).$$

The boundedness of $B(\cdot, \cdot)$, (i), can now be used to obtain

$$\sum_{i=0}^{N} B(Q_i u^h, u_i^h) \leq C \sum_{i=0}^{N} \|Q_i u^h\|_A \|u_i^h\|_A,$$

which by Lemma 1 and the Cauchy-Schwarz inequality can be bounded above by

$$CC_0 (\sum_{i=0}^{N} \|Q_i u^h\|_A^2)^{1/2} \|u^h\|_A \ .$$

We finally obtain

$$A(u^h, u^h) \ \leq \ CC_0^2 \ \sum_{i=0}^{N} A(Q_i u^h, Q_i u^h),$$

for sufficiently small H.

The proof of the other lower bound is quite similar.

Proof of Theorem 1: The upper bounds on the norms of the operators follow immediately from Lemmas 4 and 5.

To obtain the lower bounds, we first consider

$$A(u^h, Q^{(1)} u^h) = \sum_{i=0}^{N} A(u^h, Q_i u^h).$$

Using Lemma 5, we see that it suffices to show that

$$|\sum_{i=0}^{N}(A(u^h, Q_i u^h) - A(Q_i u^h, Q_i u^h))|$$

can be bounded from above by

$$CHA(u^h, u^h).$$

By the definition of the quadratic forms

$$\begin{array}{ccc} A(u^h - Q_i u^h, Q_i u^h) & = & B(u^h - Q_i u^h, Q_i u^h) \\ - \; S(u^h - Q_i u^h, Q_i u^h) & - & (\tilde{c}(u^h - Q_i u^h), Q_i u^h). \end{array}$$

By using the definition of Q_i, the first term of the right hand side is seen to vanish.

For $i = 0$, the absolute value of the second term can be bounded above by $CH^{2\gamma}A(u^h, u^h)$ using inequality (iii) and Lemma 2. We note that $S(Q_i u^h, Q_i u^h) = 0$. There remains to consider $S(u^h, \sum_1^N Q_i u^h)$. By using the inequality (iii),

$$|\sum_{i=1}^{N} S(u^h - Q_i u^h, Q_i u^h)| \leq C\|u^h\|_A \|\sum_{i=1}^{N} Q_i u^h\|_{L^2}.$$

Since, for each $x \in \Omega$, the number of terms $Q_i u^h$ that differ from zero is uniformly bounded, the second factor on the right hand side can be bounded by $C(\sum_{i=1}^{N}\|Q_i u^h\|_{L^2}^2)^{1/2}$. By an elementary estimate, which shows that the smallest eigenvalue of the Dirichlet problem for $-\triangle$ on Ω_i' is on the order of H_i^{-2}, and Lemma 5, the required inequality is established.

The third term is written as the difference of two expressions, which can be handled by exactly the same tools.

The estimate for the operator $Q^{(2)}$ is obtained similarly.

5 Numerical Results

In this section, we present some numerical results to demonstrate the behavior of our additive Schwarz algorithms for both symmetric and nonsymmetric indefinite boundary value problems in R^2. Numerical results for positive definite problems, both symmetric and nonsymmetric, have previously been given in [3],[4],[12].

We consider the problem

$$\begin{cases} Lu = f \quad \text{in} \quad \Omega = [0,1] \times [0,1], \\ u = 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

The right hand side f is always chosen so that the exact solution is $u = xe^{xy}sin(\pi x)sin(\pi y)$. The coefficients of L are specified later for each problem.

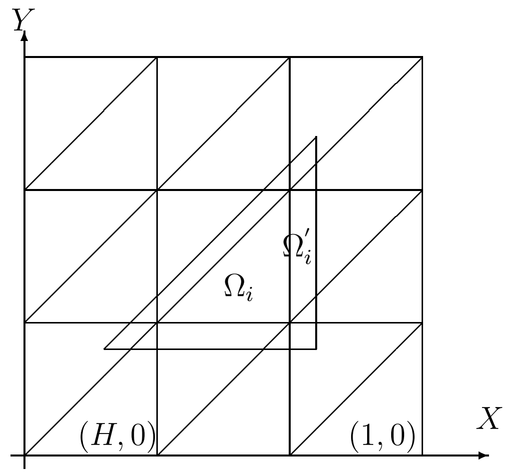

Figure 1: An extended subregion

We use a two-level subdivision of Ω as described in section 2. The subregion Ω_i' is obtained by enlarging the triangle Ω_i as in Figure 1. In this extension, the same number **ovlp** of h-level triangles are added in all directions.

In our experiments all the subproblems are solved exactly by using a band solver from LINPACK. We stop the GMRES method as soon as $\|r_i\|_A/\|r_0\|_A \leq 10^{-3}$. We work with the A-norm, since our theory so far has not been developed for any other norms. However, in our experience, the performance of the algorithm is quite comparable if we replace that norm with the 2-norm. We have found that the overall error is not substantially reduced by a more stringent stopping criterion. In our tables, the **error** denotes the difference between the computed solution and the exact solution of the continuous problem measured in the norms indicated. The programs have been run in single precision on the Multiflow computer at Yale University.

Example 1. We consider the symmetric and indefinite Helmholtz equation

$$\begin{cases} -\triangle u - \delta u &= f \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{6}$$

δ is a constant. The eigenvalues of the operator in (6) are $(i^2+j^2)\pi^2 - \delta$, where i, j are positive integers. The numerical results are given in Tables 1 and 2. Algorithms 1 and 2 given in (4) and (5), respectively, are used.

Iteration	$A-$ norm; residual	L^2 norm; error	L^∞ norm; error
1	2.50722	0.318660	0.719067
2	1.44028	0.182950	0.405074
3	0.971708	1.63224E-02	4.73967E-02
4	0.218693	7.38034E-03	2.46119E-02
5	5.54836E-02	6.13811E-03	1.94588E-02
6	3.40790E-02	5.16731E-03	1.53261E-02
7	2.60738E-02	3.71766E-03	9.73493E-03
8	1.87841E-02	2.19535E-03	6.23578E-03
9	1.03642E-02	1.67804E-03	4.90165E-03
10	6.81844E-03	1.36607E-03	4.00215E-03
11	5.02644E-03	8.28570E-04	2.39784E-03

Table 1: Convergence history for Algorithm 1 and Example 1. Here $h^{-1} = 75$, $H^{-1} = 15$, **ovlp** $= 2$ and $\delta = 16.0\pi^2$

Case #	δ	h^{-1}	H^{-1}	**ovlp**	**Algorithm 1**	**Algorithm 2**
1	$3\pi^2$	15	3	2	11	12
2		30	3	4	11	12
3		45	3	6	12	12
4		60	3	8	12	12
5		15	5	1	10	10
6		30	5	2	12	12
7		45	5	3	12	12
8		60	5	4	12	12
10	$16\pi^2$	45	15	1	10	10
11		60	15	1	11	11
12		75	15	2	11	11
13		60	5	4	44	33
14		60	10	2	17	17
15		60	20	1	8	8
16	$30\pi^2$	60	20	1	16	16
17		80	20	1	17	18

Table 2: Example 1. The last two columns give the number of GMRES iterations.

Iteration	$A-$ norm; residual	L^2 norm; error	L^∞ norm; error
1	2.99430	0.309833	0.696816
2	1.82397	0.234651	0.529782
3	1.34039	0.136604	0.313758
4	0.905845	7.27307E-02	0.172788
5	0.585598	4.46239E-02	0.108047
6	0.407054	2.78872E-02	6.71409E-02
7	0.288880	1.37858E-02	3.38832E-02
8	0.180577	8.21095E-03	2.06587E-02
9	0.129736	4.44917E-03	1.15359E-02
10	8.77408E-02	2.19873E-03	6.17071E-03
11	5.48599E-02	1.18357E-03	3.88315E-03
12	3.46359E-02	7.18704E-04	2.24515E-03
13	2.29454E-02	4.35330E-04	1.39198E-03
14	1.35519E-02	2.90249E-04	1.03428E-03
15	8.90639E-03	2.18270E-04	6.67672E-04
16	5.98530E-03	1.90636E-04	5.61312E-04
17	3.89341E-03	1.72248E-04	5.31457E-04

Table 3: Convergence history for Algorithm 1 and Example 2. Here $h^{-1} = 120$, $H^{-1} = 20$, **ovlp** $= 2$, $\eta = 16.0\pi$ and $\delta = 16.0\pi^2$

Example 2. We consider a nonsymmetric and indefinite problem

$$\begin{cases} -\triangle u - \eta(\partial u/\partial x + \partial u/\partial y) - \delta u &= f \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{7}$$

The numerical results are given in Tables 3 and 4.

We note that in a few of the experiments, the rate of convergence is unsatisfactory, but that the rate of convergence improves considerably by decreasing H. The rate of convergence varies only marginally with the parameter **ovlp**. Normally, the overall cost of the computation is smallest if **ovlp** $=$ **1**. We also note that, as expected, a smaller H is required when the parameters δ and η are increased to increase the terms that make the operators skewsymmetric and indefinite.

6 Two Other Methods

We conclude by outlining how some other results, previously analyzed for the positive definite, symmetric case, can be extended to the class of elliptic problems

Case #	parameters	h^{-1}	H^{-1}	ovlp	Algorithm 1	Algorithm 2
1	$\eta = 3\pi$	15	5	1	13	12
2		30	5	2	17	14
3	$\delta = 3\pi^2$	45	5	3	18	14
4		60	5	4	18	14
5		60	6	3	16	14
6		60	10	2	12	11
7	$\eta = 16\pi$	45	15	1	17	13
8		60	15	1	18	14
9	$\delta = 16\pi^2$	75	15	2	25	17
10		60	20	1	13	11
11		80	20	1	14	12
12		100	20	2	18	14
13		120	20	2	17	14
14	$\eta = 30\pi$	60	20	1	24	16
15		120	20	2	35	19
16	$\delta = 30\pi^2$	75	25	1	17	13
17		100	25	1	18	14
18		120	30	1	15	13

Table 4: Example 2. The last two columns give the number of GMRES iterations.

described in Section 2. We confine our discussion to problems in the plane; both of the algorithms considered here need to be modified considerably in order to obtain fast methods for problems in three dimensions.

We first consider a basic iterative substructuring method for problems in two dimensions; cf. Dryja and Widlund [8], [9]. For problems that are nonsymmetric, but positive definite, the result to be formulated has previously been obtained by Cai [3],[4].

When iterative substructuring methods are used, the region is divided into substructures and elements as in Section 2. Though originally derived differently, it has been demonstrated by Dryja and Widlund [8] that these methods can be viewed as additive Schwarz methods. Our work depends heavily on this reinterpretation of the algorithms; see [8] for detailed arguments.

In defining the partition of the finite element space into subspaces, we use the coarse space V^H introduced in Section 4. We also use subspaces corresponding to the subregions $\Omega_{ij} = \Omega_i \bigcup \Gamma_{ij} \bigcup \Omega_j$. These subregions play the same role as the Ω'_i in Section 4. Here Ω_i and Ω_j are adjacent substructures with the common edge Γ_{ij}. We note that an interior substructure is covered by three such regions. The local subspaces are $V^h_{ij} = H^1_0(\Omega_{ij}) \cap V^h$.

Compared with the case considered previously, we use less overlap in the sense that only the elements of V^H can differ from zero at the vertices of the substructures. This is reflected in a poorer bound for the constant of Lemma 1,

$$C_0^2 \leq const.(1 + log(H/h))^2;$$

cf. Dryja and Widlund [8]. Lemma 1 is modified accordingly. The rest of the proof carries over without change. In Theorem 2, we use the notation $\tilde{Q} = Q_0 + \sum Q_{ij}$.

Theorem 2 *For the iterative substructuring method, introduced as an additive Schwarz method with the subspaces V^H and V^h_{ij}, there exist constants $H_0 > 0$, $c(H_0) > 0$ and $C(H_0) > 0$, such that if $H \leq H_0$,*

$$c(H_0)(1 + log(H/h))^{-2} A(u^h, u^h) \quad \leq \quad A(u^h, \tilde{Q}u^h)$$

and

$$A(\tilde{Q}u^h, \tilde{Q}u^h) \quad \leq \quad C(H_0) A(u^h, u^h).$$

We finally show that the result, obtained by Yserentant [26] for positive definite symmetric problems, can be extended in the same way. We note that Bank and Yserentant [1] have already reported on successful numerical experiments with an accelerated variant of this algorithm for the class of elliptic problems introduced in Section 2. We also note that our algorithm is different from those proposed by Yserentant [27, 28] for indefinite and nonsymmetric problems. Thus in [27] a reduced system obtained by implicitly eliminating the nodes of the coarest mesh is solved by an iterative method.

We assume that the region Ω is a plane polygon. A coarse triangulation is introduced as before. Its triangles are recursively subdivided into four congruent triangles, a total of j times. The characteristic mesh size for the level k triangulation is h_k. As demonstrated in Yserentant [8], more complicated situations can also be considered, where the final triangulation is highly nonuniform, but to simplify our discussion, we only consider the regular case in this paper.

As shown in Dryja and Widlund [10], Yserentant's method can also be viewed as an additive Schwarz method defined by a set of subspaces. Let $I_k v \equiv I_{h_k} v$ be the linear interpolant of $v \in V^h$ onto the space of finite elements on the level k triangulation. The following identity holds

$$v = I_0 v + (I_1 v - I_0 v) + \cdots + (I_j v - I_{j-1} v) \ , \ \forall v \in V^h \ .$$

We represent V^h as

$$V^h = V_0 \oplus V_1 \oplus \cdots \oplus V_j \ ,$$

where $V_0 = V^H$ and, for $k > 0$, $V_k = R(I_k - I_{k-1})$ is the range of the operator $(I_k - I_{k-1})$. An additive Schwarz method is defined for this set of subspaces. We obtain Yserentant's method by replacing , for $k > 0$, the resulting problems on the subspaces by suitable preconditioners.

The following result holds for the family of elliptic problems introduced in Section 2. Here $\hat{Q}$ denotes the operator of the transformed equation, which corresponds to Yserentant's method.

Theorem 3 *For Yserentant's method there exist constants $H_0 > 0$, $c(H_0) > 0$ and $C(H_0) > 0$, such that if $H \leq H_0$,*

$$c(H_0) j^{-2} A(u^h, u^h) \quad \leq \quad A(u^h, \hat{Q} u^h)$$

and

$$A(\hat{Q} u^h, \hat{Q} u^h) \quad \leq \quad C(H_0) A(u^h, u^h).$$

We will only outline how this result can be established. We model our proof on that of Theorem 1. We have to show that the different lemmas hold for the spaces just introduced. It is shown in Yserentant [26] that Lemma 1 holds with $C_0 \leq C j^2$; cf. Lemmas 2.4 and 2.5 of [26]. Lemma 2 is still valid since the same coarse operator is used in all the methods considered in this paper. A counter part of Lemma 3 can be obtained as well, by using Lemma 2.4 of [26]. Lemma 4 is modified by using Lemma 2.7 of [26], a result that makes it possible to obtain a sharp upper bound in Yserentant's main theorem. Lemma 5 can be modified in a straightforward manner. One change is required in the proof of the theorem. The factor $\| \sum_{i=1}^N \hat{Q}_i u^h \|_{L^2}$ must be estimated differently, since, typically, all these terms differ from zero everywhere. By using Yserentant's tools, it is however possible to show that

$$\|\hat{Q}_i u^h\|_{L^2} \ \leq \ Ch_i \|\hat{Q}_i u^h\|_A \ .$$

Since the h_i decay geometrically, the triangle and Cauchy-Schwarz inequalities give

$$\| \sum_{i=1}^{N} \hat{Q}_i u^h\|_{L^2}^2 \ \leq \ CH^2 \sum_{i=1}^{N} \|\hat{Q}_i u^h\|_A^2 \ ,$$

and the proof can be completed.

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