COMMUNICATIONS IN COMPUTATIONAL PHYSICS
Vol. **6**, No. 2, pp. 342-353

Commun. Comput. Phys.
August 2009

A Preconditioned Recycling GMRES Solver for Stochastic Helmholtz Problems

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Received 20 October 2007; Accepted (in revised version) 6 October 2008

Available online 15 December 2008

Abstract. We present a parallel Schwarz type domain decomposition preconditioned recycling Krylov subspace method for the numerical solution of stochastic indefinite elliptic equations with two random coefficients. Karhunen-Loève expansions are used to represent the stochastic variables and the stochastic Galerkin method with double orthogonal polynomials is used to derive a sequence of uncoupled deterministic equations. We show numerically that the Schwarz preconditioned recycling GMRES method is an effective technique for solving the entire family of linear systems and, in particular, the use of recycled Krylov subspaces is the key element of this successful approach.

AMS subject classifications: 35R60, 60H15, 60H35, 65C30, 47B80, 65N55, 65M55

Key words: Recycling GMRES, domain decomposition, additive Schwarz preconditioner, stochastic Helmholtz equation.

1 Introduction

Tremendous progress has been made recently in developing reliable and fast algorithms for solving partial differential equations with uncertainty in the coefficients [1, 5–7, 10, 15, 21, 22]. We study a domain decomposition preconditioned recycling Krylov subspace technique [18] for solving some stochastic partial differential equations. In particular, we focus on a class of indefinite elliptic equations which are more sensitive to the stochastic perturbations. The method was introduced in [14] for solving the uncoupled systems of equations arising from the discretization of stochastic elliptic equations with a single

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random coefficient. In this paper, we extend the approach to a class of indefinite elliptic problems with random diffusion and reaction coefficients

$$\begin{cases} -\nabla\cdot(a(x,\omega_1)\nabla u(x,\omega))-c(x,\omega_2)u(x,\omega)=f(x) & x\in D,\ \omega_i\in\Omega, \\ u(x,\omega)=0 & x\in\partial D,\ \omega_i\in\Omega, \end{cases}$$

where

$$\omega=(\omega_1,\omega_2),\quad a(x,\omega_1)\geq\alpha>0,\quad c(x,\omega_2)\geq 0,$$

D is the domain for x, and Ω is the sample space for $\omega_i, i=1,2$. This type of Helmholtz equations appears in many important applications such as computational acoustics and is rather difficult to solve by iterative methods because of the existence of both positive and negative eigenvalues and eigenvalues that are very close to zero. Slight perturbation of the coefficients of the equations may move some of eigenvalues from positive to negative or to somewhere very close to zero. Through a large number of numerical experiments, we found that traditional preconditioning technique, which uses one matrix in the sequence to precondition another matrix in the same sequence, is not very effective. The precise reason is not clear, but it may be because the eigen bounds (for both positive and negative eigenvalues) are too different for the un-preconditioned matrix and the preconditioning matrix. On the other hand, our numerical experiments show that the recycling Krylov subspace method is very effective in this situation. We believe this is due to the fact that the recycling Krylov subspace method provides a very good initial guess for solving the next system. We mention that the idea of re-using preconditioner and the idea of re-using the Krylov subspace are not new, but to use the combination for solving this type of indefinite problems is new and the observation of the effectiveness of the method has never been reported elsewhere.

There are several approaches for solving the problems, we follow [1, 10] to use the so-called double orthogonal basis to decouple the high dimensional equation in the probability space and produce a sequence of independent systems

$$A_i x_i = b_i,\quad i=1,2,\cdots, \tag{1.1}$$

where the matrices A_i and right-hand sides b_i are somewhat related but independent from each other. Each system in the sequence is indefinite, and is rather difficult to solve using any iterative methods. We use the recently introduced recycling Krylov subspace method [18], which starts the iteration from a Krylov subspace created from a previous system. For preconditioning, we use an overlapping additive Schwarz domain decomposition method [20]. Our parallel implementation is based on the Portable Extensible Toolkit for Scientific computation (PETSc) package from Argonne National Laboratory [2].

The rest of the paper is organized as follows. In Section 2, we describe the stochastic Galerkin method including the stochastic weak formulation, the Karhunen-Loève (KL) expansion, the double orthogonal basis, and the discretization. Section 3 presents the additive Schwarz preconditioned recycling Krylov subspace method. Some experimental results are reported in Section 4.

2 Stochastic Galerkin method

We briefly review some notations [11, 16]. Given a probability space $(\Omega,\mathcal{A},\mathcal{P})$ with sample space Ω, σ-algebra $\mathcal{A}$ and probability measure $\mathcal{P}$, a real-valued random variable is a function $\xi(\omega_1)\colon \Omega\to\mathbf{R}$. The probability distribution measure of ξ is defined on the Borel sets B as $\mu(B)=\mathcal{P}(\sim^{-\infty}(\mathcal{B}))$. The mean, or expected value of $\xi(\omega_1)$, is

$$\langle\xi\rangle=\int_\Omega \xi(\omega_1)d\mathcal{P}(\omega_1)=\int_{\mathbf{R}} x d\mu(x)=\int_{\mathbf{R}} x\rho(x)dx,$$

where ρ is the probability density function of ξ. We also define the space

$$L^2(\Omega)=\left\{\xi(\omega_1)\,\middle|\,\int_\Omega |\xi|^2 d\mathcal{P}<\infty\right\}.$$

Let $D\subset R^2$ be the domain of x. A random field $a(x,\omega_1)\colon D\times\Omega\to\mathbf{R}$ is a real-valued function jointly measurable with respect to the Lebesgue measure on D and the probability measure $\mathcal{P}$ on Ω. Define the space

$$L^2(D\times\Omega)=\{u(x,\omega)|\ \langle\|u(x,\omega)\|_{L^2(D)}\rangle<\infty\}.$$

The stochastic Sobolev space $H_0^1(D\times\Omega)$ is defined analogously.

We consider a stochastic partial differential equation with stochastic diffusion and reaction coefficients as follows:

$$\begin{cases} -\nabla\cdot(a(x,\omega_1)\nabla u(x,\omega))-c(x,\omega_2)u(x,\omega)=f(x) & \text{in } D,\ \omega_i\in\Omega,\\ u(x,\omega)=0 & \text{on } \partial D,\ \omega_i\in\Omega,\end{cases}$$

where $\omega=(\omega_1,\omega_2)$. The weak form of this equation is to find $u(x,\omega)\in H_0^1(D\times\Omega)$ such that

$$\langle B[u,v]\rangle=\langle(f,v)\rangle\quad\forall v\in H_0^1(D\times\Omega),\tag{2.1}$$

where

$$B[u,v]=\int_D a(x,\omega_1)\nabla u(x,\omega)\cdot\nabla v(x)dx-\int_D c(x,\omega_2)u(x,\omega)v(x)dx,$$
$$(f,v)=\int_D f(x)v(x)dx.$$

We assume that $\beta\geq a(x,\omega_1)\geq\alpha>0$ and $c(x,\omega_2)\geq 0$ are in $L^\infty(D\times\Omega)$. Note that the condition that f is deterministic can be relaxed without introducing much difficulty.

Denoting the mean and the covariance of $a(x,\omega_1)$ as $a_0(x)=\int_\Omega a(x,\omega_1)d\mathcal{P}(\omega_1)$ and

$$\mathrm{Cov}_a(x,x')=\int_\Omega (a(x,\omega_1)-a_0(x))(a(x',\omega_1)-a_0(x))d\mathcal{P}(\omega_1)$$

respectively, the KL expansion represents $a(x,\omega_1)$ in the series form as

$$a(x,\omega_1)=a_0(x)+\sum_{j=1}^{\infty}\sqrt{\lambda_{a,j}}k_{a,j}(x)\,y_{a,j}(\omega_1), \tag{2.2}$$

where $\lambda_{a,j}$ and $k_{a,j}(x)$ are the eigenvalues and orthogonal eigenfunctions of $\mathrm{Cov}_a(x,x')$; i.e.,

$$\int_D \mathrm{Cov}_a(x,x')k_{a,j}(x')d(x')=\lambda_j k_{a,j}(x). \tag{2.3}$$

This series converges in the mean-square sense. By definition, $\mathrm{Cov}_a(x,x')$ is symmetric and positive semidefinite, and there exists a countable sequence of eigenpairs $\{(\lambda_{a,j},k_{a,j})\}$ satisfying

$$\lambda_{a,1}\geq\lambda_{a,2}\geq\cdots\geq\lambda_{a,n}\geq\cdots\to 0$$

and the eigenfunctions $\{k_{a,j}(x)\}$ are orthogonal in $L^2(D)$. Moreover, $\{y_{a,j}\}$ is a set of uncorrelated random variables with mean value zero. If the eigenfunctions are normalized, $y_{a,j}$ all have unit variance; i.e.,

$$\langle y_{a,j}(\omega_1)\rangle=0,\quad \langle y_{a,i}(\omega_1)y_{a,j}(\omega_1)\rangle=\delta_{ij}.$$

For the computation, we approximate $a(x,\omega_1)$ by a truncation of (2.2),

$$a_{M_a}(x,\omega_1)=a_0(x)+\sum_{j=1}^{M_a}\sqrt{\lambda_{a,j}}k_{a,j}(x)\,y_{a,j}(\omega_1),$$

where M_a denotes the number of terms in the truncation.

In this paper, we also assume that $\{y_{a,j}\}_{j=1}^{M_a}$ are independent. The probability density function of $y_{a,j}$ is denoted as $\rho_{a,j}$, and the joint probability density function of $y_a=(y_{a,1},\cdots,y_{a,M_a})$ is $\rho_a=\rho_{a,1}\times\cdots\times\rho_{a,M_a}$. Let $\Gamma_{a,j}$ denote the image of $y_{a,j}$ and $\Gamma_a=\Gamma_{a,1}\times\cdots\times\Gamma_{a,M_a}$. We treat $a_M: D\times\Gamma_a\to R$ as

$$a_M(x,y_a)=a_0(x)+\sum_{j=1}^{M_a}\sqrt{\lambda_{a,j}}k_{a,j}(x)\,y_{a,j}. \tag{2.4}$$

Similarly, we have the truncation of the KL expansion for $c(x,\omega_2)$:

$$c_M(x,y_c)=c_0(x)+\sum_{j=1}^{M_c}\sqrt{\lambda_{c,j}}k_{c,j}(x)\,y_{c,j}, \tag{2.5}$$

where $y_c=\{y_{c,j}\}_{j=1}^{M_c}$ are independent variables with probability density functions $\{\rho_{c,j}\}_{j=1}^{M_c}$. The image of y_c is $\Gamma_c=\Gamma_{c,1}\times\cdots\times\Gamma_{c,M_c}$. Now, we arrive at the deterministic problem,

$$\begin{cases} -\nabla\cdot(a_M(x,y_a)\nabla u(x,y))-c_M(x,y_c)u(x,y)=f(x), \\ u(x,y)=0, \end{cases} \tag{2.6}$$

where $x\in D$, $y=(y_a,y_c)\in\Gamma=\Gamma_a\times\Gamma_c$.

For $y\in\Gamma$, we use the double orthogonal polynomial function space [1, 10] to approximate $L^2(\Gamma,\rho)$, which decouples the equation in the y-space, yielding a sequence of uncoupled equations. We first construct the double orthogonal basis for y_a. For any $r\in\mathbf{N}$, the space of single-variable polynomials of degree at most r is denoted as P_r. For $\mathbf{r_a}=(r_1^a,r_2^a,\cdots,r_{M_a}^a)\in\mathbf{N}^{M_a}$, we construct the multi-variable polynomial space

$$P_{\mathbf{r_a}}:=P_{r_1^a}\otimes P_{r_2^a}\otimes\cdots\otimes P_{r_{Ma}^a}\in L^2(\Gamma_a,\rho_a). \tag{2.7}$$

For the space $P_{r_j^a},j=1,2,\cdots,M_a$, we use the double orthogonal functions, denoted as $\{\psi_{k,j}^a(t)\}_{k=0}^{r_j^a}$, as basis instead of the simple polynomial basis $\{1,t,t^2,\cdots,t^{r_j^a}\}$. We require that $\psi_{k,j}^a(t),k=0,\cdots,r_j^a$ satisfy two orthogonality conditions:

$$\begin{cases} \displaystyle\int_{\Gamma_{a,j}}\psi_{p,j}^a(t)\psi_{q,j}^a(t)\rho_{a,j}(t)dt=\delta_{p,q}, & p,q=0,\cdots,r_j^a,\\ \displaystyle\int_{\Gamma_{a,j}}t\psi_{p,j}^a(t)\psi_{q,j}^a(t)\rho_{a,j}(t)dt=C_{p,j}^a\delta_{p,q}, & p,q=0,\cdots,r_j^a, \end{cases} \tag{2.8}$$

where $\{C_{p,j}^a\}_{p=0}^{r_j}$ are nonzero constants. Next we construct a basis function of $P_{\mathbf{r}}$ by selecting one polynomial basis function from each $P_{r_j^a},j=1,\cdots,M_a$, and then multiplying these selected M_a basis functions together. So given a $\mathbf{r_a}=(r_1^a,r_2^a,\cdots,r_{M_a}^a)\in\mathbf{N}^{M_a}$, there are total $N_{y_a}=\prod_{j=1}^{M_a}(r_j^a+1)$ basis functions for $P_{\mathbf{r_a}}(y_1,y_2,\cdots,y_{M_a})$.

Let $\mathbf{i_a}=\{i_1^a,i_2^a,\cdots,i_{M_a}^a\}$. If $0\le i_j^a\le r_j^a,\forall 1\le j\le M_a$, we say that $\mathbf{i_a}\le\mathbf{r_a}$. It is obvious that there are N_{y_a} multiindex $\mathbf{i_a}$, which is less than or equal to $\mathbf{r_a}$. Each $\mathbf{i_a}$ corresponds to one basis function for $P_{\mathbf{r_a}}$. We denote all the basis functions for $P_{\mathbf{r_a}}$ as the set

$$\left\{\psi_{\mathbf{i_a}}(y_a)\,\middle|\,\psi_{\mathbf{i_a}}(y_a)=\prod_{j=1}^{M_a}\psi_{i_j^a,j}^a(y_{a,j}),\ i_j^a\in\{0,1,\cdots,r_j^a\}\right\}_{\mathbf{i_a}\le\mathbf{r_a}}. \tag{2.9}$$

Finding $\{\psi_{k,j}^a(y_{a,j})\}_{k=0}^{r_j^a}$ for spaces $P_{r_j^a}$, $j=1,\cdots,M_a$, results in an eigenproblem (cf. Section 8.7.2 in [12]). For the probability space of y, generally we do not need high order polynomials. So the computational work for these eigen problems is negligible comparing to the cost required to solve the coupled equations. Similarly, we construct the double orthogonal basis for $L^2(\Gamma_c,\rho_c)$. The basis functions are denoted as $\{\psi_{\mathbf{i_c}}(y_c)\}_{\mathbf{i_c}\le\mathbf{r_c}}$. Let $\mathbf{i}=(\mathbf{i_a},\mathbf{i_c})$, $\mathbf{r}=(\mathbf{r_a},\mathbf{r_c})$, and $\rho=\rho_a\times\rho_c$. The double orthogonal basis for $L^2(\Gamma,\rho)$ is

$$\left\{\psi_{\mathbf{i}}(y)\,|\,\psi_{\mathbf{i}}(y)=\psi_{\mathbf{i_a}}(y_a)\psi_{\mathbf{i_c}}(y_c),\ \forall\,\mathbf{i}\le\mathbf{r}\right\}_{\mathbf{i}\le\mathbf{r}}, \tag{2.10}$$

where $\mathbf{i}\le\mathbf{r}$ is defined as $\mathbf{i_a}\le\mathbf{r_a}$ and $\mathbf{i_c}\le\mathbf{r_c}$. The basis functions defined by (2.10) satisfy the following equations:

$$\int_\Gamma y_k\psi_{\mathbf{i}}(y)\psi_{\mathbf{j}}(y)\rho(y)dy=C_{i_k,k}\delta_{\mathbf{i},\mathbf{j}}, \tag{2.11}$$

where y_k is $\{y_{a,k}]\}_{k=1}^{M_a}$ or $\{y_{c,k}\}_{k=1}^{M_c}$, and $C_{i_k,k}$ is $\{C^a_{i^a_k,k}\}_{k=1}^{M_a}$ or $\{C^c_{i^c_k,k}\}_{k=1}^{M_c}$ depending on the index $\mathbf{i}$ and $\mathbf{j}$.

To see why (2.11) holds, we start with

$$\psi_{\mathbf{i}}(y)=\psi_{\mathbf{i_a}}(y_a)\psi_{\mathbf{i_c}}(y_c)=\prod_{p=1}^{M_a}\psi^a_{i^a_p,p}(y_{a,p})\prod_{p=1}^{M_c}\psi^c_{i^c_p,p}(y_{c,p}),$$
$$\psi_{\mathbf{j}}(y)=\psi_{\mathbf{j_a}}(y_a)\psi_{\mathbf{j_c}}(y_c)=\prod_{q=1}^{M_a}\psi^a_{j^a_q,q}(y_{a,q})\prod_{q=1}^{M_c}\psi^c_{i^c_q,q}(y_{c,q}).$$

Without loss of generality, let us assume y_k is $y_{a,k}$. Then

$$\begin{aligned}&\int_\Gamma y_{a,k}\psi_{\mathbf{i}}(y)\psi_{\mathbf{j}}(y)\rho(y)dy\\ =&\int_{\Gamma_a} y_{a,k}\prod_{p=1}^{M_a}\psi^a_{i^a_p,p}(y_{a,p})\prod_{q=1}^{M_a}\psi^a_{j^a_q,q}(y_{a,q})\rho_a(y_a)d(y_a)\\ &\times\int_{\Gamma_c}\prod_{p=1}^{M_c}\psi^c_{i^c_p,p}(y_{c,p})\prod_{q=1}^{M_c}\psi^c_{j^c_q,q}(y_{c,q})\rho_c(y_c)d(y_c).\end{aligned}$$

From (2.8),

$$\begin{aligned}&\int_{\Gamma_a} y_{a,k}\prod_{p=1}^{M_a}\psi^a_{i^a_p,p}(y_{a,p})\prod_{q=1}^{M_a}\psi^a_{j^a_q,q}(y_{a,q})\rho_a(y_a)d(y_a)\\ =&\prod_{q=1,q\neq k}^{M_a}\int_{\Gamma_{a,q}}\psi^a_{i^a_q,q}(y_{a,q})\psi^a_{j^a_q,q}(y_{a,q})\rho_{a,q}(y_{a,q})d(y_{a,q})\\ &\times\int_{\Gamma_{a,k}} y_{a,k}\psi^a_{i^a_k,k}(y_{a,k})\psi^a_{j^a_k,k}(y_{a,k})\rho_{a,k}(y_{a,k})d(y_{a,k})=C_{i^a_k,k}\delta_{\mathbf{i_a},\mathbf{j_a}}.\end{aligned}\tag{2.12}$$

Note that $\mathbf{i_a}=\mathbf{j_a}$ if and only if $i^a_1=j^a_1, i^a_2=j^a_2,\cdots,i^a_{M_a}=j^a_{M_a}$. Similarly, from the first equation in (2.8), we have

$$\int_{\Gamma_c}\prod_{p=1}^{M_c}\psi^c_{i^c_p,p}(y_{c,p})\prod_{q=1}^{M_c}\psi^c_{j^c_q,q}(y_{c,q})\rho_c(y_c)d(y_c)=\delta_{\mathbf{i_c},\mathbf{j_c}}.$$

Combining with (2.12), we obtain

$$\int_\Gamma y_{a,k}\psi_{\mathbf{i}}(y)\psi_{\mathbf{j}}(y)\rho(y)dy=C_{i^a_k,k}\delta_{\mathbf{i},\mathbf{j}}.$$

With the double orthogonal basis given in (2.10), Eq. (2.6) is equivalent to N_y independent deterministic equations in D as follows:

$$-\nabla\Big(a_{M,\mathbf{i}}(x)\nabla u_{M,\mathbf{i}}(x)\Big)-c_{M,\mathbf{i}}(x)u_{M,\mathbf{i}}(x)=f_{\mathbf{i}}(x)\tag{2.13}$$

with

$$\begin{cases} a_{M,\mathbf{i}}(x) := a_0(x) + \sum_{j=1}^{M_a} \sqrt{\lambda_{a,j}} k_{a,j}(x) C^a_{i^a_j,j}, \\ c_{M,\mathbf{i}}(x) := c_0(x) + \sum_{j=1}^{M_c} \sqrt{\lambda_{c,j}} k_{c,j}(x) C^c_{i^c_j,j}, \\ f_{\mathbf{i}}(x) := f(x) \cdot \int_{\Gamma_a} \psi_{\mathbf{i_a}}(t)\rho_a(t)dt \int_{\Gamma_c} \psi_{\mathbf{i_c}}(t)\rho_c(t)dt. \end{cases} \tag{2.14}$$

The statistics of the solution can be found from the approximate solutions $u_{M,\mathbf{i}}(x)$. For example, the mean of $u(x,y)$ can be approximated by,

$$\langle u_M \rangle = \sum_{\mathbf{i}\le\mathbf{r}} u_{M,\mathbf{i}}(x) \int_\Gamma \psi_{\mathbf{i}}(y)\rho(y)dy,$$

and the second order moment of the solution can calculated by

$$\int_\Gamma \Big(\sum_{\mathbf{i}\le\mathbf{r}} u_{M,\mathbf{i}}(x)\psi_{\mathbf{i}}(y) \Big)^2 \rho(y)dy = \sum_{\mathbf{i}\le\mathbf{r}} (u_{M,\mathbf{i}}(x))^2 \int_\Gamma (\psi_{\mathbf{i}}(y))^2 \rho(y)dy = \sum_{\mathbf{i}\le\mathbf{r}} (u_{M,\mathbf{i}}(x))^2.$$

3 Numerical method

The number of linear systems in the sequence (2.13) can be very large, and every system is large and highly ill-conditioned. To efficiently solve the whole set of systems, one has to be able to re-use as much computation as possible from one of the previous systems. The most straightforward idea is to simply re-use the preconditioner computed from a previous system. The idea is quite successful for regular elliptic problems. However, for this type of indefinite problem under consideration in this paper, we found, somewhat surprisingly, that re-using preconditioner is not very effective. The re-use of a Krylov subspace turns out to be the best choice. It is not understood theoretically why this approach is so good, but intuitively it seems to suggest that the Krylov subspace from a previous system carries some important eigen information that is needed for obtaining the fast convergence when solving a later system in the same sequence. We next briefly describe the recycling method and the additive Schwarz preconditioner.

There are several recycling Krylov subspace methods, see for example [4,8,9,13,17,19] and references therein. We use the Generalized Conjugate Residual with implicit inner Orthogonalization and Deflated Restarting (GCRO-DR) introduced in [18]. In our case the sequence of linear systems has different matrices and different right-hand sides.

The idea of GCRO-DR is to retain a Krylov subspace for subsequent restarted GMRES cycles, or for solving other linear systems. This approach was extended to include the flexible version of GMRES (FGMRES) in [14]. Suppose we have solved the i^{th} system with FGMRES. We retain k vectors

$$\tilde{Y}_k = [\tilde{y}_1, \tilde{y}_2, \cdots, \tilde{y}_k]. \tag{3.1}$$

Let $C_k=Q$, $U_k=\tilde{Y}_kR^{-1}$ where Q and R are from the reduced QR decomposition of $A_{i+1}\tilde{Y}_k$. U_k, $C_k\in R^{n\times k}$ satisfy

$$A_{i+1}U_k=C_k,\ C_k^HC_k=I_k. \tag{3.2}$$

Let x_0 and r_0 be the initial guess and initial residual of $A_{i+1}x_{i+1}=b_{i+1}$. We update the solution as $x=x_0+U_kC_k^Hr_0$, and set $r=r_0-C_kC_k^Hr_0$. In our numerical experiment, we find that for some systems, this updated solution x already satisfies the error tolerance so that we do not need any more iterations after the initial step. However, if the x is still not good enough, we continue to generate a Krylov subspace of dimension $m-k+1$ with $(I-C_kC_k^H)A_{i+1}$, where m is the maximum number of iterations before restarting. In the case of recycling FGMRES, it produces the Arnoldi relation

$$(I-C_kC_k^H)A_{i+1}Z_{m-k}=V_{m-k+1}\bar{H}_{m-k}, \tag{3.3}$$

where the columns of Z_{m-k} are the preconditioned orthogonal vectors V_{m-k}.

To obtain the solution of the linear system $A_ix_i=b_i$, we actually solve a preconditioned system $M_j^{-1}A_ix=M_j^{-1}b_i$, where M_j^{-1} is a one-level additive Schwarz preconditioner constructed using matrix A_j, [3]. To formally define M_j^{-1}, we need to introduce a partition of D. We first partition the domain into non-overlapping substructures D_l, $l=1,\cdots,N$. In order to obtain an overlapping decomposition of the domain, we extend each subregion D_l to a larger region D_l', i.e., $D_l\subset D_l'$. Only simple box decomposition is considered in this paper – all subdomains D_l and D_l' are rectangular and made up of integral numbers of fine mesh cells. The size of D_l is $H_x\times H_y$ and the size of D_l' is $H_x'\times H_y'$, where the H's are chosen so that the overlap, $ovlp$, is uniform in the number of fine grid cells all around the perimeter, i.e., $ovlp=(H_x'-H_x)/2=(H_y'-H_y)/2$ for every subdomain. Homogeneous Dirichlet boundary conditions are used on the subdomain boundary $\partial D_l'$. The additive Schwarz preconditioner can be written as

$$M_i^{-1}=(R_1)^T(A_i)_1^{-1}R_1+\cdots+(R_N)^T(A_i)_N^{-1}R_N. \tag{3.4}$$

Let n be the total number of mesh points and n_l' the total number of mesh points in D_l'. Then, R_l is an $n_l'\times n$ matrix that is defined as: its element $(R_l)_{i,j}$ is 1 if the integer indices $1\le i\le n_l'$ and $1\le j\le n$ belong to a mesh point in D_l', or zero otherwise. The R_l serves as a restriction matrix because its multiplication by a $n\times 1$ vector results in a smaller $n_l'\times 1$ vector by dropping the components corresponding to mesh points outside D_l'. Various inexact additive Schwarz preconditioners can be constructed by replacing the matrices A_l in (3.4) with convenient and inexpensive to compute matrices, such as those obtained with incomplete and complete factorizations. In this paper we employ the LU factorization.

4 Numerical experiment

Consider the problem (2.6) on $D=(0,1)\times(0,1)$ with zero Dirichlet boundary conditions. The mean and covariance functions for $a(x,\omega_1)$ explicitly given as

$$a_0(x)=5+\sin(\pi x_1) \text{ and } C_a(x,x')=e^{-|x-x'|^2}.$$

The covariance function for $c(x,\omega_2)$ is chosen to be the same as that of $a(x,\omega_1)$ and the mean function is $c_0(x)=5$. For the KL expansion of $a(x,\omega_1)$ and $c(x,\omega_2)$, we choose $M_a=M_c=4$ truncation terms as the approximation. This implies the dimensions for the y_a-space and y_c-space are 4 respectively. We also choose the highest degree of the polynomials for $y_{a,j}$ and $y_{c,j}$ to be $\mathbf{r}=(3,2,2,1)$. This produces total 5184 systems. At the same time, we assume that $y_{a,j}$ and $y_{c,j}$, $j=1,2,3,4$ are uniformly distributed in $\Gamma_{a,j}=[-\sqrt{3},\sqrt{3}]$. So the probability density function $\rho_{a,j}=\frac{1}{2\sqrt{3}}$. Note that this implies that the variance of $y_{a,j}$ is unit. To make the test case highly indefinite, we multiply the reaction coefficient in our numerical computation by 45. So the reaction coefficient in our numerical example is actually $45c(x,\omega_2)$. We study the following four schemes

Scheme 1. No recycling of the Krylov subspace and no recycling of the preconditioner

Scheme 2. Recycle the preconditioner only without recycling of the Krylov subspace for the entire sequence of systems

Scheme 3. Recycle the Krylov subspace and recycle the symbolic factorizations of subdomain matrices for the entire sequence of systems

Scheme 4. Recycle the Krylov subspace and recycle the preconditioner for the entire sequence of systems

We next show some numerical results obtained on an IBM BG/L using 64 processors. The mesh size is 512×512. The solution is considered to be acceptable if

$$\|b_i-A_ix_i\|\leq\max\{10^{-10},10^{-6}\|b_i\|\}.$$

The overlap of the subdomains is 8. The subdomain solver is LU factorization. FGMRES restarts every 50 iterations including the recycling Krylov subspace. Fig. 1 is the number of iterations for Scheme 1 and 2. It is obvious that the two schemes share similar situations in the number of iterations. Most of the systems require about 100 iterations to converge. Some systems do not converge as they reach the maximum number of iterations, which is 1000 in our numerical experiments. Scheme 2 has more divergent systems compared with Scheme 1.

Fig. 2 shows the number of iterations for Scheme 3 and 4. For these two schemes, we see that the number of iterations for the first system is about 60, which is shown as

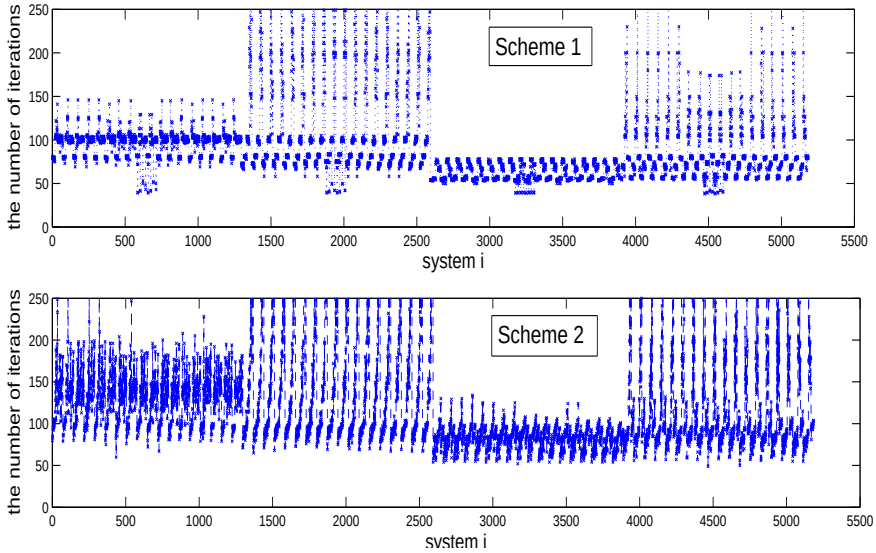

Figure 1: The number of iterations for Scheme 1 and 2.

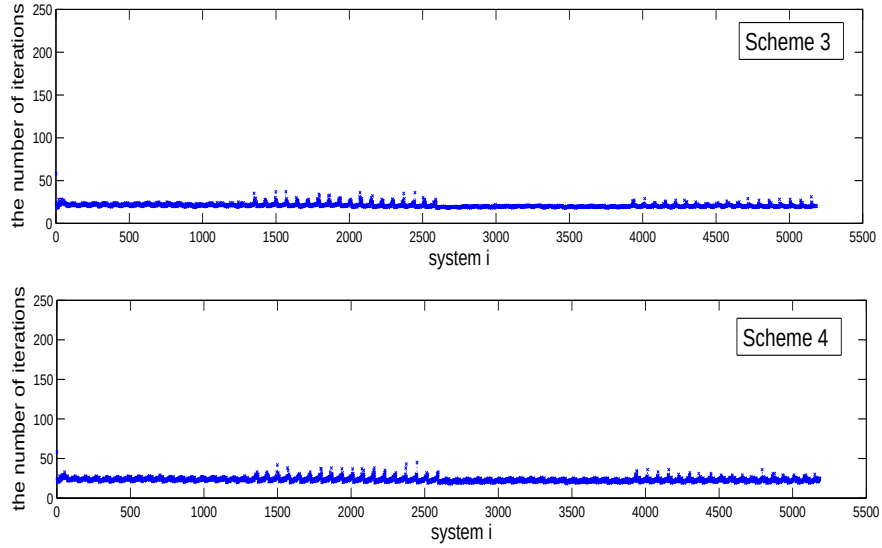

Figure 2: The number of iterations for Scheme 3 and 4.

a single mark in the figures. However, the number decreases to around 25 due to the use of the recycled Krylov subspace. Since we apply the same preconditioner to all the systems in Scheme 4, the number of iterations for Scheme 4 is a little more than Scheme 3. However, Scheme 4 takes less computing time because of the savings from recomputing the preconditioners as in Scheme 3. The computing time is shown in Table 1.

Table 1: The average number of iterations and computing time for different schemes.

	Scheme 1	Scheme 2	Scheme 3	Scheme 4
Average Iterations	91.9	130.1	20.7	23.1
Computing Time(s)	33500	42000	13630	11610

Comparing the number of iterations for different schemes, we see that the additive Schwarz preconditioned recycling Krylov subspace method is very effective in solving indefinite problems. Table 1 contains a comparison of the average number of iterations and the total computing time for different schemes. Scheme 3 is the best strategy in terms of the number of iterations. The average number of iterations in Scheme 4 is a littler more than that in Scheme 3 due to the fact that the same preconditioner is used for all systems. However, this reduces the total computing time. Overall, Scheme 4 is the winner in terms of the total computing time.

Acknowledgments

The authors would like to thank Professor Dongbin Xiu for helpful discussions. The research was supported in part by DOE under DE-FC02-04ER25595, and in part by NSF under grants CCF-0634894, and CNS-0722023.

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