OVERLAPPING NON-MATCHING GRID MORTAR ELEMENT METHODS FOR ELLIPTIC PROBLEMS*

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Abstract. In the first part of the paper, we introduce an overlapping mortar finite element methods for solving two-dimensional elliptic problems discretized on overlapping non-matching grids. We prove an optimal error bound and estimate the condition numbers of certain overlapping Schwarz preconditioned systems for the two-subdomain case. We show that the error bound is independent of the size of the overlap and the ratio of the mesh parameters. In the second part, we introduce three additive Schwarz preconditioned conjugate gradient algorithms based on the trivial and harmonic extensions. We provide estimates for the spectral bounds on the condition numbers of the preconditioned operators. We show that although the error bound is independent of the size of the overlap, the condition number does depend on it. Numerical examples are presented to support our theory.

Key words. non-matching grid, finite element, mortar projection, overlapping domain decomposition, elliptic equations, Schwarz preconditioner

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1. Introduction. The mortar element method was first developed for the purpose of coupling different discretizations in different nonoverlapping subdomains. Several studies have been carried out; see e.g., [1, 2, 3, 4, 5, 6, 7, 11, 12, 15, 16, 22, 25, 29, 30]. In this paper, we consider the case of overlapping subdomains. We provide an optimal error analysis for the two-subdomain case, and also spectral bound estimations for the Schwarz preconditioned systems. The main advantage of non-matching grid methods is that highly structured local grids and corresponding fast solvers (and software) can be used easily. To preserve the global accuracy of the discretization, the interpolation between the neighboring subdomains has to be sufficiently accurate. The mortar method provides one such interpolation scheme that passes the values of a function from one grid to another without loosing accuracy as will be shown in this paper. It is somewhat surprising that the discretization error is independent of the overlap as long as a trivial requirement is satisfied; the overlap is not smaller than the size of the coarser mesh. We also show that the error is independent of the ratio of the mesh sizes. Another interesting finding is that larger overlap can make the resulting linear system easier to precondition. We note that, independent of the development

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of mortar based methods, overlapping non-matching grids techniques have been used for more than ten years by computational engineers in many large scale simulations as a way to reduce the cost of grid generation. The methods are often referred to as the Chimera methods or overset grid methods ([13, 20, 26]).

We are interested in solving the following elliptic variational problem: Find $u^* \in H_0^1(\Omega)$, such that

$$a(u^*, v) = f(v), \quad \forall\, v \in H_0^1(\Omega), \tag{1}$$

where

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v \, dx \quad \text{and} \quad f(v) = \int_\Omega f v \, dx.$$

Here $f(x) \in L^2(\Omega)$ is a given function and $\Omega = \Omega_1 \cup \Omega_2$ an open polygonal domain in $\Re^2$. We assume that both Ω_1 and Ω_2 are open polygonal domains and that the diameters of Ω, Ω_1 and Ω_2 are of order 1. We shall introduce two independent triangulations on Ω_1 and Ω_2, respectively, and a mortar element method defined on the union of the two, generally non-matching, triangulations. We assume that u^* satisfies the following local regularity conditions:

$$u^*|_{\Omega_i} \in H^{1+\tau_i}(\Omega_i), \text{ and } 0 < \tau_i \le 1$$

for $i = 1, 2$. No global regularity of u^* is assumed.

As mentioned earlier a lot of work has been done in the area of non-overlapping non-matching grid methods. There are also several methods that use overlapping non-matching grid preconditioners for matrix problems obtained from non-overlapping discretization schemes; see [12, 15]. Some very interesting recent development in using overlapping non-matching grid methods can be found, for examples, in the papers of Kuznetsov [23], Blake [8] and Cai, Mathew and Sarkis [10]. However, to the best of our knowledge, this is the first paper that provides an optimal error analysis for the overlapping mortar element method.

To avoid unnecessary complications, we restrict our discussion to Poisson's equation with zero Dirichlet boundary condition. The extension to the smooth variable coefficient case is straightforward. The paper is organized as follows. In Section 2, we introduce some notations. The mortar element method and some implementation remarks are given in Section 3. The analysis of the method is provided in Section 4. Several technical lemmas, used in Section 4, are actually introduced and proved in Section 5. Section 6 reports several numerical experiments that are used to verify the theory on the accuracy. Three preconditioning techniques are proposed and analyzed in Section 7. Section 8 contains some numerical examples supporting the theory of the preconditioning methods. A short conclusion is given in Section 9.

2. Model cases and function spaces. In this paper, we shall focus on two model cases that have different technical difficulties. The main theorem on accuracy holds for both cases, however, different proofs are needed. Most of our results can be extended to more general cases.

Case R: The union of Ω_1 and Ω_2 is a rectangular domain, as shown in Fig. 1.

Case L: The union of Ω_1 and Ω_2 is an L-shaped domain, as shown in Fig. 2.

Before introducing the mortar element method in Ω with non-matching grids in the overlapping subdomains, we need to define some notations. First of all, let $\gamma_i = \partial\Omega_i \cap \Omega, i = 1, 2$, be the interfaces. For Case R we define δ as the distance between the two interfaces, shown in Fig. 1, and for Case L we assume $\delta = O(1)$.

• **Triangulations and finite element spaces**. For $i = 1, 2$, let

$$\mathcal{T}^{h_i} = \{K_j^{h_i}, j = 1, \ldots, M_i\}$$

be a standard finite element triangulation in Ω_i; see for example Fig. 1. Here $K_j^{h_i}$ is a triangle and h_i the mesh size. M_i is the total number of triangles. We assume that they are shape regular and quasi-uniform; see Ciarlet [14]. The two triangulations need not to match in the overlapping region. Let $V^{h_i} \equiv V^{h_i}(\Omega_i)$ be the space of continuous piecewise linear functions on $\mathcal{T}^{h_i}$ which vanishes on $\partial\Omega \cap \partial\Omega_i$. For each node $x_l^{h_i}$ in $\mathcal{T}^{h_i}$ we denote by $\phi_l^{h_i}(x)$ the usual basis function, i.e., $\phi_l^{h_i}(x) \in V^{h_i}$, and $\phi_l^{h_i}(x) = 1$ if $x = x_l^{h_i}$ and zero at all the other nodes. We define the support of a basis function by

$$supp(\phi_l^{h_i}) \equiv supp(x_l^{h_i}) \equiv \{x| \in \Omega_i \text{ and } \phi_l^{h_i}(x) \neq 0\}.$$

Note that $supp(x_l^{h_i})$ is an open set. We also need the space

$$X^h = \{(u_1, u_2)| u_i \in V^{h_i}, i = 1, 2\}.$$

We denote by $V_0^{h_i}$ as a subspace of V^{h_i} containing all functions that vanish on $\partial\Omega_i$.

• **Trace spaces.** We denote by $V^{h_i}(\gamma_i)$ the restriction of V^{h_i} on γ_i. Let us denote by $a_1^i, a_2^i, \cdots, a_{m_i}^i$ the nodes of $\mathcal{T}^{h_i}(\bar{\Omega}_i)$ on $\bar{\gamma}_i$, and also denote by a_0^i and $a_{m_i+1}^i$ the two endpoints of γ_i; see Fig. 2 (a) and Fig. 3. We assume that if a_0^i (or $a_{m_i+1}^i$) is a node of $\mathcal{T}^{h_i}(\bar{\Omega}_i)$ then $a_0^i = a_1^i$ (or $a_{m_i}^i = a_{m_i+1}^i$); see Fig. 1 and Fig. 2 (a). It is important to note that for v_i to belong to V^{h_i}, v_i must vanish at a_1^i and $a_{m_i}^i$; see Fig. 3 (a) for an example of a function in $V^{h_i}(\gamma_i)$.

• **Trivial extension operators.** For any $r^i \in V^{h_i}(\gamma_i)$, we define a function denoted by $\mathcal{E}_i r^i$ in $V^{h_i}(\Omega_i)$ satisfying: $\mathcal{E}_i r^i = r^i$ at the nodes $a_2^i, a_3^i, \cdots, a_{m_i-1}^i$, and $\mathcal{E}_i r^i$ equals to zero at the remaining nodes of $\mathcal{T}^{h_i}$.

• **Interface test function spaces**. For $i = 1, 2$, $\tilde{\mathcal{W}}_{h_i}(\gamma_i)$ denote the space of continuous piecewise linear functions on the grid $a_0^i, a_2^i, \cdots, a_{m_i-1}^i, a_{m_i+1}^i$, subject to the constraints that these continuous piecewise linear functions are constants in the intervals $[a_0^i, a_2^i]$ and $[a_{m_i-1}^i, a_{m_i+1}^i]$; see Fig. 3 (b).

• **Mortars, mortar spaces and slave nodes**. The curve γ_i has two sides. We refer to one of them as the mortar side, and the other as the non-mortar side. In most mortar element methods, see e.g. [7], the choice is rather arbitrary. In our case, we have only one choice. For γ_1, we define the $\mathcal{T}^{h_2}$ side as the mortar side and the $\mathcal{T}^{h_1}$ side as the non-mortar side. On the non-mortar side, a finite element space is defined by using the mortar projection given below by (2). A similar definition is used for γ_2. We define the mortar space $V^{h_2}(\gamma_1)$ (resp. $V^{h_1}(\gamma_2)$), as the restriction to the interface γ_1 (resp. γ_2) of the space V^{h_2} (resp. V^{h_1}). Among the points $a^i_0, a^i_1, \cdots, a^i_{m_i}, a^i_{m_i+1}$, as will be seen later, the values of the solution are known at $a^i_0, a^i_1, a^i_{m_i}$ and $a^i_{m_i+1}$ through the given boundary conditions. We shall refer to the other points, $a^i_2, a^i_3, \cdots, a^i_{m_i-1}$ as the *slave nodes* since their values are determined by the mortar projections to be defined below.

• **Mortar projections**. The *mortar projection* π_1 maps the space $V^{h_2}(\gamma_1)$ into $V^{h_1}(\gamma_1)$. Given a $\varphi \in L_2(\gamma_1)$, we set $(\pi_1\varphi) \in V^{h_1}(\gamma_1)$ to zero in the intervals $[a^1_0, a^1_1]$ and $[a^1_{m_1}, a^1_{m_1+1}]$ and determine the values of $(\pi_1\varphi)$ at the slave nodes $a^1_2, a^1_3, \cdots, a^1_{m_1-1}$ by

$$\int_{\gamma_1} (\varphi - \pi_1\varphi)\psi \, ds = 0, \qquad \forall \psi \in \tilde{\mathcal{W}}_{h_1}(\gamma_1). \tag{2}$$

Similarly, we define the mortar projection π_2 on γ_2, which maps $V^{h_1}(\gamma_2)$ into $V^{h_2}(\gamma_2)$.

• **The solution space**. We define the solution space V^h as follows:

$$V^h = \left\{(u_1, u_2)|u_i \in V^{h_i},\ i = 1, 2,\ \ u_{1|\gamma_1} = \pi_1(u_{2|\gamma_1}) \text{ and } u_{2|\gamma_2} = \pi_2(u_{1|\gamma_2})\right\}.$$

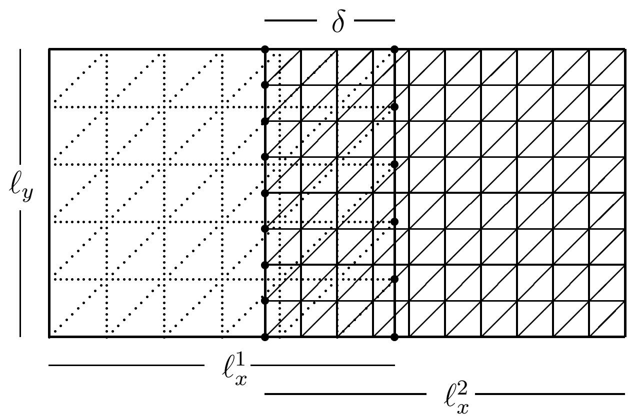

FIG. 1. *The subregions $\Omega_i, i = 1, 2,$ are rectangles $\Omega_i = \ell^i_x \times \ell_y$. ℓ^i_x, ℓ_y are of $O(1)$. δ is the size of the overlap.*

Before closing this section, we need to make an important assumption under which the mortar projections are computable.

Assumption 1. *Let a^i_k be a slave node on γ_i, then*

$$supp(a^i_k) \cap \gamma_j = \emptyset, \ \ for\ i \neq j,$$

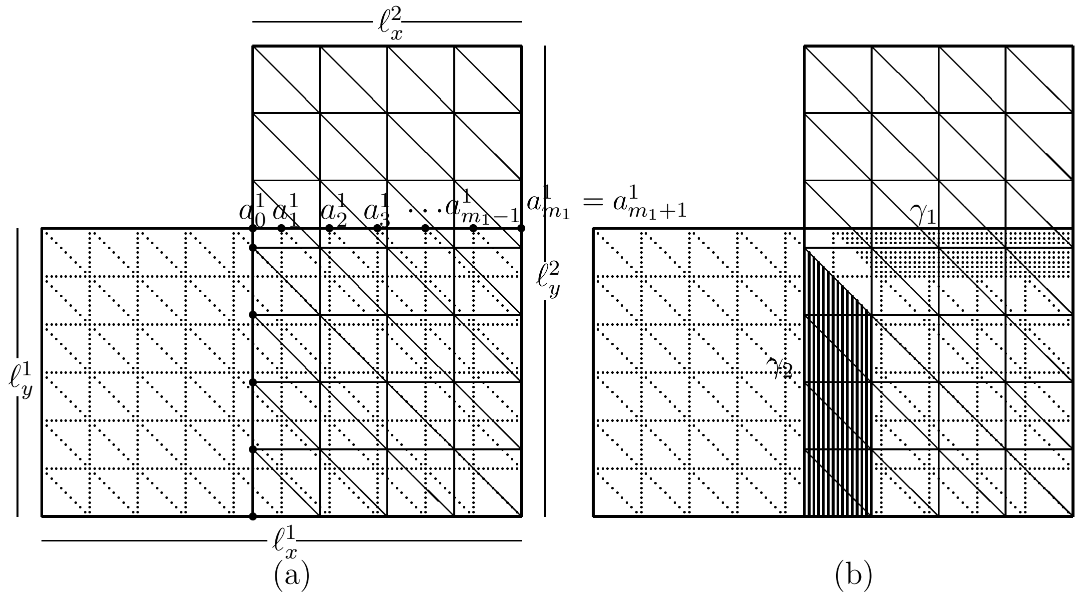

FIG. 2. *Case L, i.e. the union of Ω_1 and Ω_2 is an L-shaped region.*

and $i, j = 1, 2$.

REMARK 2.1. *For Case R, the assumption implies that $\delta \geq \max\{h_1, h_2\}$; otherwise the subdomains are not connected on the mesh level. For Case L, it means that the two darkened regions in Fig. 2 (b) do not intersect each other. Without this condition, the two mortar projections cannot be calculated independently.*

3. Overlapping mortar element methods. In this section, we introduce the overlapping mortar element method and discuss some implementation issues, such as the construction of basis functions in V^h. Our variational problem associated with (1) is defined by: Find $u = (u_1, u_2) \in V^h$, such that

$$a_h(u, v) = f_h(v) \quad \forall v = (v_1, v_2) \in V^h, \tag{3}$$

where the weighted bilinear form is defined as

$$a_h(u,v) = \int_{\Omega_1 \setminus \Omega_2} \nabla u_1 \cdot \nabla v_1 \, dx + \frac{1}{2} \int_{\Omega_1 \cap \Omega_2} \nabla u_1 \cdot \nabla v_1 \, dx +$$

$$\frac{1}{2} \int_{\Omega_1 \cap \Omega_2} \nabla u_2 \cdot \nabla v_2 \, dx + \int_{\Omega_2 \setminus \Omega_1} \nabla u_2 \cdot \nabla v_2 \, dx$$

and

$$f_h(v) = \int_{\Omega_1 \setminus \Omega_2} f v_1 \, dx + \frac{1}{2} \int_{\Omega_1 \cap \Omega_2} f v_1 \, dx +$$

$$\frac{1}{2} \int_{\Omega_1 \cap \Omega_2} f v_2 \, dx + \int_{\Omega_1 \setminus \Omega_2} f v_2 \, dx.$$

The main motivation for defining the variational problem this way is that the resulting stiffness matrix is symmetric. We will show later that the space V^h is non-empty under *Assumption* 1. We remark that for matching overlapping grids, by identifying the nodes that are in the overlapping region, (3) reduces to the usual finite element problem associated with (1). In fact, (1) is well defined for continuous functions and in this case it is equivalent to (3).

Since v_i vanishes on part of $\partial\Omega_i$, $i = 1, 2$, we can define a norm in X^h by

$$\|v\|_h^2 = a_h(v, v).$$

It is easy to see that the bilinear form $a_h(\cdot, \cdot)$ is bounded in the sense that

$$a_h(u, v) \leq \|u\|_h \|v\|_h, \quad \forall u, v \in X^h. \tag{4}$$

For our estimate of the discretization error, we assume that

$$u^* \in H^{1+\tau_1}(\Omega_1) \times H^{1+\tau_2}(\Omega_2),$$

where $0 < \tau_i \leq 1$, for $i = 1, 2$. The main result of the paper is summarized as

THEOREM 1. *Assume that Assumption 1 is true. Then, the exact solution u^* of (1) and the mortar element solution u of (3) satisfy*

$$\|u^* - u\|_h \leq C \left(h_1^{\tau_1} \|u^*\|_{H^{1+\tau_1}(\Omega_1)} + h_2^{\tau_2} \|u^*\|_{H^{1+\tau_2}(\Omega_2)} \right), \tag{5}$$

where $C > 0$ is a constant independent of h_1, h_2, h_1/h_2, h_2/h_1, and δ.

In the next few sections, we shall prove the theorem for both Case R and Case L, with slightly different techniques. We note that $V^h \subset X^h$. The selection of basis functions in V^h is not as trivial as in the usual finite element case because the matching conditions have to be satisfied. As a result of the mortar mapping, some of the basis functions, near the interfaces, are not local functions, i.e. the support of the basis function covers all the elements that intersect the interface.

Let $Z_i = \{x_l^{h_i}, l = 1, \ldots, N_0^{h_i}\}$ be the set of nodal points in Ω_i, not including boundary or interface nodes. $N_0^{h_i}$ indicates the total number of nodes in Ω_i. For each $x_l^{h_i}$, recall that $\phi_l^{h_i}(x)$ denotes the corresponding regular finite element basis function. Let $\tilde{Z}_i = \{\tilde{x}_l^{h_i}, l = 1, \ldots, \tilde{N}_0^{h_i}\} \subset Z_i$ be a subset of nodes such that $supp(x_l^{h_i}) \bigcap \gamma_j \neq \emptyset$ (for $i \neq j$). For each $x_l^{h_i} \in \tilde{Z}_i$, we define

$$\psi_l^{h_j} = \mathcal{E}_j(\pi_j(\phi_l^{h_i}|_{\gamma_j})), \quad j \neq i.$$

Then, every function $u = (u_1, u_2) \in V^h$ has a unique representation of the form

$$u_1 = \sum_{x_l^{h_1} \in Z_1} u_1(x_l^{h_1}) \phi_l^{h_1}(x) + \sum_{x_l^{h_2} \in \tilde{Z}_2} u_2(x_l^{h_2}) \psi_l^{h_1}(x)$$

and

$$u_2 = \sum_{x_l^{h_2} \in Z_2} u_2(x_l^{h_2}) \phi_l^{h_2}(x) + \sum_{x_l^{h_1} \in \tilde{Z}_1} u_1(x_l^{h_1}) \psi_l^{h_2}(x).$$

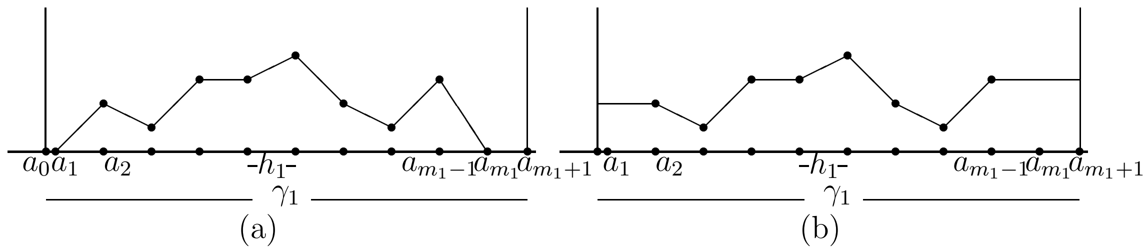

FIG. 3. *(a) A function in the space $V^{h_1}(\gamma_1)$, which is the image of π_1. (b) A test function in the space $\tilde{\mathcal{W}}_{h_1}(\gamma_1)$.*

In summary, the basis functions have the form:

$$\bar{\Omega}_1 : \begin{cases} (\phi_l^{h_1}(x),\ 0) & \text{if } x_l^{h_1} \in Z_1 \setminus \tilde{Z}_1 \\ (\phi_l^{h_1}(x),\ \psi_l^{h_1}(x)) & \text{if } x_l^{h_1} \in \tilde{Z}_1 \end{cases}$$

and

$$\bar{\Omega}_2 : \begin{cases} (0,\ \phi_l^{h_2}(x)) & \text{if } x_l^{h_2} \in Z_2 \setminus \tilde{Z}_2 \\ (\psi_l^{h_2}(x),\ \phi_l^{h_2}(x)) & \text{if } x_l^{h_2} \in \tilde{Z}_2. \end{cases}$$

Note that the interface slave nodes are not accounted for the degree of freedoms. The total degree of freedoms is $N_0^{h_1} + N_0^{h_2}$. The functions $\psi_l^{h_i}(x)$ $(i = 1, 2)$ have to be pre-calculated by solving some small linear systems of equations determined by the mortar projection. Two additional linear systems need to be solved for finding the slave values. The numbers of unknowns of these two linear systems are equal to the numbers of the slave nodes on the interfaces. In the two dimensional cases that we consider, the linear systems are always tridiagonal, symmetric and well-conditioned due to the nature of the mortar projection.

We note that two equivalent formulations for overlapping non-matching grids are given by Kuznetsov in [23]. One approach is the based on a minimization principle and the other uses Lagrange multipliers.

4. Analysis of the discretization error. To analyze the discretization error, we use the well-known Second Strang's Lemma, in Strang and Fix [27], for the non-conforming situation. Let u^* and u be the solutions of (1) and (3), respectively. We have

$$\|u^* - u\|_h \leq \inf_{v \in V^h} (\|u^* - v\|_h + \|u - v\|_h).$$

Here and below we use u^* to represent $(u^*|_{\Omega_1}, u^*|_{\Omega_2})$. Using the fact that

$$\|u - v\|_h^2 = a_h(u - v, u - v) = a_h(u^* - v, u - v) + \{f_h(u - v) - a_h(u^*, u - v)\},$$

and (4), we obtain

$$\|u-v\|_h \leq \|u^*-v\|_h + \frac{|f_h(u-v)-a_h(u^*,u-v)|}{\|u-v\|_h}$$

$$\leq \|u^*-v\|_h + \sup_{0\neq w\in V^h} \frac{|f_h(w)-a_h(u^*,w)|}{\|w\|_h}.$$

Therefore,

$$\|u^*-u\|_h \leq \inf_{v\in V^h} 2\|u^*-v\|_h + \sup_{0\neq w\in V^h} \frac{|f_h(w)-a_h(u^*,w)|}{\|w\|_h}. \tag{6}$$

In the rest of this paper, we shall refer to the first and the second term of the right-hand side of (6) as the best approximation error and the consistency error, respectively.

4.1. The best approximation error. Let us denote the subregion $\bar{\Omega}_{12}^{h_1}$ as the union of all closed simplices $\bar{K}_j^{h_1}$, where $K_j^{h_1} \in \mathcal{T}^{h_1}$ and $K_j^{h_1}$ belongs to $\Omega_1 \cap \Omega_2$. Let us assume that *Assumption 1* holds; therefore, $\Omega_{12}^{h_1}$ is a non-empty connected open subregion. Let $V^{h_1}(\Omega_{12}^{h_1})$ denote the space of continuous piecewise linear functions on $\Omega_{12}^{h_1}$ that vanish on $\partial\Omega_{12}^{h_1}\backslash\gamma_1$. Let $\mathcal{H}_{12}^{h_1}$ denote the discrete harmonic extension operator on $V^{h_1}(\Omega_{12}^{h_1})$ with boundary data on γ_1 and zero data on $\partial\Omega_{12}^{h_1}\backslash\gamma_1$.

Similarly, let us denote the subregion $\bar{\Omega}_{12}^{h_2}$ as the union of all closed simplices $\bar{K}_j^{h_2}$, where $K_j^{h_2} \in \mathcal{T}^{h_2}$ and $K_j^{h_2}$ belongs to $\Omega_2 \cap \Omega_1$. Let us assume that *Assumption 1* holds; therefore, $\Omega_{12}^{h_2}$ is a non-empty connected open subregion. Let $V^{h_2}(\Omega_{12}^{h_2})$ denote the space of continuous piecewise linear functions in $\Omega_{12}^{h_2}$ which vanishes on $\partial\Omega_{12}^{h_2}\backslash\gamma_2$. Let $\mathcal{H}_{12}^{h_2}$ denote the discrete harmonic extension operator in $V^{h_2}(\Omega_{12}^{h_2})$ with boundary data on γ_2 and zero data on $\partial\Omega_{12}^{h_2}\backslash\gamma_2$.

In the next lemma, we prove that the best approximation error is optimal. In the proof, we use several technical lemmas that will be discussed in Section 5.

LEMMA 1. *Assume Assumption 1 holds. Then, for any $u^* \in H^{1+\tau_i}(\Omega_i)$, $i=1,2$, and $0<\tau_1,\tau_2\leq 1$, there exists $v=(v_1,v_2)\in V^h$ such that*

$$|u^*-v_1|_{H^1(\Omega_1)} \leq C\left(h_1^{\tau_1}\|u^*\|_{H^{1+\tau_1}(\Omega_1)} + h_2^{\tau_2}\|u^*\|_{H^{1+\tau_2}(\Omega_2)}\right), \tag{7}$$

and

$$|u^*-v_2|_{H^1(\Omega_2)} \leq C\left(h_1^{\tau_1}\|u^*\|_{H^{1+\tau_1}(\Omega_1)} + h_2^{\tau_2}\|u^*\|_{H^{1+\tau_2}(\Omega_2)}\right). \tag{8}$$

Here the constant $C>0$ is independent of h_1, h_2, h_1/h_2, h_2/h_1, and δ.

Proof. We first construct $w=(w_1,w_2)\in X^h$. Let w_i be a continuous piecewise linear function defined in Ω_i by using the pointwise interpolation of u^* at the nodal points of $\mathcal{T}^{h_i}$. The standard interpolation theory ([14]) gives

$$\|u^*-w_i\|_{L^2(\Omega_i)} + h_i|u^*-w_i|_{H^1(\Omega_i)} \leq Ch_i^{1+\tau_i}\|u^*\|_{H^{1+\tau_i}(\Omega_i)}, \quad 0<\tau_i\leq 1. \tag{9}$$

Note however that $w \notin V^h$, in general, since $w_i, i = 1, 2$, do not vanish at the nodes $\{a_1^i\}$ and $\{a_{m_i}^i\}$. Also, w does not satisfy the matching conditions across the interfaces $\gamma_i, i = 1, 2$.

Let $z_i \in V^{h_i}$ be a continuous piecewise linear function that equals to zero at the nodes a_1^i and $a_{m_i}^i$, and to w_i at the remaining nodes of $\mathcal{T}^{h_i}$. Thus, the piecewise linear function $w_i - z_i$ is equal to $u^*(a_1^i)$ at a_1^i, and to $u^*(a_{m_i}^i)$ at $a_{m_i}^i$. Then by using Lemma 4 (to be introduced in the Technical Lemmas Section 5), we obtain, for $0 < \tau_i \leq 1$,

$$|w_i(a_1^i) - z_i(a_1^i)| = |u^*(a_1^i)| \leq C h_i^{\tau_i} \|u^*\|_{H^{1+\tau_i}(\Omega_i)}, \tag{10}$$

and

$$|w_i(a_{m_i}^i) - z_i(a_{m_i}^i)| = |u^*(a_{m_i}^i)| \leq C h_i^{\tau_i} \|u^*\|_{H^{1+\tau_i}(\Omega_i)}. \tag{11}$$

Since $w_i - z_i$ is equal to zero at all nodes of $\mathcal{T}^{h_i}$ except a_1^i and $a_{m_i}^i$, we can use (10), and (11) to obtain, for $0 < \tau_i \leq 1$,

$$\|w_i - z_i\|_{L^2(\Omega_i)} + h_i |w_i - z_i|_{H^1(\Omega_i)} \leq C h_i^{1+\tau_i} \|u^*\|_{H^{1+\tau_i}(\Omega_i)}, \tag{12}$$

and consequently, using a triangle inequality and (9), we obtain

$$\|u^* - z_i\|_{L^2(\Omega_i)} + h_i |u^* - z_i|_{H^1(\Omega_i)} \leq C h_i^{1+\tau_i} \|u^*\|_{H^{1+\tau}(\Omega_i)}. \tag{13}$$

Now $z_i \in V^{h_i}$ $(i = 1, 2)$, but $z = (z_1, z_2) \notin V^h$ because the matching conditions across the interfaces are not satisfied. To match the interface values, we need to further modify z_i. Let

$$r^1 = \pi_1(z_2(\gamma_1)) - z_1 \text{ on } \gamma_1,$$

and

$$r^2 = \pi_2(z_1(\gamma_2)) - z_2 \text{ on } \gamma_2.$$

We define the function $v = (v_1, v_2)$ as

$$v_i = z_i + \mathcal{H}_{12}^{h_i} r^i, i = 1, 2.$$

Note that *Assumption 1* is used to guarantee the existence of $\mathcal{H}_{12}^{h_i} r^i$. Note also that $\mathcal{H}_{12}^{h_1} r^1$ (resp. $\mathcal{H}_{12}^{h_2} r^2$) vanishes on γ_2 (resp. γ_1). Since v_i belongs to $V^{h_i}(\Omega_i)$, for $i = 1, 2$, and they satisfy the matching conditions, v belongs to V^h. We next show that v satisfies (7) and (8). By the triangle inequality

$$|u^* - v_i|_{H^1(\Omega_i)} \leq |u^* - z_i|_{H^1(\Omega_i)} + |\mathcal{H}_{12}^{h_i} r^i|_{H^1(\Omega_i)}. \tag{14}$$

The first term above has been estimated in (13). For the second term, we use Lemma 8 of Section 5, to obtain

$$|\mathcal{H}_{12}^{h_i} r^i|_{H^1(\Omega_i)} \leq C \left(\|r^i\|^2_{H_{00}^{1/2}(\gamma_i)} + \frac{1}{\delta} \|r^i\|^2_{L^2(\gamma_i)} \right). \tag{15}$$

We bound $\|r^1\|_{L^2(\gamma_1)}$, and similarly $\|r^2\|_{L^2(\gamma_2)}$, as follows,

$$\|r^1\|_{L^2(\gamma_1)} = \|\pi_1 z_2 - z_1\|_{L^2(\gamma_1)} = \|\pi_1 z_2 - \pi_1 z_1\|_{L^2(\gamma_1)}$$

$$\leq \|\pi_1 z_2 - \pi_1 u^*\|_{L^2(\gamma_1)} + \|\pi_1 z_1 - \pi_1 u^*\|_{L^2(\gamma_1)}.$$

A consequence of the L^2 stability of Lemma 5 is that

$$\|r^1\|_{L^2(\gamma_1)} \leq 6\|z_2 - u^*\|_{L^2(\gamma_1)} + 6\|z_1 - u^*\|_{L^2(\gamma_1)}.$$

Using *Assumption 1*, we have that $z_2 = w_2$ on γ_1. Then,

$$\|z_2 - u^*\|_{L^2(\gamma_1)} = \|w_2 - u^*\|_{L^2(\gamma_1)}.$$

According to the standard estimate for pointwise interpolation, we get, for $0 < \tau_2 \leq 1$, that

$$\|w_2 - u^*\|_{L^2(\gamma_1)} \leq C h_2^{1/2+\tau_2} \|u^*\|_{H^{1+\tau_2}(\Omega_2)}. \tag{16}$$

Thus, we have obtained

$$\|\pi_1 z_2 - \pi_1 u^*\|_{L^2(\gamma_1)} \leq C h_2^{1/2+\tau_2} \|u^*\|_{H^{1+\tau_2}(\Omega_2)}, \quad 0 < \tau_2 \leq 1. \tag{17}$$

We also have,

$$\|\pi_1 z_1 - \pi_1 u^*\|_{L^2(\gamma_1)} \leq 6\|z_1 - u^*\|_{L^2(\gamma_1)}$$

and therefore, by using a triangle inequality

$$\|z_1 - u^*\|_{L^2(\gamma_1)} \leq \|w_1 - u^*\|_{L^2(\gamma_1)}$$

$$+ \left\|u^*(a_1^1)\phi_{a_1^1}^{h_1}\right\|_{L^2(\gamma_1)} + \left\|u^*(a_{m_1}^1)\phi_{a_{m_1}^1}^{h_1}\right\|_{L^2(\gamma_1)}.$$

Together with (10), (11) and (12), we arrive at

$$\|\pi_1 z_1 - \pi_1 u^*\|_{L^2(\gamma_1)} \leq C h_1^{1/2+\tau_1} \|u^*\|_{H^{1+\tau_1}(\Omega_1)}, \quad 0 < \tau_1 \leq 1.$$

This implies

$$\|r_i\|_{L^2(\gamma_i)} \leq C \sum_{i=1}^{2} h_i^{1/2+\tau_i} \|u^*\|_{H^{1+\tau_i}(\Omega_i)}, \qquad i = 1, 2. \tag{18}$$

We next bound $\|r^1\|_{H_{00}^{1/2}(\gamma_1)}$, and similarly $\|r^2\|_{H_{00}^{1/2}(\gamma_2)}$. We use the $H_{00}^{1/2}$ stability of Lemma 5 to obtain

$$\|r^1\|_{H_{00}^{1/2}(\gamma_1)} \leq \|\pi_1 z_2 - \pi_1 u^*\|_{H_{00}^{1/2}(\gamma_1)} + \|\pi_1 z_1 - \pi_1 u^*\|_{H_{00}^{1/2}(\gamma_1)} \leq$$

$$C\ \|z_2 - u^*\|_{H_{00}^{1/2}(\gamma_1)} + 6\|z_1 - u^*\|_{H_{00}^{1/2}(\gamma_1)}.$$

Now with (13), we get

$$\|r_i\|_{H_{00}^{1/2}(\gamma_i)} \leq C \sum_{i=1}^{2} h_i^{\tau_i} \|u^*\|_{H^{1+\tau_i}(\Omega_i)}, \qquad i = 1, 2. \tag{19}$$

Finally (7) and (8) follow immediately from (14), (15), (18), (19), and the fact that δ is larger than $\max\{h_1, h_2\}$. $\square$

4.2. The consistency error. The consistency error can be estimated rather easily. For a smooth u^*, by using Green's formula and that $-\Delta u^* = f$ in the L^2 sense, we obtain

$$f_h(w) - a_h(u^*, w) = \int_{\Omega_1} (f + \Delta u^*) w_1 \, dx - \frac{1}{2} \int_{\gamma_1} \frac{\partial u^*}{\partial n}(w_1) ds +$$

$$\frac{1}{2} \int_{\gamma_2} \frac{\partial u^*}{\partial n}(w_1) ds + \int_{\Omega_2} (f + \Delta u^*) w_2 \, dx - \frac{1}{2} \int_{\gamma_2} \frac{\partial u^*}{\partial n}(w_2) ds + \frac{1}{2} \int_{\gamma_1} \frac{\partial u^*}{\partial n}(w_2) ds =$$

$$\frac{1}{2} \int_{\gamma_1 \cup \gamma_2} \frac{\partial u^*}{\partial n}[w] ds = \frac{1}{2} \int_{\gamma_1} \frac{\partial u^*}{\partial n}(w_2 - w_1) ds + \frac{1}{2} \int_{\gamma_2} \frac{\partial u^*}{\partial n}(w_1 - w_2) ds,$$

where $\frac{\partial u^*}{\partial n}$ denotes the normal derivative of u^*, with the unit vector n pointing to the outside of $\Omega_1 \cap \Omega_2$. Later, we use the density argument (Grisvard [19]) to estimate $f_h(w) - a_h(u^*, w)$ for any $u^* \in H^1(\Omega)$.

We summarize the result in the following lemma.

LEMMA 2. *Let $u^* \in H^{1+\tau_i}(\Omega_i)$, $0 \le \tau_i \le 1, i = 1, 2$. Then, there exists a constant $C > 0$ independent of δ, h_i and u^*, such that*

$$\sup_{0 \neq w \in V_0^h} \frac{|\int_{\gamma_1 \cup \gamma_2} \frac{\partial u^*}{\partial n}[w] ds|}{\|w\|_h} \le C \left(h_1^{\tau_1} \|u^*\|_{H^{1+\tau_1}(\Omega_1)} + h_2^{\tau_2} \|u^*\|_{H^{1+\tau_2}(\Omega_2)} \right).$$

Proof. We derive a bound for the consistency error on γ_1. The bound on γ_2 can be obtained in a similar way. Let $w = (w_1, w_2) \in V^h$, we have

$$\left| \int_{\gamma_1} \frac{\partial u^*}{\partial n}(w_2 - w_1) ds \right| = \left| \int_{\gamma_1} \frac{\partial u^*}{\partial n}(w_2 - \pi_1 w_2) ds \right|,$$

and by using the definition of the mortar mapping (2), we also have, $\forall \psi \in \tilde{\mathcal{W}}_{h_1}(\gamma_1)$,

$$\left| \int_{\gamma_1} \frac{\partial u^*}{\partial n}(w_2 - \pi_1 w_2) ds \right| = \left| \int_{\gamma_1} \left(\frac{\partial u^*}{\partial n} - \psi \right) (w_2 - \pi_1 w_2) ds \right|$$

$$\le \left\| \frac{\partial u^*}{\partial n} - \psi \right\|_{[H^{1/2}(\gamma_1)]'} \|w_2 - \pi_1 w_2\|_{H^{1/2}(\gamma_1)}$$

$$\le \left\| \frac{\partial u^*}{\partial n} - \psi \right\|_{[H^{1/2}(\gamma_1)]'} \left(\|w_2\|_{H^{1/2}(\gamma_1)} + \|w_1\|_{H^{1/2}(\gamma_1)} \right).$$

Applying the trace theorem for w, we deduce that

$$\left| \int_{\gamma_1} \frac{\partial u^*}{\partial n}(w_2 - w_1) ds \right| \le C \|w\|_h \inf_{\psi \in \tilde{\mathcal{W}}_{h_1}(\gamma_1)} \left\{ \left\| \frac{\partial u^*}{\partial n} - \psi \right\|_{[H^{1/2}(\gamma_1)]'} \right\}.$$

With the help of Lemma 3 (or Lemma 4.1 of Bernardi, Maday and Patera [7]), we obtain

$$\left| \int_{\gamma_1} \frac{\partial u^*}{\partial n}(w_2 - w_1) ds \right| \le C h_1^{\tau_1} \|w\|_h \left\| \frac{\partial u^*}{\partial n} \right\|_{H^{1/2+\tau_1}(\gamma_1)} \le C h_1^{\tau_1} \|w\|_h \|u^*\|_{H^{1+\tau_1}(\Omega_1)}.$$

□

5. Technical lemmas. In this section, we discuss several technical estimates. We formulate and prove some of the lemmas in a way that is more general than needed in this paper since we believe their applicabilities go beyond this paper.

The proof of the following lemma can be found in Bernardi, Maday and Patera [7], although their definition of the mortar mapping is slightly different from ours for Case L because of the two extra intervals $[a_0^i, a_1^i]$ and $[a_{m_i}^i, a_{m_i+1}^i]$. Their proof also holds here because the length of the intervals $[a_0^i, a_2^i]$ and $[a_{m_i-1}^i, a_{m_i+1}^i]$ are $O(h_i)$; we do not include the proof here.

LEMMA 3. *Let $\tilde{\pi}_i$ be the orthogonal projection from $L^2(\gamma_i)$ onto $\tilde{\mathcal{W}}_{h_i}(\gamma_i)$. Then, for any $0 \le \tau_i \le 1$, the following estimate holds for any $v \in H^{\tau_i}(\gamma_i)$,*

$$\|v - \tilde{\pi}_i v\|_{L^2(\gamma_i)} + h_i^{-1/2} \|v - \tilde{\pi}_i v\|_{[H^{1/2}(\gamma_i)]'} \le C h_i^{\tau_i} \|v\|_{H^{\tau_i}(\gamma_i)}.$$

As a consequence,

$$\inf_{\psi \in \tilde{\mathcal{W}}_{h_i}(\gamma_i)} \left\{ \|v - \psi\|_{[H^{1/2}(\gamma_i)]'} \right\} \le C h_i^{1/2+\tau_i} \|v\|_{H^{\tau_i}(\gamma_i)}.$$

Here $C > 0$ is independent of h_i.

The next lemma is useful only for Case L. Let us restrict our arguments to Ω_1, similar argument applies for Ω_2. Recall that in the definition of the finite element space $V^{h_1}(\Omega_1)$, we insist that the functions vanish at two interior points a_1^1 and $a_{m_1}^1$, which is a bit unusual in the classical finite element theory. Due to the following lemma, we show that the interior zero points do not affect the second order (or $1+\tau_i$ order) accuracy of the overall discretization.

LEMMA 4. *Let Ω_1 be a bounded open subset of $\Re^2$ with a piecewise $C^{0,1}$ boundary $\partial\Omega_1$. Assume that the aspect ratio and the size of Ω_1 are both $O(1)$. Let $\nu \subset \partial\Omega_1$ be a $C^{1,1}$ (differentiable Lipschitz) curve with end points A and B. Also let $\eta \subset \nu \cap \partial\Omega$ be an open non-empty connected curve with end points A and x_0. Then for any $u \in H^{1+\tau_1}(\Omega_1)$, $0 < \tau_1 \le 1$, that vanishes on $\partial\Omega$, we have*

$$|u(x)| \le C d_x^{\tau_1} \|u\|_{H^{1+\tau_1}(\Omega_1)}, \quad \forall x \in \nu. \tag{20}$$

Here d_x is the arc distance of the point x to η along the curve ν. The constant $C > 0$ does not depend on u, x_0 and x, but in general depends on the Lipschitz constant of $\partial\Omega_1$.

Proof. If $x \in \eta$ then $u(x) = 0$ and (20) holds trivially. Let us assume that $x \in \nu \setminus \eta$. Let $z(x)$ be a point in the interior of η such that

$$d(z(x), x_0) \le d(x, x_0) = d_x.$$

We shall first assume that u is a smooth function and then pass it to any functions in $H^{1+\tau_1}(\Omega_1)$ using the classical density argument; see e.g. Grisvard [19] or Lions and Magenes [24]. Now let $u \in C^\infty(\Omega_1)$, then

$$u(x) = u(z(x)) + \int_{z(x)}^{x} u'(s) ds.$$

Since $u(z(x)) = 0$ and $u^{'}(s) = 0$ on $s \in \eta$, we have

$$u(x) = \int_{x_0}^{x} u^{'}(s) ds.$$

Using Schwarz inequality, we have

$$|u(x)| \leq \int_{x_0}^{x} |u^{'}(s)| ds \leq d_x^{1/2} |u|_{H^1(\nu)}. \tag{21}$$

With the Fundamental Theorem of Calculus,

$$u^{'}(s) = u^{'}(z(x)) + \int_{z(x)}^{s} u^{''}(t) dt$$

and using that $u^{'}(s) = 0$ on $s \in \eta$, we get

$$u(x) = \int_{x_0}^{x} \int_{z(x)}^{s} u^{''}(t) dt ds.$$

By using the fact that $u^{''}(y) = 0$, $y \in \eta$, the Schwarz inequality, and that $d(x_0, z(x)) \leq d(x, x_0)$ we obtain

$$|u(x)| \leq C d(x, x_0)^{3/2} |u|_{H^2(\nu)}. \tag{22}$$

We obtain the estimate in $H^{1+\tau_1}(\nu)$ by interpolating the $H^1(\nu)$ estimate (21) and the $H^2(\nu)$ estimate (22) (Lions and Magenes [24]). Thus, for $0 \leq \tau_1 \leq 1$,

$$|u(x)| \leq C d_x^{1/2+\tau_1} \|u\|_{H^{1+\tau_1}(\nu)}. \tag{23}$$

With the usual density argument, the above estimate holds for any $u \in H^{1+\tau_1}(\nu)$.

Finally, to obtain (20) from (23), we consider two cases $1/2 \leq \tau_1 \leq 1$ and $0 < \tau_1 \leq 1/2$ separately.

For $1/2 \leq \tau_1 \leq 1$, we use the trace theorem for $C^{0,1}$ (differentiable Lipschitz) curve (cf. Theorem 1.5.2.1 of Grisvard [19]), which gives

$$|u(x)| \leq C d_x^{\tau_1} \|u\|_{H^{1/2+\tau_1}(\nu)} \leq C d_x^{\tau_1} \|u\|_{H^{1+\tau_1}(\Omega_1)}.$$

For $0 < \tau_1 \leq 1/2$, it is known that the continuous function space is embedded into $H^{1/2+\tau_1}(\Omega_1)$. Using that u vanishes on η, we can use the Bramble-Hilbert lemma and scaling arguments to obtain, for $0 < \tau_1 \leq 1/2$,

$$|u(x)| \leq C d_x^{\tau_1} \|u\|_{H^{1+\tau_1}(\Omega_1)} \quad \forall u \in H^{1+\tau_1}(\Omega_1).$$

The last arguments can be found in detail in the proof of Theorem 3.3 in Xu [31]. $\square$

REMARK 5.1. *We remark that we use the above lemma by taking $x_0 = a_0^1$ (or $x_0 = a_{m_1+1}^1$) and ν as an edge of an element $K_j^{h_1}$ of $\mathcal{T}^{h_1}(\bar{\Omega}_1)$ that contains a_0^1 and a_1^1. The lemma is useful only when $a_0^1 \neq a_1^1$, and therefore (using the definition of a_0^1 and a_1^1) a_0^1 belongs to the interior of ν.*

We next show the boundness of the mortar projection in two different norms. Since the mortar projection is, in some sense, close to the regular L^2 projection, the L^2 bound is rather easy to obtain. It is a bit involved to obtain its $H_{00}^{1/2}$ bound.

LEMMA 5. *The mortar mapping π_i is bounded in $L^2(\gamma_i)$, i.e.,*

$$\|\pi_i w\|_{L^2(\gamma_i)} \le \sqrt{6}\|w\|_{L^2(\gamma_i)}, \quad \forall w \in L^2(\gamma_i) \tag{24}$$

and π_i is also bounded in $H_{00}^{1/2}(\gamma_i)$, i.e.,

$$\|\pi_i w\|_{H_{00}^{1/2}(\gamma_i)} \le C\|w\|_{H_{00}^{1/2}(\gamma_i)}, \quad \forall w \in H_{00}^{1/2}(\gamma_i), \tag{25}$$

where the constant $C > 0$ is independent of h_1, h_2, h_1/h_2, h_2/h_1 and δ.

Proof. Let us consider the proof for π_1. The proof for π_2 is similar. Using (2), and taking ψ, here denoted by v, which equals to $\pi_1 w$ at the nodal points $a_2^1, a_3^1, \cdots, a_{m_1-1}^1$, we obtain,

$$\|\pi_1 w\|_{L^2(\gamma_1)}^2 \le (\pi_1 w, v)_{L^2(\gamma_1)} = (w, v)_{L^2(\gamma_1)}.$$

Using simple calculations, we have

$$\|v\|_{L^2(\gamma_1)}^2 \le 6\|\pi_1 w\|_{L^2(\gamma_1)}^2,$$

and (24) follows easily. We next estimate the $H_{00}^{1/2}$ bound. Let $w \in H_0^1(\gamma_1)$. By the triangle inequality and then the inverse inequality, we have

$$\|\pi_1 w\|_{H_{00}^{1/2}(\gamma_1)}^2 \le C\left(\frac{1}{h_1}\|\pi_1 w - \mathcal{Q}_{h_1} w\|_{L^2(\gamma_1)}^2 + \|\mathcal{Q}_{h_1} w\|_{H_{00}^{1/2}(\gamma_1)}\right). \tag{26}$$

Here $\mathcal{Q}_{h_1} : V^{h_2}(\gamma_1) \to V^{h_1}(\gamma_1)$ is the usual orthogonal L^2 projection. Note that $\pi_1 \mathcal{Q}_{h_1} w = \mathcal{Q}_{h_1} w$. Therefore, using (24) we have

$$\|\pi_1 w - \mathcal{Q}_{h_1} w\|_{L^2(\gamma_1)}^2 = \|\pi_1 w - \pi_1 \mathcal{Q}_{h_1} w\|_{L^2(\gamma_1)}^2 \le C\|w - \mathcal{Q}_{h_1} w\|_{L^2(\gamma_1)}. \tag{27}$$

The next step is to bound $\|w - \mathcal{Q}_{h_1} w\|_{L^2(\gamma_1)}$. Now we follow the proofs of Theorems 3.2 and 3.4 of Bramble and Xu [9]. Let us denote by $\mathcal{I}^{h_1}$ the usual nodal value interpolant on the grid $a_0^1, a_1^1, a_2^1, \cdots, a_{m_1}^1, a_{m_1+1}^1$. The interpolator is well-defined in $H^1(\gamma_1)$. Let us denote by $\phi_{a_i^1}$ the standard basis functions associated to the continuous piecewise linear functions on the grid $a_0^1, a_1^1, a_2^1, \cdots, a_{m_1}^1, a_{m_1+1}^1$. It is easy to see that

$$\tilde{w} = \mathcal{I}^{h_1} w - w(a_1^1)\phi_{a_1^1} - w(a_{m_1}^1)\phi_{a_{m_1}^1}$$

belongs to $V^{h_1}(\gamma_1)$. Therefore

$$\|w - \mathcal{Q}_{h_1} w\|_{L^2(\gamma_1)} \le \|w - \tilde{w}\|_{L^2(\gamma_1)}$$

$$\le \|w - \mathcal{I}^{h_1} w\|_{L^2(\gamma_1)} + \|w(a_1^1)\phi_{a_1^1}\|_{L^2(\gamma_1)} + \|w(a_{m_1}^1)\phi_{a_{m_1}^1}\|_{L^2(\gamma_1)}.$$

Since $\mathcal{I}^{h_1} w$ is well-defined for $w \in H^1(\gamma_1)$, by using a well-known result of Ciarlet [14], we obtain

$$\|w - \mathcal{I}^{h_1} w\|_{L^2(\gamma_1)} \le C h_1 |w|_{H^1(\gamma_1)}.$$

Using that w vanishes at a_0^1 and $a_{m_1+1}^1$, we have

$$|w(a_1^1)| \le C h_1^{1/2} |w|_{H^1(\gamma_1)} \text{ and } |w(a_{m_1}^1)| \le C h_1^{1/2} |w|_{H^1(\gamma_1)},$$

and then obtain

$$\|w - \mathcal{Q}_{h_1} w\|_{L^2(\gamma_1)} \le C h_1 |w|_{H^1(\gamma_1)}. \tag{28}$$

Using that $\mathcal{Q}_{h_1}$ is a L^2 projection, we have $\|w - \mathcal{Q}_{h_1} w\|_{L^2(\gamma_1)} \le 2\|w\|_{L^2(\gamma_1)}$. Then by the interpolation procedure we obtain

$$\|w - \mathcal{Q}_{h_1} w\|_{L^2(\gamma_1)} \le C h_1^{1/2} \|w\|_{H_{00}^{1/2}(\gamma_1)}. \tag{29}$$

The next step is to show that

$$|\mathcal{Q}_{h_1} w|_{H^1(\gamma_1)} \le C |w|_{H^1(\gamma_1)}. \tag{30}$$

Let $w_0 = w$ on $[a_0^1, a_1^1]$ and $[a_{m_1}^1, a_{m_1+1}^1]$, and $w_0 = w(a_1^1)\phi_{a_1^1}(x)$ on $[a_1^1, a_2^1]$ and $w_0 = w(a_{m_1}^1)\phi_{a_{m_1}^1}(x)$ on $[a_{m_1-1}^1, a_{m_1}^1]$, and zero at the remaining points of γ_1. Hence,

$$|\mathcal{Q}_{h_1} w|_{H^1(\gamma_1)}^2 \le 2\left(|\mathcal{Q}_{h_1}(w - w_0)|_{H^1(\gamma_1)}^2 + |\mathcal{Q}_{h_1} w_0|_{H^1(\gamma_1)}^2 \right).$$

By using an inverse inequality, the L^2 stability result (24), and the definition of w_0, we have

$$|\mathcal{Q}_{h_1} w_0|_{H^1(\gamma_1)}^2 \le \frac{C}{h_1^2} \|\mathcal{Q}_{h_1} w_0\|_{L^2(\gamma_1)}^2 \le \frac{C}{h_1^2} \|w_0\|_{L^2(\gamma_1)}^2 \le$$

$$\frac{C}{h_1^2}\left(\|w\|_{L^2(a_0^1,a_1^1)}^2 + \|w(a_1^1)\phi_{a_1^1}\|_{L^2(a_1^1,a_2^1)}^2 + \|w(a_{m_1}^1)\phi_{a_{m_1}^1}\|_{L^2(a_{m_1-1}^1,a_{m_1}^1)}^2 + \|w\|_{L^2(a_{m_1}^1,a_{m_1+1}^1)}^2 \right)$$

$$\le C|w|_{H^1(\gamma_1)}^2.$$

In the last inequality, we use (21), which holds for functions w that vanish at a_0^1 and $a_{m_1+1}^1$.

Note that $\mathcal{Q}_{h_1}(w - w_0) = \tilde{\mathcal{Q}}_{h_1}(w - w_0)$, where $\tilde{\mathcal{Q}}_{h_1}$ is the standard L^2 projection in the space of piecewise linear functions defined on the grids $a_1^1, a_2^1, \cdots, a_{m_1}^1$ and vanish at the end points a_1^1 and $a_{m_1}^1$. Hence by using standard results of the L^2 projection, and some previous arguments, we obtain (30) by

$$|\mathcal{Q}_{h_1}(w - w_0)|_{H^1(\gamma_1)}^2 \le C|w - w_0|_{H^1(\gamma_1)}^2 \le C|w|_{H^1(\gamma_1)}^2 +$$

$$C\frac{1}{h_1^2}\left(\|w(a_1^1)\phi_{a_1^1}\|^2_{L^2(a_1^1,a_2^1)} + \|w(a_{m_1}^1)\phi_{a_{m_1}^1}\|^2_{L^2(a_{m_1-1}^1,a_{m_1}^1)}\right) \le C|w|^2_{H^1(\gamma_1)}.$$

We then use (30), the L^2 stability of $\mathcal{Q}_{h_1}$, and an interpolation procedure to obtain

$$\|\mathcal{Q}_{h_1}w\|^2_{H^{1/2}_{00}(\gamma_1)} \le C\|w\|_{H^{1/2}_{00}(\gamma_1)}. \tag{31}$$

The inequality (25) follows from (31), (29), (27) and (26). □

To simplify the discussion of the next lemma we assume that $\Omega_1 = (0,1)\times(0,1)$ is a unit square with sides parallel to the coordinate axes. The result of the following lemma can be extended to any Lipschitz regions by using the techniques developed in e.g. Nečas [21]. Let the x-coordinate of γ_1 equal to 1. Let $\Gamma_\delta \subset \Omega_1$ be the set of points that is within a distance δ of γ_1 and define $\zeta = \partial\Gamma_\delta \cap \Omega_1$. Thus the x-coordinate of ζ equals to $(1-\delta)$.

LEMMA 6. *There exists a constant $C > 0$ independent of δ, such that*

$$\|w\|^2_{L^2(\zeta)} \le C\ (\|w\|^2_{L^2(\gamma_1)} + \delta|w|^2_{H^1(\Gamma_\delta)}), \tag{32}$$

and

$$\|w\|^2_{L^2(\zeta)} \le C\left(\delta|w|^2_{H^1(\Gamma_\delta)} + \frac{1}{\delta}\|w\|^2_{L^2(\Gamma_\delta)}\right) \tag{33}$$

hold for any $w \in H^1(\Omega_1)$.

Proof. By using the Fundamental Theorem of Calculus we have

$$w(1-\delta,y) = w(1,y) - \int_{1-\delta}^1 \frac{\partial w}{\partial x}(s,y)ds.$$

Squaring both sides and taking the integral in y from 0 to 1, we obtain

$$\int_0^1 (w(1-\delta,y))^2dy \le 2\int_0^1 (w(1,y))^2dy + 2\int_0^1 \left(\int_{1-\delta}^1 \frac{\partial w}{\partial x}(s,y)ds\right)^2 dy.$$

Now using Schwarz inequality on the last term,

$$\int_0^1 (w(1-\delta,y))^2dy \le 2\int_0^1 (w(1,y))^2dy + 2\int_0^1 \delta\left(\int_{1-\delta}^1 \left(\frac{\partial w}{\partial x}(s,y)\right)^2 ds\right) dy,$$

and (32) follows. To prove (33), we note that for $x \in (1-\delta,1)$,

$$w(1-\delta,y) = w(x,y) - \int_{1-\delta}^x \frac{\partial w}{\partial x}(s,y)ds,$$

which implies, by squaring both sides and using Schwarz inequality, that

$$(w(1-\delta,y))^2 \le 2\left(w(x,y))^2 + \delta\int_{1-\delta}^1 \left(\frac{\partial w}{\partial x}(s,y)\right)^2 ds\right).$$

The proof of (33) is now obtained by integrating this inequality over $(1-\delta, 1) \times (0,1)$. $\square$

REMARK 5.2. *A similar estimate plays a very important role in the study of the optimal convergence of the overlapping Schwarz methods with small overlap; see Dryja and Widlund [17].*

The next two lemmas are devoted to Case R. For a given overlap δ, we introduce a finite element triangulation of size $O(\delta)$ on Ω_1. More precisely, we let $\mathcal{T}^\delta(\bar{\Omega}_1)$ be a triangulation of Ω_1, which may or may not be nested with $\mathcal{T}^{h_1}(\bar{\Omega}_1)$. We assume the triangulation is quasi-uniform with size $O(\delta)$ and $V^\delta(\Omega_1)$ is the space of continuous piecewise linear functions on the triangulation $\mathcal{T}^\delta(\bar{\Omega}_1)$. We denote by γ_1^δ the set of nodal points of $\mathcal{T}^\delta(\bar{\Omega}_1)$ belonging to $\bar{\gamma}_1$. Following Dryja, Sarkis and Widlund [18], we define an interpolation operator $I_\delta^M : V^{h_1}(\Omega_1) \to V^\delta(\Omega_1)$ as follows.

DEFINITION 1. *Given $w \in V^{h_1}(\Omega_1)$, define $w_\delta = I_\delta^M w \in V^\delta(\Omega_1)$ by the values of w_δ at two types of nodes of $\mathcal{T}^\delta(\bar{\Omega}_1)$:*

- *i) For an interior nodal point $P \in \mathcal{T}^\delta(\bar{\Omega}_1)\backslash\gamma_1^\delta$, let $\tau_P \in \mathcal{T}^\delta(\bar{\Omega}_1)$ be a triangle with P as one of its vertices. We define $w_\delta(P)$ as the average of w over τ_P, i.e., $\int_{\tau_P} w dx / \int_{\tau_P} 1 dx$.*
- *ii) For a boundary nodal point $P \in \gamma_1^\delta$, let $\bar{\tau}_P \in \mathcal{T}^\delta(\bar{\Omega}_1)$ be a triangle with P as one of its vertices, and having an edge on γ_1. We define $w_\delta(P)$ as the average of w over $\bar{\tau}_P \cap \gamma_1$, i.e. the line integral $\int_{\bar{\tau}_P\cap\gamma_1} w ds / \int_{\bar{\tau}_P\cap\gamma_1} 1 ds$.*

LEMMA 7. *There exists a constant $C > 0$, independent of δ and h_1, such that*

$$\|(I - I_\delta^M)w\|_{L^2(\Omega_1)} \le C\ \delta\, |w|_{H^1(\Omega_1)}, \tag{34}$$

$$|I_\delta^M w|_{H^1(\Omega_1)} \le C\ |w|_{H^1(\Omega_1)}, \tag{35}$$

and

$$\|I_\delta^M w\|_{L^2(\gamma_1)} \le C\ \|w\|_{L^2(\gamma_1)} \tag{36}$$

hold for any $w \in V^{h_1}(\gamma_1)$.

REMARK 5.3. *A proof can be found in the paper of Dryja, Sarkis and Widlund [18]. The interpolation operator I_δ^M is used only as part of the proof of the next lemma, not in the implementation of any of the algorithms proposed in this paper.*

For the next lemma, let us assume that ζ is aligned with the h_1-grid, and let $\mathcal{H}_1^\delta$ be the h_1-discrete harmonic extension operator in $V^{h_1}(\Gamma_\delta)$ with boundary data on γ_1 and zero data on $\partial\Gamma_\delta\backslash\gamma_1$. Also, let $\mathcal{H}_1$ be the h_1-discrete harmonic extension operator in $V^{h_1}(\Omega_1)$ with boundary data on γ_1 and zero data on $\partial\Omega_1\backslash\gamma_1$.

LEMMA 8. *There exists a constant $C > 0$ independent of δ and h_1, such that*

$$|\mathcal{H}_1^\delta w|^2_{H^1(\Gamma_\delta)} \le C \left(\|w\|^2_{H^{1/2}_{00}(\gamma_1)} + \frac{1}{\delta}\|w\|^2_{L^2(\gamma_1)} \right) \tag{37}$$

for any $w \in V^{h_1}(\gamma_1)$.

Proof. Using a triangle inequality, we have

$$|\mathcal{H}_1^\delta w|^2_{H^1(\Gamma_\delta)} \le 2|\mathcal{H}_1^\delta(w - I_\delta^M w)|^2_{H^1(\Gamma_\delta)} + 2|\mathcal{H}_1^\delta I_\delta^M w|^2_{H^1(\Gamma_\delta)} = 2I_1 + 2I_2.$$

Let θ_δ be a smooth function with values equal to one on γ_1 and to zero on $\Omega_1 \backslash \Gamma_\delta$. Let I_{h_1} be the usual pointwise piecewise linear continuous interpolation operator. Using the fact that the discrete harmonic extension has minimal energy,

$$I_1 \le |I_{h_1}(\theta_\delta(\mathcal{H}_1 w - I_\delta^M \mathcal{H}_1 w))|^2_{H^1(\Omega_1)} \le$$

$$C\left(|\mathcal{H}_1 w - I_\delta^M \mathcal{H}_1 w|^2_{H^1(\Omega_1)} + \frac{1}{\delta^2}\|\mathcal{H}_1 w - I_\delta^M \mathcal{H}_1 w\|^2_{L^2(\Omega_1)}\right).$$

In the last inequality, we used the standard estimate as in the additive Schwarz theory (see e.g. [17]). Finally we use (34) and (35) to obtain

$$I_1 \le C|\mathcal{H}_1 w|^2_{H^1(\Omega_1)} \le C\|w\|^2_{H^{1/2}_{00}(\gamma_1)}.$$

Using again that the discrete harmonic extension has minimal energy, and estimate (36), we obtain

$$I_2 \le C \sum_{x_k \in \gamma_1^\delta} (I_\delta^M w)^2(x_k) \le \frac{C}{\delta}\|w\|^2_{L^2(\gamma_1)}.$$

The proof of the lemma follows immediately. ▯

REMARK 5.4. *This lemma is used only for Case R.*

6. Numerical experiments: Accuracy. To support the accuracy theory developed in the last few sections, we conduct some numerical experiments. We only consider Case R, and the problem domain is shown in Fig. 1. In all tests, we assume that the exact solution u has the form

$$u^*(x, y) = \left(\sin(\pi x) + \sin\left(\frac{\pi}{2}x\right)\right)\sin(\pi y)$$

and $\Omega = (0, 2) \times (0, 1)$. We denote $\Omega_1^0 = (0, 1) \times (0, 1)$, $\Omega_2^0 = (1, 2) \times (0, 1)$. and the computed solution $u = (u_1, u_2) \in V^h$. Let I_{h_i} be the pointwise piecewise linear interpolation operator in $\mathcal{T}^{h_i}$. The error that we report in this section is defined by

$$e = (e_1, e_2) = (I_{h_1}u^* - u_1, I_{h_2}u^* - u_2).$$

Our theory applies only to the H^1 norm, but three discrete norms L^2, L^∞ and H^1 are used to measure the numerical error. More precisely, we use

$$\|e\|_{L^2(\Omega)} = \sqrt{\|e_1\|^2_{L^2(\Omega_1^0)} + \|e_2\|^2_{L^2(\Omega_2^0)}}.$$

TABLE 1

The initial grid on Ω_1 is 6×5 and 5×4 on Ω_2. The element sizes are $h_1 = 0.2$ and $h_2 = 0.25$. $\delta = 0.45$. In row l, the number in ( ) is the ratio with the number in row $l-1$. The ratio indicates the order of the accuracy of the discretization.

	L^2	L^∞	H^1	$L^\infty(\nabla e)$
l=0	8.629D-02	0.1375	1.363	1.717
l=1	2.274D-02(3.79)	3.754D-02(3.66)	0.7108(1.92)	0.8686(1.98)
l=2	5.905D-03(3.85)	9.469D-03(3.96)	0.3569(1.99)	0.4346(2.00)
l=3	1.480D-03(3.99)	2.375D-03(3.99)	0.1785(2.00)	0.2172(2.00)
l=4	3.704D-04(4.00)	5.945D-04(3.99)	8.927D-02(2.00)	0.1086(2.00)
l=5	9.264D-05(4.00)	1.486D-04(4.00)	4.463D-02(2.00)	5.429D-02(2.00)

TABLE 2

We fix the refinement to $l = 5$, i.e, $h_1 = 0.2/32$ and $h_2 = 0.25/32$. The grids are $(160+ovlp)\times 160$ and $(128 + ovlp) \times 128$.

	L^2	L^∞	H^1	$L^\infty(\nabla e)$
$ovlp = 1$	9.159D-05	1.415D-04	4.462D-02	5.429D-02
$ovlp = 2$	9.158D-05	1.415D-04	4.463D-02	5.429D-02
$ovlp = 4$	9.170D-05	1.417D-04	4.462D-02	5.429D-02
$ovlp = 8$	9.190D-05	1.421D-04	4.462D-02	5.429D-02
$ovlp = 16$	9.220D-05	1.435D-04	4.463D-02	5.429D-02
$ovlp = 32$	9.264D-05	1.486D-04	4.463D-02	5.429D-02

Similarly, we can define $\|e\|_{H^1(\Omega)}$. $\|e\|_{L^\infty(\Omega)}$ is given as

$$\|e\|_{L^\infty(\Omega)} = \max\{\|e\|_{L^\infty(\Omega_1)}, \|e\|_{L^\infty(\Omega_2)}\}.$$

The refinement is done by simply cutting each triangle into four equal triangles. We use l to denote the level of refinement.

In the first test case, we take h_1 and h_2 close to each other. We choose $\Omega_1 = (0, 1.2) \times (0, 1)$ and $\Omega_2 = (0.75, 2) \times (0, 1)$. The overlapping size is fixed to $\delta = 0.45$. The initial mesh (i.e. $l = 0$) sizes are $h_1 = 0.2$ and $h_2 = 0.25$, which translate to two non-matching grids of 6×5 and 5×4. The results are summarized in Table 1. Five levels of uniform refinements are performed. One can see clearly that the method is of first order in $H^1(\Omega)$ and second order in $L^2(\Omega)$.

We next examine the dependence on the overlap. We fix the mesh sizes at $h_1 = 0.2/32$ and $h_2 = 0.25/32$, i.e. the refinement level $l = 5$. Let *ovlp* be an integer denoting the number of elements in the x direction in the overlapping region, we let *ovlp* go from 1 to 32. The results can be found in Table 2. As predicted in Theorem 1, the accuracy is independent of the overlap.

Table 3

We fix the overlap delta $=0.275$. *The initial grid is* 6×5 *and* 5×4. *The table below gives the error on* Ω_1 *and* Ω_2 *when we refine both grids uniformly with different level of refinement denoted by* l_{Ω_1} *and* l_{Ω_2}, *respectively.*

	L^2	L^∞	H^1	$L^\infty(\nabla e)$
	error in Ω_1			
$l_{\Omega_1}=3$, $l_{\Omega_2}=0$	3.059D-02	7.890D-02	1.538	0.6549
$l_{\Omega_1}=4$, $l_{\Omega_2}=1$	8.126D-03(3.76)	2.238D-02(3.52)	0.6592(2.33)	0.3942(1.66)
$l_{\Omega_1}=5$, $l_{\Omega_2}=2$	2.119D-03(3.83)	6.177D-03(3.62)	0.3070(2.14)	0.1417(2.78)
	error in Ω_2			
$l_{\Omega_1}=3$, $l_{\Omega_2}=0$	4.732D-02	9.488D-02	0.3460	1.2002
$l_{\Omega_1}=4$, $l_{\Omega_2}=1$	1.294D-02(3.66)	2.596D-02(3.65)	0.1754(1.97)	0.6110(1.96)
$l_{\Omega_1}=5$, $l_{\Omega_2}=2$	3.310D-03(3.91)	6.709D-03(3.87)	8.646D-02(2.03)	0.3095(1.97)

Instead of using the same level of refinement in both subdomains, we experiment with different level of refinement denoted by l_{Ω_1} and l_{Ω_2}. We also measure the error separately in Ω_1 and Ω_2. We start with the same initial mesh (6×5 and 5×4), and refine three times in each subdomain with levels equal to $l_{\Omega_1}=3,4,5$ and $l_{\Omega_2}=0,1,2$. The results are provided in Table 3.

7. Additive Schwarz preconditioners. The linear system of equations corresponding to (3) is usually large, sparse, symmetric positive definite and ill-conditioned. Preconditioning is necessary if iterative methods are used to solve it. In this section, we introduce several additive Schwarz preconditioners. A good introduction on the abstract additive Schwarz method (ASM) and its theory can be found in the book by Smith, Bjørstad and Gropp [28]. The key element of the abstract ASM theory is the introduction of a bounded decomposition of the finite element solution space V^h. Three such decompositions will be discussed in this section. Some numerical results are given at the end to support our theory.

7.1. An additive Schwarz method based on the harmonic extension (ASHE). We first introduce a method that uses discrete harmonic extensions in the overlapping region. The subspace decomposition is given by

$$V^h = \mathcal{I}_1 V_1 + \mathcal{I}_2 V_2, \quad V_1 = V_0^{h_1}(\Omega_1), \quad V_2 = V_0^{h_2}(\Omega_2),$$

where the interpolation operator $\mathcal{I}_1 : V_0^{h_1}(\Omega_1) \to V^h(\Omega)$ is given as follows: For $v_1 \in V_0^{h_1}(\Omega_1)$, we define $\mathcal{I}_1 v_1 \in V^h(\Omega)$ by

$$\mathcal{I}_1 v_1 = \begin{cases} v_1 & in \quad \Omega_1(\text{interior, zero on } \gamma_1) \\ \pi_2 v_1 & on \quad \gamma_2 \\ \mathcal{H}_{12}^{h_2}\pi_2 v_1 & in \quad \Omega_2 \end{cases}$$

and the interpolation operator $\mathcal{I}_2 : V_0^{h_2}(\Omega_2) \to V^h(\Omega)$ is given as follows: For $v_2 \in V_0^{h_2}(\Omega_2)$, we define $\mathcal{I}_2 v_2 \in V^h(\Omega)$ by

$$\mathcal{I}_2 v_2 = \begin{cases} v_2 & in \quad \Omega_2 (\text{interior, zero on } \gamma_2) \\ \pi_1 v_2 & on \quad \gamma_1 \\ \mathcal{H}_{12}^{h_1} \pi_1 v_2 & in \quad \Omega_1. \end{cases}$$

Let the bilinear forms $b_i(u_i, v_i) : V_0^{h_i}(\Omega_i) \times V_0^{h_i}(\Omega_i) \to \Re$, $i = 1, 2$, be defined by

$$b_i(u_i, v_i) = a_i(u_i, v_i) \equiv \int_{\Omega_i} \nabla u_i \cdot \nabla v_i \, dx. \tag{38}$$

The subspace projection operator $\tilde{T}_i : V^h(\Omega) \to V_0^{h_i}(\Omega_i), i = 1, 2$, satisfies

$$b_i(\tilde{T}_i u, v) = a_h(u, \mathcal{I}_i v), \qquad \forall v \in V_0^{h_i}(\Omega_i).$$

Now we define the operator $T_i = \mathcal{I}_i \tilde{T}_i : V^h(\Omega) \to V^h(\Omega)$, and let

$$T = T_1 + T_2.$$

To analyze the spectral condition of the operator T, we use the abstract ASM theory. The following lemma is a slightly modified version of the abstract ASM lemma in Smith, Bjørstad and Gropp [28], for two overlapping subregions with no coarse space.

LEMMA 9. *Suppose the following three assumptions hold:*

i) There exists a constant C_0 such that for all $u \in V^h(\Omega)$ there exists a decomposition $u = \sum_{i=1}^2 \mathcal{I}_i u_i, u_i \in V_0^{h_i}(\Omega_i)$, with

$$\sum_{i=1}^{2} b_i(u_i, u_i) \le C_0^2 a_h(u, u).$$

ii) There exist constants $\epsilon_{ij}, i, j = 1, 2$, such that

$$a_h(\mathcal{I}_i u_i, \mathcal{I}_j u_j) \le \epsilon_{ij} \, a_h(\mathcal{I}_i u_i, \mathcal{I}_i u_i)^{1/2} a_h(\mathcal{I}_j u_j, \mathcal{I}_j u_j)^{1/2}$$

$$\forall u_i \in V_0^{h_i}(\Omega_i) \quad \forall u_j \in V_0^{h_j}(\Omega_j).$$

iii) There exists a constant ω such that

$$a_h(\mathcal{I}_i u_i, \mathcal{I}_i u_i) \le \omega b_i(u_i, u_i) \quad \forall u_i \in V_0^{h_i}(\Omega_i), i = 1, 2.$$

Then, T is invertible, $a_h(Tu, v) = a_h(u, Tv), \quad \forall u, v \in V^h(\Omega)$, and

$$C_0^{-2} a_h(u, u) \le a_h(Tu, u) \le (\rho(\mathcal{E})\omega) a_h(u, u) \quad \forall u \in V^h(\Omega). \tag{39}$$

Here $\rho(\mathcal{E})$ is the spectral radius of $\mathcal{E}$, which is a 2×2 matrix made of $\{\epsilon_{ij}\}$.

We estimate the condition number of T in the next theorem. Both Case R and Case L are considered. For Case R, we define the overlapping size δ as usual, and for Case L, we assume that $\delta = O(1)$.

THEOREM 2. *Assume that Assumption 1 holds. Then,*

$$c\delta a_h(u,u) \le a_h(Tu,u) \le Ca_h(u,u), \quad \forall u \in V^h(\Omega),$$

where $c > 0$ and $C > 0$ are constants independent of h_i and δ. Therefore if the overlap is sufficiently large, i.e., $\delta = O(1)$, the preconditioner is optimal.

Proof. We follow the abstract theory stated in Lemma 9. We need only to verify the three assumptions.

Assumption i). Given $v = (v_1, v_2) \in V^h(\Omega)$, we define $u_i \in V_0^{h_i}(\Omega_i)$ as follows:

$$u_1 = v_1 - \mathcal{H}_{12}^{h_1} v_1 = v_1 - \mathcal{H}_{12}^{h_1}(\pi_1 v_2) \ \text{ in } \ \Omega_1,$$

and

$$u_2 = v_2 - \mathcal{H}_{12}^{h_2} v_2 = v_2 - \mathcal{H}_{12}^{h_2}(\pi_2 v_1) \ \text{ in } \ \Omega_2.$$

It is easy to check that $u_i \in V_0^{h_i}(\Omega_i)$ and that $v = \mathcal{I}_1 u_1 + \mathcal{I}_2 u_2$, since

$$\mathcal{I}_1 u_1 + \mathcal{I}_2 u_2 = \begin{cases} v_1 - \mathcal{H}_{12}^{h_1} v_1 & + & \mathcal{H}_{12}^{h_1}\pi_1 v_2 & = & v_1 & \text{in } \Omega_1 \\ \mathcal{H}_{12}^{h_2}\pi_2 v_1 & + & v_2 - \mathcal{H}_{12}^{h_2} v_2 & = & v_2 & \text{in } \Omega_2. \end{cases}$$

For $i = 1, 2$, we have

$$a_i(u_i,u_i) \le 2\left(a_i(v_i,v_i) + a_i(\mathcal{H}_{12}^{h_i} v_i, \mathcal{H}_{12}^{h_i} v_i)\right) \le \frac{C}{\delta} a_i(v_i,v_i). \tag{40}$$

To obtain the last inequality, we use Lemma 8 and the standard trace theorem

$$|\mathcal{H}_{12}^{h_i} v_i|^2_{H^1(\Omega_{12}^{h_i})} \le C \ \left(\|v_i\|^2_{H_{00}^{1/2}(\gamma_i)} + \frac{1}{\delta}\|v_i\|^2_{L^2(\gamma_i)}\right) \le \frac{C}{\delta} a_i(v_i,v_i).$$

Note that the above inequality holds for Case L with $\delta = O(1)$. From (40), we obtain $C_0^2 = C/\delta$, since

$$b_1(u_1,u_1) + b_2(u_2,u_2) \le \frac{C}{\delta} a_h(u,u).$$

Assumption ii). It is easy to see that $\rho(\mathcal{E}) \le 2$.

Assumption iii). We prove for $i = 1$. Let $u_1 \in V_0^{h_1}(\Omega_1)$. Then,

$$a_h(\mathcal{I}_1 u_1, \mathcal{I}_1 u_1) \le 2a_1(u_1,u_1) + 2a_2\left(\mathcal{H}_{12}^{h_2}(\pi_2 u_1), \mathcal{H}_{12}^{h_2}(\pi_2 u_1)\right).$$

To bound the second term, we again use Lemma 8, which implies that

$$|\mathcal{H}_{12}^{h_2}(\pi_2 u_1)|^2_{H^1(\Omega_{12}^{h_2})} \le C \ \left(\|\pi_2 u_1\|^2_{H_{00}^{1/2}(\gamma_2)} + \frac{1}{\delta}\|\pi_2 u_1\|^2_{L^2(\gamma_2)}\right).$$

To bound $\|\pi_2 u_1\|_{H_{00}^{1/2}(\gamma_2)}$, we apply the $H_{00}^{1/2}$ stability result of Lemma 5

$$\|\pi_2 u_1\|^2_{H_{00}^{1/2}(\gamma_2)} \le C\|u_1\|^2_{H_{00}^{1/2}(\gamma_2)} \le C a_1(u_1, u_1).$$

To bound $\|\pi_2 u_1\|_{L^2(\gamma_2)}$, we use the L^2 stability result of Lemma 5

$$\|\pi_2 u_1\|^2_{L^2(\gamma_2)} \le C\|u_1\|^2_{L^2(\gamma_2)}$$

and use the fact that u_1 vanishes on γ_1 and Lemma 6 we have

$$\|u_1\|^2_{L^2(\gamma_2)} \le C\delta\ b_1(u_1, u_1).$$

Therefore $\omega = C$, which appears in the above inequality. $\square$

REMARK 7.1. *We remark that if the overlap is sufficiently large, i.e., $\delta = O(1)$, then the algorithm is optimal in the sense that the convergence rate is independent of the mesh parameters h_1 and h_2. The large overlap condition is satisfied automatically for Case L.*

7.2. An additive Schwarz method based on the trivial extension (ASTE). We propose another additive Schwarz method in which the harmonic extension operator used in the previous subsection is replaced by a trivial zero extension. This method is computationally cheaper and easier to implement. Let us recall the definition of the trivial extension operators. For $i = 1, 2$, let $\mathcal{E}_i r^i : V^{h_i}(\gamma_i) \to V^{h_i}(\Omega_i)$ be the zero extension of r^i to Ω_i, i.e., $\mathcal{E}_i r^i = r^i$ at the nodes $a_2^i, a_3^i, \cdots, a_{m_i-1}^i$, and $\mathcal{E}_i r^i$ equals to zero at the remaining nodes of $\mathcal{T}^{h_i}$.

The subspace decomposition is given by

$$V^h = \hat{\mathcal{I}}_1 V_1 + \hat{\mathcal{I}}_2 V_2, \quad V_1 = V_0^{h_1}(\Omega_1), \quad V_2 = V_0^{h_2}(\Omega_2),$$

where the interpolation operator $\hat{\mathcal{I}}_1 : V_0^{h_1}(\Omega_1) \to V^h(\Omega)$ is given as follows: For $v_1 \in V_0^{h_1}(\Omega_1)$, we define $\hat{\mathcal{I}}_1 v_1 \in V^h(\Omega)$ by

$$\hat{\mathcal{I}}_1 v_1 = \begin{cases} v_1 & in \quad \Omega_1 \\ \pi_2 v_1 & on \quad \gamma_2 \\ \mathcal{E}_2 \pi_2 v_1 & in \quad \Omega_2 \end{cases}$$

and the interpolation operator $\hat{\mathcal{I}}_2 : V_0^{h_2}(\Omega_2) \to V^h(\Omega)$ is given as follows: For $v_2 \in V_0^{h_2}(\Omega_2)$, we define $\hat{\mathcal{I}}_2 v_2 \in V^h(\Omega)$ by

$$\hat{\mathcal{I}}_2 v_2 = \begin{cases} v_2 & in \quad \Omega_2 \\ \pi_1 v_2 & on \quad \gamma_1 \\ \mathcal{E}_1 \pi_1 v_2 & in \quad \Omega_1. \end{cases}$$

The bilinear forms $b_i(u_i, v_i) : V_0^{h_i}(\Omega_i) \times V_0^{h_i}(\Omega_i) \to \Re$, $,i = 1, 2$, are defined the same as in (38). We define the projection operator $\hat{T}_i : V^h(\Omega) \to V_0^{h_i}(\Omega_i), i = 1, 2,$ by

$$b_i(\hat{T}_i u, v) = a_h(u, \hat{\mathcal{I}}_i v), \qquad \forall v \in V_0^{h_i}(\Omega_i).$$

Now we define the operator $T_i = \hat{\mathcal{I}}_i \hat{T}_i : V^h(\Omega) \to V^h(\Omega)$, and let $T = T_1 + T_2$. The spectral bounds of T are estimated in the following theorem. Again, for Case L, we assume $\delta = O(1)$.

THEOREM 3. *Assume that Assumption 1 holds, and let $h = \min\{h_1, h_2\}$. Then,*

$$cha_h(u,u) \le a_h(Tu,u) \le C\ \frac{\delta}{h} a_h(u,u), \quad \forall u \in V^h(\Omega),$$

where $c > 0$ and $C > 0$ are constants independent of h_i and δ.

Proof. We only need to verify the assumptions in Lemma 9.

Assumption i). Given $v = (v_1, v_2) \in V^h(\Omega)$, we define $u_i \in V_0^{h_i}(\Omega_i)$ as follows:

$$u_1 = v_1 - \mathcal{E}_1 v_1 = v_1 - \mathcal{E}_1(\pi_1 v_2) \ \ in \ \ \Omega_1,$$

and

$$u_2 = v_2 - \mathcal{E}_2 v_2 = v_2 - \mathcal{E}_2(\pi_2 v_1) \ \ in \ \ \Omega_2.$$

It is easy to check that $u_i \in V_0^{h_i}(\Omega_i)$, and that $u = \hat{\mathcal{I}}_1 u_1 + \hat{\mathcal{I}}_2 u_2$. It is straightforward to show that

$$b_i(u_i, u_i) \le \frac{C}{h_i} a_i(v_i, v_i) \le \frac{C}{h_i} a_h(v, v)$$

and therefore $C_0^2 = C/h$.

Assumption ii). It is easy to see that $\rho(\mathcal{E}) \le 2$.

Assumption iii). We only discuss the case $i = 1$. Let $u_1 \in V_0^{h_1}(\Omega_1)$. Then,

$$a_h(\hat{\mathcal{I}}_1 u_1, \hat{\mathcal{I}}_1 u_1) \le 2\left(a_1(u_1, u_1) + a_2(\mathcal{E}_2(\pi_2 u_1), \mathcal{E}_2(\pi_2 u_1))\right).$$

Using an inverse inequality and the L^2 stability result of Lemma 5, we obtain

$$a_2\left(\mathcal{E}_2(\pi_2 u_1), \mathcal{E}_2(\pi_2 u_1)\right) \le \frac{C}{h_2}\ \|\pi_2 u_1\|^2_{L^2(\gamma_2)} \le \frac{C}{h_2}\ \|u_1\|^2_{L^2(\gamma_2)}.$$

Recall the fact that $u_1 = 0$ on γ_1, and using Lemma 6, we have

$$\|u_1\|^2_{L^2(\gamma_2)} \le C\delta |u_1|^2_{H^1(\Omega_1 \cap \Omega_2)}.$$

Note that for Case L, δ can be replaced by 1. Therefore,

$$a_h(\hat{\mathcal{I}}_1 u_1, \hat{\mathcal{I}}_1 u_1) \le C\ \frac{\delta}{h_2} b_1(u_1, u_1).$$

Similarly, we can get

$$a_h(\hat{\mathcal{I}}_2 u_2, \hat{\mathcal{I}}_2 u_2) \le C\ \frac{\delta}{h_1} b_2(u_2, u_2).$$

Thus, we can take $\omega = C\ \delta/h$. $\square$

Remark 7.2. *The algorithm is not optimal, and both lower and upper bounds are dependent on h and the overlapping size δ. But, the algorithm is easy to implement. A slightly improved version of the algorithm is given in the next subsection. A comparison with ASHE is given in the numerical examples section.*

Remark 7.3. *The upper bound depends on δ in a rather bad way, i.e., it increases when the overlap increases. This also shows up in the numerical examples.*

Remark 7.4. *We note however that the lower bound for Case R can be improved from Ch to $Ch/(1-\delta)$ for large overlap. For the proof we use (32) to obtain*

$$|\mathcal{E}_1 v_1|^2_{H^1(\Omega_1)} \leq C\frac{1}{h_1}\|v_1\|^2_{L^2(\gamma_1)} \leq C\frac{1}{h_1}\|v_2\|^2_{L^2(\gamma_1)} \leq C\frac{1-\delta}{h_1}\|v_2\|_{H^1(\Omega\backslash\Omega_1)}.$$

7.3. A method based on a modified trivial extension (ASTE1). Both of the upper and lower bounds of ASTE depend on the mesh parameters. Here we propose a modification of the bilinear form $b_i(\cdot,\cdot)$ and as a result the upper bound becomes independent of the mesh parameters. We assume the subspace decomposition is the same as in the previous subsection. Here we modify the bilinear forms, i.e., $b_i(u_i, v_i) : V_0^{h_i}(\Omega_i) \times V_0^{h_i}(\Omega_i) \to \Re$, $,i = 1, 2$, are now defined by:

$$b_1(u_1, u_1) \equiv \left(1 + \frac{h_1}{h_2}\right) a_1(u_1, u_1) + \frac{h_1}{h_2} \sum_{x\in D_2^{h_1}} u_1^2(x),$$

and

$$b_2(u_2, u_2) \equiv \left(1 + \frac{h_2}{h_1}\right) a_2(u_2, u_2) + \frac{h_2}{h_1} \sum_{x\in D_1^{h_2}} u_2^2(x).$$

Here $D_j^{h_i}$ $(i \neq j)$ denotes the set of mesh points x in the triangulation $\mathcal{T}^{h_i}$, such that

$$supp(x) \cap \gamma_j \neq \emptyset.$$

We define the projection operator $\tilde{T}_i : V^h(\Omega) \to V_0^{h_i}(\Omega_i), i = 1, 2$, by

$$b_i(\tilde{T}_i u, v) = a_h(u, \hat{\mathcal{I}}_i v), \quad \forall v \in V_0^{h_i}(\Omega_i).$$

Now we define the operator $T_i = \hat{\mathcal{I}}_i \tilde{T}_i : V^h(\Omega) \to V^h(\Omega)$, and let $T = T_1 + T_2$.

Theorem 4. *Assume that Assumption 1 holds, then*

$$c\left(\frac{1}{h_1} + \frac{1}{h_2}\right)^{-1} a_h(u, u) \leq a_h(Tu, u) \leq Ca_h(u, u), \quad \forall u \in V^h(\Omega),$$

where $c > 0$ and $C > 0$ are constants independent of h_i and δ.

Proof. We exam the assumptions in Lemma 9.

Assumption i) Given $v = (v_1, v_2) \in V^h(\Omega)$ we define $u_i \in V_0^{h_i}(\Omega_i)$ as follows:

$$u_1 = v_1 - \mathcal{E}_1 v_1 = v_1 - \mathcal{E}_1(\pi_1 v_2) \;\; in \;\; \Omega_1,$$

and

$$u_2 = v_2 - \mathcal{E}_2 v_2 = v_2 - \mathcal{E}_2(\pi_2 v_1) \ \ in \ \ \Omega_2.$$

We have

$$b_1(u_1,u_1) \le 2\left(1+\frac{h_1}{h_2}\right)(a_1(v_1,v_1)+a_1(\mathcal{E}_1 v_1,\mathcal{E}_1 v_1)) + \frac{h_1}{h_2}\sum_{D_2^{h_1}} v_1^2(x)$$

$$\le C\left(1+\frac{h_1}{h_2}\right)\left(a_1(v_1,v_1)+\frac{1}{h_1}||v_1||^2_{L^2(\gamma_1)}\right)$$

$$+\frac{C}{h_2}\left(\|v_1\|^2_{L^2(\gamma_2^-)}+\|v_1\|^2_{L^2(\gamma_2^+)}\right),$$

where γ_2^+ and γ_2^- are the lines parallel to γ_2 and contain the nodal points of $D_2^{h_1}$. Using the standard trace theorem, we have

$$b_1(u_1,u_1) \le C\left(\frac{1}{h_1}+\frac{1}{h_2}\right)a_1(v_1,v_1).$$

And similarly

$$b_2(u_2,u_2) \le C\left(\frac{1}{h_1}+\frac{1}{h_2}\right)a_2(v_2,v_2).$$

Adding these estimates, we get

$$b_1(u_1,u_1)+b_2(u_2,u_2) \le C\left(\frac{1}{h_1}+\frac{1}{h_2}\right)a_h(u,u).$$

Therefore, $C_0^2 = C(\frac{1}{h_1}+\frac{1}{h_2})$.

Assumption ii). $\rho(\mathcal{E}) \le 2$.

Assumption (iii). For $u_1 \in V_0^{h_1}(\Omega_1)$, and using the L^2 stability of Lemma 5

$$a_h(\hat{\mathcal{I}}_1 u_1,\hat{\mathcal{I}}_1 u_1) \le 2a_1(u_1,u_1)+C\frac{1}{h_2}||u_1||^2_{L^2(\gamma_2)}.$$

Now we use inequality (33) for a strip $D_2^{h_1}$ of width $2h_1$, i.e.,

$$||u_1||^2_{L^2(\gamma_2)} \le C\left(h_1|u_1|^2_{H^1(D_2^{h_1})}+\frac{1}{h_1}||u_1||^2_{L^2(D_2^{h_1})}\right)$$

to obtain

$$a_h(\hat{\mathcal{I}}_1 u_1,\hat{\mathcal{I}}_1 u_1) \le C\left(\left(1+\frac{h_1}{h_2}\right)a_1(u_1,u_1)+\frac{h_1}{h_2}\sum_{D_2^{h_1}} u_1^2(x)\right) = Cb_1(u_1,u_1).$$

Similarly, we have

$$a_h(\hat{\mathcal{I}}_2 u_1,\hat{\mathcal{I}}_2 u_2) \le Cb_2(u_2,u_2).$$

Thus, we obtain $\omega = C$. ▯

Remark 7.5. *Note that the bounds appear in the lemma are independent of the overlapping parameter δ, even for Case R. Numerical examples given in the next section indeed show that increasing overlap does not decrease the number of iterations.*

8. Numerical Results: Preconditioning. In this section, we present some numerical results concerning the convergence rate of the preconditioned conjugate gradient(PCG) methods. We are particularly interested in the dependence of the algorithms on the mesh parameters h_1 and h_2, and the overlapping size δ. All tests are for Case R.

In Table 4, we present the number of PCG iterations and the condition number of the preconditioned system for each of the three algorithms, plus the case when no preconditioner is used. We stop the iteration when the initial preconditioned residual is reduced by a factor of 10^{-12}. The initial grids are 6×5 and 5×4, and the grids are refined simultaneously for up to $l = 5$ times. The overlapping size is fixed at $\delta = 0.45$. It can be seen clearly that the number of iterations for ASHE stays as a constant, however all other methods have some dependence of the refinement level. The modified method ASTE1 is considerably better than ASTE.

TABLE 4

A comparison of four methods in terms of the iteration numbers and condition numbers, given in ( ). The initial grids are 6×5 and 5×4. The overlap is fixed at $\delta = 0.45$. l is the level of refinement.

	no prec	ASHE	ASTE	ASTE1
$l = 0$	27(15.8)	14(3.0)	17(3.7)	19 (3.8)
$l = 1$	60(73.5)	14(2.2)	22(6.5)	21(5.5)
$l = 2$	121(310.95)	14(2.6)	28(14.8)	26(9.4)
$l = 3$	241(1270.)	14(2.5)	37(38.2)	31(17.3)
$l = 4$	472(5132)	13(2.5)	54(118.4)	39(33.1)
$l = 5$	916(20621)	13(2.5)	85(404.4)	52(64.6)

In the second set of tests, we fix the mesh sizes and vary the overlapping parameter δ. As predicted by our theory, ASHE gets better when the overlap becomes larger. The other two preconditioners do not share this property. The results can be found in Table 5. We should mention that although ASTE and ASTE1 do not perform as well as ASHE they still have practical value since they are much easier to implement.

9. Concluding remarks. In the first part of the paper, we introduce a mortar finite element method defined on overlapping non-matching grids. An optimal accuracy theory is provided for the two-subdomain cases. When a geometrical condition is satisfied we prove that the accuracy is independent of the overlap, as well as the ratio of the subdomain mesh sizes. In the second part of the paper, we study three additive overlapping Schwarz preconditioning techniques. One of the preconditioners, based on the local harmonic extension, is optimal in the sense that the convergence rate of the corresponding PCG method is independent of the mesh parameters h_1 and h_2. Much more work needs to be done in the area of overlapping mortar element methods, such as extending the methods and theory to the case when more than two subdomains overlap, and to three dimensional problems.

TABLE 5

Verifying the overlapping size. The mesh sizes are $h_1 = 0.2/2^5$ and $h_2 = 0.25/2^5$. The actual meshes are $(160 + ovlp) \times 160$ and $(128 + ovlp) \times 128$. Note that $ovlp = 32$ is the same as $\delta = 0.45$.

	no prec	ASHE	ASTE	ASTE1
$ovlp = 1$	751(14418)	50(74.4)	61(116.0)	44(101.0)
$ovlp = 2$	759(14585)	32(27.3)	65(158.8)	53(99.0)
$ovlp = 4$	774(14937)	22(12.6)	70(230.4)	49(95.5)
$ovlp = 8$	788(15702)	17(6.1)	74(318.2)	49(88.2)
$ovlp = 16$	809(17364)	15(3.3)	79(396.4)	48(77.6)
$ovlp = 32$	916(20621)	13(2.5)	85(404.4)	52(64.6)

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